

Solutions of Selected Problems from *Probability Essentials, Second Edition*

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 2

2.1 Let's first prove by induction that $\#(2^{\Omega_n}) = 2^n$ if $\Omega = \{x_1, \dots, x_n\}$. For $n = 1$ it is clear that $\#(2^{\Omega_1}) = \#(\{\emptyset, \{x_1\}\}) = 2$. Suppose $\#(2^{\Omega_{n-1}}) = 2_{n-1}$. Observe that $2^{\Omega_n} = \{\{x_n\} \cup A, A \in 2^{\Omega_{n-1}}\} \cup 2^{\Omega_{n-1}}$ hence $\#(2^{\Omega_n}) = 2\#(2^{\Omega_{n-1}}) = 2^n$. This proves finiteness. To show that 2^Ω is a σ -algebra we check:

1. $\emptyset \subset \Omega$ hence $\emptyset \in 2^\Omega$.
2. If $A \in 2^\Omega$ then $A \subset \Omega$ and $A^c \subset \Omega$ hence $A^c \in 2^\Omega$.
3. Let $(A_n)_{n \geq 1}$ be a sequence of subsets of Ω . Then $\bigcup_{n=1}^\infty A_n$ is also a subset of Ω hence in 2^Ω .

Therefore 2^Ω is a σ -algebra.

2.2 We check if $\mathcal{H} = \bigcap_{\alpha \in A} \mathcal{G}_\alpha$ has the three properties of a σ -algebra:

1. $\emptyset \in \mathcal{G}_\alpha \forall \alpha \in A$ hence $\emptyset \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$.
2. If $B \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$ then $B \in \mathcal{G}_\alpha \forall \alpha \in A$. This implies that $B^c \in \mathcal{G}_\alpha \forall \alpha \in A$ since each \mathcal{G}_α is a σ -algebra. So $B^c \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$.
3. Let $(A_n)_{n \geq 1}$ be a sequence in \mathcal{H} . Since each $A_n \in \mathcal{G}_\alpha$, $\bigcup_{n=1}^\infty A_n$ is in \mathcal{G}_α since \mathcal{G}_α is a σ -algebra for each $\alpha \in A$. Hence $\bigcup_{n=1}^\infty A_n \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$.

Therefore $\mathcal{H} = \bigcap_{\alpha \in A} \mathcal{G}_\alpha$ is a σ -algebra.

2.3 a. Let $x \in (\bigcup_{n=1}^\infty A_n)^c$. Then $x \in A_n^c$ for all n , hence $x \in \bigcap_{n=1}^\infty A_n^c$. So $(\bigcup_{n=1}^\infty A_n)^c \subset \bigcap_{n=1}^\infty A_n^c$. Similarly if $x \in \bigcap_{n=1}^\infty A_n^c$ then $x \in A_n^c$ for any n hence $x \in (\bigcup_{n=1}^\infty A_n)^c$. So $(\bigcup_{n=1}^\infty A_n)^c = \bigcap_{n=1}^\infty A_n^c$.

b. By part-a $\bigcap_{n=1}^\infty A_n = (\bigcup_{n=1}^\infty A_n^c)^c$, hence $(\bigcap_{n=1}^\infty A_n)^c = \bigcup_{n=1}^\infty A_n^c$.

2.4 $\liminf_{n \rightarrow \infty} A_n = \bigcup_{n=1}^\infty B_n$ where $B_n = \bigcap_{m \geq n} A_m \in \mathcal{A} \forall n$ since \mathcal{A} is closed under taking countable intersections. Therefore $\liminf_{n \rightarrow \infty} A_n \in \mathcal{A}$ since \mathcal{A} is closed under taking countable unions.

By De Morgan's Law it is easy to see that $\limsup_{n \rightarrow \infty} A_n = (\liminf_{n \rightarrow \infty} A_n^c)^c$, hence $\limsup_{n \rightarrow \infty} A_n \in \mathcal{A}$ since $\liminf_{n \rightarrow \infty} A_n^c \in \mathcal{A}$ and \mathcal{A} is closed under taking complements.

Note that $x \in \liminf_{n \rightarrow \infty} A_n \Rightarrow \exists n^* \text{ s.t } x \in \bigcap_{m \geq n^*} A_m \Rightarrow x \in \bigcap_{m \geq n} A_m \forall n \Rightarrow x \in \limsup_{n \rightarrow \infty} A_n$. Therefore $\liminf_{n \rightarrow \infty} A_n \subset \limsup_{n \rightarrow \infty} A_n$.

2.8 Let $\mathcal{L} = \{B \subset \mathbf{R} : f^{-1}(B) \in \mathcal{B}\}$. It is easy to check that \mathcal{L} is a σ -algebra. Since f is continuous $f^{-1}(B)$ is open (hence Borel) if B is open. Therefore \mathcal{L} contains the open sets which implies $\mathcal{L} \supset \mathcal{B}$ since \mathcal{B} is generated by the open sets of \mathbf{R} . This proves that $f^{-1}(B) \in \mathcal{B}$ if $B \in \mathcal{B}$ and that $\mathcal{A} = \{A \subset \mathbf{R} : \exists B \in \mathcal{B} \text{ with } A = f^{-1}(B) \in \mathcal{B}\} \subset \mathcal{B}$.

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 3

3.7 a. Since $P(B) > 0$ $P(\cdot|B)$ defines a probability measure on \mathcal{A} , therefore by Theorem 2.4 $\lim_{n \rightarrow \infty} P(A_n|B) = P(A|B)$.

b. We have that $A \cap B_n \rightarrow A \cap B$ since $\mathbf{1}_{A \cap B_n}(w) = \mathbf{1}_A(w)\mathbf{1}_{B_n}(w) \rightarrow \mathbf{1}_A(w)\mathbf{1}_B(w)$. Hence $P(A \cap B_n) \rightarrow P(A \cap B)$. Also $P(B_n) \rightarrow P(B)$. Hence

$$P(A|B_n) = \frac{P(A \cap B_n)}{P(B_n)} \rightarrow \frac{P(A \cap B)}{P(B)} = P(A|B).$$

c.

$$P(A_n|B_n) = \frac{P(A_n \cap B_n)}{P(B_n)} \rightarrow \frac{P(A \cap B)}{P(B)} = P(A|B)$$

since $A_n \cap B_n \rightarrow A \cap B$ and $B_n \rightarrow B$.

3.11 Let $B = \{x_1, x_2, \dots, x_b\}$ and $R = \{y_1, y_2, \dots, y_r\}$ be the sets of b blue balls and r red balls respectively. Let $B' = \{x_{b+1}, x_{b+2}, \dots, x_{b+d}\}$ and $R' = \{y_{r+1}, y_{r+2}, \dots, y_{r+d}\}$ be the sets of d -new blue balls and d -new red balls respectively. Then we can write down the sample space Ω as

$$\Omega = \{(a, b) : (a \in B \text{ and } b \in B \cup B' \cup R) \text{ or } (a \in R \text{ and } b \in R \cup R' \cup B)\}.$$

Clearly $\text{card}(\Omega) = b(b+d+r) + r(b+d+r) = (b+r)(b+d+r)$. Now we can define a probability measure P on 2^Ω by

$$P(A) = \frac{\text{card}(A)}{\text{card}(\Omega)}.$$

a. Let

$$\begin{aligned} A &= \{\text{second ball drawn is blue}\} \\ &= \{(a, b) : a \in B, b \in B \cup B'\} \cup \{(a, b) : a \in R, b \in B\} \end{aligned}$$

$$\text{card}(A) = b(b+d) + rb = b(b+d+r), \text{ hence } P(A) = \frac{b}{b+r}.$$

b. Let

$$\begin{aligned} B &= \{\text{first ball drawn is blue}\} \\ &= \{(a, b) \in \Omega : a \in B\} \end{aligned}$$

Observe $A \cap B = \{(a, b) : a \in B, b \in B \cup B'\}$ and $\text{card}(A \cap B) = b(b+d)$. Hence

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\text{card}(A \cap B)}{\text{card}(A)} = \frac{b+d}{b+d+r}.$$

3.17 We will use the inequality $1 - x > e^{-x}$ for $x > 0$, which is obtained by taking Taylor's expansion of e^{-x} around 0.

$$\begin{aligned} P((A_1 \cup \dots \cup A_n)^c) &= P(A_1^c \cap \dots \cap A_n^c) \\ &= (1 - P(A_1)) \dots (1 - P(A_n)) \\ &\leq \exp(-P(A_1)) \dots \exp(-P(A_n)) = \exp\left(-\sum_{i=1}^n P(A_i)\right) \end{aligned}$$

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 4

4.1 Observe that

$$\begin{aligned} P(k \text{ successes}) &= \binom{n}{k} \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= C a_n b_{1,n} \cdots b_{k,n} d_n \end{aligned}$$

where

$$C = \frac{\lambda^k}{k!} \quad a_n = \left(1 - \frac{\lambda}{n}\right)^n \quad b_{j,n} = \frac{n-j+1}{n} \quad d_n = \left(1 - \frac{\lambda}{n}\right)^{-k}$$

It is clear that $b_{j,n} \rightarrow 1 \forall j$ and $d_n \rightarrow 1$ as $n \rightarrow \infty$. Observe that

$$\log\left(\left(1 - \frac{\lambda}{n}\right)^n\right) = n\left(\frac{\lambda}{n} - \frac{\lambda^2}{n^2} \frac{1}{\xi^2}\right) \text{ for some } \xi \in \left(1 - \frac{\lambda}{n}, 1\right)$$

by Taylor series expansion of $\log(x)$ around 1. It follows that $a_n \rightarrow e^{-\lambda}$ as $n \rightarrow \infty$ and that

$$|\text{Error}| = |e^{n \log(1 - \frac{\lambda}{n})} - e^{-\lambda}| \geq |n \log(1 - \frac{\lambda}{n}) - \lambda| = n \frac{\lambda^2}{n^2} \frac{1}{\xi^2} \geq \lambda p$$

Hence in order to have a good approximation we need n large and p small as well as λ to be of moderate size.

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 5

5.7 We put $x_n = P(X \text{ is even})$ for $X \sim B(p, n)$. Let us prove by induction that $x_n = \frac{1}{2}(1 + (1 - 2p)^n)$. For $n = 1$, $x_1 = 1 - p = \frac{1}{2}(1 + (1 - 2p)^1)$. Assume the formula is true for $n - 1$. If we condition on the outcome of the first trial we can write

$$\begin{aligned} x_n &= p(1 - x_{n-1}) + (1 - p)x_n \\ &= p\left(1 - \frac{1}{2}(1 + (1 - 2p)^{n-1})\right) + (1 - p)\left(\frac{1}{2}(1 + (1 - 2p)^{n-1})\right) \\ &= \frac{1}{2}(1 + (1 - 2p)^n) \end{aligned}$$

hence we have the result.

5.11 Observe that $E(|X - \lambda|) = \sum_{i < \lambda} (\lambda - i)p_i + \sum_{i \geq \lambda} (i - \lambda)p_i$. Since $\sum_{i \geq \lambda} (i - \lambda)p_i = \sum_{i=0}^{\infty} (i - \lambda)p_i - \sum_{i < \lambda} (i - \lambda)p_i$ we have that $E(|X - \lambda|) = 2 \sum_{i < \lambda} (\lambda - i)p_i$. So

$$\begin{aligned} E(|X - \lambda|) &= 2 \sum_{i < \lambda} (\lambda - i)p_i \\ &= 2 \sum_{i=1}^{\lambda-1} (\lambda - i) \frac{e^{-\lambda} \lambda^k}{k!} \\ &= 2e^{-\lambda} \sum_{i=0}^{\lambda-1} \left(\frac{\lambda^{k+1}}{k!} - \frac{\lambda^k}{(k-1)!} \right) \\ &= 2e^{-\lambda} \frac{\lambda^\lambda}{(k-1)!}. \end{aligned}$$

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 7

7.1 Suppose $\lim_{n \rightarrow \infty} P(A_n) \neq 0$. Then there exists $\epsilon > 0$ such that there are distinct A_{n_1}, A_{n_2}, \dots with $P(A_{n_k}) > 0$ for every $k \leq 1$. This gives $\sum_{k=1}^{\infty} P(A_{n_k}) = \infty$ which is a contradiction since by the hypothesis that the A_n are disjoint we have that $\sum_{k=1}^{\infty} P(A_{n_k}) = P(\cup_{n=1}^{\infty} A_{n_k}) \leq 1$.

7.2 Let $\mathcal{A}_n = \{A_\beta : P(A_\beta) > 1/n\}$. \mathcal{A}_n is a finite set otherwise we can pick disjoint $A_{\beta_1}, A_{\beta_2}, \dots$ in \mathcal{A}_n . This would give us $P(\cup_{m=1}^{\infty} A_{\beta_m}) = \sum_{m=1}^{\infty} P(A_{\beta_m}) = \infty$ which is a contradiction. Now $\{A_\beta : \beta \in B\} = \cup_{n=1}^{\infty} \mathcal{A}_n$ hence $(A_\beta)_{\beta \in B}$ is countable since it is a countable union of finite sets.

7.11 Note that $\{x_0\} = \cap_{n=1}^{\infty} [x_0 - 1/n, x_0]$ therefore $\{x_0\}$ is a Borel set. $P(\{x_0\}) = \lim_{n \rightarrow \infty} P([x_0 - 1/n, x_0])$. Assuming that f is continuous we have that f is bounded by some M on the interval $[x_0 - 1/n, x_0]$ hence $P(\{x_0\}) = \lim_{n \rightarrow \infty} M(1/n) = 0$.

Remark: In order this result to be true we don't need f to be continuous. When we define the Lebesgue integral (or more generally integral with respect to a measure) and study its properties we will see that this result is true for all Borel measurable non-negative f .

7.16 First observe that $F(x) - F(x-) > 0$ iff $P(\{x\}) > 0$. The family of events $\{\{x\} : P(\{x\}) > 0\}$ can be at most countable as we have proven in problem 7.2 since these events are disjoint and have positive probability. Hence F can have at most countable discontinuities. For an example with infinitely many jump discontinuities consider the Poisson distribution.

7.18 Let F be as given. It is clear that F is a nondecreasing function. For $x < 0$ and $x \geq 1$ right continuity of F is clear. For any $0 < x < 1$ let i^* be such that $\frac{1}{i^*+1} \leq x < \frac{1}{i^*}$. If $x_n \downarrow x$ then there exists N such that $\frac{1}{i^*+1} \leq x_n < \frac{1}{i^*}$ for every $n \geq N$. Hence $F(x_n) = F(x)$ for every $n \geq N$ which implies that F is right continuous at x . For $x = 0$ we have that $F(0) = 0$. Note that for any ϵ there exists N such that $\sum_{i=N}^{\infty} \frac{1}{2^i} < \epsilon$. So for all x s.t. $|x| \leq \frac{1}{N}$ we have that $F(x) \leq \epsilon$. Hence $F(0+) = 0$. This proves the right continuity of F for all x . We also have that $F(\infty) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$ and $F(-\infty) = 0$ so F is a distribution function of a probability on \mathbf{R} .

- a. $P([1, \infty)) = F(\infty) - F(1-) = 1 - \sum_{n=2}^{\infty} \frac{1}{2^n} = 1 - \frac{1}{2} = \frac{1}{2}$.
- b. $P([\frac{1}{10}, \infty)) = F(\infty) - F(\frac{1}{10}-) = 1 - \sum_{n=11}^{\infty} \frac{1}{2^n} = 1 - 2^{-10}$.
- c. $P(\{0\}) = F(0) - F(0-) = 0$.
- d. $P([0, \frac{1}{2})) = F(\frac{1}{2}-) - F(0-) = \sum_{n=3}^{\infty} \frac{1}{2^n} - 0 = \frac{1}{4}$.
- e. $P((-\infty, 0)) = F(0-) = 0$.
- f. $P((0, \infty)) = 1 - F(0) = 1$.

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 9

9.1 It is clear by the definition of \mathcal{F} that $X^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{B}$. So X is measurable from (Ω, \mathcal{F}) to $(\mathbf{R}, \mathcal{B})$.

9.2 Since X is both \mathcal{F} and \mathcal{G} measurable for any $B \in \mathcal{B}$, $P(X \in B) = P(X \in B)P(X \in B) = 0$ or 1 . Without loss of generality we can assume that there exists a closed interval I such that $P(I) = 1$. Let $\Lambda_n = \{t_0^n, \dots, t_{l_n}^n\}$ be a partition of I such that $\Lambda_n \subset \Lambda_{n+1}$ and $\sup_k t_k^n - t_{k-1}^n \rightarrow 0$. For each n there exists $k^*(n)$ such that $P(X \in [t_{k^*}^n, t_{k^*+1}^n]) = 1$ and $[t_{k^*(n+1)}^n, t_{k^*(n+1)+1}^n] \subset [t_{k^*(n)}^n, t_{k^*(n)+1}^n]$. Now $a_n = t_{k^*(n)}^n$ and $b_n = t_{k^*(n)}^n + 1$ are both Cauchy sequences with a common limit c . So $1 = \lim_{n \rightarrow \infty} P(X \in (t_{k^*}^n, t_{k^*+1}^n]) = P(X = c)$.

9.3 $X^{-1}(A) = (Y^{-1}(A) \cap (Y^{-1}(A) \cap X^{-1}(A)^c)^c) \cup (X^{-1}(A) \cap Y^{-1}(A)^c)$. Observe that both $Y^{-1}(A) \cap (X^{-1}(A)^c)$ and $X^{-1}(A) \cap Y^{-1}(A)^c$ are null sets and therefore measurable. Hence if $Y^{-1}(A) \in \mathcal{A}'$ then $X^{-1}(A) \in \mathcal{A}'$. In other words if Y is \mathcal{A}' measurable so is X .

9.4 Since X is integrable, for any $\epsilon > 0$ there exists M such that $\int |X| \mathbf{1}_{\{X > M\}} dP < \epsilon$ by the dominated convergence theorem. Note that

$$\begin{aligned} E[X \mathbf{1}_{A_n}] &= E[X \mathbf{1}_{A_n} \mathbf{1}_{\{X > M\}}] + E[X \mathbf{1}_{A_n} \mathbf{1}_{\{X \leq M\}}] \\ &\leq E[|X| \mathbf{1}_{\{X > M\}}] + MP(A_n) \end{aligned}$$

Since $P(A_n) \rightarrow 0$, there exists N such that $P(A_n) \leq \frac{\epsilon}{M}$ for every $n \geq N$. Therefore $E[X \mathbf{1}_{A_n}] \leq \epsilon + \epsilon \forall n \geq N$, i.e. $\lim_{n \rightarrow \infty} E[X \mathbf{1}_{A_n}] = 0$.

9.5 It is clear that $0 \leq Q(A) \leq 1$ and $Q(\Omega) = 1$ since X is nonnegative and $E[X] = 1$. Let A_1, A_2, \dots be disjoint. Then

$$Q(\cup_{n=1}^{\infty} A_n) = E[X \mathbf{1}_{\cup_{n=1}^{\infty} A_n}] = E[\sum_{n=1}^{\infty} X \mathbf{1}_{A_n}] = \sum_{n=1}^{\infty} E[X \mathbf{1}_{A_n}]$$

where the last equality follows from the monotone convergence theorem. Hence $Q(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} Q(A_n)$. Therefore Q is a probability measure.

9.6 If $P(A) = 0$ then $X \mathbf{1}_A = 0$ a.s. Hence $Q(A) = E[X \mathbf{1}_A] = 0$. Now assume P is the uniform distribution on $[0, 1]$. Let $X(x) = 2 \mathbf{1}_{[0, 1/2]}(x)$. Corresponding measure Q assigns zero measure to $(1/2, 1]$, however $P((1/2, 1]) = 1/2 \neq 0$.

9.7 Let's prove this first for simple functions, i.e. let Y be of the form

$$Y = \sum_{i=1}^n c_i \mathbf{1}_{A_i}$$

for disjoint A_1, \dots, A_n . Then

$$E_Q[Y] = \sum_{i=1}^n c_i Q(A_i) = \sum_{i=1}^n c_i E[X \mathbf{1}_{A_i}] = E_P[XY]$$

For non-negative Y we take a sequence of simple functions $Y_n \uparrow Y$. Then

$$E_Q[Y] = \lim_{n \rightarrow \infty} E_Q[Y_n] = \lim_{n \rightarrow \infty} E_P[XY_n] = E_P[XY]$$

where the last equality follows from the monotone convergence theorem. For general $Y \in \mathbb{L}^1(Q)$ we have that $E_Q[Y] = E_Q[Y^+] - E_Q[Y^-] = E_P[(XY)^+] - E_Q[(XY)^-] = E_P[XY]$.

9.8 a. Note that $\frac{1}{X}X = 1$ a.s. since $P(X > 0) = 1$. By problem 9.7 $E_Q[\frac{1}{X}] = E_P[\frac{1}{X}X] = 1$. So $\frac{1}{X}$ is Q -integrable.

b. $R : \mathcal{A} \rightarrow \mathbf{R}$, $R(A) = E_Q[\frac{1}{X} \mathbf{1}_A]$ is a probability measure since $\frac{1}{X}$ is non-negative and $E_Q[\frac{1}{X}] = 1$. Also $R(A) = E_Q[\frac{1}{X} \mathbf{1}_A] = E_P[\frac{1}{X}X \mathbf{1}_A] = P(A)$. So $R = P$.

9.9 Since $P(A) = E_Q[\frac{1}{X} \mathbf{1}_A]$ we have that $Q(A) = 0 \Rightarrow P(A) = 0$. Now combining the results of the previous problems we can easily observe that $Q(A) = 0 \Leftrightarrow P(A) = 0$ iff $P(X > 0) = 1$.

9.17. Let

$$g(x) = \frac{((x - \mu)b + \sigma)^2}{\sigma^2(1 + b^2)^2}.$$

Observe that $\{X \geq \mu + b\sigma\} \in \{g(X) \geq 1\}$. So

$$P(\{X \geq \mu + b\sigma\}) \leq P(\{g(X) \geq 1\}) \leq \frac{E[g(X)]}{1}$$

where the last inequality follows from Markov's inequality. Since $E[g(X)] = \frac{\sigma^2(1+b^2)}{\sigma^2(1+b^2)^2}$ we get that

$$P(\{X \geq \mu + b\sigma\}) \leq \frac{1}{1 + b^2}.$$

9.19

$$\begin{aligned} xP(\{X > x\}) &\leq E[X \mathbf{1}_{\{X > x\}}] \\ &= \int_x^\infty \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \end{aligned}$$

Hence

$$P(\{X > x\}) \leq \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}}$$

9.21 $h(t+s) = P(\{X > t+s\}) = P(\{X > t+s, X > s\}) = P(\{X > t+s|X > s\})P(\{X > s\}) = h(t)h(s)$ for all $t, s > 0$. Note that this gives $h(\frac{1}{n}) = h(1)^{\frac{1}{n}}$ and $h(\frac{m}{n}) = h(1)^{\frac{m}{n}}$. So for all rational r we have that $h(r) = \exp(\log(h(1))r)$. Since h is right continuous this gives $h(x) = \exp(\log(h(1))x)$ for all $x > 0$. Hence X has exponential distribution with parameter $-\log h(1)$.

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 10

10.5 Let P be the uniform distribution on $[-1/2, 1/2]$. Let $X(x) = \mathbf{1}_{[-1/4, 1/4]}$ and $Y(x) = \mathbf{1}_{[-1/4, 1/4]^c}$. It is clear that $XY = 0$ hence $E[XY] = 0$. It is also true that $E[X] = 0$. So $E[XY] = E[X]E[Y]$ however it is clear that X and Y are not independent.

10.6 a. $P(\min(X, Y) > i) = P(X > i)P(Y > i) = \frac{1}{2^i} \frac{1}{2^i} = \frac{1}{4^i}$. So $P(\min(X, Y) \leq i) = 1 - P(\min(X, Y) > i) = 1 - \frac{1}{4^i}$.

b. $P(X = Y) = \sum_{i=1}^{\infty} P(X = i)P(Y = i) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{4}} - 1 = \frac{1}{3}$.

c. $P(Y > X) = \sum_{i=1}^{\infty} P(Y > i)P(X = i) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{1}{2^i} = \frac{1}{3}$.

d. $P(X \text{ divides } Y) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^i} \frac{1}{2^{ki}} = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{1}{2^i - 1}$.

e. $P(X \geq kY) = \sum_{i=1}^{\infty} P(X \geq ki)P(Y = i) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{1}{2^{ki-1}} = \frac{2}{2^{k+1}-1}$.

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 11

11.11. Since $P\{X > 0\} = 1$ we have that $P\{Y < 1\} = 1$. So $F_Y(y) = 1$ for $y \geq 1$. Also $P\{Y \leq 0\} = 0$ hence $F_Y(y) = 0$ for $y \leq 0$. For $0 < y < 1$ $P\{Y > y\} = P\{X < \frac{1-y}{y}\} = F_X(\frac{1-y}{y})$. So

$$F_Y(y) = 1 - \int_0^{\frac{1-y}{y}} f_X(x)dx = 1 - \int_0^y \frac{-1}{z^2} f_X(\frac{1-z}{z})dz$$

by change of variables. Hence

$$f_Y(y) = \begin{cases} 0 & -\infty < y \leq 0 \\ \frac{1}{y^2} f_X(\frac{1-y}{y}) & 0 < y \leq 1 \\ 0 & 1 \leq y < \infty \end{cases}$$

11.15 Let $G(u) = \inf\{x : F(x) \geq u\}$. We would like to show $\{u : G(u) > y\} = \{u : F(Y) < u\}$. Let u be such that $G(u) > y$. Then $F(y) < u$ by definition of G . Hence $\{u : G(u) > y\} \subset \{u : F(Y) < u\}$. Now let u be such that $F(y) < u$. Then $y < x$ for any x such that $F(x) \geq u$ by monotonicity of F . Now by right continuity and the monotonicity of F we have that $F(G(u)) = \inf_{F(x) \geq u} F(x) \geq u$. Then by the previous statement $y < G(u)$. So $\{u : G(u) > y\} = \{u : F(Y) < u\}$. Now $P\{G(U) > y\} = P\{U > F(y)\} = 1 - F(y)$ so $G(U)$ has the desired distribution. **Remark: We only assumed the right continuity of F .**

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 12

12.6 Let $Z = (\frac{1}{\sigma_Y})Y - (\frac{\rho_{XY}}{\sigma_X})X$. Then $\sigma_Z^2 = (\frac{1}{\sigma_Y^2})\sigma_Y^2 - (\frac{\rho_{XY}^2}{\sigma_X^2})\sigma_X^2 - 2(\frac{\rho_{XY}}{\sigma_X\sigma_Y})\text{Cov}(X, Y) = 1 - \rho_{XY}^2$. Note that $\rho_{XY} = \mp 1$ implies $\sigma_Z^2 = 0$ which implies $Z = c$ a.s. for some constant c . In this case $X = \frac{\sigma_X}{\sigma_Y\rho_{XY}}(Y - c)$ hence X is an affine function of Y .

12.11 Consider the mapping $g(x, y) = (\sqrt{x^2 + y^2}, \arctan(\frac{x}{y}))$. Let $S_0 = \{(x, y) : y = 0\}$, $S_1 = \{(x, y) : y > 0\}$, $S_2 = \{(x, y) : y < 0\}$. Note that $\cup_{i=0}^2 S_i = \mathbf{R}^2$ and $m_2(S_0) = 0$. Also for $i = 1, 2$ $g : S_i \rightarrow \mathbf{R}^2$ is injective and continuously differentiable. Corresponding inverses are given by $g_1^{-1}(z, w) = (z \sin w, z \cos w)$ and $g_2^{-1}(z, w) = (z \sin w, -z \cos w)$. In both cases we have that $|J_{g_i^{-1}}(z, w)| = z$ hence by Corollary 12.1 the density of (Z, W) is given by

$$\begin{aligned} f_{Z,W}(z, w) &= \left(\frac{1}{2\pi\sigma^2} e^{-\frac{z^2}{2\sigma^2}} z + \frac{1}{2\pi\sigma^2} e^{-\frac{z^2}{2\sigma^2}} z \mathbf{1}_{(-\frac{\pi}{2}, \frac{\pi}{2})}(w) \mathbf{1}_{(0, \infty)}(z) \right) \\ &= \frac{1}{\pi} \mathbf{1}_{(-\frac{\pi}{2}, \frac{\pi}{2})}(w) * \frac{z}{\sigma^2} e^{-\frac{z^2}{2\sigma^2}} \mathbf{1}_{(0, \infty)}(z) \end{aligned}$$

as desired.

12.12 Let \mathcal{P} be the set of all permutations of $\{1, \dots, n\}$. For any $\pi \in \mathcal{P}$ let X^π be the corresponding permutation of X , i.e. $X_k^\pi = X_{\pi_k}$. Observe that

$$P(X_1^\pi \leq x_1, \dots, X_n^\pi \leq x_n) = F(x_1) \dots F(x_n)$$

hence the law of X^π and X coincide on a π system generating \mathcal{B}^n therefore they are equal. Now let $\Omega_0 = \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_1 < x_2 < \dots < x_n\}$. Since X_i are i.i.d and have continuous distribution $P_X(\Omega_0) = 1$. Observe that

$$P\{Y_1 \leq y_1, \dots, Y_n \leq y_n\} = P(\cup_{\pi \in \mathcal{P}} \{X_1^\pi \leq y_1, \dots, X_n^\pi \leq y_n\} \cap \Omega_0)$$

Note that $\{X_1^\pi \leq y_1, \dots, X_n^\pi \leq y_n\} \cap \Omega_0$, $\pi \in \mathcal{P}$ are disjoint and $P(\Omega_0) = 1$ hence

$$\begin{aligned} P\{Y_1 \leq y_1, \dots, Y_n \leq y_n\} &= \sum_{\pi \in \mathcal{P}} P\{X_1^\pi \leq y_1, \dots, X_n^\pi \leq y_n\} \\ &= n! F(y_1) \dots F(y_n) \end{aligned}$$

for $y_1 \leq \dots \leq y_n$. Hence

$$f_Y(y_1, \dots, y_n) = \begin{cases} n! f(y_1) \dots f(y_n) & y_1 \leq \dots \leq y_n \\ 0 & \text{otherwise} \end{cases}$$

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 14

14.7 $\varphi_X(u)$ is real valued iff $\varphi_X(u) = \overline{\varphi_X(u)} = \varphi_{-X}(u)$. By uniqueness theorem $\varphi_X(u) = \varphi_{-X}(u)$ iff $F_X = F_{-X}$. Hence $\varphi_X(u)$ is real valued iff $F_X = F_{-X}$.

14.9 We use induction. It is clear that the statement is true for $n = 1$. Put $Y_n = \sum_{i=1}^n X_i$ and assume that $E[(Y_n)^3] = \sum_{i=1}^n E[(X_i)^3]$. Note that this implies $\frac{d^3}{dx^3}\varphi_{Y_n}(0) = -i \sum_{i=1}^n E[(X_i)^3]$. Now $E[(Y_{n+1})^3] = E[(X_{n+1} + Y_n)^3] = -i \frac{d^3}{dx^3}(\varphi_{X_{n+1}}\varphi_{Y_n})(0)$ by independence of X_{n+1} and Y_n . Note that

$$\begin{aligned} \frac{d^3}{dx^3}\varphi_{X_{n+1}}\varphi_{Y_n}(0) &= \frac{d^3}{dx^3}\varphi_{X_{n+1}}(0)\varphi_{Y_n}(0) \\ &\quad + 3\frac{d^2}{dx^2}\varphi_{X_{n+1}}(0)\frac{d}{dx}\varphi_{Y_n}(0) + 3\frac{d}{dx}\varphi_{X_{n+1}}(0)\frac{d^2}{dx^2}\varphi_{Y_n}(0) \\ &\quad + \varphi_{X_{n+1}}(0)\frac{d^3}{dx^3}\varphi_{Y_n}(0) \\ &= \frac{d^3}{dx^3}\varphi_{X_{n+1}}(0) + \frac{d^3}{dx^3}\varphi_{Y_n}(0) \\ &= -i \left(E[(X_{n+1})^3] + \sum_{i=1}^n E[(X_i)^3] \right) \end{aligned}$$

where we used the fact that $\frac{d}{dx}\varphi_{X_{n+1}}(0) = iE(X_{n+1}) = 0$ and $\frac{d}{dx}\varphi_{Y_n}(0) = iE(Y_n) = 0$. So $E[(Y_{n+1})^3] = \sum_{i=1}^{n+1} E[(X_i)^3]$ hence the induction is complete.

14.10 It is clear that $0 \leq \nu(A) \leq 1$ since

$$0 \leq \sum_{j=1}^n \lambda_j \mu_j(A) \leq \sum_{j=1}^n \lambda_j = 1.$$

Also for A_i disjoint

$$\begin{aligned} \nu(\cup_{i=1}^{\infty} A_i) &= \sum_{j=1}^n \lambda_j \mu_j(\cup_{i=1}^{\infty} A_i) \\ &= \sum_{j=1}^n \lambda_j \sum_{i=1}^{\infty} \mu_j(A_i) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^n \lambda_j \mu_j(A_i) \\ &= \sum_{i=1}^{\infty} \nu(A_i) \end{aligned}$$

Hence ν is countably additive therefore it is a probability measure. Note that $\int \mathbf{1}_A d\nu(dx) = \sum_{j=1}^n \lambda_j \int \mathbf{1}_A(x) d\mu_j(dx)$ by definition of ν . Now by linearity and monotone convergence theorem for a non-negative Borel function f we have that $\int f(x) \nu(dx) = \sum_{j=1}^n \lambda_j \int f(x) d\mu_j(dx)$. Extending this to integrable f we have that $\hat{\nu}(u) = \int e^{iux} \nu(dx) = \sum_{j=1}^n \lambda_j \int e^{iux} d\mu_j(dx) = \sum_{j=1}^n \lambda_j \hat{\mu}_j(u)$.

14.11 Let ν be the double exponential distribution, μ_1 be the distribution of Y and μ_2 be the distribution of $-Y$ where Y is an exponential r.v. with parameter $\lambda = 1$. Then we have that $\nu(A) = \frac{1}{2} \int_{A \cap (0, \infty)} e^{-x} dx + \frac{1}{2} \int_{A \cap (-\infty, 0)} e^x dx = \frac{1}{2} \mu_1(A) + \frac{1}{2} \mu_2(A)$. By the previous exercise we have that $\hat{\nu}(u) = \frac{1}{2} \hat{\mu}_1(u) + \frac{1}{2} \hat{\mu}_2(u) = \frac{1}{2} \left(\frac{1}{1-iu} + \frac{1}{1+iu} \right) = \frac{1}{1+u^2}$.

14.15. Note that $E\{X^n\} = (-i)^n \frac{d^n}{dx^n} \varphi_X(0)$. Since $X \sim N(0, 1)$ $\varphi_X(s) = e^{-s^2/2}$. Note that we can get the derivatives of any order of $e^{-s^2/2}$ at 0 simply by taking Taylor's expansion of e^x :

$$\begin{aligned} e^{-s^2/2} &= \sum_{i=0}^{\infty} \frac{(-s^2/2)^i}{i!} \\ &= \sum_{i=0}^{\infty} \frac{1}{2^i i!} \frac{(-i)^{2i} (2i)!}{2^{2i} i!} s^{2i} \end{aligned}$$

hence $E\{X^n\} = (-i)^n \frac{d^n}{dx^n} \varphi_X(0) = 0$ for n odd. For $n = 2k$ $E\{X^{2k}\} = (-i)^{2k} \frac{d^{2k}}{dx^{2k}} \varphi_X(0) = (-i)^{2k} \frac{(-i)^{2k} (2k)!}{2^{2k} k!} = \frac{(2k)!}{2^{2k} k!}$ as desired.

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 15

15.1 a. $E\{\bar{x}\} = \frac{1}{n} \sum_{i=1}^n E\{X_i\} = \mu.$

b. Since X_1, \dots, X_n are independent $\text{Var}(\bar{x}) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}\{X_i\} = \frac{\sigma^2}{n}.$

c. Note that $S^2 = \frac{1}{n} \sum_{i=1}^n (X_i)^2 - \bar{x}^2.$ Hence $E(S^2) = \frac{1}{n} \sum_{i=1}^n (\sigma^2 + \mu^2) - (\frac{\sigma^2}{n} + \mu^2) = \frac{n-1}{n} \sigma^2.$

15.17 Note that $\varphi_Y(u) = \prod_{i=1}^{\alpha} \varphi_{X_i}(u) = (\frac{\beta}{\beta-iu})^{\alpha}$ which is the characteristic function of Gamma(α, β) random variable. Hence by uniqueness of characteristic function Y is Gamma(α, β).

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 16

16.3 $P(\{Y \leq y\}) = P(\{X \leq y\} \cap \{Z = 1\}) + P(\{-X \leq y\} \cap \{Z = -1\}) = \frac{1}{2}\Phi(y) + \frac{1}{2}\Phi(-y) = \Phi(y)$ since Z and X are independent and $\Phi(y)$ is symmetric. So Y is normal. Note that $P(X + Y = 0) = \frac{1}{2}$ hence $X + Y$ can not be normal. So (X, Y) is not Gaussian even though both X and Y are normal.

16.4 Observe that

$$Q = \sigma_X \sigma_Y \begin{bmatrix} \frac{\sigma_X}{\sigma_Y} & \rho \\ \rho & \frac{\sigma_Y}{\sigma_X} \end{bmatrix}$$

So $\det(Q) = \sigma_X \sigma_Y (1 - \rho^2)$. So $\det(Q) = 0$ iff $\rho = \mp 1$. By Corollary 16.2 the joint density of (X, Y) exists iff $-1 < \rho < 1$. (By Cauchy-Schwartz we know that $-1 \leq \rho \leq 1$). Note that

$$Q^{-1} = \frac{1}{\sigma_X \sigma_Y (1 - \rho^2)} \begin{bmatrix} \frac{\sigma_Y}{\sigma_X} & -\rho \\ -\rho & \frac{\sigma_X}{\sigma_Y} \end{bmatrix}$$

Substituting this in formula 16.5 we get that

$$f_{(X,Y)}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y(1-\rho^2)} \exp \left\{ \frac{-1}{2(1-\rho^2)} \left(\left(\frac{x-\mu_X}{\sigma_X} \right)^2 - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y} \right)^2 \right) \right\}.$$

16.6 By Theorem 16.2 there exists a multivariate normal r.v. Y with $E(Y) = 0$ and a diagonal covariance matrix Λ s.t. $X - \mu = AY$ where A is an orthogonal matrix. Since $Q = A\Lambda A^*$ and $\det(Q) > 0$ the diagonal entries of Λ are strictly positive hence we can define $B = \Lambda^{-1/2}A^*$. Now the covariance matrix \tilde{Q} of $B(X - \mu)$ is given by

$$\begin{aligned} \tilde{Q} &= \Lambda^{-1/2}A^*A\Lambda A^{-1/2} \\ &= I \end{aligned}$$

So $B(X - \mu)$ is standard normal.

16.17 We know that as in Exercise 16.6 if $B = \Lambda^{-1/2}A^*$ where A is the orthogonal matrix s.t. $Q = A\Lambda A^*$ then $B(X - \mu)$ is standard normal. Note that this gives $(X - \mu)^*Q^{-1}(X - \mu) = (X - \mu)^*B^*B(X - \mu)$ which has chi-square distribution with n degrees of freedom.

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 17

17.1 Let $n(m)$ and $j(m)$ be such that $Y_m = n(m)^{1/p} Z_{n(m),j(m)}$. This gives that $P(|Y_m| > 0) = \frac{1}{n(m)} \rightarrow 0$ as $m \rightarrow \infty$. So Y_m converges to 0 in probability. However $E[|Y_m|^p] = E[n(m)Z_{n(m),j(m)}] = 1$ for all m . So Y_m does not converge to 0 in L^p .

17.2 Let $X_n = 1/n$. It is clear that X_n converge to 0 in probability. If $f(x) = \mathbf{1}_{\{0\}}(x)$ then we have that $P(|f(X_n) - f(0)| > \epsilon) = 1$ for every $\epsilon \geq 1$, so $f(X_n)$ does not converge to $f(0)$ in probability.

17.3 First observe that $E(S_n) = \sum_{i=1}^n E(X_n) = 0$ and that $\text{Var}(S_n) = \sum_{i=1}^n \text{Var}(X_n) = n$ since $E(X_n) = 0$ and $\text{Var}(X_n) = E(X_n^2) = 1$. By Chebyshev's inequality $P(|\frac{S_n}{n}| \geq \epsilon) = P(|S_n| \geq n\epsilon) \leq \frac{\text{Var}(S_n)}{n^2\epsilon^2} = \frac{n}{n^2\epsilon^2} \rightarrow 0$ as $n \rightarrow \infty$. Hence $\frac{S_n}{n}$ converges to 0 in probability.

17.4 Note that Chebyshev's inequality gives $P(|\frac{S_{n^2}}{n^2}| \geq \epsilon) \leq \frac{1}{n^2\epsilon^2}$. Since $\sum_{i=1}^{\infty} \frac{1}{n^2\epsilon^2} < \infty$ by Borel Cantelli Theorem $P(\limsup_n \{|\frac{S_{n^2}}{n^2}| \geq \epsilon\}) = 0$. Let $\Omega_0 = \left(\bigcup_{m=1}^{\infty} \limsup_n \{|\frac{S_{n^2}}{n^2}| \geq \frac{1}{m}\}\right)^c$. Then $P(\Omega_0) = 1$. Now let's pick $w \in \Omega_0$. For any ϵ there exists m s.t. $\frac{1}{m} \leq \epsilon$ and $w \in (\limsup_n \{|\frac{S_{n^2}}{n^2}| \geq \frac{1}{m}\})^c$. Hence there are finitely many n s.t. $|\frac{S_{n^2}}{n^2}| \geq \frac{1}{m}$ which implies that there exists $N(w)$ s.t. $|\frac{S_{n^2}}{n^2}| \leq \frac{1}{m}$ for every $n \geq N(w)$. Hence $\frac{S_{n^2}(w)}{n^2} \rightarrow 0$. Since $P(\Omega_0) = 1$ we have almost sure convergence.

17.12 $Y < \infty$ a.s. which follows by Exercise 17.11 since $X_n < \infty$ and $X < \infty$ a.s. Let $Z = \frac{1}{c+1+Y}$. Observe that $Z > 0$ a.s. and $E_P(Z) = 1$. Therefore as in Exercise 9.8 $Q(A) = E_P(Z\mathbf{1}_A)$ defines a probability measure and $E_Q(|X_n - X|) = E_P(Z|X_n - X|)$. Note that $Z|X_n - X| \leq 1$ a.s. and $X_n \rightarrow X$ a.s. by hypothesis, hence by dominated convergence theorem $E_Q(|X_n - X|) = E_P(Z|X_n - X|) \rightarrow 0$, i.e. X_n tends to X in L^1 with respect to Q .

17.14 First observe that $|E(X_n^2) - E(X^2)| \leq E(|X_n^2 - X^2|)$. Since $|X_n^2 - X^2| \leq (X_n - X)^2 + 2|X||X_n - X|$ we get that $|E(X_n^2) - E(X^2)| \leq E((X_n - X)^2) + 2E(|X||X_n - X|)$. Note that first term goes to 0 since X_n tends to X in L^2 . Applying Cauchy Schwarz inequality to the second term we get $E(|X||X_n - X|) \leq \sqrt{E(X^2)E(|X_n - X|^2)}$, hence the second term also goes to 0 as $n \rightarrow \infty$. Now we can conclude $E(X_n^2) \rightarrow E(X^2)$.

17.15 For any $\epsilon > 0$ $P(\{|X| \leq c+\epsilon\}) \geq P(\{|X_n| \leq c, |X_n - X| \leq \epsilon\}) \rightarrow 1$ as $n \rightarrow \infty$. Hence $P(\{|X| \leq c+\epsilon\}) = 1$. Since $\{X \leq c\} = \bigcap_{m=1}^{\infty} \{X \leq c + \frac{1}{m}\}$ we get that $P\{X \leq c\} = 1$. Now we have that $E(|X_n - X|) = E(|X_n - X|\mathbf{1}_{\{|X_n - X| \leq \epsilon\}}) + E(|X_n - X|\mathbf{1}_{\{|X_n - X| > \epsilon\}}) \leq \epsilon + 2c(P\{|X_n - X| > \epsilon\})$, hence choosing n large we can make $E(|X_n - X|)$ arbitrarily small, so X_n tends to X in L^1 .

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 18

18.8 Note that $\varphi_{Y_n}(u) = \prod_{i=1}^n \varphi_{X_i}(\frac{u}{n}) = \prod_{i=1}^n e^{-\frac{|u|}{n}} = e^{-|u|}$, hence Y_n is also Cauchy with $\alpha = 0$ and $\beta = 1$ which is independent of n , hence trivially Y_n converges in distribution to a Cauchy distributed r.v. with $\alpha = 0$ and $\beta = 1$. However Y_n does not converge to any r.v. in probability. To see this, suppose there exists Y s.t. $P(|Y_n - Y| > \epsilon) \rightarrow 0$. Note that $P(|Y_n - Y_m| > \epsilon) \leq P(|Y_n - Y| > \frac{\epsilon}{2}) + P(|Y_m - Y| > \frac{\epsilon}{2})$. If we let $m = 2n$, $|Y_n - Y_m| = \frac{1}{2} |\frac{1}{n} \sum_{i=1}^n X_i - \frac{1}{n} \sum_{i=n+1}^{2n} X_i|$ which is equal in distribution to $\frac{1}{2} |U - W|$ where U and W are independent Cauchy r.v.'s with $\alpha = 0$ and $\beta = 1$. Hence $P(|Y_n - Y_m| > \frac{\epsilon}{2})$ does not depend on n and does not converge to 0 if we let $m = 2n$ and $n \rightarrow \infty$ which is a contradiction since we assumed the right hand side converges to 0.

18.16 Define f_m as the following sequence of functions:

$$f_m(x) = \begin{cases} x^2 & \text{if } |x| \leq N - \frac{1}{m} \\ (N - \frac{1}{m})x - (N - \frac{1}{m})N & \text{if } x \geq N - \frac{1}{m} \\ -(N - \frac{1}{m})x + (N - \frac{1}{m})N & \text{if } x \leq -N + \frac{1}{m} \\ 0 & \text{otherwise} \end{cases}$$

Note that each f_m is continuous and bounded. Also $f_m(x) \uparrow \mathbf{1}_{(-N,N)}(x)x^2$ for every $x \in \mathbf{R}$. Hence

$$\int_{-N}^N x^2 F(dx) = \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} f_m(x) F(dx)$$

by monotone convergence theorem. Now

$$\int_{-\infty}^{\infty} f_m(x) F(dx) = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_m(x) F_n(dx)$$

by weak convergence. Since $\int_{-\infty}^{\infty} f_m(x) F_n(dx) \leq \int_{-N}^N x^2 F_n(dx)$ it follows that

$$\int_{-N}^N x^2 F(dx) \leq \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{-N}^N x^2 F_n(dx) = \limsup_{n \rightarrow \infty} \int_{-N}^N x^2 F_n(dx)$$

as desired.

18.17 Following the hint, suppose there exists a continuity point y of F such that

$$\lim_{n \rightarrow \infty} F_n(y) \neq F(y)$$

Then there exist $\epsilon > 0$ and a subsequence $(n_k)_{k \geq 1}$ s.t. $F_{n_k}(y) - F(y) < -\epsilon$ for all k , or $F_{n_k}(y) - F(y) > \epsilon$ for all k . Suppose $F_{n_k}(y) - F(y) < -\epsilon$ for all k , observe that for $x \leq y$, $F_{n_k}(x) - F(x) \leq F_{n_k}(y) - F(x) = F_{n_k}(y) - F(y) + (F(y) - F(x)) < -\epsilon + (F(y) - F(x))$. Since f is continuous at y there exists an interval $[y_1, y)$ s.t. $|(F(y) - F(x))| < \frac{\epsilon}{2}$, hence $F_{n_k}(x) - F(x) < -\frac{\epsilon}{2}$ for all $x \in [y_1, y)$. Now suppose $F_{n_k}(y) - F(y) > \epsilon$, then for $x \geq y$, $F_{n_k}(x) - F(x) \geq F_{n_k}(y) - F(x) = F_{n_k}(y) - F(y) + (F(y) - F(x)) > \epsilon + (F(y) - F(x))$.

Now we can find an interval $(y, y_1]$ s.t. $|(F(y) - F(x))| < \frac{\epsilon}{2}$ which gives $F_{n_k}(x) - F(x) > \frac{\epsilon}{2}$ for all $x \in (y, y_1]$. Note that both cases would yield

$$\int_{-\infty}^{\infty} |F_{n_k}(x) - F(x)|^r dx > |y_1 - y| \frac{\epsilon}{2}$$

which is a contradiction to the assumption

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |F_n(x) - F(x)|^r dx = 0.$$

Therefore X_n converges to X in distribution.

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 19

19.1 Note that $\varphi_{X_n}(u) = e^{iu\mu_n - \frac{u^2\sigma_n^2}{2}} \rightarrow e^{iu\mu - \frac{u^2\sigma^2}{2}}$. By Lévy's continuity theorem it follows that $X_n \Rightarrow X$ where X is $N(\mu, \sigma^2)$.

19.3 Note that $\varphi_{X_n+Y_n}(u) = \varphi_{X_n}(u)\varphi_{Y_n}(u) \rightarrow \varphi_X(u)\varphi_Y(u) = \varphi_{X+Y}(u)$. Therefore $X_n + Y_n \Rightarrow X + Y$

SOLUTIONS TO SELECTED PROBLEMS OF CHAPTER 20

20.1 a. First observe that $E(S_n^2) = \sum_{i=1}^n \sum_{j=1}^n E(X_i X_j) = \sum_{i=1}^n X_i^2$ since $E(X_i X_j) = 0$ for $i \neq j$. Now $P\left(\frac{|S_n|}{n} \geq \epsilon\right) \leq \frac{E(S_n^2)}{\epsilon^2 n^2} = \frac{nE(X_i^2)}{\epsilon^2 n^2} \leq \frac{c}{n\epsilon^2}$ as desired.

b. From part (a) it is clear that $\frac{1}{n}S_n$ converges to 0 in probability. Also $E\left(\left(\frac{1}{n}S_n\right)^2\right) = \frac{E(X_i^2)}{n} \rightarrow 0$ since $E(X_i^2) \leq \infty$, so $\frac{1}{n}S_n$ converges to 0 in L^2 as well.

20.5 Note that $Z_n \Rightarrow Z$ implies that $\varphi_{Z_n}(u) \rightarrow \varphi_Z(u)$ uniformly on compact subset of \mathbf{R} . (See Remark 19.1). For any u , we can pick $n > N$ s.t. $\frac{u}{\sqrt{n}} < M$, $\sup_{x \in [-M, M]} |\varphi_{Z_n}(x) - \varphi_Z(x)| < \epsilon$ and $|\varphi_{Z_n}\left(\frac{u}{\sqrt{n}}\right) - \varphi_Z(0)| < \epsilon$. This gives us

$$\left| \varphi_{Z_n}\left(\frac{u}{\sqrt{n}}\right) - \varphi_Z(0) \right| = \left| \varphi_{Z_n}\left(\frac{u}{\sqrt{n}}\right) - \varphi_Z\left(\frac{u}{\sqrt{n}}\right) \right| + \left| \varphi_Z\left(\frac{u}{\sqrt{n}}\right) - \varphi_Z(0) \right| \leq 2\epsilon$$

So $\varphi_{\frac{Z_n}{\sqrt{n}}}(u) = \varphi_{Z_n}\left(\frac{u}{\sqrt{n}}\right)$ converges to $\varphi_Z(0) = 1$ for every u . Therefore $\frac{Z_n}{\sqrt{n}} \Rightarrow 0$ by continuity theorem. We also have by the strong law of large numbers that $\frac{Z_n}{\sqrt{n}} \rightarrow E(X_j) - \nu$. This implies $E(X_j) - \nu = 0$, hence the assertion follows by strong law of large numbers.



<http://www.springer.com/978-3-540-43871-7>

Probability Essentials

Jacod, J.; Protter, P.

2004, X, 254 p., Softcover

ISBN: 978-3-540-43871-7