# Chapter 2
# Vector Analysis and Coordinate Systems

## 2.1 Introduction

Vector analysis is a mathematical tool that is used in expressing and simplifying the related laws and theorems of electric and magnetic fields. The electric and magnetic fields are vector quantities. The characteristics of these fields are analysed by a set of laws known as Maxwell's equations. The basic knowledge of vectors is important to formulate Maxwell's equations and to apply in the practical field. Vector addition, subtraction, multiplication and division will be discussed in this chapter. In addition, the three most orthogonal coordinate systems, namely Cartesian, cylindrical and spherical will also be discussed to deeply understand electromagnetic fields and waves.

## 2.2 Vectors and Scalars

Knowledge of vectors and scalars is important when analysing electromagnetic fields. A vector is a quantity that has both magnitude and direction. Vectors are represented by boldface roman-type symbols (**A**). An arrow on the top of the letter often represents vector ($\overrightarrow{A}$). The magnitude of the vector is represented by $|\mathbf{A}|$ or simply $A$. Displacement, velocity, force and acceleration are examples of vectors. Different vectors with directions are shown in Fig. 2.1.

A vector field is a function that specifies a vector quantity everywhere in a region. Examples are gravitational force on a body in space and the displacement of a plane in space. A scalar is a quantity with magnitude but no direction. Length, mass, time, temperature and any real number are examples of scalar quantities. A scalar field is a function that specifies a scalar quantity everywhere in a region. Examples are temperature distribution and electric potential in a room.

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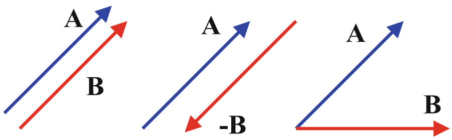



**Fig. 2.1** Vectors with directions

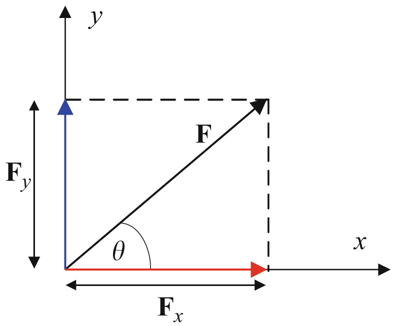



**Fig. 2.2** Vectors with directions

## 2.3 Vector Components

A vector can be resolved into two components, namely the horizontal component and the vertical component. The addition of these two components is equal to the original vector. In Fig. 2.2, a vector $\mathbf{F}$ is working at an angle of $\theta$ with the $x$-axis. The $x$-axis component of this vector is

$$\mathbf{F}_x = \mathbf{F}\cos\theta. \tag{2.1}$$

The $y$-axis component is

$$\mathbf{F}_y = \mathbf{F}\sin\theta. \tag{2.2}$$

Vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are working at angles of $\theta_1$ and $\theta_2$ with the $x$-axis, respectively, which are shown in Fig. 2.3. Here, the $x$-axis and $y$-axis components are

$$\mathbf{F}_{x1} = \mathbf{F}_1\cos\theta_1 \tag{2.3}$$

$$\mathbf{F}_{x2} = \mathbf{F}_2\cos\theta_2 \tag{2.4}$$

$$\mathbf{F}_{y1} = \mathbf{F}_1\sin\theta_1 \tag{2.5}$$

$$\mathbf{F}_{y2} = \mathbf{F}_2\sin\theta_2 \tag{2.6}$$

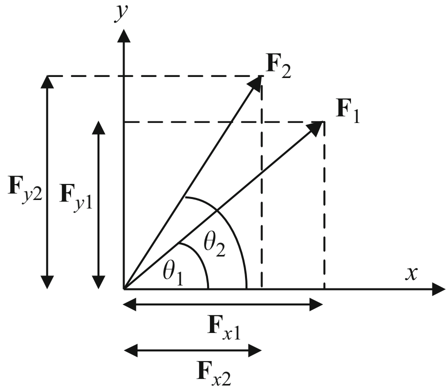
The sum of the horizontal components is

$$\mathbf{F}_x = \mathbf{F}_{x1} + \mathbf{F}_{x2}. \tag{2.7}$$

Substituting Eqs. (2.3) and (2.4) into Eq. (2.7) yields

$$\mathbf{F}_x = \mathbf{F}_1\cos\theta_1 + \mathbf{F}_2\cos\theta_2. \tag{2.8}$$

**Fig. 2.3** Two vectors with directions




The sum of the vertical components is

$$\mathbf{F}_y = \mathbf{F}_{y1} + \mathbf{F}_{y2}. \tag{2.9}$$

Substituting Eqs. (2.5) and (2.6) into Eq. (2.9) yields

$$\mathbf{F}_y = \mathbf{F}_1 \sin\theta_1 + \mathbf{F}_2 \sin\theta_2. \tag{2.10}$$

Finally, the resultant vector can be determined as

$$\mathbf{F}_r = \sqrt{\mathbf{F}_x{}^2 + \mathbf{F}_y{}^2}. \tag{2.11}$$

## 2.4 Unit Vectors

A unit vector is a vector whose magnitude is 1. Unit vectors in three directions are $\mathbf{a}_x, \mathbf{a}_y$ and $\mathbf{a}_z$ as shown in Fig. 2.4.

The magnitudes of three unit vectors are

$$\mathbf{a}_x = (1{,}0, 0), \tag{2.12}$$

$$\mathbf{a}_y = (0{,}1, 0)\ and \tag{2.13}$$

$$\mathbf{a}_z = (0{,}0, 1). \tag{2.14}$$

A general representation of a vector **A** is shown in Fig. 2.5. The unit vector $\mathbf{a}_A$ is working in the same direction as the vector **A**. The unit vector can be expressed as

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = 1, \tag{2.15}$$

Fig. 2.4 Three unit vectors

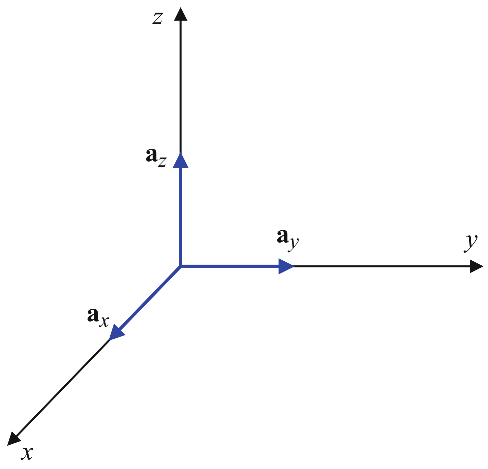
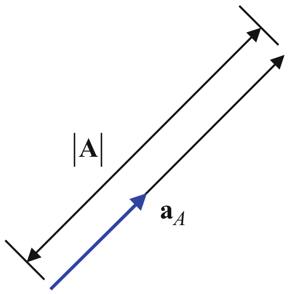



Fig. 2.5 Unit vector representation




where
$|\mathbf{A}|$ is the magnitude of the vector **A**.

## 2.5 Vector Addition

Consider that the vector **A** has three components $A_x, A_y$ and $A_z$ in the $x, y$ and z directions, respectively. According to Fig. 2.6, the vectors $A_x\mathbf{a}_x$, $A_y\mathbf{a}_y$ and $A_z\mathbf{a}_z$ are the components of the vector **A** in the $x$, $y$ and z directions, respectively. The resultant of these vectors is

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z. \tag{2.16}$$

Similarly, vectors **B** and **C** can be expressed as

$$\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z \; and \tag{2.17}$$

$$\mathbf{C} = C_x\mathbf{a}_x + C_y\mathbf{a}_y + C_z\mathbf{a}_z. \tag{2.18}$$

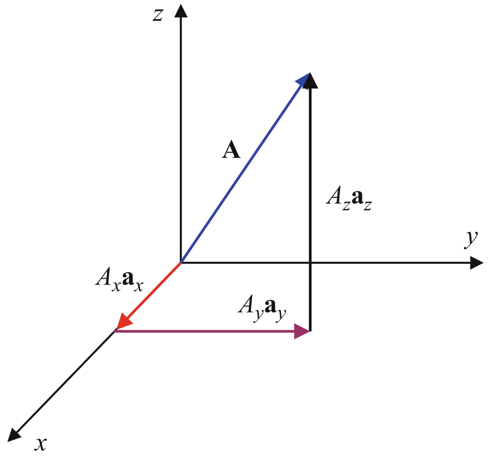



**Fig. 2.6** Three component vectors

The magnitude of the vector A can be written as

$$A = |\mathbf{A}| = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}. \tag{2.19}$$

The unit vector in the direction of the vector A is

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z}{\sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}}. \tag{2.20}$$

Vector addition can be obtained by parallelogram and nose-to-tail or head-to-tail rules. Two vectors **A** and **B** started from the same point as shown in Fig. 2.7. The resultant vector can be calculated as

$$\mathbf{R}_a = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \tag{2.21}$$

Substituting Eqs. (2.16) and (2.17) into Eq. (2.21) yields

$$\mathbf{R}_a = (A_x + B_x)\mathbf{a}_x + (A_y + B_y)\mathbf{a}_y + (A_z + B_z)\mathbf{a}_z. \tag{2.22}$$

## 2.6 Vector Subtraction

Vector subtraction is defined as the special addition of two vectors. Consider two vectors B and C for vector subtraction as shown in Fig. 2.8. The vector subtraction can be represented as

$$\mathbf{R}_s = \mathbf{B} - \mathbf{C} = \mathbf{B} + (-\mathbf{C}), \tag{2.23}$$

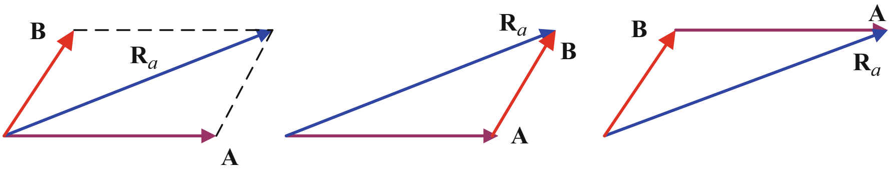
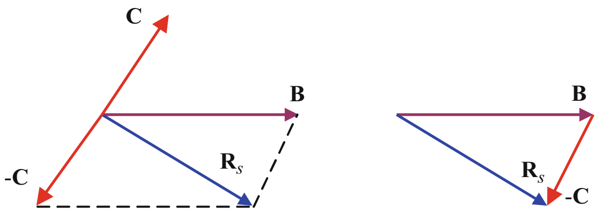



**Fig. 2.7** Parallelogram and head-to-tail rules

**Fig. 2.8** Parallelogram and head-to-tail rules




where
$-\mathbf{C}$ is the negative vector of $\mathbf{C}$.
Substituting Eqs. (2.17) and (2.18) into Eq. (2.23) provides

$$\mathbf{R}_s = (B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z) + (-C_x\mathbf{a}_x - C_y\mathbf{a}_y - C_z\mathbf{a}_z), \tag{2.24}$$

$$\mathrm{R}_s = (B_x - C_x)\mathrm{a}_x + (B_y - C_y)\mathrm{a}_y + (B_z - C_z)\mathrm{a}_z. \tag{2.25}$$

**Example 2.1** Three vectors are given by $\mathbf{A} = 4\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$, $\mathbf{B} = 2\mathbf{a}_x - 5\mathbf{a}_y - 4\mathbf{a}_z$ and $\mathbf{C} = -\mathbf{a}_x + 3\mathbf{a}_y + 6\mathbf{a}_z$, respectively. Determine the magnitude of (1) $\mathbf{R}_a = \mathbf{A} + \mathbf{B}$ and (2) $\mathbf{R}_s = \mathbf{B} - \mathbf{C}$.

**Solution**

1. The magnitude of $\mathbf{R}_a$ can be determined as

$$\mathbf{R}_a = 4\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z + 2\mathbf{a}_x - 5\mathbf{a}_y - 4\mathbf{a}_z = 6\mathbf{a}_x - 8\mathbf{a}_y - 3\mathbf{a}_z,$$

$$|\mathbf{R}_a| = \sqrt{6^2 + (-8)^2 + (-3)^2} = 10.44.$$

2. The magnitude of $\mathbf{R}_s$ can be calculated as

$$\mathbf{R}_s = 2\mathbf{a}_x - 5\mathbf{a}_y - 4\mathbf{a}_z + \mathbf{a}_x - 3\mathbf{a}_y - 6\mathbf{a}_z = 3\mathbf{a}_x - 8\mathbf{a}_y - 10\mathbf{a}_z,$$

$$|\mathbf{R}_s| = \sqrt{3^2 + (-8)^2 + (-10)^2} = 13.15.$$

**Example 2.2** A unit vector is parallel to the resultant (addition) vector of $\mathbf{A} = 2\mathbf{a}_x + 3\mathbf{a}_y + 6\mathbf{a}_z$ and $\mathbf{B} = 5\mathbf{a}_x - \mathbf{a}_y - 2\mathbf{a}_z$. Determine the unit vector.

**Solution** The resultant vector can be determined as

$$\mathbf{R}_a = \mathbf{A} + \mathbf{B} = 7\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z.$$

The unit vector can be calculated as

$$\mathbf{a}_u = \frac{\mathbf{R}_a}{|\mathbf{R}_a|} = \frac{7\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z}{\sqrt{7^2 + 2^2 + 4^2}} = 0.84\mathbf{a}_x + 0.24\mathbf{a}_y + 0.48\mathbf{a}_z.$$

**Practice Problem 2.1** Three vectors are given by $\mathbf{A} = 2\mathbf{a}_x + 5\mathbf{a}_y - 3\mathbf{a}_z, \mathbf{B} = 3\mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$ and $\mathbf{C} = \mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z$, respectively. Determine the magnitude of (1) $\mathbf{R}_a = \mathbf{A} + \mathbf{B}$ and (2) $\mathbf{R}_s = \mathbf{B} - \mathbf{C}$.

**Practice Problem 2.2** Calculate the unit vector which is parallel to the resultant (subtraction) vector of $\mathbf{A} = 3\mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + 5\mathbf{a}_y + \mathbf{a}_z$.

## 2.7 Vectors Multiplication and Division

Multiplication of a vector **A** by a positive scalar parameter k can be expressed as

$$\mathbf{R}_m = k\mathbf{A}. \tag{2.26}$$

Substituting Eq. (2.16) into Eq. (2.26) yields

$$\mathbf{R}_m = k(A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z) \tag{2.27}$$

$$\mathbf{R}_m = kA_x\mathbf{a}_x + kA_y\mathbf{a}_y + kA_z\mathbf{a}_z. \tag{2.28}$$

Division of a vector **B** by another positive scalar parameter n can be expressed as

$$\mathbf{R}_d = \frac{\mathbf{B}}{n}. \tag{2.29}$$

Substituting Eq. (2.17) into Eq. (2.29) provides

$$\mathbf{R}_d = \frac{1}{n}(B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z) \tag{2.30}$$

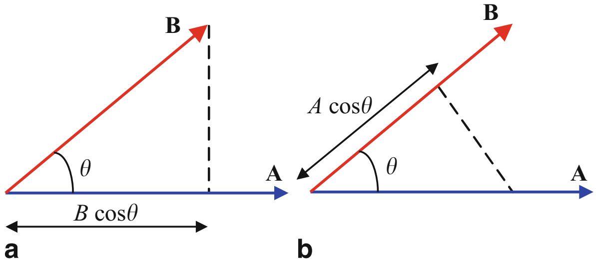
$$\mathbf{R}_d = \frac{B_x}{n}\mathbf{a}_x + \frac{B_y}{n}\mathbf{a}_y + \frac{B_z}{n}\mathbf{a}_z. \tag{2.31}$$

## 2.8 Dot Product of Two Vectors

The dot product of two vectors **A** and **B** is represented as

$$\mathbf{R}_{dot} = \mathbf{A} \bullet \mathbf{B}. \tag{2.32}$$

**Fig. 2.9** Projections of (**a**) vectors B on A (**b**) A on B




The dot product of two vectors **A** and **B** is equal to the product of the magnitudes and the cosine of the angle between them. It can be expressed as

$$\mathbf{A} \bullet \mathbf{B} = AB \cos\theta. \tag{2.33}$$

In Fig. 2.9, the vector **A** is working in the $x$-axis and the vector **B** is working at an angle $\theta$ with the vector A. The projection of the vector B on the vector A is $B\cos\theta$ as shown in Fig. 2.9a. Equation (2.32) can be modified as

$$\mathbf{A} \bullet \mathbf{B} = A(B\cos\theta) = AB\cos\theta. \tag{2.34}$$

The projection of vector **A** on vector **B** is $A\cos\theta$ as shown in Fig. 2.9b. Equation (2.32) again can be modified to

$$\mathbf{A} \bullet \mathbf{B} = (A\cos\theta)B = AB\cos\theta. \tag{2.35}$$

The angle between two vectors can be determined with

$$\cos\theta = \frac{\mathbf{A} \bullet \mathbf{B}}{AB}. \tag{2.36}$$

The dot products of unit vectors are

$$\mathbf{a}_x \bullet \mathbf{a}_x = 1.1\cos 0^\circ = 1, \tag{2.37}$$

$$\mathbf{a}_x \bullet \mathbf{a}_y = 1.1\cos 90^\circ = 0, \tag{2.38}$$

$$\mathbf{a}_y \bullet \mathbf{a}_z = 1.1\cos 90^\circ = 0, \tag{2.39}$$

$$\mathbf{a}_z \bullet \mathbf{a}_x = 1.1\cos 90^\circ = 0, \tag{2.40}$$

$$\mathbf{a}_y \bullet \mathbf{a}_y = 1.1\cos 0^\circ = 1 \tag{2.41}$$

$$\mathbf{a}_z \bullet \mathbf{a}_z = 1.1\cos 0^\circ = 1. \tag{2.42}$$

Substituting Eqs. (2.16) and (2.17) into Eq. (2.32) yields

$$\mathbf{A} \bullet \mathbf{B} = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z) \bullet (B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z), \tag{2.43}$$

$$\mathbf{A} \bullet \mathbf{B} = A_x B_x \mathbf{a}_x \bullet \mathbf{a}_x + A_y B_x \mathbf{a}_y \bullet \mathbf{a}_x + A_z B_x \mathbf{a}_z \bullet \mathbf{a}_x + A_x B_y \mathbf{a}_x \bullet \mathbf{a}_y$$
$$+ A_y B_y \mathbf{a}_y \bullet \mathbf{a}_y + A_z B_y \mathbf{a}_z \bullet \mathbf{a}_y + A_x B_z \mathbf{a}_x \bullet \mathbf{a}_z \quad (2.44)$$
$$+ A_y B_z \mathbf{a}_y \bullet \mathbf{a}_z + A_z B_z \mathbf{a}_z \bullet \mathbf{a}_z.$$

Equation (2.44) can be modified by considering unit vector properties as

$$\mathbf{A} \bullet \mathbf{B} = A_x B_x + A_y B_y + A_z B_z. \quad (2.45)$$

The dot product of two same vectors is

$$\mathbf{A} \bullet \mathbf{A} = (A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z) \bullet (A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z). \quad (2.46)$$

Equation (2.46) can be modified by applying properties of unit vectors to

$$A^2 = A_x{}^2 + A_y{}^2 + A_z{}^2. \quad (2.47)$$

**Example 2.3** Two vectors are given by $\mathbf{A} = 3\mathbf{a}_x + 2\mathbf{a}_y - 4\mathbf{a}_z$ and $\mathbf{B} = 3\mathbf{a}_x - 4\mathbf{a}_y - 5\mathbf{a}_z$, respectively. Determine the dot product of two vectors.

**Solution** The dot product can be determined as

$$\mathbf{A} \bullet \mathbf{B} = (3\mathbf{a}_x + 2\mathbf{a}_y - 4\mathbf{a}_z) \bullet (3\mathbf{a}_x - 4\mathbf{a}_y - 5\mathbf{a}_z),$$
$$\mathbf{A} \bullet \mathbf{B} = (3)(3) + (2)(-4) + (-4)(-5) = 9 - 8 + 20 = 21.$$

**Practice Problem 2.3** Determine the angle between the two vectors $\mathbf{A} = 4\mathbf{a}_x + \mathbf{a}_y - 3\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + 4\mathbf{a}_y - 3\mathbf{a}_z$.

## 2.9 Cross Product of Two Vectors

The cross product is the second kind of vector multiplication. The cross product of two vectors A and B is represented as

$$\mathbf{R}_{cross} = \mathbf{A} \times \mathbf{B}. \quad (2.48)$$

The magnitude of $\mathbf{A} \times \mathbf{B}$ is defined as the product of the magnitude of $\mathbf{A}$ and $\mathbf{B}$ and the sine of the smaller angle ($\theta$) between them. The direction of the vector $\mathbf{A} \times \mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$ as shown in Fig. 2.10. Let $\mathbf{a}_N$ be the unit vector in the direction of $\mathbf{A} \times \mathbf{B}$; then, the expression of $\mathbf{A} \times \mathbf{B}$ is

$$\mathbf{A} \times \mathbf{B} = AB \sin\theta \mathbf{a}_N. \quad (2.49)$$

In Fig. 2.10a and b, it is seen that the direction of A × B is not the same as the direction of $\mathbf{B} \times \mathbf{A}$. The direction is 180° out of phase with each other; however, the magnitude is the same. It can be expressed as

$$(\mathbf{A} \times \mathbf{B}) = -(\mathbf{B} \times \mathbf{A}). \quad (2.50)$$

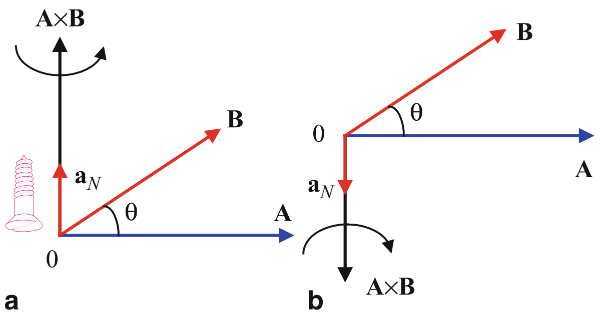



**Fig. 2.10** Vectors cross product (**a**) anticlockwise direction (**b**) clockwise direction

The properties of unit vectors for the cross product can be expressed as

$$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z, \tag{2.51}$$

$$\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x, \tag{2.52}$$

$$\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y \tag{2.53}$$

$$\mathbf{a}_x \times \mathbf{a}_x = \mathbf{a}_y \times \mathbf{a}_y = \mathbf{a}_z \times \mathbf{a}_z = 1.1 \sin 0^\circ = 0. \tag{2.54}$$

The properties of unit vectors can be determined from the cyclic permutation as shown in Fig. 2.11.

Multiplying Eqs. (2.16) and (2.17) provides

$$\mathbf{A} \times \mathbf{B} = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z) \times (B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z), \tag{2.55}$$

$$\begin{aligned} \mathbf{A} \times \mathbf{B} = {} & A_xB_x(\mathbf{a}_x \times \mathbf{a}_x) + A_yB_x(\mathbf{a}_y \times \mathbf{a}_x) + A_zB_x(\mathbf{a}_z \times \mathbf{a}_x) \\ & + A_xB_y(\mathbf{a}_x \times \mathbf{a}_y) + A_yB_y(\mathbf{a}_y \times \mathbf{a}_y) + A_zB_y(\mathbf{a}_z \times \mathbf{a}_y) \\ & + A_xB_z(\mathbf{a}_x \times \mathbf{a}_z) + A_yB_z(\mathbf{a}_y \times \mathbf{a}_z) + A_zB_z(\mathbf{a}_z \times \mathbf{a}_z), \end{aligned} \tag{2.56}$$

$$\begin{aligned} \mathbf{A} \times \mathbf{B} = {} & A_yB_x(-\mathbf{a}_z) + A_zB_x(\mathbf{a}_y) + A_xB_y(\mathbf{a}_z) + A_zB_y(-\mathbf{a}_x) \\ & + A_xB_z(-\mathbf{a}_y) + A_yB_z(\mathbf{a}_x), \end{aligned} \tag{2.57}$$

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{a}_x + (A_zB_x - A_xB_z)\mathbf{a}_y + (A_xB_y - A_yB_x)\mathbf{a}_z. \tag{2.58}$$

Equation (2.58) can be written in the determinant form as

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{2.59}$$

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