Chapter 2
The Measures of One-Way Effect, Reciprocity, and Association

Abstract To characterize the interdependent structure of a pair of two jointly second-order stationary processes, this chapter introduces the (overall as well as frequency-wise) measures of one-way effect, reciprocity, and association. Section 2.2 defines the Granger and Sims non-causality and establishes their equivalence for a general class of (not necessarily stationary) second-order processes. Sections 2.3 and 2.4 define the overall and frequency-wise one-way effect measures and provide three ways of deriving the frequency-wise measure. One is based on direct canonical factorization of the spectral density matrix. The other two are based on distributed-lag representation and innovation orthogonalization, respectively. Each approach provides a different representation of the same quantity. Section 2.5 introduces the overall and the frequency-wise measures of reciprocity and association.

Keywords Canonical factorization · Frequency-domain representation · Granger non-causality · Measure of association · Measure of one-way effect · Measure of reciprocity · Prediction improvement · Purely reciprocal component process · Sims non-causality

2.1 Prediction and Causality

2.1.1 Statement of the Problem

When we focus on a pair of time series $\{u(t), v(t)\}$ as the subject-matter series, the presence and absence of Granger causality is defined by comparing the prediction precision of the prediction of $u(t)$ using two predictor sets (i) $\{u(s), v(s), s \leq t-1\}$ and (ii) $\{u(s), s \leq t-1\}$. The series $\{v(t)\}$ is said to cause or not to cause $\{u(t)\}$ if there is or there is not improvement in prediction accuracy by the predictor set (i) compared with (ii). Absence of Granger causality implies that the addition of information attributable to v up to time $t-1$ does not improve the prediction accuracy of $u(t)$. This way of defining causality differs from the causality based on intervention, as expounded in Sect. 1.1. The point is that, regardless of how well designed an experiment is, if the knowledge of the cause detected by such an experiment does not

Y. Hosoya et al., *Characterizing Interdependencies of Multiple Time Series*, JSS Research Series in Statistics, DOI 10.1007/978-981-10-6436-4_2

help to improve the prediction of the dependent variable in real-life circumstances, the high quality of the experimental design alone is not sufficient for establishing empirical causality.

Granger causality is definable for wide varieties of time-series models, including linear or nonlinear, stationary or non-stationary, and parametric or nonparametric modes. Additionally, a variety of methods exist for measuring prediction improvement. A vast amount of literature exists on estimating and testing Granger causality using time-domain or frequency-domain representations. This book focuses on the mean-square error of the best one-step ahead prediction of second-order processes. To compare the prediction errors of the two prediction methods, it is convenient to use the difference in the log determinants of one-step ahead prediction error covariance matrices. Moreover, for the difference to have harmonic decomposition, namely for the frequency-wise contribution of the difference to be explicitly representable, it is convenient to use as the additional information not the past information $\{v(s), s \leq t-1\}$ but, instead, its one-way effect components. This one-way effect concept is particularly important in eliminating third-series confounding effects, as discussed in Chap. 3.

The cross-spectrum is a measure of the association between component series that constitutes a multivariate time series. It expresses the covariance between frequency-wise variation elements and is used to quantify short- and long-term time-series interdependence. However, the spectrum itself is not quite fit for characterizing lead or lag dependence between time-series variations. For that purpose, we need knowledge of the canonical factor of the spectral density matrix of the subject-matter time series and the allied prediction theory.

2.1.2 Terminology and Notations

The following notations and symbols are used throughout. The sets of all integers, nonnegative integers, non-positive integers, and positive integers are denoted, respectively, by $\mathbb{Z}$, $\mathbb{Z}^{0+}$, $\mathbb{Z}^{0-}$, and $\mathbb{Z}^{+}$. For a set of random variables $\{w_i, i \in \mathbb{A}\}$ with finite second moment, $H\{w_i, i \in \mathbb{A}\}$ implies the closure in the mean square of the linear hull of $\{w_i, i \in \mathbb{A}\}$ in the Hilbert space of all random variables with finite second moment. For a p-vector process $x(t)$ with finite covariance matrix and for a set of integers $\mathbb{S}$, $H\{x(t), t \in \mathbb{S}\}$ implies $H\{x_i(t), t \in \mathbb{S}, i = 1, \cdots, p\}$. For a notational economy, $H\{x(t_1 - j), y(t_2 - j);\ \ j \in \mathbb{Z}^{0+}\}$ is written simply as $H\{x(t_1), y(t_2)\}$, and $H\{x(j);\ j \in \mathbb{Z}\}$ is written as $H\{x(\infty)\}$. See Appendix A.1 for a brief introduction of Hilbert space.

We identify an information set of variables with the Hilbert subspace generated by the set of variables. Thus, we identify the linear prediction of a variable using the information set of predictor variables with the orthogonal projection of the predicted variable onto the Hilbert space generated by the predictor variables, where the projection is the best linear predictor and the accompanying prediction error is the residual (or perpendicular) of the projection. Namely, we translate the linear prediction problems to those of projections onto Hilbert subspaces.

This book primarily follows the standard notations and the basic framework of the theory; however, the one-way effect concept requires projections of random vectors onto special Hilbert subspaces. Let $\{u(t); t \in \mathbb{Z}\}$ and $\{v(t); t \in \mathbb{Z}\}$ ($\mathbb{Z}$: the set of all integers) be, respectively, real p_1- and p_2-dimensional second-order processes with mean 0 defined on a common probability space; a stochastic process with finite covariances is termed a second-order process. Denote by H the Hilbert subspace spanned by all the component random variables of $\{u(t), v(t); t \in \mathbb{Z}\}$ in the Hilbert space of all random variables with finite second moment. In this book, the projection of a random vector $z = \{z_j; j = 1, \ldots, r\}$ onto $H\{\cdot\}$, a subspace of H, implies component-wise projection. Namely, if $\tilde{z}_j$ is the projection of z_j onto $H\{\cdot\}$, then the projection of z onto $H\{\cdot\}$ implies the vector $\tilde{z}$, whose jth component is $\tilde{z}_j$. If each component z_j of a vector z belongs to $H\{\cdot\}$, the random vector z is said to belong to $H\{\cdot\}$. The difference $z - \tilde{z}$ is termed the residual of the projection of z onto $H\{\cdot\}$. If each component z_j is orthogonal to $H\{\cdot\}$, z is said to be orthogonal to $H\{\cdot\}$ and denoted as $z \perp H\{\cdot\}$. Two subspaces H_1 and H_2, which are orthogonal to each other, are denoted as $H_1 \perp H_2$. The orthogonal complement of $H\{\cdot\}$ in $H \equiv H\{u(\infty), v(\infty)\}$ is denoted as $H\{\cdot\}^{\perp}$. The subscripts and prime attached to u and v denote the subspace to which a concerned projection is related, and the upper bar denotes the projection. For example, for a second-order stationary process $\{u(t), v(t)\}$, $u_{-1,.}(t)$, $u_{-1,-1}(t)$, and $u_{-1,0}(t)$ are the residuals of the projection of $u(t)$ onto $H\{u(t-1)\}$, $H\{u(t-1), v(t-1)\}$, and $H\{u(t-1), v(t)\}$, respectively, and $\bar{u}_{-1,.}(t)$, $\bar{u}_{-1,-1}(t)$, and $\bar{u}_{-1,0}(t)$ are the corresponding projections. The prime denotes that $u(t)$ and $v(t)$ are projected subspaces of $H\{u(\infty), v_{0,-1}(\infty)\}$ and $H\{u_{-1,0}(\infty), v(\infty)\}$, respectively, such that $u'_{-1,-1}(t)$ and $u'_{.,\infty}(t)$ are the residuals of the projection of $u(t)$ onto $H\{u(t-1), v_{0,-1}(t-1)\}$, and $H\{v_{0,-1}(\infty)\}$, whereas $v'_{-1,-1}(t)$ and $v'_{\infty,.}(t)$ are the projections of $v(t)$ onto $H\{u_{-1,0}(t-1), v(t-1)\}$ and $H\{u_{-1,0}(\infty)\}$.

To quantify the contribution of the process $\{v(t)\}$ to the improvement of the one-step ahead prediction of $\{u(t)\}$, Geweke (1982) introduced the log-ratio

$$F_{v \to u} = \log[\det\{Cov(u_{-1,.}(t))\} / \det\{Cov(u_{-1,-1}(t))\}], \tag{2.1}$$

and termed it the measure of linear feedback. To represent the frequency-wise contribution of the process $\{v(t)\}$ to $\{u(t)\}$ when they jointly constitute an autoregressive process, he proposed a nonnegative function $F_{v \to u}(\lambda)$, showing under certain conditions that $F_{v \to u}$ can be represented as the integral of $F_{v \to u}(\lambda)$ over the frequency domain. In contrast, Akaike (1968) introduced the measure termed relative power contribution (RPC), which quantifies the frequency-wise contribution of a component noise process to an observation series in a multivariate autoregressive process. Through a suitable extension, his RPC measure is related to the one-way effect measure, as is shown in Sect. 2.5. [See Granger (1969) and Pierce (1979) for other proposals of measurements of interdependency.]

Because $\{v_{0,-1}(t); t \in \mathbb{Z}\}$ can be regarded as the proper innovation process contained in $\{v(t)\}$, Sect. 2.3 of this chapter proposes as the (overall) measure of the one-way effect of the process $\{v(t)\}$ to $\{u(t)\}$ the log-ratio

$$M_{v\to u} = \log[\det\{Cov(u_{-1,.}(t))\}/\det\{Cov(u'_{-1,-1}(t))\}].$$

In contrast to $F_{v\to u}$, the measure $M_{v\to u}$ has a natural decomposition into a frequency-wise measure $M_{v\to u}(\lambda)$, which is introduced in (2.16) such that

$$M_{v\to u} = \frac{1}{2\pi}\int_{-\pi}^{\pi} M_{v\to u}(\lambda)d\lambda$$

in general circumstances (Theorem 2.2). Section 2.4 constructs $M_{v\to u}(\lambda)$ by two other approaches and, thus, provides alternative representations. Theorem 2.3 shows that one of the constructed measures $\tilde{M}_{v\to u}(\lambda)$ is equal to $M_{v\to u}(\lambda)$. Moreover, in that section, $\tilde{M}_{v\to u}(\lambda)$ is shown to be equal to Geweke's $F_{v\to u}(\lambda)$ for the autoregressive process; thus, $M_{v\to u}(\lambda)$ is an extension of the latter.

Section 2.5 introduces the measures of association and reciprocity in the frequency domain and shows that the measure of association is decomposed into the sum of the measures of the one-way effect and of reciprocity [Theorem 2.5]. Section 2.5 also provides a sufficient condition for the corresponding overall measures to be equal to the Gel'fand–Yaglom measure (2.34) and the Geweke measure (2.1), respectively [Theorem 2.6]. Remark 2.3 discusses the relationship of the one-way effect measure with Akaike's RPC measure. Section 2.6 provides two examples for illustration purposes. One example is the case in which a process is not invertible but $M_{v\to u}(\lambda)$ is measured. The other example illustrates a situation in which $F_{v\to u} > M_{v\to u}$.

Regarding further notations used in this book, $\{x(t)\}$ denotes the process $\{x(t); t \in \mathbb{Z}\}$ unless otherwise specified. For a specified partition of a $(p_1 + p_2) \times (p_1 + p_2)$ matrix A

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

always implies that A_{11} is a $p_1 \times p_1$ submatrix. Two random variables that are equal with probability 1 are identified such that the a.e. notation is omitted. For a random vector x or for a pair of random vectors x and y, $Cov(x)$ and $Cov(x, y)$ indicate the variance-covariance matrix of x and $vec(x, y)$, respectively. The identity matrix of order p is denoted by I_p. A^* indicates the conjugate transpose if A is a complex matrix and the simple transpose if A is a real matrix. Sometimes, A' is also used to denote the transpose of a real matrix A. The trace of a square matrix C is denoted by $tr\, C$, and the determinant is denoted by $\det C$. The sum of two vector subspaces H_1 and H_2 is denoted by $H_1 \oplus H_2$. The symbol $\equiv$ is used for definitions.

2.2 Defining Non-causality

This section expounds on the concepts of Granger's non-causality and Sims's counterpart explicitly for a pair of vector-valued second-order processes [See Granger (1963, 69), Sims (1972), Hosoya (1977), and Florens and Mouchart (1982) for related literature.]. Let $\{u(t)\}$ and $\{v(t)\}$ be, respectively, real p_1- and p_2-vector (not necessarily stationary) stochastic processes with mean 0 and finite second-order moments defined in a common probability space. The Granger condition for non-causality in terms of the mean-square prediction error is defined as follows. [Granger throughout attributed the idea to N. Wiener.]

Definition 2.1 The process $\{v(t)\}$ does not cause $\{u(t)\}$ if the projection of $u(t)$ onto $H\{u(t-1), v(t-1)\}$ belongs to $H\{u(t-1)\}$ for all $t \in \mathbb{Z}$.

Lemma 2.1 *This Granger condition is equivalent to the relationship*

$$u_{-1,.}(t) \perp H\{u(t-1), v(t-1)\}, \quad t \in \mathbb{Z},$$

where $u_{-1,.}(t)$ *denotes the residual of the projection of* $u(t)$ *onto* $H\{u(t-1)\}$.

Proof Suppose that $u_{-1,.}(t) \perp H\{u(t-1), v(t-1)\}$. Because the projection of $u_{-1,.}(t)$ onto $H\{u(t-1), v(t-1)\}$ is 0 and because the projection of $\bar{u}_{-1,.}(t)$ onto $H\{u(t-1), v(t-1)\}$ is $\bar{u}_{-1,.}(t)$ itself, the projection of $u(t)$ onto $H\{u(t-1), v(t-1)\}$ is equal to $\bar{u}_{-1,.}(t)$, which belongs to $H\{u(t-1)\}$. Hence, the Granger condition follows. To prove the reverse implication, suppose that $\bar{u}_{-1,-1}(t) = \bar{u}_{-1,.}(t)$, whence we have $u_{-1,-1}(t) = u_{-1,.}(t)$. Because $u_{-1,-1}(t) \perp H\{u(t-1), v(t-1)\}$, the assertion $u_{-1,.}(t) \perp H\{u(t-1), v(t-1)\}$ follows. □

Denote by $\bar{v}_{0,.}(t)$ the projection of $v(t)$ onto $H\{u(t)\}$, and set $v_{0,.}(t) = v(t) - \bar{v}_{0,.}(t)$. The decomposition $v(t) = \bar{v}_{0,.}(t) + v_{0,.}(t)$, $t \in \mathbb{Z}$ is the Sims distributed-lag representation of $v(t)$ by the process $\{u(t)\}$, where we have $\bar{v}_{0,.}(t) \in H\{u(t)\}$ and $v_{0,.}(t) \in H\{u(t)\}^{\perp}$. See (1.4) for the autoregressive distributed-lag representation.

Definition 2.2 The Sims condition for $\{v(t)\}$ not causing $\{u(t)\}$ implies that $v_{0,.}(t)$ is orthogonal to $H\{u(\infty)\}$ for all t, where $v_{0,.}(t)$ is the error term in the Sims distributed-lag representation of $v(t)$ by $u(t-j)$, $j \in \mathbb{Z}^{0+}$, and $H\{u(\infty)\}$ denotes the completion of the linear hull of the set $\{u(t), t \in \mathbb{Z}\}$.

Theorem 2.1 *The Granger condition is equivalent to the Sims condition.*

Proof First, we show that the Sims condition follows from the Granger condition. Because $v_{0,.}(t) \perp H\{u(t)\}$, the problem is to show that $u(t+p) \perp v_{0,.}(t)$ for any $p \geq 1$ and $t \in \mathbb{Z}$. Set $h(t+p-1) = \bar{u}_{-1,.}(t+p)$, and set $\varepsilon(t+p) \equiv u_{-1,.}(t+p)$. For $i = 2, \ldots, p$, let $h(t+p-i)$ be the projection of $h(t+p-i+1)$ onto $H\{u(t+p-i)\}$ and set the residual as $\varepsilon(t+p-i+1) = h(t+p-i+1) - h(t+p-i)$. Then, by iterative projection we have

$$\begin{aligned}u(t+p) &= h(t+p-1)+\varepsilon(t+p)\\ &= h(t+p-2)+\{\varepsilon(t+p)+\varepsilon(t+p-1)\}\\ &= h(t)+\sum_{i=0}^{p-1}\varepsilon(t+p-i)\equiv h(t)+\xi(t+p).\end{aligned}$$

In view of Lemma 2.1, the Granger non-causality implies

$$\varepsilon(t+p)\perp H\{u(t+p-1), v(t+p-1)\}.$$

Denote by $J\{u(t+p-1)\}$ the vector space spanned singly by $u(t+p-1)$. Then, we have $h(t+p-1)\equiv \bar{u}_{-1,.}(t+p)\in J\{u(t+p-1)\}\oplus H\{u(t+p-2)\}$; namely, $h(t+p-1)$ is expressed as

$$h(t+p-1)=c_1u(t+p-1)+l(t+p-2), \tag{2.2}$$

where c_1 is a constant, $p_1\times p_1$ is a matrix, and $l(t+p-2)\in H\{u(t+p-2)\}$. By projecting $h(t+p-1)$ onto $H\{u(t+p-2)\}$, we have

$$h(t+p-1)=h(t+p-2)+\varepsilon(t+p-1)$$

where $\varepsilon(t+p-1)$ is the projection residuals of $c_1u(t+p-1)$; hence, it follows from the Granger condition and (2.2) that

$$\varepsilon(t+p-1)\perp H\{u(t+p-2), v(t+p-2)\}.$$

By repeating a similar argument, we can conclude that $h(t)\in H\{u(t)\}$ and $\xi(t+p)\perp H\{u(t), v(t)\}$. Because then $v_{0,.}(t)\in H\{u(t), v(t)\}$, we have the orthogonality $\xi(t+p)\perp v_{0,.}(t)$. Then, because $h(t)\in H\{u(t)\}$ and $v_{0,.}(t)\perp H\{u(t)\}$, we have $v_{0,.}(t)\perp h(t)$. Consequently, the relation $v_{0,.}(t)\perp u(t+p)$ follows.

To prove the reverse implication, suppose that the distributed-lag representation of $v(t)$ satisfies the Sims condition; namely, suppose that the subspace $H\{v(t-1)\}$ is included in the vector sum $H\{u(t-1)\}\oplus H\{u(\infty)\}^{\perp}$. If $g\in H\{v(t-1)\}$, it is a limit in the mean-square distance of the finite sum $\sum_{j=0}^{p}c_jv(t-j-1)\in H\{v(t-1)\}\subset H\{u(t-1)\}\oplus H\{u(\infty)\}^{\perp}$. Hence, we have $H\{u(t-1), v(t-1)\}\subset H\{u(t-1)\}\oplus H\{u(\infty)\}^{\perp}$. Because $u_{-1,.}(t)\in H\{u(\infty)\}$, we have $u_{-1,.}(t)\perp H\{u(\infty)\}^{\perp}$. Supposing that $h(t)$ is an element of $H\{v(t-1)\}$, set $h(t)=l(t)+\lambda(t)$ where $l(t)\in H\{u(t-1)\}$ and $\lambda(t)\in H\{u(\infty)\}^{\perp}$. The relations $u_{-1,.}(t)\perp l(t)$ and $u_{-1,.}(t)\perp\lambda(t)$ hold, but they imply that $u_{-1,.}(t)\perp h(t)$. Because $h(t)$ is arbitrary, $u_{-1,.}(t)\perp H\{v(t-1)\}$. Thus, $u_{-1,.}(t)$ is orthogonal to $H\{u(t-1), v(t-1)\}$. It follows then from Lemma 2.1 that the Granger condition holds for the process $\{u(t), v(t)\}$. □

Corollary 2.1 $\bar{u}'_{-1,-1}(t)\in H\{u(t-1)\}$ *if and only if* $v_{0,-1}(t)\in H(u(\infty))^{\perp}$.

Proof The Granger condition that $\{v_{0,-1}(t)\}$ does not cause $\{u(t)\}$ is the left-hand side condition, whereas the Sims condition is given by $v_{0,-1}(t) \in H\{u(t)\} \oplus H(u(\infty))^{\perp}$. However, because the projection of $v_{0,-1}(t)$ onto $H\{u(t)\}$ is 0, the corollary follows. □

The next corollary asserts that $\{v(t)\}$ does not cause $\{u(t)\}$ if and only if $\{v_{0,-1}(t)\}$ does not cause $\{u(t)\}$ under the assumption of pure non-determinism.

Corollary 2.2 *Suppose that* $\{v(t)\}$ *is purely non-deterministic, namely* $\bigcap_{j=0}^{\infty} H\{v(t-j)\} = \{0\}$. *Then,* $\bar{u}_{-1,-1}(t)$ *belongs to* $H\{u(t-1)\}$ *if and only if* $\bar{u}'_{-1,-1}(t)$ *belongs to* $H\{u(t-1)\}$.

Proof Because $H\{u(t-1), v_{0,-1}(t-1)\} \subset H\{u(t-1), v(t-1)\}$, the necessity is evident.

The sufficiency is shown as follows. Corollary 2.1 implies that $v_{0,-1}(t) \in H^{\perp}\{u(\infty)\}$ and, by definition, $\bar{v}_{0,-1}(t) \in \{u(t), v(t-1)\}$. Therefore, we have

$$v(t) = \bar{v}_{0,-1}(t) + v_{0,-1}(t) \in H\{u(t), v(t-1)\} \oplus H^{\perp}\{u(\infty)\}. \tag{2.3}$$

Hence

$$H\{u(t), v(t)\} \oplus H^{\perp}\{u(\infty)\} = H\{u(t), v(t-1)\} \oplus H^{\perp}\{u(\infty)\}.$$

Repeating the same argument, we have

$$H\{u(t), v(t)\} \oplus H^{\perp}\{u(\infty)\} = H\{u(t), v(t-j)\} \oplus H^{\perp}\{u(\infty)\} \tag{2.4}$$

for any positive integer j. Thanks to the assumption, $\bigcap_{j=0}^{\infty} H\{v(t-j)\} = \{0\}$. It then follows from (2.4) that

$$H\{u(t), v(t-1)\} \oplus H^{\perp}\{u(\infty)\} \subset H\{u(t)\} \oplus H^{\perp}\{u(\infty)\},$$

whence in view of (2.3) we have $v(t) \in H\{u(t)\} \oplus H^{\perp}\{u(\infty)\}$. Accordingly, Corollary 2.2 follows from Corollary 2.1. □

2.3 The One-Way Effect Measure

The one-way effect from one process to another is quantitatively characterized using the prediction theory of stationary processes. Let $\{u(t), v(t),\ t \in \mathbb{Z}\}$ be a zero mean jointly second-order non-deterministic full-rank stationary process where the $u(t)$ and $v(t)$ are real p_1 and p_2 vectors, respectively. Suppose also that the process has the $p \times p$ spectral density matrix

$$h(\lambda) = \begin{bmatrix} h_{11}(\lambda) & h_{12}(\lambda) \\ h_{21}(\lambda) & h_{22}(\lambda) \end{bmatrix}, \quad -\pi < \lambda \leq \pi,$$

where ($p = p_1 + p_2$) and $h_{11}(\lambda)$ is the $p_1 \times p_1$ spectral density of $\{u(t)\}$. Namely, the second-order stationarity implies that the covariance matrix

$$E\begin{bmatrix} u(t+s) \\ v(t+s) \end{bmatrix} [u(t)^*, v(t)^*] = V(s)$$

does not depend on t, for any $t, s \in \mathbb{Z}$. The matrix $V(s)$ is termed serial covariance matrix, and we assume it is representable as

$$V(s) = \int_{-\pi}^{\pi} h(\lambda) e^{i\lambda s} d\lambda, \quad s \in \mathbb{Z}$$

by means of an Hermite matrix-valued function $h(\lambda)$. The matrix $h(\lambda)$ is termed the spectral density matrix of the process $\{u(t), v(t)\}$.

Assumption 2.1 The spectral density matrix of the process $\{u(t), v(t)\}$ satisfies the Szegö condition

$$\int_{-\pi}^{\pi} \log \det h(\lambda) d\lambda > -\infty. \tag{2.5}$$

A process is said to have maximal rank if it has a full-rank spectral density matrix $a.e.$ on $(-\pi, \pi]$. The Szegö condition is necessary and sufficient for $\{u(t), v(t)\}$ being purely non-deterministic and $\det h(\lambda) > 0$ $a.e.$ with respect to the Lebesgue measure on the torus, see Rozanov (1967, p. 73). The condition is assumed throughout the sequel. Under the condition (2.5), $h(\lambda)$ is known to have a factorization such that

$$h(\lambda) = \frac{1}{2\pi} \Gamma(e^{-i\lambda}) \Gamma(e^{-i\lambda})^*, \tag{2.6}$$

where $\Gamma(e^{-i\lambda})$ is the boundary value $\lim_{\mu \to 1-} \Gamma(\mu e^{-i\lambda})$ of a $p \times p$ matrix-valued function $\Gamma(z)$, which is analytic, and $\det \Gamma(z)$ has no zeros inside the unit disk $\{z : |z| < 1\}$ of the complex plane. Let $\Gamma_1(z)$ be another such function satisfying those same conditions of $\Gamma(z)$ including the boundary condition (2.6). If there is no $\Gamma_1(z)$ such that $\Gamma_1(0)\Gamma_1(0)^* - \Gamma(0)\Gamma(0)^*$ is positive definite, the factor $\Gamma(z)$ is said maximal.

Such a factorization (2.6) by means of a maximal $\Gamma(z)$ is said to be a canonical factorization in the sequel. Denote the covariance matrix of the one-step ahead prediction error of the process $\{u(t), v(t)\}$ by

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

where $\Sigma = Cov\{u_{-1,-1}(t), v_{-1,-1}(t)\}$. Thanks to the Szegö condition, Σ is positive definite. For those discussions and the next lemma, see Rozanov (1967, pp. 71–77).

Lemma 2.2 *$\Gamma(z)$ is maximal if and only if*

$$\det\{\Gamma(0)\Gamma(0)^*\} = \det \Sigma = (2\pi)^p \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det h(\lambda)d\lambda\}. \quad (2.7)$$

We define $M_{v\to u}$ as the (overall) measure of one-way effect from v to u by

$$M_{v\to u} = \log[\det\{Cov(u_{-1,\cdot}(t))\}/\det\{Cov(u'_{-1,-1}(t))\}]. \quad (2.8)$$

Because $\det Cov(u_{-1,-1}(t)) \le \det Cov(u'_{-1,-1}(t))$, we have the next lemma.

Lemma 2.3 *Let $F_{v\to u}$ be the Geweke measure given by*

$$F_{v\to u} \equiv \log[\det\{Cov(u_{-1,\cdot}(t))\}/\det\{Cov(u_{-1,-1}(t))\}].$$

We have $0 \le M_{v\to u} \le F_{v\to u}$. The equality $M_{v\to u} = F_{v\to u}$ holds if and only if $\bar{u}_{-1,-1}(t)$ belongs to $H\{u(t-1), v_{0,-1}(t-1)\}$.

Example 2.2 of Sect. 2.6 provides the case in which $M_{v\to u} < F_{v\to u}$. As is seen in the construction of $v_{0,-1}(t)$, $M_{v\to u}$ quantifies the extent of the proper contribution by the component of $\{v(t)\}$, which does not contain the feedback effect from $\{u(t)\}$.

The following three lemmas are preliminary to the construction of the frequency-wise measure $M_{v\to u}(\lambda)$.

Lemma 2.4 *The process $\{u(t), v_{0,-1}(t)\}$ has the maximal rank and is purely non-deterministic. Additionally, we have*

$$v_{0,-1}(t) = v_{-1,-1}(t) - \Sigma_{21}\Sigma_{11}^{-1}u_{-1,-1}(t). \quad (2.9)$$

Moreover, $\{v_{0,-1}(t)\}$ is a white noise process with covariance matrix $\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$.

Proof In the decomposition

$$\begin{aligned} v(t) &= \bar{v}_{-1,-1}(t) + v_{-1,-1}(t) \\ &= \{v_{-1,-1}(t) - \Sigma_{21}\Sigma_{11}^{-1}u_{-1,-1}(t)\} + \{\Sigma_{21}\Sigma_{11}^{-1}u_{-1,-1}(t) + \bar{v}_{-1,-1}(t)\}, \end{aligned}$$

we have $\{v_{-1,-1}(t) - \Sigma_{21}\Sigma_{11}^{-1}u_{-1,-1}(t)\} \perp H\{u_{-1,-1}(t), u(t-1), v(t-1)\}$ and $\{\Sigma_{21}\Sigma_{11}^{-1}u_{-1,-1}(t) + \bar{v}_{-1,-1}(t)\} \in H\{u(t), v(t-1)\}$. By the uniqueness of the projection residual, we have

$$v_{0,-1}(t) = v_{-1,-1}(t) - \Sigma_{21}\Sigma_{11}^{-1}u_{-1,-1}(t).$$

Next, suppose that a scalar-valued process $\{s(t)\}$ is given by

$$\begin{aligned} s(t) &= \alpha^* u(t) + \beta^* v_{0,-1}(t) \\ &= \alpha^* \bar{u}_{-1,-1}(t) + \{\gamma^* u_{-1,-1}(t) + \beta^* v_{-1,-1}(t)\}, \end{aligned}$$

where α, β are real p_1 and p_2 vectors, respectively, not all zero, and $\gamma^* \equiv \alpha^* - \beta^* \Sigma_{21} \Sigma_{11}^{-1}$. That the process $\{u(t), v_{0,-1}(t)\}$ has maximal rank is proven if the spectral density $h_s(\lambda)$ of the process $\{s(t)\}$ is positive $a.e.$. The first sum is orthogonal to the last two sums, whereas the sum of those last two constitutes a white noise with positive variance because γ and β cannot be simultaneously zero unless α and β are all simultaneously zero. Therefore, the spectral density $h_s(\lambda)$ is the sum of the spectral density $\alpha^* \bar{u}_{-1,-1}(t)$ and the spectral density of the sum of the last two terms, which is a positive constant. Therefore, $h_s(\lambda) > 0$. The proposition that the process $\{u(t), v_{0,-1}(t)\}$ is purely non-deterministic follows from the relations

$$\bigcap_{j=0}^{\infty} H\{u(t-j), v_{0,-1}(t-j)\} \subset \bigcap_{j=0}^{\infty} H\{u(t-j), v(t-j)\} = \{0\}.$$

Because $\{u_{-1,-1}(t), v_{-1,-1}(t)\}$ is a white noise process, $\{v_{0,-1}(t)\}$ is also a white noise process in view of (2.9). □

Let $\tilde{h}$ and $\breve{h}$ be the spectral densities of the joint processes $\{u(t), v_{0,-1}(t)\}$ and $\{u_{-1,0}(t), v(t)\}$, and denote the partitioned matrices by

$$\tilde{h}(\lambda) = \begin{pmatrix} \tilde{h}_{11}(\lambda) & \tilde{h}_{12}(\lambda) \\ \tilde{h}_{21}(\lambda) & \tilde{h}_{22}(\lambda) \end{pmatrix}; \quad \breve{h}(\lambda) = \begin{pmatrix} \breve{h}_{11}(\lambda) & \breve{h}_{12}(\lambda) \\ \breve{h}_{21}(\lambda) & \breve{h}_{22}(\lambda) \end{pmatrix}. \tag{2.10}$$

In view of the construction, it is evident that $\tilde{h}_{11}(\lambda) = h_{11}(\lambda)$, $\tilde{h}_{22}(\lambda) = (\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})/(2\pi) \equiv \Sigma_{22:1}/(2\pi)$, $\tilde{h}_{21}(\lambda) = \tilde{h}_{12}(\lambda)^*$; $\breve{h}_{11}(\lambda) = (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})/(2\pi) \equiv \Sigma_{11:2}/(2\pi)$ and $\breve{h}_{22}(\lambda) = h_{22}(\lambda)$, $\breve{h}_{21}(\lambda) = \breve{h}_{12}(\lambda)^*$. Expressions for $\tilde{h}_{12}(\lambda)$ and $\breve{h}_{21}(\lambda)$ are given in Lemma 2.5, where we note that the inverse $\Gamma(e^{-i\lambda})^{-1}$ exists $a.e.$ because h has maximal rank.

Lemma 2.5 *$\tilde{h}_{12}(\lambda)$ and $\breve{h}_{12}(\lambda)$ are represented as:*

$$\tilde{h}_{12}(\lambda) = h_{1\cdot}\Gamma(e^{-i\lambda})^{-1*}\Gamma(0)^*(-\Sigma_{21}\Sigma_{11}^{-1}, I_{p_2})^*, \tag{2.11}$$
$$\breve{h}_{12}(\lambda) = (I_{p_1}, -\Sigma_{12}\Sigma_{22}^{-1})\Gamma(0)\Gamma(e^{-i\lambda})^{-1}h_{\cdot 2}(\lambda), \tag{2.12}$$

where $h_{1\cdot}(\lambda)$ is a $p_1 \times (p_1 + p_2)$ matrix that consists of the first p_1 rows of $h(\lambda)$ and $h_{\cdot 2}(\lambda)$ denotes the $(p_1 + p_2) \times p_2$ matrix consisting of the last p_2 columns of $h(\lambda)$.

Proof Set $A \equiv (-\Sigma_{21}\Sigma_{11}^{-1}, I_{p_2})$. Denote by $\Phi_u(d\lambda)$ and $\Phi_v(d\lambda)$ the orthogonal-increment random spectral measures of the process $\{u(t)\}$ and $\{v(t)\}$; namely, we have

$$u(t) = \int_{-\pi}^{\pi} e^{i\lambda t}\Phi_u(d\lambda) \quad \text{and} \quad v(t) = \int_{-\pi}^{\pi} e^{i\lambda t}\Phi_v(d\lambda).$$

Because $\Gamma(e^{-i\lambda})$ is invertible, the innovation process for $\{u(t), v(t)\}$ is given by

$$\begin{bmatrix} u_{-1,-1}(t) \\ v_{-1,-1}(t) \end{bmatrix} = \int_{-\pi}^{\pi} e^{i\lambda t}\Gamma(0)\Gamma(e^{-i\lambda})^{-1}\begin{bmatrix} \Phi_u(d\lambda) \\ \Phi_v(d\lambda) \end{bmatrix};$$

whence the one-way effect component of $v(t)$ is represented by

$$\begin{aligned} v_{0,-1}(t) &= v_{-1,-1}(t) - \Sigma_{21}\Sigma_{11}^{-1}u_{-1,-1}(t) \\ &= \int_{-\pi}^{\pi} e^{i\lambda t} A\Gamma(0)\Gamma(e^{-i\lambda})^{-1}\begin{bmatrix} \Phi_u(d\lambda) \\ \Phi_v(d\lambda) \end{bmatrix}. \end{aligned}$$

Then, the covariance matrix $Cov(u(t), v_{0,-1}(s))$ is expressed in the Fourier transform as

$$\begin{aligned} Cov\{u(t), v_{0,-1}(s)\} &= \int_{-\pi}^{\pi} e^{i\lambda(t-s)}E\{\Phi_u(d\lambda)\begin{bmatrix} \Phi_u(d\lambda) \\ \Phi_v(d\lambda) \end{bmatrix}^*\}\Gamma(e^{-i\lambda})^{-1*}\Gamma(0)^*A^* \\ &= \int_{-\pi}^{\pi} e^{i\lambda(t-s)}\{h_{11}(\lambda), h_{12}(\lambda)\}\Gamma(e^{-i\lambda})^{-1*}\Gamma(0)^*A^* d\lambda, \end{aligned} \tag{2.13}$$

where $\{h_{11}(\lambda), h_{12}(\lambda)\}$ is the $p_1 \times (p_1 + p_2)$ upper submatrix of $h(\lambda)$. The expression (2.11) follows from (2.13). The expression (2.12) is derived from a parallel argument. □

Set $h_{11:2}(\lambda) = h_{11}(\lambda) - 2\pi\tilde{h}_{12}(\lambda)\Sigma_{22:1}^{-1}\tilde{h}_{21}(\lambda)$ where $\Sigma_{22:1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$; then we have:

Lemma 2.6 *The covariance matrix $Cov(u'_{-1,-1}(t))$ has the following decomposability property in the frequency domain:*

$$\det Cov(u'_{-1,-1}(t)) = (2\pi)^{p_1}\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\det h_{11:2}(\lambda)d\lambda\right\}. \tag{2.14}$$

Proof Because $u'_{-1,-1}(t) \perp v_{0,-1}(t)$, we have

$$\det Cov\{u'_{-1,-1}(t), v_{0,-1}(t)\} = \det Cov\{u'_{-1,-1}(t)\}\det Cov\{v_{0,-1}(t)\}.$$

Then, in view of Lemma 2.4, we have

$$\begin{aligned}
&\log\det Cov\{u'_{-1,-1}(t), v_{0,-1}(t)\}\\
&= \log\left[(2\pi)^{p_1+p_2}\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\det\tilde{h}(\lambda)d\lambda\right\}\right]\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\log\det\tilde{h}_{22}(\lambda) + \log\det\left\{h_{11}(\lambda) - \tilde{h}_{12}(\lambda)\tilde{h}_{22}^{-1}(\lambda)\tilde{h}_{21}(\lambda)\right\}\right]d\lambda\\
&\quad + (p_1 + p_2)\log(2\pi). \qquad (2.15)
\end{aligned}$$

Because $\{v_{0,-1}(t)\}$ is a white noise process with the spectral density $\tilde{h}_{22}(\lambda) = \Sigma_{22:1}/(2\pi)$ such that $\det Cov\{v_{0,-1}(t)\} = \det \Sigma_{22:1}$. The relation (2.14) follows from (2.15). □

Using the decomposition of Lemma 2.6, define the measure $M_{v\to u}(\lambda)$ of the one-way effect from v to u at frequency λ by

$$M_{v\to u}(\lambda) = \log\{\det h_{11}(\lambda)/\det h_{11:2}(\lambda)\},\ -\pi < \lambda \leq \pi. \qquad (2.16)$$

Note that the expression $h_{11:2}(\lambda)$ in the denominator in (2.16) denotes the spectral density matrix of the process $\{u'_{\cdot,\infty}(t)\}$. It is evident that $M_{v\to u}(\lambda) \geq 0$. Moreover, since

$$\det Cov(u_{-1,\cdot}(t)) = (2\pi)^{p_1}\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\det h_{11}(\lambda)d\lambda\right\},$$

the foregoing arguments imply that the next theorem holds for the overall measure of one-way effect defined in (2.8).

Theorem 2.2 *We have*

$$M_{v\to u} = \frac{1}{2\pi}\int_{-\pi}^{\pi} M_{v\to u}(\lambda)d\lambda, \qquad (2.17)$$

where $M_{v\to u} = 0$ *if and only if* $\bar{u}_{-1,-1}(t)$ *belongs to* $H\{u(t-1)\}$.

Remark 2.1 Note that the joint process $\{u_{-1,0}(t), v_{0,-1}(t)\}$ of one-way effects has the same information as $\{u(t), v(t)\}$ in the sense that $H\{u_{-1,0}(t), v_{0,-1}(t)\} = H\{u(t), v(t)\}, t \in \mathbb{Z}$.

This is observed as follows. It follows from Lemma 2.4 and the corresponding relation for $u_{-1,0}(t)$ that

$$\begin{bmatrix} u_{-1,0}(t) \\ v_{0,-1}(t) \end{bmatrix} = \begin{bmatrix} I_{p_1} & -\Sigma_{12}\Sigma_{22}^{-1} \\ -\Sigma_{21}\Sigma_{11}^{-1} & I_{p_2} \end{bmatrix}\begin{bmatrix} u_{-1,-1}(t) \\ v_{-1,-1}(t) \end{bmatrix}$$

where the matrix on the right-hand side is non-singular because

$$\det \begin{bmatrix} I_{p_1} & -\Sigma_{12}\Sigma_{22}^{-1} \\ -\Sigma_{21}\Sigma_{11}^{-1} & I_{p_2} \end{bmatrix} = \det \Sigma_{22}^{-1} \det\{\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\}$$
$$= \det \Sigma_{11}^{-1} \det \Sigma_{22}^{-1} \det \Sigma > 0.$$

Consequently, we have

$$H\{u_{-1,0}(t), v_{0,-1}(t)\} = H\{u_{-1,-1}(t), v_{-1,-1}(t)\} = H\{u(t), v(t)\},$$

where the second equality follows from Assumption 2.1, which implies the invertibility of $\Gamma(e^{-i\lambda})\Gamma(0)^{-1}$.

2.4 Alternative Methods for Deriving $M_{v\to u}(\lambda)$

2.4.1 Distributed-Lag Representation Approach

Because $\{u(t)\}$ does not cause $\{v_{0,-1}(t)\}$, we have in view of Definition 2.2 the distributed-lag representation under Assumption 2.1:

$$u(t) = \sum_{j=0}^{\infty} \Pi(j) v_{0,-1}(t-j) + n(t), \quad t \in \mathbb{Z} \tag{2.18}$$

where the $\Pi(j)$ are $p_1 \times p_2$ matrices and $\{n(t)\}$ is a possibly serially correlated stationary process that is orthogonal to the process $\{v_{0,-1}(t)\}$, whence $h_{11}(\lambda)$, the spectral density of $\{u(t)\}$, is decomposed as

$$h_{11}(\lambda) = h^{(1)}(\lambda) + h^{(2)}(\lambda),$$

where

$$h^{(1)}(\lambda) = \frac{1}{2\pi}\left(\sum_{j=0}^{\infty} \Pi(j) e^{-ij\lambda}\right) \Sigma_{22:1} \left(\sum_{j=0}^{\infty} \Pi(j) e^{-ij\lambda}\right)^{*}.$$

The log-ratio

$$\tilde{M}_{v\to u}(\lambda) = \log\{\det h_{11}(\lambda)/\det h^{(2)}(\lambda)\} \tag{2.19}$$

and its integration over $[-\pi, \pi]$ are interpretable as measuring, respectively, the frequency-wise and overall projection improvement from the use of the information $H\{v_{0,-1}(t-j), j \in \mathbb{Z}^+\}$. In fact, we have the following equivalence:

Theorem 2.3 $\tilde{M}_{v\to u}(\lambda) = M_{v\to u}(\lambda),\ -\pi < \lambda \le \pi.$

Proof It follows then from the Sims representation (2.18) that $n(t)$ is the projection residual of $u(t)$ onto $H\{v_{0,-1}(s);\quad s \in \mathbb{Z}\}$ and is representable using $\tilde{h}$ defined in (2.10) as

$$n(t) = \int_{-\pi}^{\pi} e^{it\lambda}\{\Phi_u(d\lambda) - \tilde{h}_{12}(\lambda)\tilde{h}_{22}^{-1}(\lambda)\Phi_{v_{0,-1}}(d\lambda)\}. \tag{2.20}$$

[See Whittle (1984) for such a spectral regression representation as (2.20).] In view of (2.18) and $H\{v_{0,-1}(t-1), u(t-1)\} = H\{v_{0,-1}(t-1), n(t-1)\}$, the residual of $u(t)$ projected onto $H\{v_{0,-1}(t-1), u(t-1)\}$ is equal to the residual of $n(t)$ projected onto $H\{n(t-1)\}$. Therefore, we have

$$\det \tilde{\Sigma}_{11} = (2\pi)^{p_1} \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det Cov\{\Phi_u(d\lambda) - \tilde{h}_{12}(\lambda)\tilde{h}_{22}^{-1}(\lambda)\Phi_{v_{0,-1}}(d\lambda)\}\right], \tag{2.21}$$

where $\tilde{\Sigma}_{11}$ denotes the covariance matrix of the one-step ahead prediction error of $n(t)$ by its own past; whereas for $u(t)$ we have the relation

$$\det \Sigma_{11} = (2\pi)^{p_1} \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det Cov\{\Phi_u(d\lambda)\}\right]. \tag{2.22}$$

The comparison of (2.21) and (2.22) implies that the prediction improvement by the additional information of $v_{0,-1}(t)$ is given by

$$\tilde{M}_{v\to u} = \log\{\det \Sigma_{11}/\det \tilde{\Sigma}_{11}\}$$

and that the frequency-wise reduction of the variability is given in view of (2.21) and (2.22) by

$$\begin{aligned}
\tilde{M}_{v\to u}(\lambda) &= \log[\det Cov\{\Phi_u(d\lambda)\}/\det Cov\{\Phi_{\Phi_u}(d\lambda) - \tilde{h}_{12}\tilde{h}_{22}^{-1}(\lambda)\Phi_{v_{0,-1}}(d\lambda)\}] \\
&= \log[\det h_{11}(\lambda)/\det\{h_{11}(\lambda) - \tilde{h}_{12}(\lambda)\tilde{h}_{22}^{-1}(\lambda)\tilde{h}_{21}(\lambda)\}] \\
&\equiv \log[\det h_{11}(\lambda)/\det h_{11:2}(\lambda)] \\
&= M_{v\to u}(\lambda). \quad \square
\end{aligned}$$

2.4.2 *Innovation Orthogonalization Approach*

The one-way effect measure is constructed in this subsection using the moving average (2.25) below on which the innovation accounting of Sims (1980) is based. The one-way effect measure obtained by the MA representation obtained coincides with

Geweke's measure of linear feedback if the joint process $\{u(t), v(t)\}$ is a stationary autoregressive process. Suppose that the spectral density $h(\lambda)$ has a canonical factorization

$$h(\lambda) = \frac{1}{2\pi}\Gamma(e^{-i\lambda})\Gamma(e^{-i\lambda})^*,$$

as in (2.6) of the previous section. Choose symmetric positive definite matrices $\Sigma_{11}^{1/2}$ and $\Sigma_{22:1}^{1/2}$ such that $\Sigma_{11}^{1/2}\Sigma_{11}^{1/2} = \Sigma_{11}$ and $\Sigma_{22:1}^{1/2}\Sigma_{22:1}^{1/2} = \Sigma_{22:1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$. Set

$$\Sigma_{(1)}^{1/2} = \begin{bmatrix} \Sigma_{11}^{1/2} & 0 \\ 0 & \Sigma_{22:1}^{1/2} \end{bmatrix}; \quad \text{hence} \quad \Sigma_{(1)} = \Sigma_{(1)}^{1/2}\Sigma_{(1)}^{1/2} = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{bmatrix}.$$

Moreover, define the $(p_1 + p_2) \times (p_1 + p_2)$ matrix $A_{(1)}$

$$A_{(1)} \equiv \begin{bmatrix} I_{p_1} & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I_{p_2} \end{bmatrix}$$

and set

$$\Gamma^{\dagger}(z) = \Gamma(z)\Gamma(0)^{-1}A_{(1)}^{-1}\Sigma_{(1)}^{1/2}. \tag{2.23}$$

Lemma 2.7 *$\Gamma^{\dagger}(z)$ is a maximal analytic function in the unit disk such that $\Gamma^{\dagger}(0)\Gamma^{\dagger}(0)^* = \Sigma$, and the spectral density h has a canonical factorization*

$$h(\lambda) = \frac{1}{2\pi}\Gamma^{\dagger}(e^{-i\lambda})\Gamma^{\dagger}(e^{-i\lambda})^*.$$

Proof The analyticity of $\Gamma^{\dagger}(z)$ is evident in view of the construction, and also the maximality follows from the assumption $\Gamma^{\dagger}(0)\Gamma^{\dagger}(0)^* = \Sigma$. It follows from (2.23) and $A_{(1)}^{-1}\Sigma_{(1)}A_{(1)}^{-1*} = \Sigma$ that

$$\frac{1}{2\pi}\Gamma^{\dagger}(e^{-i\lambda})\Gamma^{\dagger}(e^{-i\lambda})^* = \frac{1}{2\pi}\Gamma(e^{-i\lambda})\Gamma(0)^{-1}\Sigma\Gamma(0)^{-1*}\Gamma(e^{-i\lambda})^* = h(\lambda). \qquad \square$$

Define $\bar{\Gamma}(z)$ by $\bar{\Gamma}(z) = \Gamma^{\dagger}(z)\Sigma_{(1)}^{-1/2}$. Since $\Gamma^{\dagger}(z)$ is analytic inside the unit circle and so is $\bar{\Gamma}(z)$, the latter has the expansion

$$\bar{\Gamma}(z) = \sum_{j=0}^{\infty}\bar{\Gamma}[j]z^j \tag{2.24}$$

which is convergent on $\{z : |z| < 1\}$ and square integrable *a.e.* on $-\pi < \lambda \le \pi$ for $z = exp(-i\lambda)$. The matrix $\bar{\Gamma}(e^{-i\lambda})$ is said a frequency response function whereas its time-domain counterpart is the moving average representation by means of the real-matrix coefficient $\bar{\Gamma}[j]$ in (2.24)

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \sum_{j=0}^{\infty} \bar{\Gamma}[j] \begin{bmatrix} \varepsilon_1(t-j) \\ \varepsilon_2(t-j) \end{bmatrix} \equiv \sum_{j=0}^{\infty} \begin{bmatrix} \bar{\Gamma}_{11}[j] & \bar{\Gamma}_{12}[j] \\ \bar{\Gamma}_{21}[j] & \bar{\Gamma}_{22}[j] \end{bmatrix} \begin{bmatrix} \varepsilon_1(t-j) \\ \varepsilon_2(t-j) \end{bmatrix} \tag{2.25}$$

where $\{\varepsilon_1(t), \varepsilon_2(t)\}$ is a white noise process with mean 0 and covariance matrix $\Sigma_{(1)}$, and the coefficient $\bar{\Gamma}[j]$ is termed the impulse response function. [See for more detailed expositions on the derivation of moving average representations on the basis of Hilbert space arguments see Rozanov (1967, pp. 28–43) and Hannan (1970, pp. 157–163).] Since

$$\Gamma^{\dagger}(e^{-i\lambda})\Gamma^{\dagger}(e^{-i\lambda})^* = \bar{\Gamma}(e^{-i\lambda})\Sigma_{(1)}\bar{\Gamma}(e^{-i\lambda})^*$$

$$= \begin{pmatrix} \bar{\Gamma}_{11}(e^{-i\lambda}) & \bar{\Gamma}_{12}(e^{-i\lambda}) \\ \bar{\Gamma}_{21}(e^{-i\lambda}) & \bar{\Gamma}_{22}(e^{-i\lambda}) \end{pmatrix} \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{pmatrix} \begin{pmatrix} \bar{\Gamma}_{11}(e^{-i\lambda})^* & \bar{\Gamma}_{21}(e^{-i\lambda})^* \\ \bar{\Gamma}_{12}(e^{-i\lambda})^* & \bar{\Gamma}_{22}(e^{-i\lambda})^* \end{pmatrix},$$

we have

$$h_{11}(\lambda) = \frac{1}{2\pi}\{\bar{\Gamma}_{11}(e^{-i\lambda})\Sigma_{11}\bar{\Gamma}_{11}(e^{-i\lambda})^* + \bar{\Gamma}_{12}(e^{-i\lambda})\Sigma_{22:1}\bar{\Gamma}_{12}(e^{-i\lambda})^*\}.$$

Now, define the measure $\bar{M}_{v\to u}(\lambda)$ by

$$\bar{M}_{v\to u}(\lambda) = \log\left[\det h_{11}(\lambda)/\det\{\frac{1}{2\pi}\bar{\Gamma}_{11}(e^{-i\lambda})\Sigma_{11}\bar{\Gamma}_{11}(e^{-i\lambda})^*\}\right]. \tag{2.26}$$

Notice that the definition does not depend on the choice of $\Sigma_{(1)}^{1/2}$ and $\Gamma(e^{-i\lambda})$. The measure $\bar{M}_{v\to u}(\lambda)$ thus defined in (2.26) turns out to be equivalent to $M_{v\to u}(\lambda)$ of (2.16).

Theorem 2.4 $\bar{M}_{v\to u}(\lambda) = M_{v\to u}(\lambda)$.

Proof It follows from (2.25) and the maximality of $\bar{\Gamma}(z)$ that the residuals $u_{-1,-1}(t)$ and $v_{-1,-1}(t)$ have a representation

$$\begin{bmatrix} u_{-1,-1}(t) \\ v_{-1,-1}(t) \end{bmatrix} = \begin{bmatrix} \bar{\Gamma}_{11}[0] & \bar{\Gamma}_{12}[0] \\ \bar{\Gamma}_{21}[0] & \bar{\Gamma}_{22}[0] \end{bmatrix} \begin{bmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{bmatrix}.$$

Hence,

$$\begin{aligned} v_{0,-1}(t) &= v_{-1,-1}(t) + A_{21}u_{-1,-1}(t) \\ &= \{\bar{\Gamma}_{21}[0] + A_{21}\bar{\Gamma}_{11}[0]\}\varepsilon_1(t) + \{\bar{\Gamma}_{22}[0] + A_{21}\bar{\Gamma}_{12}[0]\}\varepsilon_2(t). \end{aligned} \tag{2.27}$$

Because $\bar{\Gamma}[0] = A_{(1)}^{-1}$ in view of (2.23), we have

$$\begin{bmatrix} I_{p_1} & 0 \\ A_{21} & I_{p_2} \end{bmatrix} \begin{bmatrix} \bar{\Gamma}_{11}[0] & \bar{\Gamma}_{12}[0] \\ \bar{\Gamma}_{21}[0] & \bar{\Gamma}_{22}[0] \end{bmatrix} = I_p \tag{2.28}$$

It follows from (2.27) and (2.28) that $v_{0,-1}(t) = \varepsilon_2(t)$. Therefore, the residual of the projection of $u(t)$ onto $H\{v_{0,-1}(\infty)\} = H\{\varepsilon_2(\infty)\}$ is equal to $\sum_{j=0}^{\infty} \bar{\Gamma}_{11}[j]\varepsilon_1(t-j)$ in view of (2.25). It follows from the definition of $M_{v \to u}(\lambda)$ in (2.16) that

$$M_{v \to u}(\lambda) = \log \frac{\det h_{11}(\lambda)}{\det\{\frac{1}{2\pi} \sum_{j=0}^{\infty} \bar{\Gamma}_{11}[j] e^{-ij\lambda}\} \Sigma_{11} \{\frac{1}{2\pi} \sum_{j=0}^{\infty} \bar{\Gamma}_{11}[j] e^{-ij\lambda}\}^*} = \bar{M}_{v \to u}(\lambda).$$

This equality holds because $h_{11:2}(\lambda) = h_{11}(\lambda) - 2\pi h_{12}(\lambda) \Sigma_{22:1}^{-1} h_{21}(\lambda)$ in the bracket on the right-hand side of (2.16) is a spectral density of the residual process of the projection $u(t)$ onto $H\{v_{0,-1}(\infty)\}$. □

Suppose that the process $\{u(t), v(t)\}$ has the stationary autoregressive representation

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \sum_{j=1}^{\infty} H[j] \begin{bmatrix} u(t-j) \\ v(t-j) \end{bmatrix} + \begin{bmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{bmatrix}$$

where $\{\varepsilon_1(t), \varepsilon_2(t)\}$ is a white noise process with covariance matrix Σ. Set $H(z) = I_{p_1+p_2} - \sum_{j=1}^{\infty} H[j] z^j$, and define $\Gamma_{(1)}(z) \equiv H(z)^{-1} A_{(1)}^{-1} \Sigma_{(1)}^{1/2}$. Then, $\Gamma_{(1)}(z)$ is nothing but a version of the canonical factor $\Gamma(e^{-i\lambda})$ in (2.6) and, thus, possesses the properties prescribed in Lemma 2.2. Geweke's measure of linear feedback is then constructed by (2.26) on the basis of this version. Chapter 3 expounds on the construction of the measure for the vector ARMA model.

2.5 Measures of Association and Reciprocity

This subsection introduces measures of association and reciprocity as additional measures to characterize the interdependence structure of the processes $\{u(t)\}$ and $\{v(t)\}$. This subsection also shows that the measure of association is decomposed into the measures of one-way effect and reciprocity.

Recall that $\bar{u}'_{\cdot,\infty}(t)$ and $u'_{\cdot,\infty}(t)$ are the projections of $u(t)$ onto $H\{v_{0,-1}(\infty)\}$ and its residual and that $\bar{v}'_{\infty,\cdot}(t)$ and $v'_{\infty,\cdot}(t)$ are defined similarly; hence, we have

$$u(t) = u'_{\cdot,\infty}(t) + \bar{u}'_{\cdot,\infty}(t); \quad v(t) = v'_{\infty,\cdot}(t) + \bar{v}'_{\infty,\cdot}(t). \tag{2.29}$$

Lemma 2.8 *For the decomposition (2.29), we have*

$$u'_{.,\infty}(t) \in H\{u_{-1,-1}(t)\}; \quad \bar{u}'_{.,\infty} \in H\{v_{0,-1}(t-1)\},$$

and

$$v'_{\infty,.}(t) \in H\{v_{-1,-1}(t)\}; \quad \bar{v}'_{\infty,.}(t) \in H\{u_{-1,0}(t-1)\}.$$

Proof The proof for $\{u(t)\}$ is shown by the following steps:

a. We have the equalities $H\{u(t), v(t)\} = H\{u_{-1,-1}(t), v_{-1,-1}(t)\} = H\{u_{-1,-1}(t)\} \oplus H\{v_{0,-1}(t)\}$ and, hence, $u(t) \in H\{u_{-1,-1}(t)\} \oplus H\{v_{0,-1}(t)\}$. However, because $u'_{.,\infty}(t)$ is the projection residual of $u(t)$ onto $H\{v_{0,-1}(\infty)\}$, $u'_{.,\infty}(t) \perp H\{v_{0,-1}(t)\}$. Therefore, we have $u'_{.,\infty}(t) \in H\{u_{-1,-1}(t)\}$.
b. Because $\bar{u}'_{.,\infty}(t)$ is the projection of $u(t)$ onto $H\{v_{0,-1}(\infty)\}$; however, because $u(t-j) \perp v_{0,-1}(t),\ j \geq 0$, we have $\bar{u}'_{.,\infty}(t) \in H\{v_{0,-1}(t-1)\}$.

A similar train of arguments holds for $\{v(t)\}$. □

Denote the joint spectral density of the process $\{u'_{.,\infty}(t), v'_{\infty,.}(t)\}$ by

$$h'(\lambda) = \begin{bmatrix} h'_{11}(\lambda) & h'_{12}(\lambda) \\ h'_{21}(\lambda) & h'_{22}(\lambda) \end{bmatrix}.$$

The next lemma is straightforward in view of the construction, and the proof is omitted. [See Lemma 2.5. for allied notations.]

Lemma 2.9 *The spectral densities $h'_{11}(\lambda), h'_{22}(\lambda)$ of the processes $\{u'_{.,\infty}(t)\}$ and $\{v'_{\infty,.}(t)\}$, respectively, are given as follows:*

$$h'_{11}(\lambda) \equiv h_{11}(\lambda) - 2\pi \tilde{h}_{12}(\lambda) \Sigma_{22:1}^{-1} \tilde{h}_{21}(\lambda);$$
$$h'_{22}(\lambda) \equiv h_{22}(\lambda) - 2\pi \check{h}_{21}(\lambda) \Sigma_{11:2}^{-1} \check{h}_{12}(\lambda).$$

Define the measure of association at frequency λ between the two processes $\{u(t)\}$ and $\{v(t)\}$ by

$$M_{u,v}(\lambda) \equiv \log[\det h_{11}(\lambda) \det h_{22}(\lambda) / \det h'(\lambda)] \tag{2.30}$$

and define the measure of reciprocity at frequency λ by

$$M_{u.v}(\lambda) \equiv \log[\det h'_{11}(\lambda) \det h'_{22}(\lambda) / \det h'(\lambda)]. \tag{2.31}$$

It follows from the nonnegative definiteness of $h_{11}(\lambda) - h'_{11}(\lambda)$ and $h_{22}(\lambda) - h'_{22}(\lambda)$ that $M_{u,v}(\lambda) \geq 0$. It is evident that $M_{u.v}(\lambda) \geq 0$. Define the corresponding overall measure $M_{u,v}(\lambda)$ and $M_{u.v}(\lambda)$ by the integration of the respective frequency-wise measure,

$$M_{u,v} \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} M_{u,v}(\lambda)d\lambda; \quad M_{u.v} \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} M_{u.v}(\lambda)d\lambda.$$

The following equalities hold between those measures and the measures of the one-way effect.

Theorem 2.5 *We have the equality*

$$M_{u,v}(\lambda) = M_{u\to v}(\lambda) + M_{u.v}(\lambda) + M_{v\to u}(\lambda) \tag{2.32}$$

and, consequently,

$$M_{u,v} = M_{u\to v} + M_{u.v} + M_{v\to u}. \tag{2.33}$$

Proof In view of $M_{v\to u}(\lambda) = \log\{\det h_{11}(\lambda)/\det h'_{11}(\lambda)\}$, the relation (2.32) follows from the equality

$$\begin{aligned}&\log\{\det h_{11}(\lambda)\det h_{22}(\lambda)/\det h'(\lambda)\}\\ &= \log\{\det h_{11}(\lambda)/\det h'_{11}(\lambda)\} + \log\{\det h'_{11}(\lambda)\det h'_{22}(\lambda)/\det h'(\lambda)\}\\ &\quad + \log\{\det h_{22}(\lambda)/\det h'_{22}(\lambda)\}.\end{aligned}$$

The relation (2.33) is then evident. □

To characterize the interdependency between second-order stationary processes $\{u(t)\}$ and $\{v(t)\}$, Gel'fand and Yaglom (1956) introduced the measure of information $\tilde{M}_{u,v}$ defined by

$$\begin{aligned}\tilde{M}_{u,v} &= \log[\det\{Cov(u_{-1,.}(t))\}\det\{Cov(v_{.,-1}(t))\}/\det\{Cov(u_{-1,-1}(t), v_{-1,-1}(t))\}]\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\log[\det h_{11}(\lambda)\det h_{22}(\lambda)/\det h(\lambda)]d\lambda.\end{aligned} \tag{2.34}$$

Geweke (1982) termed the quantity $\tilde{M}_{u.v} \equiv \log\{\det \Sigma_{11}\cdot\det \Sigma_{22}/\det \Sigma\}$ the measure of instantaneous feedback; see also Theorem 3.3 of Sect. 3.3.1. Note that $\tilde{M}_{u,v} = \tilde{M}_{u.v}$ if $\{u(t), v(t)\}$ is a white noise process. The measures of association and reciprocity coincide with those measures under certain conditions.

Theorem 2.6 *(i) If $H\{u'_{.,\infty}(t), v'_{\infty,.}(t)\} = H\{u(t), v(t)\}, t \in \mathbb{Z}$, we have $M_{u,v} = \tilde{M}_{u,v}$. (ii) If $H\{u'_{.,\infty}(t)\} = H\{u_{-1,-1}(t)\}$ and $H\{v'_{\infty,.}(t)\} = H\{v_{-1,-1}(t)\}$, we have $M_{u,v} = \tilde{M}_{u,v}$ and $M_{u.v} = \tilde{M}_{u.v}$.*

Proof Suppose that $H\{u'_{.,\infty}(t), v'_{\infty,.}(t)\} = H\{u(t), v(t)\}$. Then, the residual of the projection of $u'_{.,\infty}(t)$ onto $H\{u'_{.,\infty}(t-1), v'_{\infty,.}(t-1)\}$ is $u_{-1,-1}(t)$ because $u(t) = u'_{.,\infty}(t) + \bar{u}'_{.,\infty}(t)$ and $\bar{u}'_{.,\infty}(t) \in H\{v_{0,-1}(t-1)\} \subset H\{u(t-1), v(t-1)\} \subset H\{u'_{.,\infty}(t-1), v'_{\infty,.}(t-1)\}$. Similarly, the residual of the projection of $v'_{\infty,.}(t)$ onto $H\{u'_{.,\infty}(t-1), v'_{\infty,.}(t-1)\}$ is $v_{-1,-1}(t)$. Hence, the one-step ahead prediction errors of the process $\{u(t), v(t)\}$ and of the process $\{u'_{.,\infty}, v'_{\infty,.}(t)\}$ are the same, whence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det h'(\lambda)d\lambda = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det h(\lambda)d\lambda. \tag{2.35}$$

Consequently, the equality $M_{u,v} = \tilde{M}_{u,v}$ follows. Regarding proposition (ii), note that the assumptions imply that $H\{u'_{.,\infty}(t), v'_{\infty,.}(t)\} = H\{u(t), v(t)\}$, whence (2.35) and $M_{u,v} = \tilde{M}_{u,v}$ hold. Moreover, because then $H\{u'_{.,\infty}(t), v_{0,-1}(t)\} = H\{u_{-1,-1}(t), v_{-1,-1}(t)\}$, it follows that $u'_{-1,-1}(t) = u_{-1,-1}(t)$ and $Cov\{u'_{-1,-1}(t)\} = \Sigma_{11}$. In the same way, we have $Cov\{v'_{-1,-1}(t)\} = \Sigma_{22}$. Hence, the equality $M_{u.v} = \tilde{M}_{u.v}$ follows from (2.30) and (2.31). □

See Theorem 3.3 for another proof of $M_{u.v} = \tilde{M}_{u.v}$ in (ii) of Theorem 2.6 under a general condition.

Set $A = (-\Sigma_{21}\Sigma_{11}^{-1}, I_{p_2})$; $\quad B = (I_{p_1}, -\Sigma_{12}\Sigma_{22}^{-1})$, where

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

is the covariance of the one-step ahead prediction error of the process $\{u(t), v(t)\}$. Also set $\Sigma_{11:2} \equiv \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ and $\Sigma_{22:1} \equiv \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$. As is seen in the preceding arguments, evaluation of the frequency-wise measures of interdependence requires an explicit representation of $h'(\lambda)$.

Theorem 2.7 *The spectral density $h'(\lambda)$ is represented as follows:*

$$\begin{aligned}
h'_{11}(\lambda) &= h_{11}(\lambda) - 2\pi h_{1\cdot}(\lambda)\Gamma(e^{-i\lambda})^{-1*}\Gamma(0)^* A^* \Sigma_{22:1}^{-1} A\Gamma(0)\Gamma(e^{-i\lambda})^{-1} h_{\cdot 1}(\lambda), \\
h'_{22}(\lambda) &= h_{22}(\lambda) - 2\pi h_{2\cdot}(\lambda)\Gamma(e^{-i\lambda})^{-1*}\Gamma(0)^* B^* \Sigma_{11:2}^{-1} B\Gamma(0)\Gamma(e^{-i\lambda})^{-1} h_{\cdot 2}(\lambda), \\
h'_{12}(\lambda) &= h_{12}(\lambda) - 2\pi h_{1\cdot}(\lambda)\Gamma(e^{-i\lambda})^{-1*}\Gamma(0)^* (A^* \Sigma_{22:1}^{-1} A + B^* \Sigma_{11:2}^{-1} B)\Gamma(0)\Gamma(e^{-i\lambda})^{-1} h_{\cdot 2}(\lambda) \\
&\quad + 2\pi \tilde{h}_{12}(\lambda)\Sigma_{22:1}^{-1}(-\Sigma_{21} + \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})\Sigma_{11:2}^{-1}\check{h}_{12}(\lambda).
\end{aligned} \tag{2.36}$$

Proof The expression (2.36) is the corrected version of Theorem 4.3 of Hosoya (1991), and the proof is omitted. Theorem 3.3 of Chap. 3 presents another representation and the derivation. □

Remark 2.2 In general, the measure of association $M_{u,v}$ is not equal to the Gel'fand–Yaglom measure $\tilde{M}_{u,v}$ given in (2.34). Example 2.2 of Sect. 2.6 provides a case in which $\tilde{M}_{u,v} > M_{u,v}$.

Remark 2.3 The notion of Akaike (1968) of relative power contribution (RPC) can be employed in an extended form for the purpose of representation of $M_{v\to u}(\lambda)$. Let $\bar{h}'_{11}(\lambda)$ be the spectral density of the process $\{\bar{u}'_{.,\infty}(t)\}$; hence, $\bar{h}'_{11}(\lambda) = h_{11}(\lambda) - h'_{11}(\lambda)$, and let $r_1(\lambda) \geq \cdots \geq r_{p_1}(\lambda)$ be the eigenvalues of the matrix $\bar{h}'_{11}(\lambda)h_{11}(\lambda)^{-1}$. Evidently, all of the $r_j(\lambda)$ are nonnegative real values and $r_1(\lambda) \leq 1$. Define $RPC_{v\to u}(\lambda)$, the relative power contribution of $\{v(t)\}$ to $\{u(t)\}$, by the diagonal $p_1 \times p_1$ matrix with $r_j(\lambda)$ for the (j, j) element. Because

$$M_{v\to u}(\lambda) = \log\{\det h_{11}(\lambda)/\det h'_{11}(\lambda)\},$$

the one-way frequency-wise measure is related to Akaike's RPC by

$$M_{v\to u}(\lambda) = -\log\det\{I_{p_1} - RPC_{v\to u}(\lambda)\}.$$

See Tanokura and Kitagawa (2015) for an allied study.

Remark 2.4 The overall measures of association, one-way effect, and reciprocity are defined on the basis of only the one-step ahead prediction error of the processes $\{u(t), v(t)\}$, $\{u'_{\cdot,\infty}(t), v'_{\infty,\cdot}(t)\}$ or of their component processes. Also, the processes $\{u_{-1,0}(t)\}$ and $\{v_{0,-1}(t)\}$ are defined in terms of the projection errors. Therefore, as long as we work with these prediction error-related concepts, the assumption of stationarity of the original process can be dispensed with; see also Theorem 2.1 of Sect. 2.2. In particular, the decomposition $M_{u,v} = M_{u\to v} + M_{u\cdot v} + M_{v\to u}$ is still valid for non-stationary second-order processes. Also valid is the proposition that $M_{v\to u}$ is the log-ratio of $\det Cov(u_{-1,\cdot}(t))$ to $\det Cov(u'_{-1,-1}(t))$, which equals the log-ratio of the determinants of the one-step ahead prediction errors of $\{u(t)\}$ and $\{u'_{\cdot,\infty}(t)\}$.

2.6 Examples

This section shows two cases of the bivariate moving average process. The first case is the non-invertible process for which $M_{v\to u}(\lambda)$ is evaluated. The second case is the situation in which $F_{v\to u} > M_{v\to u}$; hence, the two measures diverge.

Example 2.1 Consider a bivariate process $\{u(t), v(t)\}$, $t \in \mathbb{Z}$, which is generated by

$$\begin{aligned} u(t) &= \varepsilon(t) - \varepsilon(t-1) + a\eta(t-1) + b\eta(t-2) \\ v(t) &= \eta(t) + c\eta(t-1) \end{aligned}$$

where we assume that $|c| \le 1$ and $\{\varepsilon(t), \eta(t)\}$ is a white noise process such that $E(\varepsilon(t)) = E(\eta(t)) = 0$, $Var(\varepsilon(t)) = Var(\eta(t)) = 1$, and $Cov(\varepsilon(t), \eta(t)) = 0$. Setting

$$\Gamma(z) = \begin{pmatrix} 1-z & az+bz^2 \\ 0 & 1+cz \end{pmatrix},$$

we have the spectral density matrix of the $\{u(t), v(t)\}$, which is given by

$$h(\lambda) = \frac{1}{2\pi}\Gamma(e^{-i\lambda})\Gamma(e^{-i\lambda})^*$$
$$= \frac{1}{2\pi}\begin{bmatrix} |1-e^{-i\lambda}|^2 + |a+be^{-i\lambda}|^2 & (ae^{-i\lambda}+be^{-2i\lambda})(1+ce^{-i\lambda})^* \\ (ae^{-i\lambda}+be^{-2i\lambda})^*(1+ce^{-i\lambda}) & |1+ce^{-i\lambda}|^2 \end{bmatrix}$$

where $\Gamma(z)$ is a canonical factor. Thus, we have the equality

$$(2\pi)^2 \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det h(\lambda)d\lambda\right\} = |\Gamma(0)|^2.$$

Because $\Gamma(z)$ is maximal, $\{\varepsilon(t), \eta(t)\}$ constitutes an innovation process for $\{u(t), v(t)\}$; namely, we have $\varepsilon(t) = u_{-1,-1}(t)$ and $\eta(t) = v_{-1,-1}(t)$. Then, because $v_{0,-1}(t) = \eta(t)$ in view of Lemma 2.4, by applying the distributed-lag approach of Sect. 2.4.1, the spectral density of the residual process $\{n(t)\}$ is given by $h^{(2)}(\lambda) = |1-e^{-i\lambda}|^2/(2\pi)$. Hence, it follows from (2.19) that

$$M_{v\to u}(\lambda) = \log\frac{|1-e^{-i\lambda}|^2 + |a+be^{-i\lambda}|^2}{|1-e^{-i\lambda}|^2}.$$

Therefore, $\{v(t)\}$ does not cause $\{u(t)\}$ if and only if $a = b = 0$.

Example 2.2 Suppose that a bivariate process $\{u(t), v(t)\}$, $t \in \mathbb{Z}$, is generated by

$$u(t) = \varepsilon(t) + a\varepsilon(t-1) + b\eta(t-1)$$
$$v(t) = \eta(t) + c\varepsilon(t-1) + d\eta(t-1)$$

where the white noise process $\{\varepsilon(t), \eta(t)\}$ is specified as in Example 2.1. Set

$$\Gamma(z) = \begin{pmatrix} 1+az & bz \\ cz & 1+dz \end{pmatrix}.$$

Suppose that the parameters satisfy that $ad = -1$ and $bc = -3/2$, then the zeros of $\det\Gamma(z)$ are the roots of the quadratic equation

$$\det\Gamma(z) = z^2/2 + (a+d)z + 1 = 0.$$

Suppose, for example, that $b = 0.1$, and $0.7 \le a \le 1.6$, then all the zeros z_0 of $\det\Gamma(z)$ satisfy $1.04 \le |z_0| \le 1.47$; hence, they are outside of the unit circle. If all zeros of $\det\Lambda(z)$ are outside the unit circle, the spectral density $h(\lambda)$ of the process $\{u(t), v(t)\}$ is given in terms of the canonical factor $\Gamma(e^{-i\lambda})$ by

$$h(\lambda) = \frac{1}{2\pi}\Gamma(e^{-i\lambda})\Gamma(e^{-i\lambda})^*.$$

For such a canonical factor $\Gamma(z)$, we have $v_{0,-1}(t) = \eta(t)$, and thus $M_{v\to u}(\lambda)$ is expressed as

$$M_{v\to u}(\lambda) = \log \frac{|1 + ae^{-i\lambda}|^2 + b^2}{|1 + ae^{-i\lambda}|^2},$$

whereas the Geweke measure is given by

$$F_{v\to u} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\{|1 + ae^{-i\lambda}|^2 + b^2\}d\lambda - \log Var(\varepsilon(t))$$

because $H\{u(t), v(t)\} = H\{\varepsilon(t), \eta(t)\}$. Moreover, because $\log Var(\varepsilon(t)) = 0$ and

$$\int_{0}^{\pi} \log\{|1 + 2a\cos(\lambda) + a^2|\}d\lambda = 2\pi \log|a|,$$

if $a = 1.5$ for example, the difference between the two measures is equal to

$$F_{v\to u} - M_{v\to u} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\{|1 + a^{-i\lambda}|^2\}d\lambda = 2\log|a| = 2\log 1.5 > 0.$$

References

Akaike, H. (1968). On the use of a linear model for the identification of feedback system. *Annals of the Institute of Statistical Mathematics*, *20*, 425–439.

Florens, J. P., & Mouchart, M. (1982). A note on noncausality. *Econometrica*, *50*, 583–592.

Gel'fand, I. M., & Yaglom, A. M. (1959). Calculation of the amount of information about a random function contained in another such function. *American Mathematical Society Translation Series*, *2*(12), 199–246.

Geweke, J. (1982). Measurement of linear dependence and feedback between multiple time series. *Journal of the American Statistical Association*, *77*, 304–324.

Granger, C. W. J. (1963). Economic process involving feedback. *Information and Control*, *6*, 28–48.

Granger, C. W. J. (1969). Investigating causal relations by econometric methods and cross-spectral methods. *Econometrica*, *37*, 424–438.

Hannan, E. J. (1970). *Multiple Time Series*. New York: Wiley.

Hosoya, Y. (1977). On Granger condition for non-causality. *Econometrica*, *45*, 1735–1736.

Hosoya, Y. (1991). The decomposition and measurement of the interdependency between second-order stationary processes. *Probability Theory and Related Fields*, *88*, 429–444.

Pierce, D. A. (1979). R^2 measures for time series. *Journal of American Statistical Association*, *74*, 901–910.

Rozanov, Y. A. (1967). *Stationary Random Processes*. San Francisco: Holden Day.

Sims, C. A. (1972). Money, income and causality. *American Economic Review*, *62*, 540–552.

Tanokura, Y., & Kitagawa, G. (2015). *Indexation and Causation of Financial Markets: Nonstationary Time Series Method*. Tokyo: Springer.

Whittle, P. (1984). *Prediction and Regulation by Linear Least-Square Method* (2nd ed.). Oxford: Basil Blackwell Publisher.

http://www.springer.com/978-981-10-6435-7

Characterizing Interdependencies of Multiple Time
Series
Theory and Applications
Hosoya, Y.; Oya, K.; Takimoto, T.; Kinoshita, R.
2017, X, 133 p. 32 illus., Softcover
ISBN: 978-981-10-6435-7