In this chapter, we first introduce the Killing form and prove that the semisimplicity of a finite-dimensional Lie algebra over $\mathbb{C}$ is equivalent to the nondegeneracy of its Killing form. Then we use the Killing form to derive the decomposition of a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ into a direct sum of simple ideals. Moreover, it is showed that a derivation of such a Lie algebra must be an inner derivation. Furthermore, we study the completely reducible modules of a Lie algebra and prove the Weyl’s theorem of complete reducibility. The equivalence of the complete reducibility of real and complex modules is also given. Cartan’s root-space decomposition of a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ is derived. In particular, we prove that such a Lie algebra is generated by two elements. The completely reducibility of finite-dimensional modules of $sl(2, \mathbb{C})$ plays an important role in proving the properties of the corresponding root systems.

Section 2.1 gives the relation between the Killing form and the semisimplicity of Lie algebra. In Sect. 2.2, we define a module of a Lie algebra and discuss the complete reducibility. Moreover, the dual module and tensor module are introduced. The lifting from a given representation to an oscillator representation is given. In Sect. 5.3, we prove the equivalence of real and complex complete reducibility. In particular, we find the connection between real and complex finite-dimensional irreducible modules. The Weyl’s theorem is proved in Sect. 2.4 by using Schur’s Lemma. In Sect. 2.5, we derive the Cartan’s root space decomposition. Finally in Sect. 2.6, we use finite-dimensional representations of $sl(2, \mathbb{C})$ to study the properties of roots and root subspaces of a finite-dimensional semisimple Lie algebra, and prove that the algebra is generated by two elements.

2.1 Killing Form

In this section, we assume that Lie algebras are finite dimensional. Moreover, we define the Killing form of a Lie algebra and use it to characterize semisimple Lie algebras.
Let $G$ be a Lie algebra. Define
\[ \kappa(u, v) = \text{Tr} \ \text{ad} \ u \ \text{ad} \ v \quad \text{for} \quad u, v \in G. \] (2.1.1)

Then $\kappa(\cdot, \cdot)$ is a symmetric bilinear form that is called the Killing form. Moreover,
\[
\kappa([u, v], w) = \text{Tr} \ \text{ad} \ [u, v] \ \text{ad} \ w = \text{Tr} \ [\text{ad} \ u, \text{ad} \ v] \ \text{ad} \ w \\
= \text{Tr} (\text{ad} \ u \ \text{ad} \ v - \text{ad} \ v \ \text{ad} \ u) \ \text{ad} \ w = \text{Tr} (\text{ad} \ u \ \text{ad} \ v \ \text{ad} \ w - \text{ad} \ v \ \text{ad} \ u \ \text{ad} \ w) \\
= \text{Tr} (\text{ad} \ u \ \text{ad} \ v \ \text{ad} \ w) - \text{Tr} (\text{ad} \ v \ \text{ad} \ u \ \text{ad} \ w) \\
= \text{Tr} (\text{ad} \ u \ \text{ad} \ w \ \text{ad} \ v) - \text{Tr} (\text{ad} \ v \ \text{ad} \ w \ \text{ad} \ u) \\
= \text{Tr} \ \text{ad} \ u \ [\text{ad} \ v, \text{ad} \ w] = \kappa(u, [v, w])
\] (2.1.2)

for $u, v, w \in G$. Thus $\kappa$ is associative (invariant).

**Lemma 2.1.1** Let $\mathcal{I}$ be an ideal of a Lie algebra $\mathcal{G}$. If $\kappa(\cdot, \cdot)$ is the Killing form of $\mathcal{G}$, then $\kappa(\cdot, \cdot | \mathcal{I})$ is the Killing form of $\mathcal{I}$.

**Proof** Assume that $\mathcal{I}$ is a proper nonzero ideal. Take subspace $\mathcal{K}$ of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{K} \oplus \mathcal{I}$. (2.1.3)

Let $\mathcal{S}_1$ be a basis of $\mathcal{K}$ and let $\mathcal{S}_2$ be a basis of $\mathcal{I}$. Then $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ forms a basis of $\mathcal{G}$. For each $u \in \mathcal{I}$, the matrix form
\[
\text{ad}_{\mathcal{G}}u = \begin{pmatrix} 0 & A_u \\ 0 & B_u \end{pmatrix}
\] (2.1.4)

with respect to the basis $\mathcal{S}$ and the matrix form $\text{ad}_{\mathcal{I}}u = B_u$ with respect to $\mathcal{S}_2$. Thus
\[
\text{ad}_{\mathcal{G}}u \ \text{ad}_{\mathcal{G}}v = \begin{pmatrix} 0 & A_u \\ 0 & B_u \end{pmatrix} \begin{pmatrix} 0 & A_v \\ 0 & B_v \end{pmatrix} = \begin{pmatrix} 0 & A_u B_v \\ 0 & B_u B_v \end{pmatrix}
\] for $u, v \in \mathcal{G}$. (2.1.5)

Hence
\[
\kappa(u, v) = \text{Tr} \ \text{ad}_{\mathcal{G}}u \ \text{ad}_{\mathcal{G}}v = \text{Tr} B_u B_v = \text{Tr} \ \text{ad}_{\mathcal{I}}u \ \text{ad}_{\mathcal{I}}v
\] (2.1.6)

for $u, v \in \mathcal{I}$. \qed

The radical of a symmetric bilinear form $\beta(\cdot, \cdot)$ on a vector space $V$ is defined by
\[
\text{Rad} \ \beta = \{u \in V \mid \beta(u, v) = 0 \text{ for any } v \in V\}. \tag{2.1.7}
\]

If $\text{Rad} \ \beta = \{0\}$, we call $\beta(\cdot, \cdot)$ nondegenerate. Recall the radical of a Lie algebra $\mathcal{G}$ is the unique maximal solvable ideal. The algebra $\mathcal{G}$ is called semisimple if $\text{Rad} \ \mathcal{G} = \{0\}$. In the rest of this section, we assume the base field $\mathbb{F} = \mathbb{C}$. 
2.1 Killing Form

**Theorem 2.1.2** Let \( \mathcal{G} \) be a Lie algebra. Then \( \mathcal{G} \) is semisimple if and only if the Killing form is nondegenerate.

**Proof** Assume that \( \mathcal{G} \) is semisimple. For any \( u \in \text{Rad } \kappa \) and \( v, w \in \mathcal{G} \), we have

\[
\kappa([u, v], w) = \kappa(u, [v, w]) = 0.
\]  
(2.1.8)

Hence \([u, v] \in \text{Rad } \kappa\). So \( \text{Rad } \kappa \) is an ideal of \( \mathcal{G} \). By Cartan’s Criterion, \( \text{ad } \text{Rad } \kappa \) is solvable, and so is \( \text{Rad } \kappa \) because \( \ker \text{ad} = Z(\mathcal{G}) = \{0\} \). Hence \( \text{Rad } \kappa \subset \text{Rad } \mathcal{G} = \{0\} \).

Suppose that \( \text{Rad } \kappa = \{0\} \) and \( \mathcal{R} = \text{Rad } \mathcal{G} \neq \{0\} \). Let \( n \) be the minimal positive integer such that \( \mathcal{R}^{(n)} = \{0\} \). Then \( \mathcal{R}^{(n-1)} \) is an abelian ideal of \( \mathcal{G} \). For any element \( u \in \mathcal{R}^{(n-1)} \) and \( v, w \in \mathcal{G} \), we have

\[
(\text{ad } u \text{ ad } v)^2 (w) = [u, [v, [u, [v, w]]]] \in [\mathcal{R}^{(n-1)}, \mathcal{R}^{(n-1)}] = \{0\}.
\]  
(2.1.9)

Thus

\[
(\text{ad } u \text{ ad } v)^2 = 0 \implies \text{Tr ad } u v = 0 \sim \kappa(u, v) = 0.
\]  
(2.1.10)

So \( \mathcal{R}^{(n-1)} \subset \text{Rad } \kappa = \{0\} \), which leads to a contradiction. Therefore, \( \text{Rad } \mathcal{G} = \{0\} \). \( \square \)

**Theorem 2.1.3** For any semisimple Lie algebra \( \mathcal{G} \), there exist ideals \( \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k \) of \( \mathcal{G} \), which are simple Lie algebras, such that

\[
\mathcal{G} = \bigoplus_{i=1}^{k} \mathcal{G}_i.
\]  
(2.1.11)

The set \( \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k\} \) enumerates all simple ideals of \( \mathcal{G} \).

**Proof** Let \( \mathcal{G}_1 \) be a minimal nonzero ideal of \( \mathcal{G} \). Since \( \mathcal{G}_1 \) is not solvable by the semisimplicity of \( \mathcal{G} \), \( \dim \mathcal{G}_1 > 1 \). Set

\[
\mathcal{G}' = \{u \in \mathcal{G} | \kappa(u, v) = 0 \text{ for any } v \in \mathcal{G}_1\}.
\]  
(2.1.12)

For \( u \in \mathcal{G}' \), \( v \in \mathcal{G}_1 \) and \( w \in \mathcal{G} \), we have \([w, v] \in \mathcal{G}_1 \) and

\[
\kappa([u, w], v) = \kappa(u, [w, v]) = 0.
\]  
(2.1.13)

Hence \([u, w] \in \mathcal{G}' \). So \( \mathcal{G}' \) is an ideal of \( \mathcal{G} \). Since \( \kappa(\mathcal{G}_1, \mathcal{G}_1) \) is also the Killing form of \( \mathcal{G}_1 \) by Lemma 2.1.1, the radical of the Killing form of \( \mathcal{G}_1 \) is exactly \( \mathcal{G}' \cap \mathcal{G}_1 \). If \( \mathcal{G}' \cap \mathcal{G}_1 \neq \{0\} \), then it is an ideal of \( \mathcal{G} \) included in \( \mathcal{G}_1 \). By minimality, \( \mathcal{G}' \cap \mathcal{G}_1 = \mathcal{G}_1 \), which implies that the Killing form of \( \mathcal{G}_1 \) is zero. By Cartan’s Criterion, \( \mathcal{G}_1 \) is a solvable ideal, which is absurd. Hence \( \mathcal{G}' \cap \mathcal{G}_1 = \{0\} \). Thus \( \kappa(\mathcal{G}_1, \mathcal{G}_1) \) is nondegenerate. Therefore, we have
\[ \mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}'. \] (2.1.14)

Observe that \([\mathcal{G}_1, \mathcal{G}'] \subset \mathcal{G}_1 \cap \mathcal{G} = \{0\}.\) Thus any ideals of \(\mathcal{G}_1\) and \(\mathcal{G}'\) are ideals of \(\mathcal{G}\). In particular, \(\mathcal{G}_1\) is a simple Lie algebra and \(\mathcal{G}'\) is semisimple if \(\mathcal{G}' \neq \{0\}\). Now we work on \(\mathcal{G}'\). By induction, there exist simple ideals \(\mathcal{G}_2, \ldots, \mathcal{G}_k\) of \(\mathcal{G}'\) such that \(\mathcal{G}' = \bigoplus_{i=2}^{k} \mathcal{G}_i\). So we have (2.1.11).

If \(\mathcal{L}\) is any simple ideal of \(\mathcal{G}\), then \([\mathcal{L}, \mathcal{L}]\) is a nonzero ideal of \(\mathcal{L}\). Hence \([\mathcal{L}, \mathcal{L}] = \mathcal{L}\), which implies \(\mathcal{L} = [\mathcal{G}, \mathcal{L}]\). Now

\[ \mathcal{L} = [\mathcal{G}, \mathcal{L}] = \left[ \bigoplus_{i=1}^{k} \mathcal{G}_i, \mathcal{L} \right]. \] (2.1.15)

So \([\mathcal{G}_i, \mathcal{L}] \neq \{0\}\) for some \(i \in \{1, 2, \ldots, k\}\). But \([\mathcal{G}_i, \mathcal{L}]\) is an ideal of \(\mathcal{G}\) that is also an ideal of \(\mathcal{G}_i\) and \(\mathcal{L}\). Thus \(\mathcal{L} = \mathcal{[G}_i, \mathcal{L}] = \mathcal{G}_i\). \(\square\)

Let \(\mathcal{I}\) be a nonzero ideal of \(\mathcal{G}\). Suppose

\[ u = \sum_{i=1}^{k} u_i \in \mathcal{I}. \] (2.1.16)

If \(u_i \neq 0\), then \([u, \mathcal{G}_i] = [u_i, \mathcal{G}_i] \neq 0\), which implies \(\{0\} \neq \mathcal{G}_i \cap \mathcal{I} = \mathcal{G}_i\). Therefore, \(\mathcal{I}\) is a direct sum of some simple ideals.

**Theorem 2.1.4** If \(\mathcal{G}\) is a semisimple Lie algebra, then \(\text{Der } \mathcal{G} = \text{ad } \mathcal{G}\).

**Proof** We define a symmetric associative bilinear form \(\beta\) on \(\text{Der } \mathcal{G} \subset \text{End } \mathcal{G}\) by

\[ \beta(d_1, d_2) = \text{Tr} d_1 d_2 \quad \text{for } d_1, d_2 \in \text{Der } \mathcal{G}. \] (2.1.17)

Note that

\[ (\text{ad } \mathcal{G})^\perp = \{d \in \text{Der } \mathcal{G} \mid \beta(d, \text{ad } u) = 0 \text{ for any } u \in \mathcal{G}\} \] (2.1.18)

is an ideal of \(\text{Der } \mathcal{G}\) because \(\text{ad } \mathcal{G}\) is an ideal of \(\text{Der } \mathcal{G}\). Since \(\beta(\text{ad } \mathcal{G}, \text{ad } \mathcal{G}) = \kappa(\cdot, \cdot)\), the nondegenerate Killing form of \(\mathcal{G}\), we have \(\text{ad } \mathcal{G} \cap (\text{ad } \mathcal{G})^\perp = \{0\}\). Thus

\[ \text{Der } \mathcal{G} = \text{ad } \mathcal{G} \oplus (\text{ad } \mathcal{G})^\perp. \] (2.1.19)

For any \(d \in (\text{ad } \mathcal{G})^\perp\) and \(u \in \mathcal{G}\), we have \([d, \text{ad } u] = \text{ad } d(u) \in \text{ad } \mathcal{G} \cap (\text{ad } \mathcal{G})^\perp = \{0\}\). Thus \(d(u) \in Z(\mathcal{G}) = \{0\}\). So \(d = 0\). Hence \((\text{ad } \mathcal{G})^\perp = \{0\}\). Therefore, \(\text{Der } \mathcal{G} = \text{ad } \mathcal{G}\) by (2.1.19). \(\square\)
2.1 Killing Form

For any element $u$ in a semisimple Lie algebra $\mathcal{G}$, there exist elements $u_s, u_n \in \mathcal{G}$ such that $\text{ad} \ u = \text{ad} \ u_s + \text{ad} \ u_n$ is the Jordan–Chevalley decomposition of $\text{ad} \ u$. Since $Z(\mathcal{G}) = \{0\}$, we have $u = u_s + u_n$, which is called the abstract Jordan decomposition of $u$.

As an exercise, prove that a finite-dimensional Lie algebra $\mathcal{G}$ is solvable if and only if $[\mathcal{G}, \mathcal{G}] \subset \text{Rad} \kappa(\cdot, \cdot)$.

2.2 Modules

In this section, we define a module of a Lie algebra and discuss the complete reducibility. Moreover, the dual module and tensor module are introduced. The lifting from a given representation to an oscillator representation is given.

Recall that a representation $\nu$ of $\mathcal{G}$ on a vector space $V$ is a Lie algebra homomorphism from $\mathcal{G}$ to $\mathfrak{gl}(V)$. It is more convenient to work on the elements of $\mathcal{G}$ directly acting on $V$. We denote $\xi(u) = \nu(\xi)(u)$ for $\xi \in \mathcal{G}, u \in V$.

The conditions for $\nu$ to be a homomorphism becomes

\[
\xi_1(au + bv) = a\xi_1(u) + b\xi_1(v), \quad (a\xi_1 + b\xi_2)(u) = a\xi_1(u) + b\xi_2(u), \quad (2.2.2)
\]

\[
\xi_1(\xi_2(u)) - \xi_2(\xi_1(u)) = [\xi_1, \xi_2](u) \quad (2.2.3)
\]

for $a, b \in \mathbb{F}, \xi_1, \xi_2 \in \mathcal{G}$ and $u, v \in V$. We call $V$ a $\mathcal{G}$-module. Conversely, a map from $\mathcal{G} \times V \to V$: $(\xi, u) \mapsto \xi(u)$ satisfying (2.2.2) and (2.2.3) gives a representation $\nu$ of $\mathcal{G}$ on $V$ defined by (2.2.1). A submodule $U$ of a $\mathcal{G}$-module $V$ is a subspace such that $\mathcal{G}(U) \subset U$. A trivial $\mathcal{G}$-module $V$ is a vector space with the $\mathcal{G}$-action: $\xi(u) = 0$ for any $\xi \in \mathcal{G}$ and $u \in V$.

A homomorphism $\phi$ from a $\mathcal{G}$-module $V$ to a $\mathcal{G}$-module $W$ is a linear map from $V$ to $W$ such that

\[
\phi(\xi(u)) = \xi(\phi(u)) \quad \text{for} \ \xi \in \mathcal{G}, \ u \in V. \quad (2.2.4)
\]

In this case, $\phi(V)$ forms a submodule of $W$ and $\ker \phi$ is a submodule of $V$. Moreover, $\phi(V) \cong V/\ker \phi$. The map $\phi$ is called an isomorphism if it is bijective. A $\mathcal{G}$-module $V$ is called irreducible if it has exactly two submodules $\{0\}$ and $V$. We do not view $\{0\}$ as an irreducible $\mathcal{G}$-module. A module is called completely reducible if it is direct sum of irreducible submodules. A complement of a submodule $W$ of $V$ is a submodule $W'$ such that $V = W \oplus W'$.

**Lemma 2.2.1** A module $V$ is completely reducible if and only if any submodule has a complement.
Proof Suppose that $V \neq \{0\}$ and any submodule of $V$ has a complement. Take any nonzero minimal submodule $W$ of $V$, which must be irreducible. If $V = W$, it is done. Otherwise, it has a nonzero complement $W'$. Suppose that $W_2$ is any nonzero proper submodule of $W'$. Then $W_1 + W_2$ is a module. By assumption, there exists a complement $U$ of $W_1 + W_2$. This $U$ may not be in $W'_1$. Define the projection $P : V \rightarrow W'_1$ by

$$P(u + u') = u' \quad \text{for } u \in W_1, \ u' \in W'_1.$$ (2.2.5)

Then $P$ is a Lie algebra module homomorphism. In fact, $P(U)$ is a submodule of $W'_1$.

If $u \in P(U) \cap W_2$, then there exists $v \in W_1$ such that $v + u \in U$. But $v + u \in W_1 + W_2$. Thus $v + u = 0$, or equivalently, $u = v = 0$. Hence $P(U) \cap W_2 = \{0\}$. Moreover, any element $w \in W'_1$ can be written as $w = w_1 + w_2 + w'$ with $w_1 \in W_1$, $w_2 \in W_2$ and $w' \in U$. Note $w' - P(w') \in W_1$. We have

$$w - w_2 - P(w') = w_1 + w_2 - P(w') \in W_1 \cap W'_1 = \{0\};$$ (2.2.6)

that is, $w = w_2 + P(w')$. Hence

$$W'_1 = W_2 \oplus P(U),$$ (2.2.7)

or equivalently, any submodule of $W'_1$ has a complement in $W'_1$. By induction, $W'_1 = \bigoplus_{i=2}^k W_i$ is a direct sum of irreducible submodules $W_2, W_3, \ldots, W_k$ in $W'_1$. So is $V = \bigoplus_{i=1}^k W_i$.

Assume that $V = \bigoplus_{i=1}^k W_i$ is a direct sum of irreducible submodules $W_1, W_2, \ldots, W_k$ in $V$ and $U$ is any nonzero proper submodule of $V$. Then there exists some $W_i \not\subset U$. Now

$$\bar{V} = V/W_i = \bigoplus_{i \neq j} (W_j + W_i)/W_i$$ (2.2.8)

is completely reducible. By induction, $(U + W_i)/W_i$ has a complement $U'/W_i$ in $\bar{V}$, where $U'$ is a submodule of $V$ containing $W_i$. If $u \in U \cap U'$, then $u + W_i \in [(U + W_i)/W_i] \cap (U'/W_i) = W_i$. So $u \in W_i \cap U = \{0\}$. That is, $U \cap U' = \{0\}$. For any element $v \in V$, there exists $v_1 \in U$ and $v_2 \in U'$ such that

$$v + W_i = (v_1 + W_i) + (v_2 + W_i) = v_1 + v_2 + W_i;$$ (2.2.9)

equivalently, there exists $w \in W_i$ such that

$$v = v_1 + v_2 + w.$$ (2.2.10)

But $W_i \subset U'$. We have $v_2 + w \in U'$. Thus $v \in U + U'$. Therefore, $V = U \oplus U'$; that is, $U'$ is a complement of $U$. □
Recall that $V^*$ denotes the space of linear functions on $V$. Suppose that $V$ is a module of a Lie algebra $\mathcal{G}$. We define an action of $\mathcal{G}$ on $V^*$ by

$$\xi(f)(u) = -f(\xi(u)) \quad \text{for} \quad \xi \in \mathcal{G}, \ f \in V^*, \ u \in V.$$  \hfill (2.2.11)

Note

$$\begin{align*}
[\xi_1, \xi_2](f)(u) &= f(-[\xi_1, \xi_2](u)) = f(-\xi_1(\xi_2(u)) + \xi_2(\xi_1(u))) \\
&= -f(\xi_1(\xi_2(u))) + f(\xi_2(\xi_1(u))) = -\xi_1(f)(\xi_2(u)) + \xi_2(f)(\xi_1(u)) \\
&= -\xi_2(\xi_1(f))(u) + \xi_1(\xi_2(f))(u) = (\xi_1(\xi_2(f)) - \xi_2(\xi_1(f)))(u) \quad \text{for} \quad \xi_1, \xi_2 \in \mathcal{G}, \ f \in V^* \text{ and } u \in V. \tag{2.2.12}
\end{align*}$$

Thus $V^*$ forms a $\mathcal{G}$-module, which is called the dual (contragredient) module of $V$. In general, $V \not\cong V^*$ as $\mathcal{G}$-modules. It is in general a difficult problem of determining if they are isomorphic.

A bilinear form $\beta : V \times V \to \mathbb{F}$ on a $\mathcal{G}$-module $V$ is called $\mathcal{G}$-invariant if

$$\beta(\xi(u), v) = -\beta(u, \xi(v)) \quad \text{for} \quad \xi \in \mathcal{G}, \ u, v \in V.$$  \hfill (2.2.14)

In fact, we have:

**Lemma 2.2.2** A finite-dimensional module $V$ of a Lie algebra $\mathcal{G}$ is isomorphic to its dual module $V^*$ if and only if it has a nondegenerate invariant bilinear form.

**Proof** Suppose that $V$ is isomorphic to $V^*$ via $\sigma$. We define a bilinear form $\beta$ on $V$ by

$$\beta(u, v) = \sigma(u)(v) \quad \text{for} \quad u, v \in V.$$  \hfill (2.2.15)

Then

$$\beta(\xi(u), v) = \sigma(\xi(u))(v) = \xi(\sigma(u))(v) = -\sigma(u)(\xi(v)) = -\beta(u, \xi(v)). \quad \text{for} \quad \xi \in \mathcal{G} \text{ and } u, v \in V. \tag{2.2.16}$$

for $\xi \in \mathcal{G}$ and $u, v \in V$. So $\beta$ is a nondegenerate invariant bilinear form.

Assume that $V$ has a nondegenerate invariant bilinear form $\beta$. Equation (2.2.15) define a linear map $\sigma : V \to V^*$ and (2.2.16) implies that $\sigma$ is a module homomorphism. The nondegeneracy implies that $\sigma$ is bijective. □

Next, we want to lift a given representation to an oscillator (differential-operator) representation. Let $\{u_i \mid i \in I\}$ be a basis of a module $V$ of a Lie algebra $\mathcal{G}$. For any $\xi \in \mathcal{G}$, we write

$$\xi(u_i) = \sum_{j \in I} \nu_{i,j}(\xi) u_j.$$

(2.2.17)
Then \( \nu_{i,j} \in \mathcal{G}^* \). We define an action of \( \mathcal{G} \) on the polynomial algebra \( \mathbb{F}[x_i \mid i \in I] \) by:

\[
\xi(f) = \sum_{i,j \in I} \nu_{i,j}(\xi) x_j \partial_{x_i}(f).
\]

By (2.2.3) and (2.2.17), we have

\[
\sum_{j \in I} (\nu_{i,j}(\xi_2)\nu_{j,k}(\xi_1) - \nu_{i,j}(\xi_1)\nu_{j,k}(\xi_2)) = \nu_{i,k}([\xi_1, \xi_2]) \quad \text{for} \quad \xi_1, \xi_2 \in \mathcal{G}.
\]

On the other hand,

\[
\begin{align*}
&\left[ \sum_{i,j \in I} \nu_{i,j}(\xi_1)x_j \partial_{x_i}, \sum_{r,s \in I} \nu_{r,s}(\xi_2)x_s \partial_{x_r} \right] \\
= &\sum_{i,j,r,s \in I} \nu_{i,j}(\xi_1)\nu_{r,s}(\xi_2)x_j \partial_{x_i} - \sum_{i,j,r,s \in I} \nu_{i,j}(\xi_1)\nu_{r,s}(\xi_2)x_s \partial_{x_r} \\
= &\sum_{i,j \in I} \nu_{i,j}(\xi_1)\nu_{r,i}(\xi_2)x_j \partial_{x_r} - \sum_{i,j \in I} \nu_{r,i}(\xi_1)\nu_{i,j}(\xi_2)x_j \partial_{x_r} \\
= &\sum_{r,j \in I} \nu_{r,j}([\xi_1, \xi_2])x_j \partial_{x_r}
\end{align*}
\]

for \( \xi_1, \xi_2 \in \mathcal{G} \). Thus \( \mathbb{F}[x_i \mid i \in I] \) becomes a \( \mathcal{G} \)-module. It is important to understand this module structure from the information of the module \( V \). For instance, any one-dimensional trivial submodule gives an invariant of the corresponding Lie group. To find invariants is equivalent to solve the system of partial differential equations:

\[
\sum_{i,j \in I} \nu_{i,j}(\xi)x_i \partial_{x_j}(f) = 0, \quad \xi \in \mathcal{G}.
\]

The above defined module is isomorphic to the “symmetric tensor of \( V \).”

Let \( V \) and \( W \) be modules of a Lie algebra \( \mathcal{G} \). We define an action of \( \mathcal{G} \) on \( V \otimes W \) by

\[
\xi(u \otimes v) = \xi(u) \otimes v + u \otimes \xi(v) \quad \text{for} \quad \xi \in \mathcal{G}, \; u \in V, \; v \in W.
\]

Note

\[
(\xi_1\xi_2 - \xi_2\xi_1)(u \otimes v) \\
= \xi_1\xi_2(u \otimes v) - \xi_2\xi_1(u \otimes v) \\
= \xi_1(\xi_2(u) \otimes v + u \otimes \xi_2(v)) - \xi_2(\xi_1(u) \otimes v + u \otimes \xi_1(v))
\]
\[
\begin{aligned}
\xi_1(\xi_2(u)) \otimes v + \xi_1(u) \otimes \xi_2(v) + \xi_2(u) \otimes \xi_1(v) + u \otimes \xi_1(\xi_2(v)) \\
- (\xi_2(\xi_1(u)) \otimes v + \xi_2(u) \otimes \xi_1(v) + \xi_1(u) \otimes \xi_2(v)) + u \otimes \xi_2(\xi_1(v)) \\
= [\xi_1, \xi_2](u) \otimes v + u \otimes [\xi_1, \xi_2](v) = [\xi_1, \xi_2](u \otimes v)
\end{aligned}
\]

(2.2.23)

for \( \xi_1, \xi_2 \in \mathcal{G} \), \( u \in V \) and \( v \in W \). Thus \( V \otimes W \) becomes a \( \mathcal{G} \)-module.

Denote by \( \text{Hom}_C(V, W) \) the space of all linear maps from \( V \) to \( W \). Define \( \tau : V^* \otimes W \to \text{Hom}_C(V, W) \) by

\[
\tau \left( \sum_{i=1}^{k} f_i \otimes v_i \right) (u) = \sum_{i=1}^{k} f_i(u) v_i \quad \text{for} \quad u \in V.
\]

(2.2.24)

If \( V \) and \( W \) are modules of a Lie algebra \( \mathcal{G} \) and \( \dim W < \infty \), then \( \tau \) is a linear isomorphism, \( V^* \otimes W \) forms a \( \mathcal{G} \)-module and \( \tau \) induces a \( \mathcal{G} \)-module structure on \( \text{Hom}_C(V, W) \). In fact,

\[
\xi(g)(u) = \xi(g(u)) - g(\xi(u)) \quad \text{for} \quad \xi \in \mathcal{G}, \ g \in \text{Hom}_C(V, W), \ u \in V.
\]

(2.2.25)

As an exercise, write out the module structure of \( \text{Hom}_C(V, W) \) for \( \mathfrak{sl}(2, \mathbb{F}) \) if (1) \( V \) is the adjoint module and \( W \) is the 2-dimensional module of defining \( \mathfrak{sl}(2, \mathbb{F}) \); (2) exchange the positions of two modules. Are these two modules irreducible?

### 2.3 Real and Complex Modules

In this section, we prove the equivalence of real and complex complete reducibility. In particular, we find the connection between real and complex finite-dimensional irreducible modules.

Let \( V \) be a module of a Lie algebra \( \mathcal{G} \) and let \( U \) be a nonzero proper submodule of \( V \). We say that \( V \) splits at \( U \) if \( U \) has a complementary submodule in \( V \).

**Lemma 2.3.1** A \( \mathcal{G} \)-module \( V \) splits at a submodule \( U \) if and only if there exists a \( \mathcal{G} \)-module homomorphism \( \tau : V \to U \) such that \( \tau|_U = \text{Id}_U \).

**Proof** If \( V \) splits at a submodule \( U \), then \( U \) has a complement \( U' \) such that \( V = U \oplus U' \). The projection \( \tau \) defined by

\[
\tau(u + u') = u \quad \text{for} \quad u \in U, \ u' \in U',
\]

(2.3.1)

is the required homomorphism. Conversely, if there exist a \( \mathcal{G} \)-module homomorphism \( \tau : V \to U \) such that \( \tau|_U = \text{Id}_U \), then

\[
v = \tau(v) + (1 - \tau)(v) \quad \text{for} \quad v \in V,
\]

(2.3.2)
which implies that $(1 - \tau)(V)$ is a complement of $U$ because $1 - \tau$ is a $G$-module homomorphism from $V$ to $V$. □

Let $V$ be a vector space over $\mathbb{R}$ with a basis $\{v_i \mid i \in I\}$. The complexification of $V$ is a vector space over $\mathbb{C}$ with $\{v_i \mid i \in I\}$ as a basis; that is,

$$V_{\mathbb{C}} = \sum_{i \in I} \mathbb{C}v_i.$$  \hfill (2.3.3)

Moreover, for $u = \sum_{i \in I} a_i v_i \in V$ and $c \in \mathbb{C}$, we define

$$cu = \sum_{i \in I} a_i c v_i.$$  \hfill (2.3.4)

Suppose that $G$ is a finite-dimensional real Lie algebra. The Lie bracket on $G_{\mathbb{C}}$ defined by

$$\left[ \sum_{i \in I} a_i u_i, \sum_{j \in I} b_j v_j \right] = \sum_{i, j \in I} a_i b_j [u_i, v_j], \quad u_i, v_j \in G, \quad a_i, b_j \in \mathbb{C}. \quad (2.3.5)$$

In particular, $G$ is a real subalgebra of $G_{\mathbb{C}}$. In fact, $G_{\mathbb{C}} = G + \sqrt{-1}G$. For the Killing form $\kappa_{G}$ of $G$ and the Killing form $\kappa_{G_{\mathbb{C}}}$ of $G_{\mathbb{C}}$, Rad $\kappa_{G_{\mathbb{C}}}$ is a solvable ideal of $G_{\mathbb{C}}$ because Rad $\kappa_{G_{\mathbb{C}}}$ is. Thus $G$ is semisimple if and only if $\kappa_{G}$ is nondegenerate. By the proof of Theorem 2.1.3, $G$ is a direct sum of simple ideals if it is semisimple. Moreover, $G$ is semisimple if and only if $G_{\mathbb{C}}$ is.

If $V$ is a finite-dimensional (real) $G$-module, then $V_{\mathbb{C}}$ becomes a (complex) $G_{\mathbb{C}}$-module with the action:

$$\left( \sum_{i} a_i \xi_i \right) \left( \sum_{j} b_j v_j \right) = \sum_{i, j} a_i b_j \xi_i (v_j), \quad \text{for } a_i, b_j \in \mathbb{C}, \xi_i \in G, \ v_j \in V.$$  \hfill (2.3.6)

We can also view $V_{\mathbb{C}}$ as a real $G$-module. In fact, $V_{\mathbb{C}} = V + \sqrt{-1}V$ is a decomposition of real $G$-submodules. As real $G$-modules, $V \cong \sqrt{-1}V$.

**Theorem 2.3.2** The space $V$ is a completely reducible real $G$-module if and only if $V_{\mathbb{C}}$ is a completely reducible complex $G_{\mathbb{C}}$-module.

**Proof** Suppose that $V$ is a completely reducible real $G$-module. Then $V = \bigoplus_{i=1}^{k} V_i$ is a direct sum of real irreducible $G$-submodules. Moreover,

$$V_{\mathbb{C}} = \bigoplus_{i=1}^{k} (V_i \oplus \sqrt{-1}V_i)$$  \hfill (2.3.7)
is a direct sum of $2k$ real irreducible $\mathcal{G}$-submodules. So $V_\mathbb{C}$ is a completely reducible real $\mathcal{G}$-module. Let $U$ be a complex $\mathcal{G}_\mathbb{C}$-submodule of $V_\mathbb{C}$. Note that $U$ is also a real $\mathcal{G}$-submodule of $V_\mathbb{C}$. Thus there exists a real $\mathcal{G}$-module homomorphism $\tau : V_\mathbb{C} \to U$ such that $\tau|_U = \text{Id}_U$. Define

$$\tilde{\tau}(v) = \frac{1}{2}(\tau(v) - \sqrt{-1}\tau(\sqrt{-1}v)) \quad \text{for} \quad v \in V. \quad (2.3.8)$$

Note

$$\tilde{\tau}(\sqrt{-1}v) = \frac{1}{2}(\tau(\sqrt{-1}v) + \sqrt{-1}\tau(v))$$
$$= \frac{1}{2}\sqrt{-1}(-\sqrt{-1}\tau(\sqrt{-1}v) + \tau(v)) = \sqrt{-1}\tilde{\tau}(v) \quad (2.3.9)$$

for $v \in V$. So $\tilde{\tau}$ is a $\mathbb{C}$-linear map from $V_\mathbb{C}$ to $U$. Moreover,

$$\xi(\tilde{\tau}(v)) = \frac{1}{2}(\tau(\xi(v)) - \sqrt{-1}\tau(\sqrt{-1}\xi(v))) = \tilde{\tau}(\xi(v)) \quad (2.3.10)$$

for $v \in V$ and $\xi \in \mathcal{G}$ by (2.3.6). Thus $\tilde{\tau}$ is a $\mathcal{G}_\mathbb{C}$-module homomorphism due to $\mathcal{G}_\mathbb{C} = \mathbb{C}\mathcal{G}$ by (2.3.4). Furthermore, for $u \in U$, we have $\sqrt{-1}u \in U$ because $U$ is a complex $\mathcal{G}_\mathbb{C}$-submodule of $V_\mathbb{C}$. Hence $\tau(u) = u$ and $\tau(\sqrt{-1}u) = \sqrt{-1}u$. Thus

$$\tilde{\tau}(u) = \frac{1}{2}(u - \sqrt{-1}(\sqrt{-1}u)) = u; \quad (2.3.11)$$

that is, $\tilde{\tau}|_U = \text{Id}_U$. By the above lemma, $V_\mathbb{C}$ is completely reducible.

Conversely, we assume that $V_\mathbb{C}$ is a completely reducible complex $\mathcal{G}_\mathbb{C}$-module. Let $W$ be a real $\mathcal{G}$-submodule of $V$. Then $W_\mathbb{C}$ forms a $\mathcal{G}_\mathbb{C}$-submodule. There exists a $\mathcal{G}_\mathbb{C}$-module homomorphism $\bar{\tau} : V_\mathbb{C} \to W_\mathbb{C}$ such that $\bar{\tau}|_{W_\mathbb{C}} = \text{Id}_{W_\mathbb{C}}$. So $\bar{\tau}|_V$ is a real $\mathcal{G}$-module homomorphism from $V$ to $W_\mathbb{C}$ because $\mathcal{G} \subset \mathcal{G}_\mathbb{C}$. But in general, $\bar{\tau}(V) \not\subset W$. Define

$$\nu(w_1 + \sqrt{-1}w_2) = w_1 \quad w_1, w_2 \in W. \quad (2.3.12)$$

Then $\nu : W_\mathbb{C} \to W$ is a real $\mathcal{G}$-module homomorphism. Therefore, $\tau = \nu\bar{\tau}$ is a real $\mathcal{G}$-module homomorphism from $V$ to $W$ and $\tau|_W = \text{Id}_W$. By the above lemma, $V$ is a completely reducible real $\mathcal{G}$-module. \hfill $\square$

Let $M = \bigoplus_{i=1}^k M_i$ be a completely reducible module of a Lie algebra $\mathcal{G}$ over an arbitrary field, where $M_i$'s are irreducible $\mathcal{G}$-submodules. For $i \in \overline{1, k}$, we define the projection $\mathcal{P}_i : M \to M_i$ by

$$\mathcal{P}_i \left( \sum_{r=1}^k u_r \right) = u_i \quad \text{for} \quad u_r \in M_r. \quad (2.3.13)$$
Then $\mathcal{P}_i$’s are $\mathcal{G}$-module homomorphisms. Let $U$ be any irreducible $\mathcal{G}$-submodule of $M$. If $\mathcal{P}_i(U) \neq \{0\}$, then $\mathcal{P}_i|_U$ is a $\mathcal{G}$-module isomorphism from $U$ to $M_i$ by the irreducibility of $U$ and $M_i$; in particular, $\mathcal{P}_i(U) = M_i$. Set
\[ I = \{ r \in \mathbb{N} | \mathcal{P}_r(U) \neq \{0\} \}. \tag{2.3.14} \]

For $r, s \in I$, the map $\mathcal{P}_s(\mathcal{P}_r|_U)^{-1}$ is a $\mathcal{G}$-module isomorphism from $M_r$ to $M_s$. Re-indexing $\{M_1, M_2, \ldots, M_k\}$ if necessary, we can assume $1 \in I$. Now
\[ u = \sum_{r \in I} \mathcal{P}_r(u) = \mathcal{P}_1(u) + \sum_{1 \neq r \in I} [\mathcal{P}_r(\mathcal{P}_1|_U)^{-1}](\mathcal{P}_1(u)) \quad \text{for } u \in U. \tag{2.3.15} \]

Thus
\[ U = \left\{ v + \sum_{1 \neq r \in I} [\mathcal{P}_r(\mathcal{P}_1|_U)^{-1}](v) \mid v \in M_1 \right\}. \tag{2.3.16} \]

Suppose that $M = \bigoplus_{j=1}^{k'} V_j$ is another decomposition of irreducible $\mathcal{G}$-submodules. Then $\mathcal{P}_1(V_j) \neq \{0\}$ for some $i \in \mathbb{N}$. Re-indexing $\{M_1, M_2, \ldots, M_k\}$ if necessary, we can assume $\mathcal{P}_1(V_j) \neq \{0\}$. So $\mathcal{P}_1|_{V_j}$ is a $\mathcal{G}$-module isomorphism from $V_j$ to $M_1$. Moreover,
\[ V = V_1 \oplus \bigoplus_{i=2}^{k} M_i. \tag{2.3.17} \]

In particular,
\[ \bigoplus_{i=2}^{k} (M_i + V_1)/V_1 = V/V_1 = \bigoplus_{r=2}^{k'} (V_i + V_1)/V_1 \tag{2.3.18} \]
as $\mathcal{G}$-modules. By induction on $k$, we have $k - 1 = k' - 1$; equivalently, $k = k'$. This shows that although a completely reducible $\mathcal{G}$-module may have more than one decompositions into irreducible submodules, the numbers of irreducible submodules in the decompositions are equal.

Let $V$ be a finite-dimensional real irreducible module of a real Lie algebra $\mathcal{G}$. Then $V_C = \bigoplus_{i=1}^{k} W_i$ is a direct sum of irreducible $\mathcal{G}_C$-submodules. Since $W_i$ are $\mathcal{G}$-modules and $V_C = V \oplus \sqrt{-1}V$ is a direct sum of two irreducible real $\mathcal{G}$-submodules, $W_i$ contain a real $\mathcal{G}$-submodule isomorphic to $V$. Let $M$ be a finite-dimensional irreducible $\mathcal{G}_C$-module. Take a minimal nonzero real $\mathcal{G}$-submodule $U$ of $M$. Then $U$ is an irreducible real $\mathcal{G}$-module and so is $\sqrt{-1}U$. If $U \cap \sqrt{-1}U \neq \{0\}$, then $U = \sqrt{-1}U$. So $U$ is also a $\mathcal{G}_C$-submodule. Hence $M = U$ is an irreducible real $\mathcal{G}$-module. Assume $U \cap \sqrt{-1}U = \{0\}$. Then $U + \sqrt{-1}U$ is a $\mathcal{G}_C$-submodule, and so $M = U + \sqrt{-1}U$.

Denote by $\mathcal{P}$ the corresponding projection from $M$ to $U$. Let $U'$ be any nonzero proper $\mathcal{G}$-submodule such that $\mathcal{P}(U') \neq \{0\}$. Then $\mathcal{P}(U') = U$ by the irreducibility of $U$. If $\ker \mathcal{P}|_{U'} = \{0\}$, then $U' \cong U$ is an irreducible real $\mathcal{G}$-module.
Suppose $\ker \mathcal{P}|_U \neq \{0\}$. Since $\ker \mathcal{P} = \sqrt{-1}U$, we have $U' \cap \sqrt{-1}U \neq \{0\}$. By the irreducibility of $\sqrt{-1}U$, $U' \supset \sqrt{-1}U$. The facts $\mathcal{P}(U') = U$ and $M = U + \sqrt{-1}U$ imply $U' = M$, which contradicts the properness of $U'$. Note that any nonzero proper $\mathcal{G}$-submodule $U'$ such that $\mathcal{P}(U') = \{0\}$ must be a submodule of $\sqrt{-1}U$. So $U' = \sqrt{-1}U$ by the irreducibility of $\sqrt{-1}U$. In summary, we have:

**Theorem 2.3.3** Let $\mathcal{G}$ be a real Lie algebra. Any finite-dimensional irreducible real $\mathcal{G}$-module must be a real $\mathcal{G}$-submodule of a finite-dimensional irreducible complex $\mathcal{G}_C$-module. A finite-dimensional irreducible complex $\mathcal{G}_C$-module is either an irreducible real $\mathcal{G}$-module or any of its proper $\mathcal{G}$-submodules is an irreducible real $\mathcal{G}$-module.

Thus we can find all finite-dimensional irreducible real $\mathcal{G}$-modules from finite-dimensional irreducible complex $\mathcal{G}_C$-modules.

### 2.4 Weyl’s Theorem

In this section, we prove the Weyl’s theorem of complete reducibility.

Assume the base field $\mathbb{F} = \mathbb{C}$. We will prove that any finite-dimensional module of a finite-dimensional semisimple Lie algebra is completely reducible. A representation is called **irreducible** if the associated module is irreducible. First we have:

**Lemma 2.4.1** (Schur) Let $\mathcal{G}$ be a Lie algebra and let $\phi : \mathcal{G} \to \mathfrak{gl}(V)$ be a finite-dimensional irreducible representation. If $T \in \mathfrak{gl}(V)$ commutes with every element in $\phi(\mathcal{G})$, then $T = \lambda \text{Id}_V$ for some $\lambda \in \mathbb{C}$.

**Proof** Since we assume $\mathbb{F} = \mathbb{C}$ and $V$ is finite-dimensional, $T$ has an eigenvector $u$, whose eigenvalue is denoted as $\lambda$. Denote by $\mathbb{Z}_+$ the set of positive integers. Set

$$U = \text{Span}\{u, \xi_1(\cdots (\xi_i(u)) \cdots) \mid i \in \mathbb{Z}_+, \xi_j \in \mathcal{G}\}. \tag{2.4.1}$$

Then $U$ forms a nonzero submodule of $V$. The irreducibility of $V$ implies $V = U$. Since

$$T(\xi_1(\cdots (\xi_i(u)) \cdots)) = \xi_1(T(\xi_2(\cdots (\xi_i(u)) \cdots)))
= \xi_1(\cdots (\xi_i(T(u))) \cdots)
= \lambda \xi_1(\cdots (\xi_i(u)) \cdots), \tag{2.4.2}$$

We have $T = \lambda \text{Id}_V$. □

Suppose that $\phi : \mathcal{G} \to \mathfrak{gl}(V)$ is a finite-dimensional faithful ($\ker \phi = \{0\}$) representation of a semisimple Lie algebra $\mathcal{G}$. Define

$$\beta(\xi, \zeta) = \text{Tr} \phi(\xi)\phi(\zeta) \quad \text{for} \quad \xi, \zeta \in \mathcal{G}. \tag{2.4.3}$$
By Cartan’s Criterion, \( \phi(\text{Rad } \beta) \) is a solvable ideal of \( \phi(\mathcal{G}) \), which is semisimple. Thus \( \text{Rad } \beta = \{0\} \), and there exists an orthonormal basis \( \{\xi_1, \xi_2, \ldots, \xi_n\} \) of \( \mathcal{G} \) with respect to \( \beta \). Set
\[
\Omega_\phi = \sum_{i=1}^{n} \phi(\xi_i)^2 \in \text{End } V,
\]
which is called a \textit{Casimir operator}. Write
\[
[\xi_i, \xi_j] = \sum_{k=1}^{n} c_{i,j}^k \xi_k.
\]
Then
\[
c_{i,j}^k = \beta([\xi_i, \xi_j], \xi_k) = -\beta(\xi_j, [\xi_i, \xi_k]) = -c_{i,k}^j.
\]
Thus
\[
[\phi(\xi_i), \Omega_\phi] = \left[ \phi(\xi_i), \sum_{j=1}^{n} \phi_j(u)^2 \right] = \sum_{j=1}^{n} [\phi(\xi_i), \phi(\xi_j)^2]
\]
\[
= \sum_{j=1}^{n} (\phi([\xi_i, \xi_j])\phi(\xi_j) + \phi(\xi_j)[\phi(\xi_i), \phi(\xi_j)])
\]
\[
= \sum_{j=1}^{n} (\phi([\xi_i, \xi_j])\phi(\xi_j) + \phi(\xi_j)\phi([\xi_i, \xi_j]))
\]
\[
= \sum_{j,k=1}^{n} c_{i,j}^k (\phi(\xi_k)\phi(\xi_j) + \phi(\xi_j)\phi(\xi_k))
\]
\[
= \sum_{j,k=1}^{n} c_{i,j}^k \phi(\xi_k)\phi(\xi_j) + \sum_{j,k=1}^{n} c_{i,j}^k \phi(\xi_j)\phi(\xi_k)
\]
\[
= \sum_{j,k=1}^{n} c_{i,k}^j \phi(\xi_j)\phi(\xi_k) + \sum_{j,k=1}^{n} c_{i,j}^k \phi(\xi_j)\phi(\xi_k)
\]
\[
= \sum_{j,k=1}^{n} (c_{i,j}^k + c_{i,k}^j) \phi(\xi_j)\phi(\xi_k) = 0.
\]
So we have
\[
\phi(u)\Omega_\phi = \Omega_\phi \phi(u) \quad \text{for } u \in \mathcal{G}.
\]
If \( \phi \) is irreducible, then \( \Omega_\phi = \lambda_\phi \text{Id}_V \) for some \( \lambda_\phi \in \mathbb{C} \) by Lemma 2.4.1. In fact,
\[
\lambda_\phi \dim V = \text{Tr } \Omega_\phi = \sum_{i=1}^{n} \text{Tr } \phi(\xi_i)^2 = \sum_{i=1}^{n} \beta(\xi_i, \xi_i) = n.
\]
Hence
\[ \lambda_\phi = n/ \dim V = \dim \mathcal{G}/ \dim V. \quad (2.4.10) \]

Furthermore, we have:

**Lemma 2.4.2** Let \( \phi : \mathcal{G} \to \mathfrak{gl}(V) \) be a finite-dimensional representation of a semisimple Lie algebra \( \mathcal{G} \). Then \( \phi(\mathcal{G}) \in \mathfrak{sl}(V) \). In particular, one-dimensional module of \( \mathcal{G} \) must be trivial.

**Proof** This follows from \( \phi(\mathcal{G}) = \phi([\mathcal{G}, \mathcal{G}]) = [\phi(\mathcal{G}), \phi(\mathcal{G})] \) and the fact \( \Tr [A, B] = 0 \) for any \( A, B \in \mathfrak{gl}(V) \). \( \square \)

**Lemma 2.4.3** Let \( \phi : \mathcal{G} \to \mathfrak{gl}(V) \) be a finite-dimensional representation of a semisimple Lie algebra \( \mathcal{G} \). Suppose that \( V \) has an irreducible submodule \( U \) of codimension one. Then \( V \) splits at \( U \).

**Proof** If \( \phi(\mathcal{G}) = \{0\} \), the lemma holds trivially. Suppose \( \phi(\mathcal{G}) \neq \{0\} \). Since \( \ker \phi \) is either \( \{0\} \) or a direct sum of some of simple ideals of \( \mathcal{G} \), \( \phi(\mathcal{G}) \cong \mathcal{G}/\ker \phi \) is semisimple. Replacing \( \mathcal{G} \) by \( \phi(\mathcal{G}) \), we may assume that \( \phi \) is faithful. By Lemma 2.4.2, \( V/U \) is a trivial module. Thus
\[ \xi(V) \subset U \quad \text{for} \quad \xi \in \mathcal{G}. \quad (2.4.11) \]

If \( U \) is a trivial submodule, then \( V \) is two-dimensional and \( \phi(\mathcal{G}) \) is isomorphic to the Lie algebra of strict up-triangular matrices, which is absurd. Thus \( \phi(\mathcal{G})|_U \neq \{0\} \) is semisimple.

Expression (2.4.11) implies
\[ \beta(\xi, \zeta) = \Tr \phi(\xi)\phi(\zeta) = \Tr \phi(\xi)|_U \phi(\zeta)|_U \quad \text{for} \quad \xi, \zeta \in \mathcal{G}. \quad (2.4.12) \]

This shows
\[ \Omega_{\phi|_U} = (\dim \phi(\mathcal{G})|_U/ \dim U) \Id_U. \quad (2.4.13) \]

Hence
\[ \tau = \frac{\dim U}{\dim \phi(\mathcal{G})|_U} \Omega_{\phi} \quad (2.4.14) \]

is a module homomorphism from \( V \) to \( U \) with \( \tau|_U = \Id_U \). So \( V \) splits at \( U \) by Lemma 2.3.1. \( \square \)

**Lemma 2.4.4** Let \( \phi : \mathcal{G} \to \mathfrak{gl}(V) \) be a finite-dimensional representation of a semisimple Lie algebra \( \mathcal{G} \). Suppose that \( V \) has a submodule \( U \) of codimension one. Then \( V \) splits at \( U \).

**Proof** We prove it by induction on \( \dim V \). It holds for \( \dim V = 1 \). Suppose that it holds for \( \dim V < k \). Consider the case \( \dim V = k \). We assume \( U \neq \{0\} \). If \( U \) is irreducible, this is the above lemma. Suppose that \( U \) has a nonzero proper submodule.
Then $U/W$ is a $G$-module of $V/W$ with codimension one. By induction, there exists a submodule $W' \supset W$ of $V$ such that

$$V/W = W'/W \oplus U/W.$$  \hfill (2.4.15)

Note $\dim W'/W = 1$. By induction, there exists a one-dimensional submodule $U'$ of $W'$ such that $W' = U' \oplus W$. According to (2.4.15), $U' \not\subset U$. Hence $U'$ is a complement of $U$ in $V$. \hfill $\square$

**Theorem 2.4.5** (Weyl) Any nonzero finite-dimensional module $V$ of a semisimple Lie algebra $G$ is completely reducible.

**Proof** Suppose that $U$ is a nonzero proper submodule of $V$. Then $\text{Hom}_C(V, U)$ forms a $G$-module. If $f \in \text{Hom}_C(V, U)$ such that $f|_U = a\text{Id}_U$ with $a \in C$, then

$$\xi(f)(u) = \xi(f(u)) - f(\xi(u)) = \xi(au) - a\xi(u) = 0 \quad \text{for} \quad u \in U.$$  \hfill (2.4.16)

Thus

$$W = \{ f \in \text{Hom}_C(V, U) \mid f|_U \in C\text{Id}_U \}, \quad W_1 = \{ f \in \text{Hom}_C(V, U) \mid f|_U = 0 \}$$  \hfill (2.4.17)

are submodules. Moreover, $\dim W/W_1 = 1$. By Lemma 2.4.4, $W_1$ has a complement $W'_1$ in $W$, which is a one-dimensional trivial module. Hence there exists $\tau \in W'_1$ such that $\tau|_U = \text{Id}_U$ and $\xi(\tau) = 0$ for any $\xi \in G'$; that is,

$$0 = \xi(\tau)(v) = \xi(\tau(v)) - \tau(\xi(v)) \quad \text{for} \quad v \in V.$$  \hfill (2.4.18)

So $\tau$ is a $G'$-homomorphism. By Lemma 2.3.1, $V$ splits at $U$. Therefore, $V$ is completely reducible by Lemma 2.2.1. \hfill $\square$

**Theorem 2.4.6** If $\mathcal{G} \subset gl(V)$ is a semisimple Lie subalgebra and $\dim V < \infty$, then $\mathcal{G}$ includes the semisimple part and nilpotent part of its any element. In particular, the abstract and usual Jordan decompositions coincide.

**Proof** By the above theorem, $V = \bigoplus_{i=1}^k V_i$ is a direct sum of irreducible submodules. Set

$$\mathcal{L} = \{ T \in gl(V) \mid [T, \mathcal{G}] \subset \mathcal{G}; \quad T(V_i) \subset V_i, \quad \text{Tr} T|_{V_i} = 0, \quad i = 1, 2, \ldots, k \}.$$  \hfill (2.4.19)

Since $\mathcal{G}|_{V_i}$ is a homomorphic image of the semisimple Lie algebra $\mathcal{G}$, it is a semisimple Lie algebra. Thus we have $\text{Tr} \xi|_{V_i} = 0$ for $\xi \in \mathcal{G}$ and $i = 1, \ldots, k$ by Lemma 2.4.2. Hence $\mathcal{G} \subset \mathcal{L}$. For any $T \in \mathcal{L}$, $\text{ad}_\mathcal{G} T \in \text{Der} \mathcal{G} = \text{ad} \mathcal{G}$ (cf. Theorem 2.1.4). So there exists an element $\xi' \in \mathcal{G}$ such that $\text{ad}_\mathcal{G} T = \text{ad}_\mathcal{G} \xi'$. Set

$$T_1 = T - \xi'.$$  \hfill (2.4.20)
Then
\[ [T_1, \xi] = 0 \quad \text{for} \quad \xi \in \mathcal{G}. \quad (2.4.21) \]

In particular, \( T_1 |_{V_i} \) is a constant map \( \lambda \text{Id}_{V_i} \) by Schur’s Lemma (Lemma 2.4.1). Now \( 0 = \text{Tr} \ T_1 |_{V_i} = \lambda(\dim V_i) \) by (2.4.19), or equivalently, \( \lambda = 0 \). Hence \( T_1 = 0 \), that is, \( T = \xi_i' \in \mathcal{G} \). This shows \( \mathcal{L} \subseteq \mathcal{G} \).

Given \( \xi \in \mathcal{G} \). Let \( \xi = \xi_s + \xi_n \) be the usual Jordan–Chevalley decomposition of \( \xi \). Recall that \( \xi_n \) is a polynomial of \( \xi \) without constant. Hence \( \xi_n(V_i) \subseteq V_i \). Since \( \text{ad}_{\mathfrak{gl}(V)} \xi = \text{ad}_{\mathfrak{gl}(V)} \xi_s + \text{ad}_{\mathfrak{gl}(V)} \xi_n \) is also a Jordan–Chevalley decomposition, we have \( [\xi_n, \mathcal{G}] \subseteq \mathcal{G} \). Note that \( \xi_n|_{V_i} \) is nilpotent because \( \xi_n \) is. So \( \text{Tr} \ \xi_n|_{V_i} = 0 \); that is, \( \xi_n \in \mathcal{L} \subseteq \mathcal{G} \). Moreover, \( \xi_s = \xi - \xi_n \in \mathcal{G} \). Note that the usual Jordan–Chevalley decomposition \( \xi = \xi_s + \xi_n \) implies that \( \text{ad}_\mathcal{G} \xi_s \) is semisimple, \( \text{ad}_\mathcal{G} \xi_n \) is nilpotent and \( [\text{ad}_\mathcal{G} \xi_s, \text{ad}_\mathcal{G} \xi_n] = 0 \). By the uniqueness of the Jordan–Chevalley decomposition, \( \xi = \xi_s + \xi_n \) is also the abstract Jordan–Chevalley decomposition. \( \Box \)

**Corollary 2.4.7** Let \( \phi : \mathcal{G} \rightarrow \mathfrak{gl}(V) \) be a finite-dimensional representation of a semisimple Lie algebra \( \mathcal{G} \). Given \( \xi \in \mathcal{G} \). If \( \xi = \xi_s + \xi_n \) is the abstract Jordan–Chevalley decomposition, then \( \phi(\xi) = \phi(\xi_s) + \phi(x_n) \) is the usual Jordan–Chevalley decomposition.

**Proof** If \( \phi(\mathcal{G}) \neq \{0\} \), it is semisimple. Moreover, the usual Jordan–Chevalley decomposition \( \phi(\xi) = \phi(\xi_s) + \phi(\xi_n) \) is also the abstract Jordan–Chevalley decomposition of \( \phi(\xi) \) by the above theorem. If \( \xi = \xi_s + \xi_n \) is the abstract Jordan–Chevalley decomposition, then \( \text{ad}_\xi = \text{ad}_\xi_s + \text{ad}_\xi_n \) is the usual Jordan–Chevalley decomposition. Since \( \phi(\mathcal{G}) \cong \mathcal{G} / \ker \phi \), \( \text{ad}_\phi(\xi_s) \) is semisimple and \( \text{ad}_\phi(\xi_n) \) is nilpotent. Moreover, \( [\text{ad}_\phi(\xi_s), \text{ad}_\phi(\xi_n)] = \text{ad}_\phi([\xi_s, \xi_n]) = 0 \). Hence \( \text{ad}_\phi(\xi) = \text{ad}_\phi(\xi_s) + \text{ad}_\phi(\xi_n) \) is the Jordan–Chevalley decomposition of \( \phi(\xi) \) by the uniqueness. Thus \( \phi(\xi) = \phi(\xi_s) + \phi(\xi_n) \) is the abstract Jordan–Chevalley decomposition of \( \phi(\xi) \). Therefore, \( \phi(\xi_s) = \phi(\xi) \) and \( \phi(\xi_n) = \phi(\xi) \).

Let \( \mathcal{G} \) be a simple Lie algebra and let \( \beta, \gamma \) be two symmetric associative bilinear forms on \( \mathcal{G} \). If \( \beta \neq 0 \), then \( \gamma = a \beta \) for some \( a \in \mathbb{C} \) (exercise).

We can identify \( (\mathcal{G} \otimes \mathcal{G})^* \) with the space of bilinear forms on \( \mathcal{G} \). An associative form \( \beta \) is an element in \( (\mathcal{G} \otimes \mathcal{G})^* \) such that \( \mathcal{G}(\beta) = \{0\} \) (exercise), where we view \( \mathcal{G} \) as the adjoint module of \( \mathcal{G} \).

### 2.5 Root Space Decomposition

In this section, we prove the Cartan’s root space decomposition of a semisimple Lie algebra.

We always assume that \( \mathcal{G} \) is a finite-dimensional semisimple Lie algebra over \( \mathbb{C} \). Recall the abstract Jordan decomposition of any element \( \xi = \xi_s + \xi_n \) in \( \mathcal{G} \), where \( \xi_s, \xi_n \in \mathcal{G} \) and \( \text{ad} \xi = \text{ad} \xi_s + \text{ad} \xi_n \) is the usual Jordan decomposition. By Engel’s Theorem, there exist elements in \( \mathcal{G} \) that possess nonzero semisimple parts. Thus \( \mathcal{G} \).
has nonzero semisimple elements. A Lie subalgebra of $\mathcal{G}$ consisting of semisimple elements is called toral.

**Lemma 2.5.1** A toral subalgebra is abelian.

**Proof** Let $\mathcal{T}$ be a toral subalgebra of $\mathcal{G}$. Given $\xi \in \mathcal{T}$. The operator $\text{ad}_\mathcal{T} \xi$ is diagonalizable. Suppose that $\zeta$ is an eigenvector of $\text{ad}_\mathcal{T} \xi$ with eigenvalue $\lambda$; that is $[\xi, \zeta] = \lambda \zeta$. On the other hand, the fact that $\text{ad}_\mathcal{T} \zeta$ is diagonalizable implies that $\mathcal{T}$ has a basis $\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$ consisting eigenvectors of $\text{ad}_\mathcal{T} \zeta$ with corresponding eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. In particular, we can take $\zeta_1 = \zeta$ with $\lambda_1 = 0$. Write

$$\xi = \sum_{i=1}^{n} a_i \zeta_i, \quad a_i \in \mathbb{C}. \quad (2.5.1)$$

Then

$$\lambda \zeta = [\xi, \zeta] = \left[ \sum_{i=1}^{n} a_i \zeta_i, \zeta \right] = -\sum_{i=2}^{n} \lambda_i a_i \zeta_i. \quad (2.5.2)$$

By linear independence of $\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$, $\lambda = 0$. Thus all the eigenvalues of $\text{ad}_\mathcal{T} \xi$ are 0. Hence $\text{ad}_\mathcal{T} u \xi = 0$. $\square$

Let $H$ be a maximal toral subalgebra of $\mathcal{G}$. For instance, we take the algebra of all diagonal elements in $\mathfrak{sl}(n, \mathbb{C})$. Since $\{\text{ad}_h \mid h \in H\}$ are commuting diagonalizable operators, by linear algebra,

$$\mathcal{G} = \bigoplus_{\alpha \in H^*} \mathcal{G}_\alpha, \quad \mathcal{G}_\alpha = \{ \xi \in \mathcal{G} \mid [h, \xi] = \alpha(h)\xi \text{ for } h \in H \}. \quad (2.5.3)$$

**Lemma 2.5.2** For $\alpha, \beta \in H^*$, $[\mathcal{G}_\alpha, \mathcal{G}_\beta] \subset \mathcal{G}_{\alpha+\beta}$. In particular, for any $u \in \mathcal{G}_\alpha$ with $\alpha \neq 0$, $\text{ad} u$ is nilpotent. Moreover,

$$\kappa(\mathcal{G}_\alpha, \mathcal{G}_\beta) = \{0\} \quad \text{for } \alpha, \beta \in H^*, \alpha \neq -\beta, \quad (2.5.4)$$

which implies that $\kappa(\mathcal{G}_\alpha, \mathcal{G}_0)$ is nondegenerate.

**Proof** For $h \in H$, $\xi \in \mathcal{G}_\alpha$ and $\zeta \in \mathcal{G}_\beta$, we have:

$$[h, [\xi, \zeta]] = [[h, \xi], \zeta] + [\xi, [h, \zeta]] = (\alpha(h) + \beta(h))[\xi, \zeta] = (\alpha + \beta)(h)[\xi, \zeta]. \quad (2.5.5)$$

Thus $[\mathcal{G}_\alpha, \mathcal{G}_\beta] \subset \mathcal{G}_{\alpha+\beta}$. Suppose that $\alpha, \beta \in H^*$ such that $\alpha \neq -\beta$. Given $h \in H$, $\xi \in \mathcal{G}_\alpha$ and $\zeta \in \mathcal{G}_\beta$, we get

$$\alpha(h)\kappa(\xi, \zeta) = \kappa([h, \xi], \zeta) = -\kappa(\xi, [h, \zeta]) = -\beta(h)\kappa(\xi, \zeta). \quad (2.5.6)$$
Thus
\[(\alpha + \beta)(h)\kappa(\xi, \zeta) = 0 \quad \text{for any } h \in H. \tag{2.5.7}\]

So \(\kappa(\xi, \zeta) = 0\); that is, (2.5.4) holds. For \(0 \neq \xi \in \mathcal{G}_0\), we have
\[\kappa(\xi, \mathcal{G}_0) = \{0\} \quad \text{for } 0 \neq \alpha \in H^*. \tag{2.5.8}\]

Since \(\kappa\) is nondegenerate, \(\kappa(\xi, \mathcal{G}_0) \neq \{0\}\). So \(\kappa(|\mathcal{G}_0|, |\mathcal{G}_0|)\) is nondegenerate. \(\square\)

**Lemma 2.5.3** \(\mathcal{G}_0 = H\).

**Proof** For any \(\xi \in \mathcal{G}_0\), we have the abstract Jordan decomposition \(\xi = \xi_s + \xi_n\). Note \(\text{ad}\xi(H) = [\xi, H] = \{0\}\). Since \(\text{ad}\xi_s\) is a polynomial of \(\text{ad}\xi\) without constant, we have \([\xi_s, H] = \{0\}\). Thus \(H + \mathbb{C}\xi_s\) forms a toral subalgebra. But \(H\) is a maximal toral subalgebra, we have \(\xi_s \in H\), which implies \(\xi_n = \xi - \xi_s \in \mathcal{G}_0\) because \(H \subset \mathcal{G}_0\).

Since \(\text{ad}\xi_s|_{\mathcal{G}_0} = 0\) and \(\text{ad}\xi_n\) is nilpotent, \(\text{ad}|_{\mathcal{G}_0}\) is nilpotent. By Engel’s Theorem, \(\mathcal{G}_0\) is a nilpotent Lie algebra.

Suppose that \(h \in H\) satisfies \(\kappa(h, H) = \{0\}\). For any \(\xi \in \mathcal{G}_0\), let \(\xi = \xi_s + \xi_n\) be the abstract Jordan decomposition. Since \([\text{ad}\, h, \text{ad}\, \xi_n] = \text{ad}\, [h, \xi_n] = \{0\}\), \(\text{ad}\, h\) ad \(\xi_n\) is a nilpotent element. Thus \(\kappa(h, \xi_n) = 0\). So \(\kappa(h, \xi) = \kappa(h, \xi_s) + \kappa(h, \xi_n) = 0\); that is \(h \in \text{Rad}\, \kappa(|\mathcal{G}_0|, |\mathcal{G}_0|) = \{0\}\). Therefore, \(\kappa(|H|, |H|)\) is nondegenerate. For any \(h \in H\) and \(\xi_1, \xi_2 \in \mathcal{G}_0\),
\[\kappa(h, [\xi_1, \xi_2]) = \kappa([h, \xi_1], \xi_2) = 0. \tag{2.5.9}\]

Hence
\[\kappa(H, [\mathcal{G}_0, \mathcal{G}_0]) = \{0\}. \tag{2.5.10}\]

If \([\mathcal{G}_0, \mathcal{G}_0] \neq \{0\}\), we take \(0 \neq \xi \in Z(\mathcal{G}_0) \cap [\mathcal{G}_0, \mathcal{G}_0]\). Let \(\xi = \xi_s + \xi_n\) be the abstract Jordan decomposition. Then \(\xi_s \in H\) and \(\kappa(H, \xi_n) = \{0\}\). Hence
\[\kappa(H, \xi_s) = \kappa(H, \xi) = \{0\}. \tag{2.5.11}\]

by (2.5.10). The nondegeneracy of \(\kappa(|H|, |H|)\) implies \(\xi_s = 0\). So \(\xi\) is ad-nilpotent. If \([\mathcal{G}_0, \mathcal{G}_0] = \{0\}\) and \(H \neq \mathcal{G}_0\), then \(u_n\) is a nonzero ad-nilpotent element for any \(u \in \mathcal{G}_0 \setminus H\). In summary, if \(H \neq \mathcal{G}_0\), we have a nonzero ad-nilpotent element \(\xi \in Z(\mathcal{G}_0)\).

For any \(\xi' \in \mathcal{G}_0\), \(\text{ad}\, \xi'\) ad \(\xi\) is nilpotent. So \(\kappa(\xi', \xi) = 0\); that is, \(\xi \in \text{Rad}\, \kappa(|\mathcal{G}_0|, |\mathcal{G}_0|) = \{0\}\), which leads a contradiction. Therefore \(H = \mathcal{G}_0\). \(\square\)

Set
\[\Phi = \{\alpha \in H^* \mid \alpha \neq 0, \mathcal{G}_0 \neq \{0\}\}. \tag{2.5.12}\]

The elements of \(\Phi\) are called *roots* of \(\mathcal{G}\) and \(\Phi\) is called the *root system* of \(\mathcal{G}\). Since \(\kappa(|H|, |H|)\) is nondegenerate, for any \(\alpha \in H^*\), there exists a unique \(t_\alpha \in H\) such that
\[\alpha(h) = \kappa(t_\alpha, h) \quad \text{for } h \in H. \tag{2.5.13}\]
Moreover, \( t_{-\alpha} = -t_{\alpha} \).

**Lemma 2.5.4** (a) \( \Phi \) spans \( H^* \).

(b) If \( \alpha \in \Phi \), then \(-\alpha \in \Phi \).

(c) Let \( \alpha \in \Phi \), \( \xi \in \mathcal{G}_\alpha \), \( \zeta \in \mathcal{G}_{-\alpha} \). Then \( [\xi, \zeta] = \kappa(\xi, \zeta) t_{\alpha} \). In particular, \( [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}] \supseteq \mathbb{C} t_{\alpha} \).

(d) If \( \alpha \in \Phi \) and \( 0 \not= \zeta_{\alpha} \in \mathcal{G}_{-\alpha} \), there exists \( \zeta_{\alpha} \in \mathcal{G}_{-\alpha} \) such that \( \{ \xi_{\alpha}, t_{\alpha}, \zeta_{\alpha} \} \) spans a Lie subalgebra isomorphic to \( sl(2, \mathbb{C}) \).

**Proof** (a) If \( \Phi \) does not span \( H^* \), then there exists \( 0 \neq h \in H \) such that \( \alpha(h) = 0 \) for any \( \alpha \in \Phi \). By (2.5.3), \( h \in Z(\mathcal{G}) \), which is absurd.

(b) By (2.5.4), \( \kappa(\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}) \neq \{0\} \) if \( \alpha \in \Phi \).

(c) By Lemmas 2.5.2 and 2.5.3, \( [\xi, \zeta] \in H \). For any \( h \in H \), we have

\[
\kappa(h, [\xi, \zeta]) = \kappa([h, \xi], \zeta) = \alpha(h) \kappa(\xi, \zeta) = \kappa(h, t_{\alpha}) \kappa(\xi, \zeta),
\]

or equivalently,

\[
\kappa(h, [\xi, \zeta] - \kappa(\xi, \zeta) t_{\alpha}) = \kappa(h, [\xi, \zeta]) - \kappa(h, t_{\alpha}) \kappa(\xi, \zeta) = 0.
\]

Since \( \kappa|_H \) is nondegenerate, we have \( [\xi, \zeta] - \kappa(\xi, \zeta) t_{\alpha} = 0 \). \( \{ \mathcal{G}_\alpha, \mathcal{G}_{-\alpha} \} \supseteq \mathbb{C} t_{\alpha} \) because \( \kappa(\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}) \neq \{0\} \).

(d) By (2.5.3) and the nongeneracy of \( \kappa \), there exist \( \zeta \in \mathcal{G}_{-\alpha} \) such that \( \kappa(\zeta_{\alpha}, \zeta) = 1 \). So

\[
[\xi_{\alpha}, \zeta] = t_{\alpha}
\]

by (c). If \( \alpha(t_{\alpha}) = 0 \), then

\[
[t_{\alpha}, \xi_{\alpha}] = \alpha(t_{\alpha}) \xi_{\alpha} = 0 = -\alpha(t_{\alpha}) \zeta = [t_{\alpha}, \zeta].
\]

Thus \( \mathcal{N} = \mathbb{C} \xi_{\alpha} + \mathbb{C} t_{\alpha} + \mathbb{C} \zeta \) forms a solvable Lie subalgebra of \( \mathcal{G} \) and \( t_{\alpha} \in [\mathcal{N}, \mathcal{N}] \). By Lie’s theorem, \( \text{ad} t_{\alpha} \) is nilpotent. But \( t_{\alpha} \in H \) is ad-semisimple. Hence \( \text{ad} t_{\alpha} = 0 \). Since \( Z(\mathcal{G}) = \{0\} \), we have \( t_{\alpha} = 0 \). Thereby, \( \alpha = 0 \), which is absurd. Hence \( 0 \not= \alpha(t_{\alpha}) = \kappa(t_{\alpha}, t_{\alpha}) \).

Set

\[
\frac{2}{\alpha(t_{\alpha})} t_{\alpha}, \quad \zeta_{\alpha} = \frac{2}{\alpha(t_{\alpha})} \zeta.
\]

Then

\[
[\xi_{\alpha}, \zeta_{\alpha}] = h_{\alpha}, \quad [h_{\alpha}, \xi_{\alpha}] = 2 \xi_{\alpha}, \quad [h_{\alpha}, \zeta_{\alpha}] = -2 \zeta_{\alpha}.
\]

Hence

\[
\mathcal{G}_\alpha = \mathbb{C} \xi_{\alpha} + \mathbb{C} h_{\alpha} + \mathbb{C} \zeta_{\alpha}
\]
forms a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ via the linear map determined by
\[
\xi_\alpha \mapsto E_{1,2}, \quad h_\alpha \mapsto E_{1,1} - E_{2,2}, \quad \zeta_\alpha \mapsto E_{2,1}.
\] (2.5.21)
By (2.5.18), $h_{-\alpha} = -h_\alpha$. \qed

**Remark 2.5.5** In [X7, X8], six families of infinite-dimensional simple Lie algebra $\mathcal{G}$ without any toral subalgebra $H$ such that $\mathcal{G}_0 = H$ (cf. (2.5.3)) were constructed.

### 2.6 Properties of Roots and Root Subspaces

In this section, we use finite-dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$ to study the properties of roots and root subspaces of finite-dimensional semisimple Lie algebra $\mathcal{G}$. In particular, we prove that such a Lie algebra is generated by two elements.

Again we assume the base field $\mathbb{F} = \mathbb{C}$. For convenience, we also take $\partial_x = d/dx$.

For any nonnegative integer $n$, set
\[
V(n) = \sum_{i=0}^{n} \mathbb{C}x^i \subset \mathbb{C}[x].
\] (2.6.1)

Recall
\[
\mathfrak{sl}(2, \mathbb{C}) = \mathbb{C}E_{1,2} + \mathbb{C}(E_{1,1} - E_{2,2}) + \mathbb{C}E_{2,1}.
\] (2.6.2)

We define an action of $\mathfrak{sl}(2, \mathbb{C})$ on $V(n)$ by
\[
E_{1,2}|_{V(n)} = x^2 \partial_x - nx, \quad E_{2,1}|_{V(n)} = -\partial_x, \quad (E_{1,1} - E_{2,2})|_{V(n)} = 2x \partial_x - n.
\] (2.6.3)

It can be verified that $V(n)$ forms an irreducible $\mathfrak{sl}(2, \mathbb{C})$-module (exercise). For convenience, we denote
\[
e = E_{1,2}, \quad f = E_{2,1}, \quad h = E_{1,1} - E_{2,2}.
\] (2.6.4)

For any $\mathfrak{sl}(2, \mathbb{C})$-module $W$, we denote
\[
W_a = \{ w \in W \mid h(w) = aw \} \quad \text{for} \quad a \in \mathbb{C}.
\] (2.6.5)

For instance,
\[
V(n)_{2i-n} = \mathbb{C}x^i, \quad V(n) = \bigoplus_{i=0}^{n} V(n)_{2i-n}.
\] (2.6.6)
Theorem 2.6.1 Any \((n + 1)\)-dimensional irreducible \(s\ell(2, \mathbb{C})\)-module is isomorphic to \(V(n)\). A finite-dimensional \(s\ell(2, \mathbb{C})\)-module \(W\) is a direct sum of \(\dim W_0 + \dim W_1\) irreducible submodules.

Proof Let \(V\) be any \((n + 1)\)-dimensional irreducible \(s\ell(2, \mathbb{C})\)-module. Note that \(\mathcal{B} = \mathbb{C}h + \mathbb{C}e\) forms a solvable Lie subalgebra of \(s\ell(2, \mathbb{C})\). By Lie’s Theorem, there exists a common eigenvector \(v\) of \(h\) and \(e\) in \(V\); that is,

\[
h(v) = \lambda v, \quad e(v) = \mu v, \quad \lambda, \mu \in \mathbb{C}. \tag{2.6.7}
\]

But

\[
2\mu v = 2e(v) = [h, e](v) = h(e(v)) - e(h(v)) = \mu \lambda v - \lambda \mu v = 0. \tag{2.6.8}
\]

Thus \(\mu = 0\); that is, \(e(v) = 0\).

Recall that \(\mathbb{N}\) is the set of nonnegative integers. Note

\[
h(f^i(v)) = (\lambda - 2i)f^i(v) \quad \text{for} \quad i \in \mathbb{N}. \tag{2.6.9}
\]

Moreover,

\[
e(f^{i+1}(v)) = e(f((f^i(v)))) = [e, f](f^i(v)) + f(e(f^i(v)))
\]

\[
= (\lambda - 2i)f^i(v) + f(e(f^i(v))). \tag{2.6.10}
\]

By induction, we have

\[
e(f^{i+1}(v)) = \left[(i + 1)\lambda - 2 \sum_{r=0}^{i} r\right]f^i(v) = (i + 1)(\lambda - i)f^i(v). \tag{2.6.11}
\]

The above equation shows

\[
f^i(v) \neq 0 \quad \text{if} \quad \lambda \notin \{0, 1, \ldots, i - 1\}. \tag{2.6.12}
\]

On the other hand, (2.6.9) implies that

\[
\{v, f(v), \ldots, f^i(v) \mid \lambda \notin \{0, 1, \ldots, i - 1\}\} \tag{2.6.13}
\]

is a set of eigenvectors of \(h\) with distinct eigenvalues. So it is linearly independent. Since \(\dim V = n + 1, \lambda \in \{0, 1, \ldots, n\}\). By (2.6.11), \(e(f^{\lambda+1}(v)) = 0\). Hence

\[
\sum_{i=1}^{\infty} \mathbb{C}f^{\lambda+i}(v). \tag{2.6.14}
\]
forms a proper $sl(2, \mathbb{C})$ submodule of $V$. The irreducibility of $V$ implies

$$f^{\lambda+1}(v) = 0. \quad (2.6.15)$$

Now $\sum_{i=0}^{\lambda} \mathbb{C} f^i(v)$ forms a nonzero submodule of $V$. Thus

$$V = \sum_{i=0}^{\lambda} \mathbb{C} f^i(v) \quad (2.6.16)$$

and $\{v, f(v), \ldots, f^\lambda(v)\}$ is a basis by $(2.6.9)$. Therefore, $\lambda = n$.

By $(2.6.3)$ and $(2.6.4)$,

$$f^i(x^n) = (-1)^i n(n-1) \cdots (n-i+1)x^{n-i} \quad \text{for } i \in \{0, 1, \ldots, n\}. \quad (2.6.17)$$

Define a linear map $\sigma : V(n) \to V$ by

$$\sigma(f^i(x^n)) = f^i(v) \quad \text{for } i \in \{0, 1, \ldots, n\}. \quad (2.6.18)$$

The Eqs. $(2.6.9)$, $(2.6.11)$ and $(2.6.15)$ show that the linear map $\sigma$ is $sl(2, \mathbb{C})$-module isomorphism.

Expressions $(2.6.5)$ and $(2.6.6)$ imply that

$$\dim V_0 = 0, \quad \dim V_1 = 1 \quad \text{if } n \in 2\mathbb{N} + 1 \quad (2.6.19)$$

and

$$\dim V_0 = 1, \quad \dim V_1 = 0 \quad \text{if } n \in 2\mathbb{N}. \quad (2.6.20)$$

Now Weyl’s Theorem tell us that any finite-dimensional $sl(2, \mathbb{C})$-module $W$ is a direct sum of $\dim W_0 + \dim W_1$ irreducible submodules. $\square$

Let $\mathcal{G}$ be a finite-dimensional semisimple Lie algebra and let $H$ be a maximal toral subalgebra of $\mathcal{G}$. Then we have the root space decomposition:

$$\mathcal{G} = H + \sum_{\alpha \in \Phi} \mathcal{G}_\alpha. \quad (2.6.21)$$

where $\Phi$ is the root system of $\mathcal{G}$. Recall Lemma 2.5.4.

**Lemma 2.6.2** (a) $\dim \mathcal{G}_\alpha = 1$ and $\mathbb{C}\alpha \cap \Phi = \{\alpha, -\alpha\}$ for any $\alpha \in \Phi$. In particular, $\mathcal{L}_\alpha = \mathcal{G}_\alpha + \mathbb{C} h_\alpha + \mathcal{G}_{-\alpha}$ (cf. $(2.5.20)$).

(b) If $\alpha, \beta \in \Phi$, then $\beta(h_\alpha) \in \mathbb{Z}$ (which is called Cartan integer) and $\beta - \beta(h_\alpha)\alpha \in \Phi$.

(c) If $\alpha, \beta, \alpha + \beta \in \Phi$, then $[\mathcal{G}_\alpha, \mathcal{G}_\beta] = \mathcal{G}_{\alpha + \beta}$.
(d) Let $\alpha, \beta \in \Phi, \beta \neq \pm \alpha$. Let $r, q$ be (respectively) the largest integers such that $\beta - r\alpha$ and $\beta + q\alpha$ are roots. Then all $\beta + i\alpha \in \Phi$ for $-r \leq i \leq q$.

(e) $G$ is generated by $\{\mathscr{G}_\alpha \mid \alpha \in \Phi\}$.

Proof. Denote $\dim H = n$. Let $\alpha \in \Phi$ be any element. Recall the Lie subalgebra $\mathcal{S}_\alpha$ defined in (2.5.20). By the above lemma, all the eigenvalues of $\text{ad} h_\alpha$ are integers. If $a\alpha \in \Phi$, then

$$a\alpha(h_\alpha) = 2a \in \mathbb{Z} \implies a \in \frac{\mathbb{Z}}{2}. \tag{2.6.22}$$

Note that $\mathcal{S} = \mathcal{S}_\alpha$ is generated by $\mathcal{S}_\alpha$-submodules. Thus $\mathcal{S}$ is a direct sum of one-dimensional irreducible $\mathcal{S}_\alpha$-submodules. Set

$$H' = \{h \in H \mid \alpha(h) = 0\}. \tag{2.6.24}$$

For any $h \in H'$, we have

$$[h, \xi_\alpha] = \alpha(h)\xi_\alpha = 0, \ [h, \zeta_\alpha] = -\alpha(h)\zeta_\alpha = 0, \ [h, h_\alpha] = 0 \tag{2.6.25}$$

(cf. Lemma 2.5.4). Since $\mathcal{S}_\alpha = \mathbb{C}\xi_\alpha + \mathbb{C}h_\alpha + \mathbb{C}\zeta_\alpha$, $H'$ is a direct sum of $(n - 1)$ one-dimensional irreducible ad $\mathcal{S}_\alpha$-submodules. Thus $\mathcal{S}_\alpha + H'$ is a direct sum of $n$ irreducible ad $\mathcal{S}_\alpha$-submodules. Therefore, the complement of $\mathcal{S}_\alpha + H'$ in $\mathcal{L}$ must be zero; that is, $\mathcal{L} = H' + \mathcal{S}_\alpha$. So $\mathcal{S}_\alpha = \mathcal{G}_\alpha + \mathbb{C}h_\alpha + \mathcal{H}_\alpha$, $\dim \mathcal{G}_\alpha = \dim \mathbb{C}\xi_\alpha = 1$ and $\mathbb{Z}\alpha \cap \Phi = \{\alpha, -\alpha\}$. In particular, $2\alpha \notin \Phi$ for any $\alpha \in \Phi$. This shows $\alpha/2 \notin \Phi$; that is, $\mathcal{G}_{\alpha/2} = \{0\}$. Since $\mathcal{L} = \sum_{m \in \mathbb{Z}} \mathcal{G}_{m\alpha + \alpha/2}$ forms an ad $\mathcal{S}_\alpha$-submodule, the eigenvalues of $\text{ad} h_\alpha$ in $\mathcal{L}$ are odd integers and the eigenspace of the eigenvalue 1 is $\mathcal{G}_{\alpha/2} = \{0\}$, we must have $\mathcal{L} = \{0\}$ by the above lemma. According to (2.6.22), we get $\mathbb{C}\alpha \cap \Phi = \{\alpha, -\alpha\}$ for any $\alpha \in \Phi$. This proves (a).

Let $\alpha, \beta \in \Phi, \beta \neq \pm \alpha$. Then

$$\mathcal{K} = \sum_{i \in \mathbb{Z}} \mathcal{G}_{\beta + i\alpha} \tag{2.6.26}$$

forms an ad $\mathcal{S}_\alpha$-submodule of $G$. Since $\mathcal{G}_\beta$ is an eigenspace of $\text{ad} h_\alpha$ with the eigenvalue $\beta(h_\alpha)$, we must have $\beta(h_\alpha) \in \mathbb{Z}$. By (a), $\dim \mathcal{G}_{\beta + i\alpha} \leq 1$. If $\beta + i\alpha \in \Phi$, then $\mathcal{G}_{\beta + i\alpha}$ is an eigenspace of $\text{ad} h_\alpha$ with the eigenvalue $\beta(h_\alpha) + 2i$. Thus

$$\dim \{\xi \in \mathcal{K} \mid [h_\alpha, \xi] = 0\} + \dim \{\zeta \in \mathcal{K} \mid [h_\alpha, \zeta] = \zeta\} \leq 1. \tag{2.6.27}$$

By the above lemma, $\mathcal{K}$ is an irreducible ad $\mathcal{S}_\alpha$-module. Let $r, q$ be (respectively) the largest integers such that $\beta - r\alpha$ and $\beta + q\alpha$ are roots. Then all $\beta + i\alpha \in \Phi$ for $-r \leq i \leq q$ by (2.6.6) and the above lemma. This proves (d). If $\alpha + \beta \in \Phi$, then
\[ [\xi_\alpha, \mathcal{G}_\beta] = \mathcal{G}_{\alpha + \beta} \] by (2.6.11). So (c) holds. Expression (2.6.6) shows
\[ - (\beta(h_\alpha) - 2r) = \dim \mathcal{X} - 1 = \beta(h_\alpha) + 2q, \] (2.6.28)
which implies \( \beta(h_\alpha) = r - q \). Furthermore,
\[ \beta - \beta(h_\alpha) \alpha = \beta - (r - q) \alpha = \beta + q \alpha - r \alpha \in \Phi. \] (2.6.29)
Thus (b) holds. Since \( \Phi \) spans \( H^* \), \( \{t_\alpha \mid \alpha \in \Phi \} \) spans \( H \). The equations in (2.5.18) and (2.5.19) yield (e). □

**Proposition 2.6.3** The semisimple Lie algebra \( \mathcal{G} \) is generated by two elements.

**Proof** Denote
\[ \Psi = \{ \alpha - \beta \mid \alpha, \beta \in \Phi, \alpha \neq \beta \}. \] (2.6.30)
Take
\[ h_0 \in H \setminus \bigcup_{\gamma \in \Psi} \{ h \in H \mid \gamma(h) = 0 \}. \] (2.6.31)
For each \( \alpha \in \Phi \), we pick \( 0 \neq \xi_\alpha \in \mathcal{G}_\alpha \). Then \( \{\xi_\alpha \mid \alpha \in \Phi \} \) are eigenvectors of \( \text{ad} \, h_0 \) with distinct eigenvalues \( \{\alpha(h_0) \mid \alpha \in \Phi \} \). Set
\[ \xi = \sum_{\alpha \in \Phi} \xi_\alpha. \] (2.6.32)
Let \( \mathcal{L} \) be the Lie subalgebra of \( \mathcal{G} \) generated by \( h_0 \) and \( \xi \). Then
\[ \text{ad} \, h_0(\mathcal{L}) \subset \mathcal{L}, \quad \text{ad} \, h_0(\xi) \in \mathcal{L}. \] (2.6.33)
By Lemma 1.3.1,
\[ \xi_\alpha \in \mathcal{L} \quad \text{for} \quad \alpha \in \Phi. \] (2.6.34)
By Lemma 2.6.2 (e), \( \mathcal{L} = \mathcal{G} \). □

Define
\[ (\alpha, \beta) = \kappa(t_\alpha, t_\beta) \quad \text{for} \quad \alpha, \beta \in H^*. \] (2.6.35)
(cf. (2.5.13)). Then \( (\cdot, \cdot) \) is a nondegenerate symmetric bilinear form on \( H^* \). By the arguments above (2.5.18), \( (\alpha, \alpha) = \alpha(t_\alpha) \neq 0 \) for \( \alpha \in \Phi \). Moreover, we set
\[ \langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \quad \text{for} \quad \alpha \in \Phi, \beta \in H^*. \] (2.6.36)
By (2.5.18) and Lemma 2.6.2 (b),
\[ (\alpha, \beta) \in \mathbb{Z}, \quad \beta - \langle \beta, \alpha \rangle \alpha \in \Phi \quad \text{for} \quad \alpha, \beta \in \Phi. \] (2.6.37)
Since $\Phi$ spans $H^*$, $\Phi$ contains a basis $\{\alpha_1, \ldots, \alpha_n\}$ of $H^*$. For any $\beta \in \Phi$, $\beta = \sum_{i=1}^n c_i \alpha_i$ with $c_i \in \mathbb{C}$. Note

$$\sum_{i=1}^n c_i \langle \alpha_i, \alpha_j \rangle = \langle \beta, \alpha_j \rangle, \quad j = 1, 2, \ldots, n. \quad (2.6.38)$$

View $\{c_1, c_2, \ldots, c_n\}$ as unknowns. The coefficients in the above linear systems are integers. The coefficient determinant

$$|\langle (\alpha_i, \alpha_j) \rangle_{n \times n}| = \frac{2^n}{\prod_{r=1}^n (\alpha_r, \alpha_r)} |\langle (\alpha_i, \alpha_j) \rangle_{n \times n}| \neq 0 \quad (2.6.39)$$

by the nondegeneracy of $(\cdot, \cdot)$ on $H^*$. Solving (2.6.38) for $\{c_i\}$ in terms of $\{\langle \alpha_i, \alpha_j \rangle, \langle \beta, \alpha_j \rangle\}$, we obtain

$$c_i \in \mathbb{Q}, \quad i = 1, 2, \ldots, n. \quad (2.6.40)$$

Thus the $\mathbb{Q}$-subspace

$$\Gamma = \sum_{\alpha \in \Phi} \mathbb{Q}\alpha \quad (2.6.41)$$

of $H^*$ is $n$-dimensional.

By (2.6.21),

$$(\lambda, \mu) = \text{Tr } \text{ad} t_{\lambda} \text{ad} t_{\mu} = \sum_{\alpha \in \Phi} \alpha(t_{\lambda}) \alpha(t_{\mu}) = \sum_{\alpha \in \Phi} (\alpha, \lambda)(\alpha, \mu) \quad \text{for } \lambda, \mu \in H^*. \quad (2.6.42)$$

In particular,

$$(\beta, \beta) = \sum_{\alpha \in \Phi} (\alpha, \beta)^2 \quad \text{for } \beta \in \Phi, \quad (2.6.43)$$

equivalently,

$$\frac{1}{(\beta, \beta)} = \frac{1}{4} \sum_{\alpha \in \Phi} \left( \frac{2(\alpha, \beta)}{(\beta, \beta)} \right)^2 = \frac{1}{4} \sum_{\alpha \in \Phi} (\alpha, \beta)^2 \in \mathbb{Q}. \quad (2.6.44)$$

So $$(\beta, \beta) \in \mathbb{Q}.$$ For $\alpha, \beta \in \phi$, we have $\alpha - (\alpha, \beta) \beta \in \Phi$ by the above lemma. Hence

$$\langle \alpha, \beta \rangle (\alpha, \beta) = \frac{1}{2} \left[ (\alpha, \alpha) + (\alpha, \beta)^2 (\beta, \beta) - (\alpha - (\alpha, \beta) \beta, \alpha - (\alpha, \beta) \beta) \right] \in \mathbb{Q}. \quad (2.6.45)$$

Therefore,

$$(\alpha, \beta) \in \mathbb{Q} \quad \text{for } \alpha, \beta \in \Phi. \quad (2.6.46)$$
Thus $(\cdot, \cdot)$ is a positive definite $\mathbb{Q}$-valued symmetric $\mathbb{Q}$-bilinear form on $\Gamma$ by (2.6.42) and (2.6.46). Extend $(\cdot, \cdot)$ on

$$\mathcal{E} = \Gamma_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Q} \Gamma$$

$\mathbb{R}$-bilinearly. Then $\mathcal{E}$ is isomorphic to the Euclidean space $\mathbb{R}^n$. Now we obtain a main theorem.

**Theorem 2.6.4** Let $\mathcal{G}$ be a finite-dimensional semisimple Lie algebra. There exists a maximal toral subalgebra $H$, whose dimension denoted by $n$. With respect to $H$, $\mathcal{G}$ has a root space decomposition (Cartan decomposition) (2.6.21), where $\dim \mathcal{G}_\alpha = 1$ and if $\alpha, \beta, \alpha + \beta \in \Phi$, then $[\mathcal{G}_\alpha, \mathcal{G}_\beta] = \mathcal{G}_{\alpha+\beta}$. Moreover, $\Phi$ can be identified with a finite subset of the Euclidean space $\mathbb{R}^n$ satisfying: (a) $\Phi$ spans $\mathbb{R}^n$ and $0 \notin \Phi$; (b) $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$ for $\alpha \in \Phi$; (c) If $\alpha, \beta \in \Phi$, then $2(\beta, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ and $\beta - [2(\beta, \alpha)/(\alpha, \alpha)]\alpha \in \Phi$.

As an exercise, find all finite-dimensional real irreducible modules for $o(3, \mathbb{R})$ and $o(4, \mathbb{R})$ defined in (1.2.22).

**References**


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