Chapter 2
Spectral Methods in the Theory of Wave Propagation

2.1 Equations of Motion of an Ideal Fluid. Small-Amplitude Waves

Analysis of wave phenomena at the interface between two media is one of the topical problems in many areas of physics and mechanics. This chapter deals with studying the wave motion of the fluid surface when the density of the medium above the surface is much smaller than the density of the fluid under consideration. An example of such a system is the surface of sea or ocean. We limit our consideration to the wave dynamics caused by only one external factor—*force of gravity*. In hydrodynamics such waves are called *gravity waves*. These are the waves with a length of one metre and longer. On smaller scales, surface tension and viscosity start to play an essential role. Amongst the variety of surface waves, we put our focus on two-dimensional steady periodic waves that propagate along a given direction.

2.1.1 Euler and Laplace Equations with Boundary Conditions

In the general model of continuous medium, the fluid motion is described by the Navier–Stokes equation [40]. In the case of *ideal fluid*, when the effects due to viscosity and heat conduction can be ignored, the Navier–Stokes equation reduces to the Euler equation describing the conservation of the momentum of a fluid particle. For the fluid motion in the gravity field, the Euler equation is written as

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} + \boldsymbol{g}. \tag{2.1}$$

Here, $\boldsymbol{v}$ is the flow velocity vector field, p is the pressure inside the fluid, ρ is the fluid density and $\boldsymbol{g}$ is the acceleration due to gravity. This equation is essentially nonlinear owing to the term $(\boldsymbol{v}\nabla)\boldsymbol{v}$ that is a part of the material derivative $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\boldsymbol{v}\nabla)$. Another fundamental law of motion of an ideal fluid is the continuity equation

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$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{v}) = 0, \tag{2.2}$$

which is a result of the mass conservation law.

The equations of motion of an ideal fluid (2.1) and (2.2) are the basic equations for deriving the equations of motion of the fluid surface. Hereafter, we assume that fluid is incompressible ($\rho = \text{const}$) and the fluid motion is potential (irrotational). The condition of potential flow rot $\boldsymbol{v} = 0$ means that one can introduce the velocity potential $\boldsymbol{\Phi}$ such as $\boldsymbol{v} = \nabla \boldsymbol{\Phi}$. Then the continuity equation (2.2) for the velocity $\boldsymbol{v}$ is transformed to the *Laplace equation*

$$\Delta \boldsymbol{\Phi} = 0 \tag{2.3}$$

for the velocity potential. The Laplace equation describes the fluid motion in the entire flow domain. This equation should be supplemented with boundary conditions at each boundary, depending on the problem geometry:

$$\nabla F \cdot \boldsymbol{v} + \frac{\partial F}{\partial t} = 0, \tag{2.4}$$

where $F(\boldsymbol{r}, t) = 0$ is the equation of the boundary and $\boldsymbol{r} = \{x, y, z\}$ is the radius vector. Condition (2.4) implies that no particles cross the boundary. Such conditions are called kinematic.

We consider the flow domain that is unbounded in the horizontal xz-plane, whereas in the vertical xy-plane it is bounded by a flat bed from the bottom and a free surface from the top (Fig. 2.1). Then the kinematic boundary conditions at the moving free surface $y = \eta(x, t)$ and motionless bottom $y = -h$ are written as

$$\eta_t + \boldsymbol{\Phi}_x \eta_x - \boldsymbol{\Phi}_y = 0, \quad y = \eta(x, t); \tag{2.5}$$

$$\boldsymbol{\Phi}_y = 0, \quad y = -h. \tag{2.6}$$

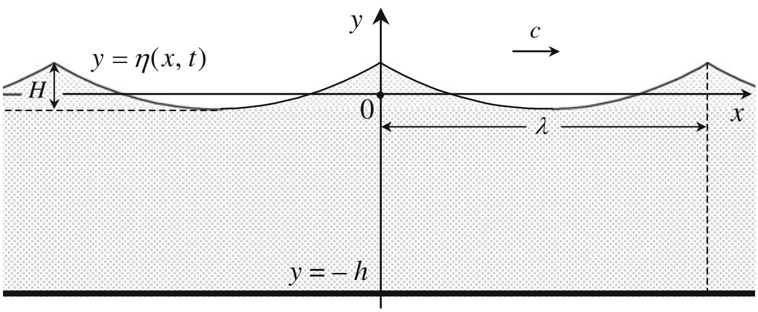

Fig. 2.1 Laboratory reference frame

Here, we assume the fluid motion to be planar (two-dimensional), i.e. such that wave parameters do not change along the wavefront parallel to the z-axis. The shape of the free surface is described by the unknown function $y = \eta(x,\ t)$ of the horizontal coordinate and time. It can be determined from the Euler equation that is reduced by integration to the Bernoulli equation in the case of incompressible fluid and irrotational motion, namely,

$$\Phi_t + \frac{1}{2}\left(\Phi_x^2 + \Phi_y^2\right) + \frac{p}{\rho} + gy = C(t), \tag{2.7}$$

where $C(t)$ is an integration constant. The pressure p is assumed to be constant and was therefore included in the integration constant. Since the velocity potential Φ is defined to within an arbitrary function of time, the integration constant $C(t)$ can be put equal to zero without loss of generality. This is equivalent to the redefinition of the velocity potential as

$$\widetilde{\Phi} = \Phi - \int C(t)\mathrm{d}t.$$

When Eq. (2.7) is written at the boundary $y = \eta(x,\ t)$, it is called the dynamic boundary condition. Laplace equation (2.3) and boundary conditions (2.5), (2.6) and (2.7) form the closed system of equations, which is usually referred to as the *canonical model* of hydrodynamics.

2.1.2 Stationary Fluid Motion

In the conservative system such as an ideal fluid, the fluid motion can become *stationary* after all the transient processes have finished. In this case, the parameters of surface waves and their shape do not change with time. The condition of stationary motion for progressive waves establishes an unambiguous correspondence between the coordinate x, which is parallel to the direction of wave propagation, and time t in terms of wave phase

$$\theta = kx - \omega t,$$

where k is the wave number, $\omega = c\,k$ is the wave frequency and c is the wave phase speed, so that

$$\frac{\partial}{\partial t} = -c\,\frac{\partial}{\partial x}\,. \tag{2.8}$$

In this case, any external initial perturbation affects only the wavelength λ and the wave energy (or amplitude), which serve as free parameters of the problem and can be determined from the relevant initial conditions.

Amongst a variety of stationary waves, we restrict our attention only to periodic waves. The periodicity condition implies that the following condition holds true for any function f describing the spatial and temporal parameters of the flow:

$$f(x,\ y,\ t) = f(x+\lambda,\ y,\ t) = f(x,\ y,\ t+T), \tag{2.9}$$

where $\lambda = 2\pi/k$ is the wavelength (spatial period) and T is the temporal period. In the case of condition (2.8), we have $\lambda = cT$ and $f(x,\ y,\ t) = f(\theta,\ y)$.

Depending on the ratio between the fluid depth and wavelength, the problem of propagation of two-dimensional stationary waves has two limiting cases: (i) $h/\lambda \to \infty$ (infinite depth and deep water waves) and (ii) $h/\lambda \to 0$ (solitary waves). In the latter case, the waves are not periodic (their wavelength is infinite), and they are beyond the scope of our consideration.

Let us determine the integration constant C in the Bernoulli equation (2.7) (in view of conditions (2.8) and (2.9), it is not arbitrary any longer). This constant does not depend on time when the flow is stationary, and it can be determined by averaging the Bernoulli equation at the bottom. To this end, let us find the mass of a fluid column between the free surface and the bottom, the column base having dimensions equal to the wavelength λ along the x-axis and length L along the z-axis parallel to the wavefront:

$$M = L\int_0^{\lambda}\int_{-h}^{\eta(\theta)} \rho\,\mathrm{d}x\,\mathrm{d}y = L\int_0^{\lambda} \rho\big(h+\eta(\theta)\big)\mathrm{d}x = \rho S h + \rho S\overline{\eta}, \tag{2.10}$$

where $S = L\lambda$ is the column base area and

$$\overline{\star} = \frac{1}{\lambda}\int_0^{\lambda} \star\ \mathrm{d}x \tag{2.11}$$

designates averaging over the spatial period. Note that $\overline{\eta} = 0$ if $y = 0$ is the wave *mean level* that separates the crests and the troughs into two domains of equal areas. Assuming that there is no external flow in and out of the domain filled with fluid, the column mass M remains constant in the course of wave propagation. Hence, the depth $h + \overline{\eta}$ corresponds to the *still fluid* depth, when there is no waves and flows. Since we have $\overline{\eta} = 0$ relative to the wave mean level, the mean level of a stationary wave coincides with the still fluid level, provided that the total mass of fluid remains constant.

Next we determine the fluid pressure at the bottom. In the case of infinite depth $h = \infty$, the motion of surface waves does not produce any flows at the bottom, so that $\Phi_x\,|_{y=-\infty} = 0$. From boundary condition (2.6) we have $\Phi_y\,|_{y=-\infty} = 0$, therefore it follows from the Bernoulli equation (2.7) that pressure is constant at infinite depth. In the case of finite depth, the motion of surface waves sets the entire fluid in motion, although it subsides with depth. The fluid velocity at the bottom (and at any other horizontal level located inside the domain filled with fluid) is a periodic function with spatial period λ, i.e.

$$\overline{\Phi_x}\,|_{y=\mathrm{const}} = 0, \tag{2.12}$$

although the unaveraged velocity is not constant, namely, $\Phi_x\,|_{y=-h} \neq \mathrm{const}$. Therefore, the pressure at the bottom is not constant but its average over the spatial period is constant at the entire bottom and is determined by the fluid mass above the bottom and the atmospheric pressure p_0 at the surface. Hence, taking into account relation (2.10) we have

$$\overline{p}\,|_{y=-h} = p_0 + \frac{Mg}{S} = p_0 + \rho\, g(h + \overline{\eta}). \tag{2.13}$$

Next, by averaging the Bernoulli equation (2.7) at the bottom over the spatial period and taking into account conditions (2.6), (2.12), and (2.13), we get the value of constant C:

$$C = \frac{p_0}{\rho} + \frac{1}{2}\,\overline{\Phi_x^2}\,|_{y=-h} + g\overline{\eta}. \tag{2.14}$$

Assuming the atmospheric pressure constant at the entire free surface and equal to the mean pressure p_0 and taking into account that $\Phi_t = -c\Phi_x$, the dynamic boundary condition can be rewritten as

$$\frac{1}{2}\left(\left(c - \Phi_x\right)^2 + \Phi_y^2\right) + gy = \frac{c^2}{2} + \frac{\overline{\Phi_x^2}\,|_{y=-h}}{2} + g\overline{\eta} = B, \quad y = \eta(x,\ t), \tag{2.15}$$

where the constant B is called the *Bernoulli constant*. If $y = 0$ is the wave mean level, we have $\overline{\eta} = 0$ and then $B = c^2/2$ in the case of infinite depth. This value of the Bernoulli constant is often used in the case of finite depth as well. To this end, one has to set $\overline{\eta} = -\frac{1}{2g}\,\overline{\Phi_x^2}\,|_{y=-h}$, which means that the coordinate origin $y = 0$ needs to be translated upwards by distance $\frac{1}{2g}\,\overline{\Phi_x^2}\,|_{y=-h}$ relative to the still fluid level.

2.1.3 Wave Energy, Momentum and Power

The energy of a unit fluid volume is a sum of volume densities of the kinetic and potential energies:

$$\mathscr{E} = \frac{\rho v^2}{2} + \rho\varepsilon,$$

where ε is the internal energy of the unit fluid mass [40]. For the external gravity field, we have $\varepsilon = gy$. Let us find the total energy of the fluid column between the free surface and the bottom, the column base having dimensions equal to the wavelength λ along the x-axis and length L along the z-axis:

$$\mathscr{H}_{\text{total}} = L \int_0^\lambda \int_{-h}^{\eta(\theta)} \left(\frac{\rho v^2}{2} + \rho g y \right) \mathrm{d}x \, \mathrm{d}y.$$

When there is no wave motion, the energy of this fluid column is

$$\mathscr{H}_{\text{still}} = L \int_0^\lambda \int_{-h}^{\bar{\eta}} \rho g y \, \mathrm{d}x \, \mathrm{d}y.$$

Hence, the contribution of the wave energy to the energy of the fluid column under consideration is

$$\mathscr{H}_{\text{wave}} = \mathscr{H}_{\text{total}} - \mathscr{H}_{\text{still}} = L \int_0^\lambda \int_{-h}^{\eta(\theta)} \frac{\rho v^2}{2} \, \mathrm{d}x \, \mathrm{d}y + L \int_0^\lambda \int_{\bar{\eta}}^{\eta(\theta)} \rho g y \, \mathrm{d}x \, \mathrm{d}y.$$

The ratio of this energy to the area $L\lambda$ of the column base is usually called the *wave energy density* (energy per unit horizontal area):

$$E = \frac{\mathscr{H}}{\lambda L} = K + U = \frac{\rho}{2\lambda} \int_0^\lambda \int_{-h}^{\eta(\theta)} \left(\Phi_x^2 + \Phi_y^2 \right) \mathrm{d}x \, \mathrm{d}y + \frac{\rho g}{2\lambda} \int_0^\lambda \left(\eta^2(\theta) - \bar{\eta}^2 \right) \mathrm{d}x, \quad (2.16)$$

where K and U are the densities of kinetic and potential energies, respectively.

The momentum of a unit fluid volume (fluid flow density) is $\rho \boldsymbol{v}$ [40]. Then the flow density of the fluid being transported along the direction of wave propagation is ρv_x. Similarly to the wave energy density, the *wave momentum density* (momentum per unit horizontal area) is defined as

$$I = \frac{\rho}{\lambda} \int_0^\lambda \int_{-h}^{\eta(\theta)} \Phi_x \, \mathrm{d}x \, \mathrm{d}y. \quad (2.17)$$

In what follows, we omit the words "per unit horizontal area" when referring to the wave energy and momentum.

The energy flux transferred by the wave is defined as the derivative of the total energy with respect to time:

$$\mathscr{P}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \mathscr{E} \mathrm{d}V. \quad (2.18)$$

It can be shown that [74]

$$\mathscr{P}(t) = \rho c L \int_{-h}^{\eta(\theta)} \Phi_x^2 \, \mathrm{d}y. \tag{2.19}$$

The wave power is defined as the energy flux averaged over the wave period that we write normalised to the unit wavefront length:

$$P = \frac{1}{LT} \int_0^T \mathscr{P} \mathrm{d}t. \tag{2.20}$$

2.1.4 General Solution of the Laplace Equation

We search for a partial solution of the Laplace equation (2.3) by the method of separation of variables:

$$\Phi(x,\ y,\ t) = X(x - c\,t)\ Y(y) \ \Rightarrow\ \frac{X''}{X} = -\frac{Y''}{Y} = -\kappa^2.$$

Then we have

$$X(x - c\,t) = X_1 \mathrm{e}^{\mathrm{i}\kappa(x - c\,t)} + X_2 \mathrm{e}^{-\mathrm{i}\kappa(x - c\,t)},$$
$$Y(y) = Y_1 \mathrm{e}^{\kappa y} + Y_2 \mathrm{e}^{-\kappa y}.$$

Since the solution should be periodic with respect to coordinate x with period $\lambda = 2\pi/k$, we need to set $\kappa = n\,k$, n being a natural number. In addition, the following conditions should hold true.

1. Boundary condition (2.6) at the bottom implying that

$$Y_1 \mathrm{e}^{-n\,kh} - Y_2 \mathrm{e}^{n\,kh} = 0 \quad \Leftrightarrow \quad Y_1 = Y_2 \mathrm{e}^{2n\,kh}.$$

2. Transition to the limit at $h \to \infty$. To this end, the function $Y(y)$ should be bounded ($Y_2 \to 0$) and should not depend on depth h at $h \to \infty$. These conditions are satisfied if we introduce the scaling as follows

$$Y_2 = \frac{A_2 \mathrm{e}^{-n\,kh}}{\mathrm{e}^{n\,kh} + \mathrm{e}^{-n\,kh}} \ \Rightarrow\ Y_1 = \frac{A_2 \mathrm{e}^{n\,kh}}{\mathrm{e}^{n\,kh} + \mathrm{e}^{-n\,kh}}\ .$$

The constant A_2 can be included in the function X by redefinition $C_1 = X_1 \cdot A_2$ and $C_2 = X_2 \cdot A_2$.

3. The velocity potential $\boldsymbol{\Phi}$ should be a real function:

$$\boldsymbol{\Phi}^* = \boldsymbol{\Phi} \quad \Rightarrow \quad C_1^* = C_2,$$

where * designates the complex conjugate. Thus,

$$\begin{aligned}
\boldsymbol{\Phi}(x,\ y,\ t) &= X(x - c\,t)\ Y(y);\\
X(x) = C_n \mathrm{e}^{\mathrm{i}n\,kx} + C_n^* \mathrm{e}^{-\mathrm{i}n\,kx} &= \left(\operatorname{Re} C_n\ \cos(n\,kx) - \operatorname{Im} C_n\ \sin(n\,kx)\right);\\
Y(y) = T_n\,\mathrm{e}^{n\,ky} + T_{-n}\,\mathrm{e}^{-n\,ky} &= \frac{\cosh\left(n\,k(y+h)\right)}{\cosh(n\,kh)};\\
T_n = \frac{\mathrm{e}^{n\,kh}}{\mathrm{e}^{n\,kh} + \mathrm{e}^{-n\,kh}} &= \frac{1}{2}\left(1 + \tanh(n\,kh)\right).
\end{aligned} \tag{2.21}$$

The general solution of the Laplace equation is a linear combination of all partial solutions ($n = 1, 2, 3, \ldots$):

$$\boldsymbol{\Phi}(x,\ y,\ t) = \sum_{n=1}^{\infty}\left(C_n \mathrm{e}^{\mathrm{i}n\,k(x-ct)} + C_n^* \mathrm{e}^{-\mathrm{i}n\,k(x-ct)}\right)\left(T_n \mathrm{e}^{n\,ky} + T_{-n}\mathrm{e}^{-n\,ky}\right). \tag{2.22}$$

For symmetric waves we have $\boldsymbol{\Phi}_y(\theta) = -\boldsymbol{\Phi}_y(-\theta)$, so that

$$C_n^* = -C_n, \tag{2.23}$$

i.e. the coefficients C_n must be real in this case.

2.1.5 Small-Amplitude Waves (Linear Approximation)

Let us find the conditions when the nonlinearity of the boundary conditions can be neglected. To this end, we estimate the magnitudes of the terms present in the Euler equation (2.1):

$$|(\boldsymbol{v}\nabla)\boldsymbol{v}| \simeq \frac{v\delta v}{\delta r}, \qquad \left|\frac{\partial \boldsymbol{v}}{\partial t}\right| \simeq \frac{\delta v}{\delta t},$$

where δv, δr and δt are typical variations of the velocity, distance and time, respectively. To get an estimate related to the wave motion, we can set δr equal to the wavelength λ and δt equal to the wave period T. Then we have

$$\frac{|(\boldsymbol{v}\nabla)\boldsymbol{v}|}{\left|\dfrac{\partial \boldsymbol{v}}{\partial t}\right|} \simeq \frac{vT}{\lambda} = \frac{v}{c} \simeq \frac{H}{\lambda} \equiv A.$$

Therefore, the nonlinear term $|(\boldsymbol{v}\nabla)\boldsymbol{v}|$ is small as compared to the linear term $|\frac{\partial \boldsymbol{v}}{\partial t}|$ when $H \ll \lambda$ ($A \ll 1$), i.e. the wave amplitude is much smaller than the wavelength. The waves for which the contribution of nonlinearity is negligibly small (it happens at $A \to 0$) are called *linear waves* or *waves of infinitesimal amplitude*. Let us consider the main properties of linear waves.

The wave speed c and wave profile can be found from the boundary conditions at the free surface. In the linear approximation, all the derivatives at the free surface $y = \eta(x, t)$ are written at the zero level $y = 0$ (since the next terms of the corresponding Taylor series around the undisturbed level $y = 0$ have a higher order of smallness with respect to the parameter A). Then the kinematic and dynamic boundary conditions (2.5) and (2.15) take the following form:

$$\eta_x = -\frac{1}{c}\,\Phi_y\,|_{y=0}, \tag{2.24}$$

$$\eta = \frac{c}{g}\,\Phi_x\,|_{y=0}. \tag{2.25}$$

Eliminating the wave profile from these equations, we get

$$\left(\frac{c^2}{g}\,\Phi_{xx} + \Phi_y\right)\Big|_{y=0} = 0. \tag{2.26}$$

Substituting the solution (2.22) of the Laplace equation at $n = 1$ (linear approximation) in this equation, we obtain a relation that expresses the dependence of wave speed (or frequency) on wave number (or length):

$$c^2 = \frac{g}{k}\,\tanh(kh) \quad \Leftrightarrow \quad \omega^2 = gk\,\tanh(kh). \tag{2.27}$$

This expression specifies the relationship between the temporal and spatial parameters of the wave. Such a relationship is called the wave *dispersion* [40]. The function $\tanh(kh)$ rapidly tends to unity with increasing kh. For example, at $kh = 20$ we have $1 - \tanh(kh) \approx 8.5 \cdot 10^{-18}$ and at $kh = 100$ we have $1 - \tanh(kh) \approx 2.8 \cdot 10^{-87}$. Thus, even the case $kh = 20$ can be regarded as the infinite-depth case (deep fluid) to within high accuracy ($\approx 10^{-15}\%$). Needless to say the same conclusion holds true for larger kh.

The free surface profile in the linear approximation is found from boundary condition (2.25) with (2.21) taken into account:

$$\eta = \frac{\mathrm{i}ck}{g}\left(C_1 \mathrm{e}^{\mathrm{i}k(x-c\,t)} + C_1^* \mathrm{e}^{-\mathrm{i}k(x-c\,t)}\right) = \\ -\frac{2ck}{g}\left(\mathrm{Re}\,C_1 \sin\big(k(x-c\,t)\big) + \mathrm{Im}\,C_1 \cos\big(k(x-c\,t)\big)\right). \tag{2.28}$$

For *symmetric* waves we have $\eta(x - c\,t) = \eta(-x + c\,t)$, so that $\operatorname{Re} C_1 = 0$. Then we finally get

$$\eta = a\,\cos(kx - \omega\,t), \quad a = -\frac{2c\,k}{g}\,\operatorname{Im} C_1; \tag{2.29}$$

$$\boldsymbol{\Phi} = \frac{ga}{\omega}\,\sin(kx - \omega\,t)\big(T_1^+ \mathrm{e}^{ky} + T_1^- \mathrm{e}^{-ky}\big) = ac\,\sin(kx - \omega\,t)\,\frac{\cosh\big(k(y+h)\big)}{\sinh(kh)}\,. \tag{2.30}$$

In the linear approximation, the parameter a is the height $\eta(0)$ of the wave above the still fluid level, which is equal to a half of the wave height:

$$a = \frac{H}{2} = \frac{\lambda A}{2} = \frac{\pi A}{k}. \tag{2.31}$$

For the velocity distribution we have

$$\begin{aligned} \boldsymbol{\Phi}_x &= \frac{\mathrm{d}x}{\mathrm{d}\,t} = ack\,\cos(kx - \omega\,t)\,\frac{\cosh\big(k(y+h)\big)}{\sinh(kh)}\,; \\ \boldsymbol{\Phi}_y &= \frac{\mathrm{d}y}{\mathrm{d}\,t} = ack\,\sin(kx - \omega\,t)\,\frac{\sinh\big(k(y+h)\big)}{\sinh(kh)}\,. \end{aligned} \tag{2.32}$$

Accordingly, the wave momentum density I calculated by formula (2.17) is equal to zero in the linear approximation with respect to the small parameter ak. Thus, *linear waves do not cause the mass transfer*.

The wave density energy calculated by formula (2.16) has the following form:

$$E = K + U = \frac{\rho g a^2}{2}, \quad K = U = \frac{\rho g a^2}{4}, \tag{2.33}$$

i.e. the kinetic and potential energies make the equal contribution to the total wave energy in the linear approximation.

The distribution of the kinetic energy over depth can be calculated by the formula

$$\begin{aligned} \overline{\mathscr{E}}_K = \frac{1}{2\pi}\int_0^{2\pi} \mathscr{E}_K \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho v^2}{2}\mathrm{d}\theta = \frac{\rho g}{k}\frac{a^2}{2}\,\frac{\cosh\big(2k(y+h)\big)}{\sinh(2kh)} \\ = \Big| h \gg \lambda \Big| = \frac{\rho g}{k}\frac{a^2}{2}\exp(2ky), \end{aligned}$$

where $\mathscr{E}_K$ is the volume density of the kinetic energy. It can be seen that the major part of the wave's kinetic energy is concentrated in the near-surface layer of thickness $\lambda/2$. This fact is used as the basis for the operation of apparatuses that take the wave power off and convert it to the mechanical work or electric energy for subsequent use. Wave energy conversion methods and apparatuses are considered in detail in Ref. [59].

The wave power calculated by formula (2.20) is

$$P = \frac{\rho g a^2}{4}\, c \left(1 + \frac{2kh}{\sinh(2kh)}\right) = E\, c_g, \tag{2.34}$$

where

$$c_g = \frac{c}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right) = \frac{\mathrm{d}\omega}{\mathrm{d}k} \tag{2.35}$$

is the wave *group speed* by definition. Formula (2.34) means that the wave energy is transferred with group speed, which is always smaller than the phase speed ($c_g \to c$ only at $h \to 0$). Thus, the wave energy transfer always lags behind the wavefront motion. In the case of infinite depth, the group speed is twice as small as the phase speed:

$$c_g|_{h\to\infty} = \frac{c}{2}.$$

A more detailed insight into the theory of linear surface waves can be found in Refs. [39, 79, 92].

2.1.6 Dimensionless Equations and Parameters

For further analysis, we rewrite the equations of canonical model in dimensionless form. The units of length r, time t and mass m are selected such that in dimensionless form we could get

$$\widetilde{k} = \widetilde{g} = \widetilde{\rho} = 1,$$

where the tilde refers to dimensionless units. To this end, all the distances are normalised to the reciprocal of wave number k and time is normalised to the reciprocal of frequency $\omega_0 = \sqrt{g\,k}$, which is the linear wave frequency in the case of infinite depth. Then the relationship between the dimensionless and dimensional variables is given by the relations

$$\begin{gathered} \widetilde{r} = r\,k, \quad \widetilde{h} = h\,k, \quad \widetilde{t} = t\,\omega_0, \\ \widetilde{v} = \frac{v}{c_0}, \quad \widetilde{\nabla} = \frac{\nabla}{k}, \quad \widetilde{\Phi} = \frac{k}{c_0}\,\Phi, \quad \widetilde{m} = \frac{k^3}{\rho}\,m, \quad \widetilde{E} = \frac{k^2}{\rho g}\,E, \quad \widetilde{I} = \frac{k^2}{\rho\omega_0}\,I, \end{gathered} \tag{2.36}$$

where $c_0 = \omega_0\,/\,k$ is the linear wave phase speed in the case of infinite depth. In this case we have

$$\widetilde{\lambda} = 2\pi, \quad \widetilde{\omega} = \widetilde{c}, \quad \widetilde{\theta} = \widetilde{x} - \widetilde{c}\,\widetilde{t}, \quad \frac{\partial}{\partial\widetilde{x}} = \frac{\partial}{\partial\widetilde{\theta}}.$$

In terms of dimensionless variables (tildes are omitted henceforth), the equations of canonical model are

$$\boldsymbol{\Phi}_{\theta\theta} + \boldsymbol{\Phi}_{yy} = 0, \quad -h \leqslant y \leqslant \eta(\theta); \tag{2.37}$$

$$\frac{1}{2}\left(\left(c - \boldsymbol{\Phi}_\theta\right)^2 + \boldsymbol{\Phi}_y^2\right) + \eta = B, \quad y = \eta(\theta), \quad B = \frac{c^2}{2} + \frac{1}{2}\,\overline{\boldsymbol{\Phi}_\theta^2}\,|_{y=-h} + \overline{\eta}\,; \tag{2.38}$$

$$\left(c - \boldsymbol{\Phi}_\theta\right)\eta_\theta + \boldsymbol{\Phi}_y = 0, \quad y = \eta(\theta); \tag{2.39}$$

$$\boldsymbol{\Phi}_y = 0, \quad y = -h. \tag{2.40}$$

This system of equations at fixed depth h defines the unknown velocity potential $\boldsymbol{\Phi}(\theta,\ y)$, free surface profile $\eta(\theta)$ and phase speed c as functions of only one dimensionless parameter—wave amplitude $A = H/(2\pi)$ (H being the wave height):

$$A = \frac{\eta(0) - \eta(\pi)}{2\pi}. \tag{2.41}$$

The Bernoulli equation (2.38) yields another expression:

$$A = \frac{q^2(\pi) - q^2(0)}{4\pi}, \tag{2.42}$$

where $q(0)$ and $q(\pi)$ are the velocities of fluid particles at the wave crest and trough, respectively, in the reference frame where the fluid is motionless. We refer to such a reference frame as the *intrinsic wave reference frame*.

The velocity potential and phase speed being determined, the particle trajectories can be found from a system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \boldsymbol{\Phi}_\theta\,(\theta,\ y)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \boldsymbol{\Phi}_y\,(\theta,\ y)\,. \tag{2.43}$$

Wave energy (2.16) and momentum (2.17) have the following form in dimensionless variables:

$$E = K + U = \frac{1}{4\pi}\int_0^{2\pi}\int_{-h}^{\eta(\theta)}\left(\boldsymbol{\Phi}_\theta^2 + \boldsymbol{\Phi}_y^2\right)\mathrm{d}\theta\,\mathrm{d}y + \frac{1}{4\pi}\int_0^{2\pi}\left(\eta^2(\theta) - \overline{\eta}^2\right)\mathrm{d}\theta. \tag{2.44}$$

$$I = \frac{1}{2\pi}\int_0^{2\pi}\int_{-h}^{\eta(\theta)}\boldsymbol{\Phi}_\theta \mathrm{d}\theta\,\mathrm{d}y. \tag{2.45}$$

2.1.7 Complex Potential

Kinematic boundary condition (2.4) means that the velocity of fluid particles at each boundary is directed along the tangent to the surface. The lines in the fluid flow at which the velocity vector is always directed along the tangent are called *streamlines* [42]. The equations for streamlines are found from the condition for the vectors $\boldsymbol{v}$ and $\mathrm{d}\boldsymbol{r}$ to be parallel:

$$\boldsymbol{v} \times \mathrm{d}\boldsymbol{r} = 0. \tag{2.46}$$

In the intrinsic wave reference frame and in the case of planar motion, the equations for streamlines can be rewritten as

$$\frac{\mathrm{d}\theta}{\phi_\theta} = \frac{\mathrm{d}y}{\phi_y} \quad \Leftrightarrow \quad -\phi_y\,\mathrm{d}\theta + \phi_\theta\,\mathrm{d}y = 0. \tag{2.47}$$

Here, the function $\phi(\theta, y)$ is the velocity potential in the intrinsic wave reference frame, its relationship to the velocity potential $\Phi(\theta, y)$ in the laboratory reference frame (which is motionless with respect to the bottom) is expressed as

$$\phi(\theta,\ y) = \Phi(\theta,\ y) - c\theta. \tag{2.48}$$

Then it follows from the Laplace equation that

$$\frac{\partial\phi_\theta}{\partial\theta} = \frac{\partial\left(-\phi_y\right)}{\partial y}.$$

This relation implies that there exists some function $\psi(\theta,\ y)$ such that

$$\phi_\theta = \psi_y, \quad \phi_y = -\psi_\theta. \tag{2.49}$$

The function $\psi(\theta, y)$ is called the *stream function* and equation (2.49) represent the well-known Cauchy–Riemann conditions for the complex function $w = \phi + \mathrm{i}\psi$ to be an analytic function of complex argument $\zeta = \theta + \mathrm{i}y$. The function $w(\zeta)$ is called the *complex potential* [40]. In this case, the expression $-\phi_y\,\mathrm{d}\theta + \phi_\theta\,\mathrm{d}y$ is the total differential of function ψ, so that the condition (2.47) at the streamline yields

$$\mathrm{d}\psi = 0 \quad \Leftrightarrow \quad \psi(\theta,\ y) = \mathrm{const}, \tag{2.50}$$

the constant in the right-hand side being different for different streamlines. The stream function $\psi(\theta, y)$ is constant along any streamline. In terms of velocity potential, the streamlines are determined from a system of differential equations:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \phi_\theta = \Phi_\theta\left(\theta,\ y\right) - c, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \phi_y = \Phi_y\left(\theta,\ y\right). \tag{2.51}$$

Note that streamlines (2.51) and particle trajectories (2.43) are different curves. Streamlines specify the velocity direction of different fluid particles in the successive points of the space at a *fixed time moment* (Euler variables), whereas trajectories specify the velocity direction of selected particles at *successive time moments* (Lagrange variables) [40].

The analytic function $w(\zeta)$ unambiguously maps the points of the complex plane $\zeta = \theta + \mathrm{i}y$ (which we call the *physical plane*) onto the complex plane $w = \phi + \mathrm{i}\psi$ (which we call the *inverse plane*). Such a mapping is called *conformal* [42]. The geometry of boundaries that enclose the flow domain are often much simpler in the inverse plane because the stream function ψ is constant at each boundary in the case of stationary velocity fields. Therefore, often it is more convenient to solve the planar problems in the inverse plane by taking the velocity potential ϕ and stream function ψ as independent variables and the spatial coordinates θ and y as functions of ϕ and ψ.

The squared velocity in the intrinsic wave reference frame is expressed in terms of the complex potential as

$$\left(c - \boldsymbol{\Phi}_\theta\right)^2 + \boldsymbol{\Phi}_y^2 = \left|\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right|^2, \tag{2.52}$$

inasmuch as

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = \frac{\partial\phi}{\partial\theta} + \mathrm{i}\frac{\partial\psi}{\partial\theta} = \phi_\theta - \mathrm{i}\phi_y \quad \Rightarrow \phi_\theta = \mathrm{Re}\left(\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right), \quad \phi_y = -\,\mathrm{Im}\left(\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right). \tag{2.53}$$

The quantity $\frac{\mathrm{d}w}{\mathrm{d}\zeta}$ is called the *complex velocity* [40]. In terms of complex variables, the equations for streamlines (2.51) are written as

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \left(\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right)^*. \tag{2.54}$$

We define the stream function $\boldsymbol{\Psi}(\theta,\ y)$ in the laboratory reference frame such that the Cauchy–Riemann conditions might hold true:

$$\boldsymbol{\Phi}_\theta = \boldsymbol{\Psi}_y, \quad \boldsymbol{\Phi}_y = -\boldsymbol{\Psi}_\theta. \tag{2.55}$$

To this end, it is sufficient to put $\boldsymbol{\Psi}(\theta,\ y) = \psi(\theta,\ y) + cy$. The function

$$W = \boldsymbol{\Phi} + \mathrm{i}\boldsymbol{\Psi} = w + c\zeta$$

is the complex potential in the laboratory reference frame. Then

$$\begin{aligned} \boldsymbol{\Phi} &= \frac{1}{2}\,(W + W^*) \equiv -\mathrm{i}c\,(R - R^*), \\ \boldsymbol{\Psi} &= \frac{1}{2\mathrm{i}}\,(W - W^*) \equiv c\,(R + R^*), \end{aligned} \tag{2.56}$$

where we introduced the complex function

$$R = \frac{\mathrm{i}W^*}{2c},$$

for the sake of convenience. The Cauchy–Riemann conditions (2.55) imply that

$$R_\theta = \mathrm{i}R_y \quad \Leftrightarrow \quad R(\theta,\ y) = R(y + \mathrm{i}\theta). \tag{2.57}$$

Any complex function R that satisfies condition (2.57) also identically satisfies the Laplace equation $\Delta R = 0$. In this case, in view of relation (2.56), the Laplace equation for the velocity potential $\Delta\Phi = 0$ and stream function

$$\Delta\Psi = 0 \tag{2.58}$$

are also satisfied identically.

With relations (2.56) and (2.57) taken into account, the kinematic boundary condition (2.39) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(R(\theta,\ \eta) + R^*(\theta,\ \eta) - \eta\Big) = 0$$

or

$$\Psi(\theta,\ \eta) - c\eta = \psi(\theta,\ \eta) = \mathrm{const}\,. \tag{2.59}$$

The latter equation implies that the free surface $y = \eta(\theta)$ is a streamline. Since the stream function is defined to within an arbitrary constant, as it follows from the Cauchy–Riemann conditions (2.55), the integration constant in Eq. (2.59) is also arbitrary. However, this arbitrary constant is conventionally selected such that the stream function at the free surface might be equal to zero, $\psi = 0$, whence we have $\mathrm{const} = 0$. Specifying const is this way, we unambiguously define the stream function.

The boundary condition (2.40) at the bottom implies that the bottom is a streamline, much like the free surface, i.e. $\psi(\theta,\ -h) = \mathfrak{const}$. The integration constant can be found by calculating the *fluid mass flux* Q through a curve connecting the arbitrary points 1 and 2 at the bottom and free surface, respectively. The orientation of the normal vector $\boldsymbol{n}$ to the curve at each point is selected such that the angle between the vector $\boldsymbol{n}$ and the direction of circulation around the curve $\mathrm{d}\boldsymbol{l}$ (by absolute value $\mathrm{d}l$ is the length element of the curve) might be $+\frac{\pi}{2}$, i.e. the shortest turn from the vector $\boldsymbol{n}$ to the vector $\mathrm{d}\boldsymbol{l}$ might be accomplished counterclockwise. Then we have

$$Q = \int_1^2 (\boldsymbol{v}\cdot\boldsymbol{n})\,\mathrm{d}l = \int_1^2 \big(-\phi_y\,\mathrm{d}\theta + \phi_\theta\,\mathrm{d}y\big) = \int_1^2 \mathrm{d}\psi = \big(\psi_2 - \psi_1\big). \tag{2.60}$$

In this case, the mass flux depends neither on the curve shape nor on the positions of points 1 and 2. Since we set $\psi = 0$ at the free surface, formula (2.60) implies that $\mathfrak{const} = -Q$. In terms of the velocity potential, the flux Q is defined as

$$Q = \int_{-h}^{\eta(\theta)} \left(\boldsymbol{\Phi}_\theta - c\right) \mathrm{d}y. \tag{2.61}$$

The stream function in the intrinsic wave reference frame monotonically increases from zero at the free surface to $|Q|$ at the bottom, $\psi|_{y=-h} > 0$ $(Q < 0)$, provided that $\boldsymbol{\Phi}_\theta < c$ in the entire flow domain (wave propagates faster than fluid particles).

The mass flux Q and the stream function at the bottom can be expressed in terms of the wave momentum I. Taking into account the Cauchy–Riemann conditions (2.55), we obtain from formula (2.45) that

$$I = \frac{1}{2\pi} \int_0^{2\pi} \left(\int_{-h}^{\eta(\theta)} \Psi_y(\theta,\ y)\, \mathrm{d}y \right) \mathrm{d}\theta = \overline{\Psi}\,|_{y=\eta(\theta)} - \overline{\Psi}\,|_{y=-h}$$
$$= c\,\overline{\eta} - \left(\psi\,|_{y=-h} - c\,h\right).$$

Therefore,

$$\Psi(\theta,\ -h) = c\overline{\eta} - I \Leftrightarrow \psi(\theta,\ -h) = -Q = c(\overline{\eta} + h) - I. \tag{2.62}$$

By its absolute value, the quantity $Q_0 = c(\overline{\eta} + h)$ is the mass flux that originates in the intrinsic wave reference frame due to its motion with respect to the motionless laboratory reference frame (for the observer in the intrinsic reference frame fluid seems to move from right to left with speed c). The total mass flux Q differs from the "zero" flux Q_0 by the value equal to the wave momentum I. Thus, the wave momentum I is determined by the flow induced by the wave in the direction of its propagation. This flow is often called the *drift flow* because it is caused by the drift of fluid particles in the direction of wave propagation or the *near-surface flow* because the drift velocity of particles rapidly decreases with depth [39]. Sometimes it is called the *Stokes flow* in honour of Sir George Gabriel Stokes, who was the first to establish these properties of surface waves [80].

Since

$$-Q = c(\overline{\eta} + h) - I = c\left(\overline{\eta} + h - \frac{I}{c}\right) \equiv cd, \tag{2.63}$$

the parameter d is the depth of uniform flow with mass flux Q that moves with velocity c from right to left in the intrinsic wave reference frame. The depth d is called the *undisturbed depth* [10] and it determines the stream function at the bottom:

$$\frac{\psi\,|_{y=-h}}{c} = d. \tag{2.64}$$

In this case, the depth h measured with respect to the mean wave level $(\overline{\eta} = 0)$ is larger than the undisturbed depth d by the value

$$h - d = \frac{I}{c} > 0. \tag{2.65}$$

The wave's kinetic energy K can be expressed in terms of the wave momentum I (or in terms of the Stokes flow) as

$$K = \frac{cI}{2}. \tag{2.66}$$

This relation was first derived by Levi-Cività in 1924 (see Ref. [43]).

The general solution to the Laplace equation (2.37) in the physical plane that satisfies the boundary condition (2.40) at the bottom is given by expansion (2.22) which has the following form in terms of dimensionless variables:

$$\Phi(\theta,\ y) = \sum_{n=1}^{\infty} \Big(C_n e^{in\theta} + C_n^* e^{-in\theta} \Big) \Big(T_n e^{ny} + T_{-n} e^{-ny} \Big), \tag{2.67}$$

$$T_n = \frac{e^{nh}}{e^{nh} + e^{-nh}} = \frac{1}{2}\big(1 + \tanh(nh)\big). \tag{2.68}$$

The general solution (2.67) to the Laplace equation for the velocity potential can be rewritten for the stream function by taking into account the Cauchy–Riemann conditions (2.55):

$$\Psi(\theta,\ y) = (c\overline{\eta} - I) + \sum_{n=1}^{\infty} \Big(iC_n e^{in\theta} - iC_n^* e^{-in\theta} \Big) \Big(T_n e^{ny} - T_{-n} e^{-ny} \Big), \tag{2.69}$$

where the integration constant is determined from the value of the stream function at the bottom (2.62), inasmuch as $(T_n e^{-nh} - T_{-n} e^{nh}) = 0$ for any n. Then the complex potential $W = \Phi + i\Psi$ in the *physical plane* can be written in the form of Fourier series for a periodic function of complex variable ζ with period $\lambda = 2\pi$:

$$\frac{W(\zeta)}{c} = 2i\xi_0 + 2i \sum_{n=1}^{\infty} \Big(\xi_n^* T_n e^{-in\zeta} - \xi_n T_{-n} e^{in\zeta} \Big),$$
$$\xi_0 = \frac{1}{2}\Big(\overline{\eta} - \frac{I}{c}\Big),\ \xi_n = \frac{iC_n}{c}. \tag{2.70}$$

For the symmetric waves, the coefficients ξ_n are real in view of condition (2.23). The complex potential in the intrinsic wave reference frame is

$$\frac{w(\zeta)}{c} = -\zeta + B_0 + i \sum_{n=1}^{\infty} \Big(B_n^* e^{-in\zeta} - e^{-2nh} B_n e^{in\zeta} \Big),$$
$$B_0 = 2i\xi_0,\ B_n = 2\xi_n T_n. \tag{2.71}$$

In the *inverse plane*, the complex potential is an independent variable and the solution is specified by the inverse expansion as follows [10]

$$\zeta(w) = -\frac{w}{c} + \mathrm{i}a_0 + \mathrm{i}\sum_{n=1}^{\infty}\left(a_n\, \mathrm{e}^{\mathrm{i}n\,w/c} - \mathrm{e}^{-2n\,d} a_n^*\, \mathrm{e}^{-\mathrm{i}n\,w/c}\right),\; a_0 = \left(\overline{\eta} - \frac{I}{c}\right), \quad (2.72)$$

where the undisturbed depth d appears from the condition (2.64) at the bottom. Note that the period $\lambda = 2\pi$ in the physical plane corresponds to the period $c\lambda$ in the inverse plane. In solving the problem in the inverse plane, the zero level $y = 0$ is often selected at the undisturbed depth level to set the free term a_0 equal to zero ($\overline{\eta} = I/c$). The coefficients a_n should be real for symmetric waves.

2.2 Stokes Waves and Methods of Their Calculation

2.2.1 Stokes Waves

Linear waves represent the solutions of Eqs. (2.37)–(2.40) at infinitesimal amplitudes ($A \to 0$). Their shape is described by a cosine curve and in terms of dimensionless variables their phase speed is $c = \tanh h$ ($c = 1$ in the case of infinite depth). When the wave amplitude can no longer be assumed infinitesimal, the nonlinearity of boundary conditions needs to be taken into account. In the case of finite but small amplitudes ($A \ll 1$), one can make use of typical perturbation techniques in terms of parameter A. For example, the coefficients of expansion (2.72) can be expressed as $a_n \sim a_1^n$. Stokes [80] (1847) was the first to construct the perturbation theory for gravity waves on deep water in the physical plane. Later he used expansions in the inverse plane to extend this theory to the case of finite depth [81]. Stokes established the following properties of finite-amplitude gravity waves [80–82].

1. The phase speed of nonlinear waves depends on their amplitude. In the case of deep water, this dependence is written as [39]

$$c^2 = 1 + a^2 + \frac{5}{4}\,a^4 + \dots, \quad (2.73)$$

where a is the first-harmonic amplitude of the wave profile:

$$\eta = a\,\cos\theta + \left(\frac{1}{2} + \frac{17}{24}\,a^2\right)a^2\cos 2\theta\; + \frac{3}{8}\,a^3\cos 3\theta + \frac{1}{3}\,a^4\cos 4\theta + \dots\,. \quad (2.74)$$

For finite depth, the nonlinear dispersion relation (2.73) is much more cumbersome [92].

2. Finite-amplitude waves cause the mass transfer in the direction of wave propagation (particle trajectories are not closed curves in contrast to linear waves), i.e. the wave momentum $I \neq 0$.

3. Wave crests sharpen as the amplitude a increases and the troughs flatten.

4. There exists a limiting wave of the maximum amplitude (height), and the crests of this limiting wave form a 120° cusp. No stationary waves higher than the limiting wave can exist in the framework of the canonical model.

The one-parametric family of waves in amplitude a (or parameter A) that is approximately described by expansions (2.73) and (2.74) (or similar expansions in the case of finite depth) is usually referred to as *Stokes waves* [92]. It follows from formula (2.74) that the Stokes waves are symmetric both with respect to their crests ($\theta = 0$) and with respect to their troughs ($\theta = \pi$). This fact was first proved for small-amplitude waves by Levi-Cività [41] with the use of conformal transforms. Later Garabedian [27] used the variational approach and symmetrisation to show rigorously that the canonical model does not admit the existence of asymmetric waves with equal crests and equal troughs.

The Bernoulli equation (2.38) results in the following expression for the velocity of fluid particles at the surface in the intrinsic wave reference frame:

$$\Phi_\theta - c = \pm\sqrt{2\left(B - \eta\right) - \Phi_y^2}\,. \tag{2.75}$$

In particular, the particle velocity at the wave crest is

$$q(0) \equiv \left(\Phi_\theta\,|_{y=\eta(0)} - c\right) = \pm\sqrt{2\left(B - \eta(0)\right)}\,, \tag{2.76}$$

inasmuch as $\Phi_y\,|_{y=\eta(0)} = 0$. The minus sign corresponds to the family of Stokes waves. The plus sign corresponds to a family of irregular gravity waves described in Ref. [54].

Formula (2.76) implies that for the Stokes waves the fluid particle velocity at the wave crest increases from the value $q(0) = -c$ for the infinitesimal amplitude to the limiting value $q(0) = 0$. The wave with $q(0) = 0$ is called the *limiting Stokes wave*. The condition $(\Phi_\theta - c) < 0$ holds true in the entire flow domain, except for one point at the wave crest where $(\Phi_\theta - c) \equiv q(0) = 0$. This is a singular point at which the flow is motionless in the intrinsic wave reference frame. Such points are also referred to as *stagnation* or *critical* points [40]. It follows from the kinematic boundary condition (2.39) that the derivative η_θ is undetermined at the surface point where $(\Phi_\theta - c) = 0$ and $\Phi_y = 0$, i.e. it exhibits a discontinuity there. Therefore, the wave profile $y = \eta(\theta)$ forms a cusp at the point where $q(0) = 0$.

2.2.2 Steep Stokes Waves and Their Main Properties

Stokes performed calculations for the case of infinite depth with accuracy to within $O(A^5)$ and obtained results for the case of finite depth with accuracy to within $O(A^3)$. Stokes' results were confirmed by Lord Rayleigh [68]. Much later Wilton [94] could extend the infinite-depth expansions to the $O(A^{10})$ order (although with errors made in the eighth order) and De [16] published the results valid for an arbitrary depth with accuracy to within $O(A^5)$. This was a practical limit that could be achieved in the pre-computer era. The crests of Stokes waves sharpen as the wave amplitude increases and the wave profile becomes steeper. This makes the problem essentially nonlinear and much more terms are required in the perturbation expansions for the correct description of wave parameters.

Schwartz [72] was the first who used computer arithmetics to construct the Stokes expansions with accuracy to within $O(A^{48})$ for the case of finite depth and $O(A^{117})$ for the case of infinite depth. He used Fourier expansions in the inverse plane to derive a *nonlinear* system of algebraic equations for the unknown expansion coefficients a_n and wave speed c. Since solving those equations directly was not possible at that time, Schwartz used the perturbation techniques to expand the unknown parameters in power series in terms of small parameter ε:

$$a_n = \sum_{k=0}^{\infty} a_{nk}\, \varepsilon^{n+2k}, \quad c^2 = \sum_{k=0}^{\infty} \gamma_k\, \varepsilon^{2k}. \tag{2.77}$$

The coefficients a_{nk} and γ_k were calculated from the corresponding recurrence formulas. By setting $\varepsilon = a_1$ (a_1 being the first-harmonic amplitude), Schwartz found that the convergence of amplitude expansions (2.77) deteriorates at sufficiently large a_1, so that they eventually diverged. To increase the convergence radius of the perturbation series, Schwartz made use of Padé approximations. He showed that the divergence of perturbation expansions in terms of the first-harmonic amplitude a_1 was caused by the fact that a_1 is not a monotonic function of wave height A and attains a maximum before the maximum amplitude is reached. In the case of infinite depth, this maximum was found at $A \approx 0.13$. It is the position of this maximum that determines the convergence radius of amplitude expansions. Since the wave height A monotonically grows up to the limiting value, the convergence of the expansions can be improved by setting $\varepsilon \sim A$. Indeed, by making use of the expansions in terms of $\varepsilon = 2A$, Schwartz could reach nearly the limiting amplitude, although he had still to use Padé approximations to improve the convergence. For the case of infinite depth, he got the following estimate of the maximum wave amplitude: $A_{\max} = 0.1412$.

Longuet-Higgins [43] recalculated Schwartz's expansions for the case of infinite depth in terms of parameter $\upsilon = \varepsilon^2 = 1 - \frac{q^2(0)\, q^2(\pi)}{c^2\, c_0^2}$, where $q(0)$ and $q(\pi)$ are the fluid particle velocities at the wave crest and trough, respectively, and c_0 is the phase speed of linear waves. These expansions turned out to converge faster than Schwartz's expansions in terms of parameter $\varepsilon = 2A$. The new parameter υ also brings another advantage consisting in the fact that its variation range is known beforehand: it increases from $\upsilon = 0$ for linear waves to $\upsilon = 1$ for the limiting wave. Longuet-Higgins obtained expansions with accuracy to within $O(\upsilon^{40})$ and found that not only the first-harmonic amplitude but also the phase speed c, total energy E, potential energy U, kinetic energy K and momentum I of Stokes waves are not monotonic functions of their amplitude A and attain their maximum values before the limiting

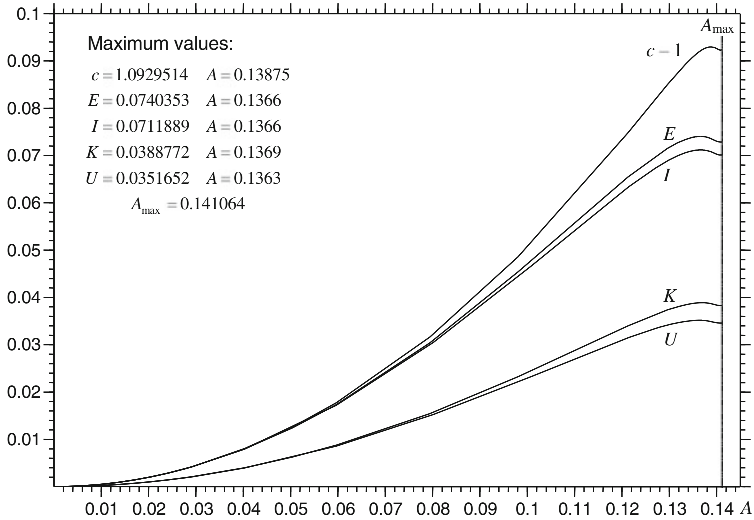

Fig. 2.2 Phase speed c, total energy E, potential energy U, kinetic energy K and momentum I of Stokes waves as functions of their amplitude A

amplitude is reached. The corresponding plots are shown in Fig. 2.2. Thus, *the wave of maximum height is not the fastest wave and does not possess the maximum energy and momentum*, at it had been assumed since the time of Stokes. Cokelet [10] proved that this result holds true for the case of other (finite) depths. To this end, he constructed Schwartz's expansions in terms of parameter $\varepsilon^2 = 1 - \frac{q^2(0)\,q^2(\pi)}{c^4}$ with accuracy to within $O(\varepsilon^{110})$. Cokelet found that many other integral wave parameters attain their maximums before the limiting amplitude is reached. Note that the maximum of the phase speed was first detected in Ref. [71], but the authors of that work did not make any definite conclusion in regard to the existence of the wave speed maximum because of insufficient accuracy of their calculations. Later Longuet-Higgins and Fox [49–51] constructed asymptotic expansions for waves close to the 120°-cusped wave (almost highest waves) and showed that these dependences oscillate infinitely as the limiting wave is approached. Longuet-Higgins [43] also rigorously proved the following relations that are valid for an arbitrary depth:

$$\mathrm{d}E = \mathrm{d}(K + U) = c\,\mathrm{d}I, \;\; \mathrm{d}\mathscr{L} = \mathrm{d}(K - U) = I\,\mathrm{d}\,c, \tag{2.78}$$

where $\mathscr{L}$ is the Lagrange function. Thus, the extremums of the momentum and total energy coincide, as well as the extremums of the phase speed and Lagrangian.

Dallaston and McCue [13] used computer symbolic arithmetics to recalculate Schwartz's and Cokelet's power expansions in exact symbolic form to within $O(\varepsilon^{300})$ and numerically to within $O(\varepsilon^{846})$. Further, by using Padé approximations they could compute the positions of several next extremums in the phase speed and energy of

Stokes waves at approaching the maximum amplitude: their first local minimum and second local maximum.

Higher extremums can be found with the use of contour integral methods that deal with solving integrodifferential equations. The majority of these equations are derived in the following way. Since the complex potential is an analytic function in the domain Ω filled with fluid, its value at any point ζ_0 inside the selected domain $G \in \Omega$ can be expressed in terms of its value at the boundary Γ of that domain with the use of the Cauchy integral [84]:

$$w(\zeta_0) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{w(\zeta)}{\zeta - \zeta_0} \, \mathrm{d}\zeta. \tag{2.79}$$

In the case of periodic waves, the contour Γ is selected such that the entire wave span from the bottom to the free surface might be encompassed: $\Gamma = \Gamma_s + \Gamma_- + \Gamma_b + \Gamma_+$. Here, Γ_s are Γ_b are the parts of the contour Γ that belong to the free surface and bottom, respectively, Γ_+ and Γ_- are its vertical side parts that connect the bottom and the surface and are spaced by the wave period from one another. The integral over the horizontal bottom Γ_b is always easy to calculate, and the integrals over Γ_+ and Γ_- cancel one another out due to periodicity condition. Thus, the Cauchy integral (2.79) expresses the complex potential at any point ζ_0 inside the fluid in terms of its value at the free surface Γ_s. The integral equation is obtained when the point ζ_0 is selected at the free surface. Although the Cauchy integral is divergent in conventional sense in this case, the quantity $w(\zeta_0)$ is equal to twice the principal value of the Cauchy integral [84].

Contour integral methods are the most accurate methods for calculating two-dimensional surface waves. As an example, let us consider one contour integral approach that was proposed by Tanaka in Ref. [86] for solitary waves and then extended to the case of periodic waves on finite depth [87]. Based on the equation

$$\frac{\mathrm{d}\left(\frac{1}{3}\, q^3\right)}{\mathrm{d}\phi} = \sin\vartheta, \quad \{\theta,\ y\} \in \Gamma_s\,, \tag{2.80}$$

that expresses the relationship between the absolute value of the fluid particle velocity q and the angle of inclination ϑ of the free surface to the horizontal axis, Tanaka derived an integral equation that defines $\ln q$ as a function of ϑ. This integral equation utilises one parameter defined as

$$\upsilon = 1 - \frac{q(0)}{q(\pi)}. \tag{2.81}$$

For linear waves, we have $\upsilon = 0$, and for the limiting wave we have $\upsilon = 1$. The obtained integral equation was solved by the method of successive approximations. To concentrate the computational mesh around the crest, Tanaka used the nonlinear transformation

Table 2.1 Extremums of the phase speed c of Stokes waves calculated by Tanaka's method in the case of infinite depth

Extremum c	υ	A	c
First max	0.8637	0.138753	1.092951384
First min	0.9685	0.140920	1.092276839
Second max	0.9927	0.141056	1.092285150
Second min	0.9985	0.141063	1.092285047

$$\phi = \phi' - \gamma \sin \phi', \quad 0 \leqslant \gamma < 1, \tag{2.82}$$

that was earlier proposed in Ref. [6]. This transformation dramatically increased the computational accuracy, and the effect was the largest with parameter γ selected closer to unity. Tanaka's method allows the second local maximum of the wave phase speed to be located with high accuracy even at $N = 1000$ nodes, and the second local minimum can be computed at $N = 3500$ (Table 2.1).

In this context, we also need to mention the results obtained by Maklakov in Ref. [58]. He also calculated the second maximum and the second minimum of the wave phase speed and other integral wave parameters and reached the amplitude equal to 99.99997% of the limiting value. However, he did not tabulate those results. To this end, Maklakov used a modified contour integral method when no more than $N = 2000$ of mesh nodes were sufficient to achieve the required accuracy, which gives some advantage in practical computations. Later Dyachenko et al. [22] used another realisation of the conformal mapping and rational Padé approximations to calculate the parameters of Stokes waves on deep water with quadruple precision when approaching the limiting form with a 120° cusp. They could confidently tabulate the positions of the first local minimum and second local maximum of the phase speed. Lushnikov et al. [57] made an auxiliary conformal mapping in the numerical algorithm proposed in Ref. [22] that dramatically speeded up the numerical convergence by adapting the numerical grid for resolving singularities. As a result, they could compute the next two extremums of the wave phase speed: the second local minimum and the third local maximum.

2.2.3 Spectral Methods of Calculating the Stokes Waves

Expansions (2.71) and (2.72) represent the *exact* solution of the canonical problem in the form of Fourier series. In practice, these series are truncated to contain a finite number of terms. Such truncated series represent *approximate* solutions. After collecting the coefficients at the like powers of linearly independent functions ($\exp(\mathrm{i}\,nf)$ in our case), this technique results in a fully nonlinear system of algebraic equations for the corresponding Fourier coefficients of expansion (2.70) in the physical plane (first Stokes method) or expansion (2.72) in the inverse plane (second Stokes method). The advantage of the latter expansions is that the free surface in the inverse plane is known and given by the horizontal line $\psi = 0$. Sometimes instead of solving the system of equations for the Fourier coefficients directly, the free surface is subdivided into a certain number of points (that are called *collocation* points) at which the boundary conditions need to be satisfied exactly. As a result, one gets a system of algebraic equations for the unknown coefficients at the collocation points. It is the approach that was used by Chappelear [5] to calculate the first nine terms in series (2.67). He computed several steep wave profiles and compared his results with the fifth-order Stokes expansions.

Longuet-Higgins [44] established simple quadratic relations between the coefficients a_n of expansion (2.72) and used them instead of solving a more cumbersome system of cubic equations that is obtained when Fourier series (2.72) are substituted directly in the Bernoulli equation. This approach, which is usually referred to as the *Longuet-Higgins method*, is described in detail in Refs. [45, 46, 70] for infinite depth and in Refs. [47, 60] for finite depth.

The methods of calculating the Stokes waves in the physical plane include the modification of the first Stokes method performed by Rienecker and Fenton in Ref. [69] in which the coefficients of the Fourier series for the velocity potential are calculated by the collocation method and the Hamiltonian method implemented by Zufiria in Ref. [104]. Rienecker and Fenton carried out the calculations with up to $N = 64$ terms in their expansions, and Zufiria took into account up to $N = 72$ terms. The accuracy of their calculations was sufficient to evaluate only the first local maximum of the wave phase speed. The results of Rienecker and Fenton were extended by Zhao et al. [103] who improved the iterative stability of their numerical scheme by a relaxation technique. Tao et al. [88] developed another method of calculating the progressive waves in the case of finite depth with the use of operator expansions employing homotopy analysis and Padé approximations.

Lukomsky et al. [53, 54] constructed a general realisation of the method of Fourier expansions in the physical plane with the use of explicit relations between the Fourier coefficients of the velocity potential and free surface profile. In this approach, the truncated Fourier series for the velocity potential in the physical plane is written as

$$R(\theta,\ y) = \xi_0 + \sum_{n=1}^{N} \left(\xi_n T_n e^{n(y+i\theta)} - \xi_n^* T_{-n} e^{-n(y+i\theta)} \right),$$
$$\xi_0 = \frac{1}{2}\left(\overline{\eta} - \frac{I}{c}\right), \quad \Phi = -ic(R - R^*), \tag{2.83}$$

where the coefficients T_n are defined by relations (2.68). The truncated Fourier series for the free surface profile is given by

$$\eta(\theta) = \sum_{n=-M}^{M} \eta_n \exp(in\theta), \quad \eta_{-n} = \eta_n^*. \tag{2.84}$$

Expansions (2.83) and (2.84) are substituted in the dynamic and kinematic boundary conditions (2.38) and (2.39), which, after collecting the coefficients at the like powers of exponential functions, reduce to a system of nonlinear algebraic equations for the unknown coefficients ξ_n, η_n and the phase speed c. This system of equations is solved numerically by Newton's method with the use of Fast Fourier Transform for calculating the Fourier harmonics of exponential functions. For the convergence of the method, M should be several times larger than N. (Employing different truncation numbers in the Fourier series of the velocity potential and free surface profile was originally proposed in Ref. [104].) The coefficients ξ_n and η_n are complex-valued in the general case, but for the symmetric waves they are real. This method could provide accurate data on the first local maximum of the wave phase speed and locate its first minimum [53]. The calculations were performed up to $N = 250$ and $M = 4N$.

The use of ordinary Fourier expansions (2.83) and (2.84) is limited by the fact that the expansion coefficients decrease with their number more and more slowly as the wave crests sharpen with increasing of the wave amplitude. Hence, the number of coefficients required for accurate calculating of the waves approaching the limiting amplitude grows up enormously beyond any practical limits. In view of this, Lukomsky and Gandzha [55] proposed a modified spectral method optimised for the efficient calculation of steep waves on deep water. The idea of this method was to select a more suitable set of functions in the expansion of the velocity potential. To this end, Lukomsky and Gandzha used the Euler method for the summation of series described in Ref. [30]. The resulting expansion of the velocity potential is written as ($h = \infty$)

$$R(\theta,\ y;\ y_0) = \sum_{n=0}^{N} \frac{\alpha_n}{\left(\exp(-y_0) - \exp\left(-y - \mathrm{i}\theta\right)\right)^n}\,, \tag{2.85}$$

where y_0 is a free parameter used to accelerate the convergence of the series for the velocity potential. Fractional expansions (2.85) reduce to ordinary Fourier expansions (2.83) at $y_0 = \infty$. Expansions (2.85) and (2.83) are equivalent for all y_0 at $N = \infty$, but fractional expansion (2.85) converges mush faster at $y_0 \simeq 1$ and, therefore, needs less terms to maintain the same computational accuracy. The convergence of the series for the free surface profile was accelerated by the nonlinear transformation of the horizontal scale that was earlier proposed in Ref. [6]:

$$\theta(\chi;\ \gamma) = \chi - \gamma\ \sin\chi\,,\ \ 0 < \gamma < 1. \tag{2.86}$$

Then the corresponding expansion in the transformed space is written as

$$\eta\left(\chi;\ \gamma\right) = \sum_{n=-M}^{M} \eta_n^{(\gamma)}\ \exp(\mathrm{i}n\chi),\ \ \eta_{-n}^{(\gamma)} = \eta_n^{(\gamma)}. \tag{2.87}$$

As compared to expansion (2.84), much smaller number of terms is required at $\gamma \to 1$ to keep the same computational accuracy. Further steps are similar to the method of ordinary Fourier expansions. In practical calculations, the free parameters y_0 and γ

of the method as well as the ratio between the numbers N and M are selected such that the overall numerical mismatch would be the smallest. The advantage over the ordinary Fourier expansions was from one to ten orders of magnitude, depending on the wave amplitude [55].

Expansions (2.85) and (2.83) or any other analytic function

$$R(\theta,\ y) = R(y + \mathrm{i}\theta)$$

satisfy the Laplace equation (2.37) identically. The last fact was used by Clamond [8, 9] to introduce the so-called renormalisation principle which allows the velocity potential in the whole flow domain to be reconstructed from the form of the velocity potential at the bottom (or any other fluid level). Applying the renormalisation to the first approximation of cnoidal waves on shallow water, Clamond [9] obtained the velocity potential that identically coincided with the first term ($N = 1$) of fractional expansion (2.85). This first term proved to yield more accurate estimates than the fifth-order Stokes amplitude approximation.

2.3 The Limiting Stokes Wave with a Corner at the Crest and Its Calculation

The limiting wave with $q(0) = 0$ was first considered by Stokes is his pioneer work [82]. He showed that when the fluid flow at the wave crest is motionless with respect to the wave crest forms a 120° angle. Let us reproduce the proof of this statement. It follows from Eq. (2.76) that for the wave with $q(0) = 0$ the crest elevation above the wave mean level is

$$\eta(0) = B = \frac{c^2}{2} + \frac{1}{2}\,\overline{\Phi_x^2}\,|_{y=-h}. \tag{2.88}$$

The origin of the intrinsic wave reference frame is selected to lie at the wave crest, i.e. the level $y = 0$ is translated by $\eta(0)$ upwards. In this reference frame, the Bernoulli equation (2.38) is written in terms of the complex potential with the use of Eq. (2.52) as follows

$$\left|\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right|^2 + 2\,\mathrm{Im}\,\zeta = 0, \quad \zeta \in \Gamma, \tag{2.89}$$

where $\zeta = \theta + \mathrm{i}\big(y - \eta(0)\big)$ and Γ is the surface. Consider the complex potential of the flow described by a power function:

$$w(\zeta) = \phi + \mathrm{i}\psi = \mathscr{A}\,\zeta^n. \tag{2.90}$$

Such a complex potential describes the flow into a $\frac{\pi}{n}$ corner. Substituting this expression in Eq. (2.89) yields the unknown n and $\mathscr{A}$. In particular, for the exponent n the only admissible value is

$$n = \frac{3}{2},$$

which indeed corresponds to a 120° angle. Accordingly, for the complex potential of the flow into a 120° corner we get

$$w(\zeta) = \mathrm{i}\frac{2}{3}\,(\mathrm{i}\zeta)^{\frac{3}{2}}, \tag{2.91}$$

the inverse relation being

$$\zeta(w) = -\mathrm{i}\Big(\frac{3}{2\mathrm{i}}\,w\Big)^{\frac{2}{3}}. \tag{2.92}$$

Thus, the analytic function $\zeta(w)$ has a singularity of order $\frac{2}{3}$. The corresponding complex velocity is written as

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = -\big(\mathrm{i}\,\zeta\big)^{\frac{1}{2}} = -\Big(\frac{3}{2\mathrm{i}}\,w\Big)^{\frac{1}{3}}. \tag{2.93}$$

Stokes' solution (2.91) and (2.92) is valid only in the infinitesimal vicinity of wave crest. Therefore, the obtained corner flow is usually referred to as *local*. It represents only the first term of the expansion about the corner. The second term of such an expansion about the singular point was found by Grant [29]:

$$\zeta(w) = -\mathrm{i}\Big(\frac{3}{2\mathrm{i}}\,w\Big)^{\frac{2}{3}} + \mathrm{i}\gamma\,(-\mathrm{i}w)^{2\mu} + \dots, \tag{2.94}$$

and higher terms were calculated by Norman [64] in the form of a power series in terms of the exponent μ that satisfies the transcendental equation

$$\tan\pi\mu = -\frac{2+3\mu}{3\sqrt{3}\mu}. \tag{2.95}$$

The first root of this equation is $\mu = 0.73467287\dots$. Hence, the expansion about the singular point $q(0) = 0$ is not a regular power expansion but includes irrational exponents, i.e. the singularity formed by the stagnation point at the crest has an irregular nature.

Stokes' local solution (2.92) and expansions (2.94) satisfy the Laplace and Bernoulli equations but do not satisfy the asymptotic at the bottom, which has the following form:

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = -c, \quad \psi = c\,d. \tag{2.96}$$

To make this condition hold, one needs to employ linear combinations of the solutions about the corner or to construct approximate expansions that take into account the singularity at the crest in one way or another. This task turns out to be more simple when the layer occupied by fluid in the complex plane w is transformed into a unit circle:

$$w = -\mathrm{i}c \ln u, \quad u = \rho \exp(\mathrm{i}s). \tag{2.97}$$

In this case, the velocity potential and stream function are written as

$$\phi = c\, s, \quad \psi = -c \ln \rho. \tag{2.98}$$

In the transformed space, the fluid occupies the circular domain bounded by the circles $\rho = 1$ (surface) and $\rho = \rho_0 = \mathrm{e}^{-d}$ (bottom). The wave crest and trough have the phases $s = 0$ and $s = \pm\pi$, respectively (Fig. 2.3). In the case of infinite depth, we have $\rho_0 = 0$ and the bottom is transformed into the point $u = 0$.

The singularity $\zeta = 0$ ($w = 0$) is located at the point $u = 1$ in the complex plane u, and Stokes' local solution (2.92), (2.93) has the following form:

$$\zeta = -\mathrm{i}\left(\frac{3}{2}\, c\,(1-u)\right)^{\frac{2}{3}}, \ \frac{\mathrm{d}w}{\mathrm{d}\zeta} = -\left(\frac{3}{2}\, c\,(1-u)\right)^{\frac{1}{3}}. \tag{2.99}$$

With these asymptotics taken into account, Michell [61] proposed the following approximation for the wave with a singularity of order $\frac{2}{3}$ in the case of infinite depth:

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = -c\,(1-u)^{\frac{1}{3}} \sum_{n=0}^{\infty} b_n\, u^n. \tag{2.100}$$

The asymptotic (2.96) at the bottom $u = 0$ is satisfied identically by setting $b_0 = 1$. All other unknown coefficients b_n and the wave speed c can be found from a system of algebraic equations that is obtained by substituting expansion (2.100) in the Bernoulli equation (2.89).

Michell calculated the first three coefficients (assuming that $b_n \sim b_1^n$) and obtained the estimates $c^2 \approx 1.2$ and $A_{\max} \approx 0.142$, i.e. the speed of the wave of limiting height is about 1.09 times higher than the linear wave speed. Later Havelock [32] calculated one additional (the fourth) coefficient and obtained $A_{\max} \approx 0.1418$. Nekrasov [63] confirmed Michell's results with the use of the inverse expansion

$$\frac{\mathrm{d}\zeta}{\mathrm{d}w} = -c^{-1}(1-u)^{-\frac{1}{3}} \sum_{n=0}^{\infty} \alpha_n\, u^n, \quad \alpha_0 = 1. \tag{2.101}$$

Davies [15] used a different approach and obtained $A_{\max} \approx 0.1443$. Michell's calculations could essentially be refined after the advent of the computer era. Yamada [95] could compute the first twelve coefficients of Nekrasov's expansion (2.101) and got $A_{\max} \approx 0.1412$. These calculations were also performed in the case of finite depth in Ref. [96].

Olfe and Rottman [65] formalised Michell's method and derived a general system of nonlinear algebraic equations for the coefficients b_n of expansion (2.100) and the phase speed c:

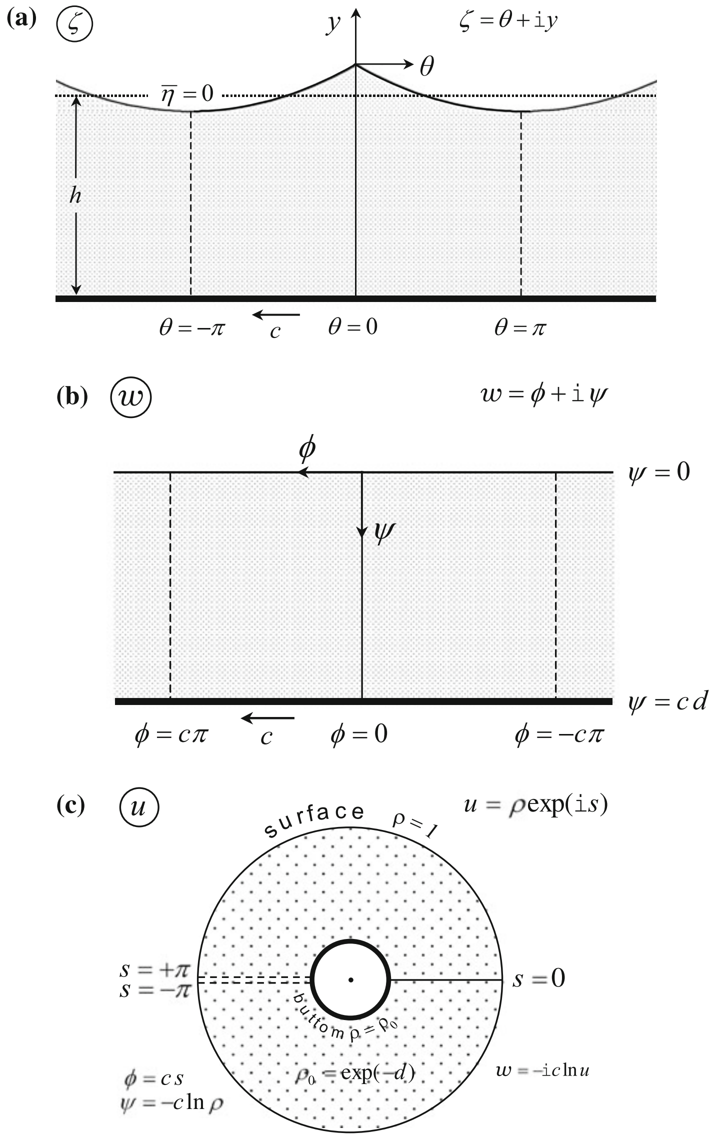

Fig. 2.3 Direct and inverse planes: **a** physical plane, **b** inverse plane of the complex potential, **c** inverse plane transformed into a unit circle

$$(3n+2)A_n - (3n+1)A_{n+1} - c^{-2}F_n = 0,\ n = \overline{0,\,N}, \tag{2.102}$$

where N is the number of the coefficients that were taken into account and

$$A_n = \frac{1}{2}\sum_{n_1=0}^{n} B_{n_1} B_{n-n_1} + \sum_{n_1=1}^{N-n} B_{n_1} B_{n+n_1},$$

$$B_n = \sum_{n_1=0}^{N-n} b_{n_1}\, b_{n+n_1},$$

$$F_n = \frac{18\sqrt{3}}{\pi}\sum_{n_1=0}^{N} \frac{(6n_1+1)\, b_{n_1}}{9(2n+1)2-(6n_1+1)^2},\ b_0 = 1.$$

The wave amplitude A can be found from expression (2.42) using relation (2.53):

$$A = 2^{\frac{2}{3}}\frac{c^2}{4\pi}\left(\sum_{n=0}^{N}(-1)^n\, b_n\right)^2. \tag{2.103}$$

With $N = 120$ coefficients, Olfe and Rottman obtained the estimate $A_{max} \approx 0.141061$. The results of computations for higher N are given in Table 2.2. It can be seen that the rate of convergence of Michell's expansions drops with increasing N because they do not make allowance for the irrational exponents found by Grant in Ref. [29].

Michell's method received further development in Ref. [24]. The convergence of Michell's expansion (2.100) was drastically improved with the use of nonlinear transformation of the u-plane into a new ξ-plane:

$$\xi \equiv \exp(\mathrm{i}\chi) = \frac{u-\gamma}{1-\gamma u}, \quad 0 \leqslant \gamma < 1. \tag{2.104}$$

This transformation, which was originally proposed by Tanaka in Ref. [85], maps a unit circle of the u-plane onto a unit circle of the ξ-plane with stretching the crest region while contracting the trough region (Fig. 2.4). The rate of stretch and contraction is intensified as the parameter γ approaches unity. In the ξ-plane, the infinite depth is located at the level $\xi = -\gamma$ and χ stands for the complex potential.

Table 2.2 Speed c and amplitude A of the limiting Stokes wave on infinite depth calculated from the system of equations (2.102) of the Michell method

N	A	c	
100	0.14106002	1.09227679	$b_1 = 0.04119$
200	0.14106250	1.09228173	$b_2 = 0.01251$
300	0.14106307	1.09228335	$b_3 = 0.00605$
400	0.14106327	1.09228405	$b_4 = 0.00360$
500	0.14106337	1.09228441	$b_5 = 0.00240$

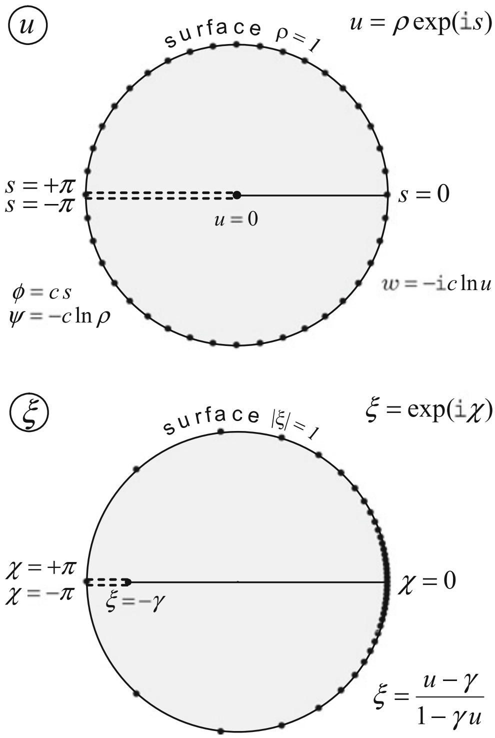

Fig. 2.4 Tanaka's transformation of the inverse plane. Reprinted from [24]

One can easily prove that asymptotic (2.99) remains unchanged in the ξ-plane:

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} \sim (1-\xi)^{1/3}. \tag{2.105}$$

So, the Michell expansion in the ξ-plane is written as

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = \left(1-\exp(\mathrm{i}\chi)\right)^{\nu} \sum_{n=0}^{N} b_n \exp(\mathrm{i}n\chi), \quad \nu \equiv 1/3, \tag{2.106}$$

where N is the number of coefficients taken into account in the approximate solution. The boundary condition at infinite depth (2.96) takes the form

$$(1+\gamma)^{\nu} \sum_{n=0}^{N} b_n (-\gamma)^n = -c. \tag{2.107}$$

Further procedure is not much more awkward than that elaborated in Refs. [61, 65]. The corresponding system of nonlinear algebraic equations for the coefficients b_n of expansion (2.106) and phase speed c is

$$-\gamma(3n+4)A_{n+2} + \big(\gamma(3n+5) - (1+\gamma^2)(3n+1)\big)A_{n+1} + \\ \big((1+\gamma^2)(3n+2) - \gamma(3n-2)\big)A_n + \gamma(3n-1)A_{n-1} + \\ c(1-\gamma^2)F_n = 0,\ n = \overline{0,\,N}, \quad (2.108)$$

where

$$A_n = \sum_{n_1=0}^{n} B_{n_1} B_{n-n_1} + 2\sum_{n_1=1}^{N-n} B_{n_1} B_{n+n_1},\ n = \overline{0,\,N},$$

$$A_n = \sum_{n_1=n-N}^{N} B_{n_1} B_{n-n_1},\ n = \overline{N+1,\,2N}, \quad A_n \equiv 0,\ |n| > 2N, \quad A_{-n} = A_n,$$

$$B_n = \sum_{n_1=0}^{N-n} b_{n_1}\, b_{n+n_1}, \quad F_n = \frac{36\sqrt{3}}{\pi} \sum_{n_1=0}^{N} \frac{(6n_1+1)\, b_{n_1}}{9(2n+1)^2 - (6n_1+1)^2}.$$

The wave amplitude A is found by the formula

$$A = \frac{1}{4\pi}\left|\left(\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right)_{\chi=\pi}\right|^2 = \frac{2^{2\nu}}{4\pi}\left(\sum_{n=0}^{N}(-1)^n\, b_n\right)^2. \quad (2.109)$$

The wave profile $y = \eta(\theta)$ is found in parametric form from the following integrals:

$$\theta(\chi) = \int_0^{\chi} \frac{\partial\theta}{\partial\chi'}\mathrm{d}\chi' = \int_0^{\chi} \frac{\partial\theta}{\partial\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\chi'}\mathrm{d}\chi' = \int_0^{\chi} \mathrm{Re}\left(\left(\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right)^{-1}\right)\frac{\mathrm{d}\phi}{\mathrm{d}\chi'}\mathrm{d}\chi', \quad (2.110)$$

$$y(\chi) = \int_0^{\chi} \frac{\partial y}{\partial\chi'}\mathrm{d}\chi' = \int_0^{\chi} \frac{\partial y}{\partial\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\chi'}\mathrm{d}\chi' = \int_0^{\chi} \mathrm{Im}\left(\left(\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right)^{-1}\right)\frac{\mathrm{d}\phi}{\mathrm{d}\chi'}\mathrm{d}\chi', \quad (2.111)$$

where χ ranges from 0 (crest) to π (trough) and

$$\frac{\mathrm{d}\phi}{\mathrm{d}\chi} = \frac{c(1-\gamma^2)\xi}{(\xi+\gamma)(1+\gamma\xi)}, \quad \xi = \exp(\mathrm{i}\chi). \quad (2.112)$$

The kinetic and potential energies of the wave can be calculated as

$$T = \frac{cI}{2} = \frac{c}{2}\,\frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\eta(\theta)}(\phi_\theta + c)\,\mathrm{d}y\,\mathrm{d}\theta \\ = -\frac{c}{2\pi}\int_0^{\pi c}(y|_{\psi=0} + \overline{\eta})\,\mathrm{d}\phi = -\frac{c^4}{4} + \frac{c}{4\pi}\int_0^{\pi}\left|\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right|^2\frac{\mathrm{d}\phi}{\mathrm{d}\chi}\,\mathrm{d}\chi, \quad (2.113)$$

$$
\begin{aligned}
U = \frac{1}{4\pi}\int_0^{2\pi}\left(y^2\,|_{\psi=0} - \overline{\eta}^2\right)\mathrm{d}\theta \\
= \frac{1}{8\pi}\int_0^{\pi}\left(c^4 - \left|\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right|^4\right)\mathrm{Re}\left(\left(\frac{\mathrm{d}w}{\mathrm{d}\zeta}\right)^{-1}\right)\frac{\mathrm{d}\phi}{\mathrm{d}\chi}\,\mathrm{d}\chi,
\end{aligned}
\tag{2.114}
$$

where I is the wave momentum.

Based on the above-described approach, Gandzha and Lukomsky [24] obtained the most accurate values of the amplitude, phase speed and energy of the limiting Stokes wave:

$$c = 1.09228504859, \tag{2.115a}$$

$$A = 0.14106348398, \tag{2.115b}$$

$$U = 0.03456832591, \tag{2.115c}$$

$$T = 0.03829219689. \tag{2.115d}$$

These results are in agreement with the asymptotics derived earlier in Ref. [58] with the use of the theory of almost highest waves.

The method of calculating the limiting Stokes wave in the case of finite depth was developed by Willians [93] with Grant's asymptotic taken into account explicitly. Among approximate approaches to describing the limiting Stokes waves, we can mention a simple analytic approximation for the wave profile proposed by Longuet-Higgins [48] and more complex approximations for a broad range of depths presented by Karabut in Ref. [38].

2.4 High-Order Nonlinear Schrödinger Equation and Split-Step Fourier Technique

In this section, we briefly consider the evolution of narrow-band wave trains on the surface of an ideal fluid. In the case of finite depth, the evolution of surface gravity waves and associated fluid flows is governed by the following set of Euler equations derived in Sect. 2.1:

$$\Phi_{xx} + \Phi_{yy} = 0, \quad -h \leqslant y \leqslant \eta(x, t); \tag{2.116a}$$

$$\Phi_t + \frac{1}{2}\left(\Phi_x^2 + \Phi_y^2\right) + g\eta = 0, \quad y = \eta(x, t); \tag{2.116b}$$

$$\eta_t - \Phi_y + \eta_x \Phi_x = 0, \quad y = \eta(x, t); \tag{2.116c}$$

$$\Phi_y = 0, \quad y = -h. \tag{2.116d}$$

Here, the indices designate the partial derivatives with respect to the corresponding variables. In the case of slowly modulated wave trains and small-amplitude approximation, the unknown free-surface displacement and velocity potential can be expressed in the form of Fourier expansions around the carrier frequency and wave number:

$$\begin{pmatrix} \eta(x, t) \\ \boldsymbol{\Phi}(x, y, t) \end{pmatrix} = \sum_{n=-\infty}^{\infty} \begin{pmatrix} \eta_n(x, t) \\ \boldsymbol{\Phi}_n(x, y, t) \end{pmatrix} \exp\big(\mathrm{i}n(kx - \omega t)\big),$$
$$\eta_{-n} \equiv \eta_n^*, \quad \boldsymbol{\Phi}_{-n} \equiv \boldsymbol{\Phi}_n^*, \tag{2.117}$$

with $\eta_0 \sim \varepsilon^2$, $\boldsymbol{\Phi}_0 \sim \varepsilon$, $\eta_n \sim \varepsilon^n$, $\boldsymbol{\Phi}_n \sim \varepsilon^n$, $n \geqslant 1$, ε being a formal small parameter related to the smallness of wave amplitude as compared to the carrier wavelength $\lambda \equiv \frac{2\pi}{k}$. The carrier frequency ω and wave number k are connected by the linear dispersion relation derived in Sect. 2.1.5:

$$\omega^2 = gk \tanh(kh) \equiv gk\sigma, \quad \sigma \equiv \tanh(kh). \tag{2.118}$$

Further, by exploiting various perturbation techniques, the initial set of equations (2.116) can be reduced to two coupled equations for the first-harmonic amplitude η_1 of the wave profile and the mean flow expressed by the zero harmonic of the velocity potential: $\overline{\boldsymbol{\Phi}} = \boldsymbol{\Phi}_0$. In particular, it was done by Brinch-Nielsen and Jonsson [4] and then independently by Gramstad and Trulsen [28] and Craig et al. [11] in the Hamiltonian (canonical) form preserving momentum. Further simplification can be obtained when the mean flow is expanded into power series in terms of formal small parameter ε:

$$\boldsymbol{\Phi}_0 = \varepsilon \boldsymbol{\Phi}_0^{(1)} + \varepsilon^2 \boldsymbol{\Phi}_0^{(2)} + O(\varepsilon^3). \tag{2.119}$$

This expansion is valid only in the case of finite depth, and its limitations are well discussed in Ref. [4]. The use of multi-scale expansions (see Sect. 1.6) allows the functions $\boldsymbol{\Phi}_0^{(n)}$ to be expressed in terms of the first-harmonic amplitude η_1, so that two coupled equations for η_1 and $\boldsymbol{\Phi}_0$ might be reduced to a single evolution equation for η_1. It is the approach that was first utilised by Hasimoto and Ono [31] to derive the nonlinear Schrödinger equation (NLSE) for the first-harmonic amplitude η_1 of the wave profile:

$$\mathrm{i}\left(\frac{\partial \eta_1}{\partial t} + c_g \frac{\partial \eta_1}{\partial x}\right) + \left(\frac{\omega''}{2}\frac{\partial^2 \eta_1}{\partial x^2} + 4\omega k^2 a_{0,0,0}\, |\eta_1|^2 \eta_1\right) = 0, \tag{2.120}$$

where c_g is the wave group speed defined by formula (2.35) and ω'' is the second derivative of the carrier frequency with respect to k. The coefficient $a_{0,0,0}$ is defined in terms of dimensionless water depth kh:

$$\begin{aligned} a_{0,0,0} = &- \frac{1}{16\sigma^4 \nu}\Big(\left(\sigma^2 - 1\right)^2 \left(9\sigma^4 - 10\sigma^2 + 9\right) k^2 h^2 \\ &+ 2\sigma\left(3\sigma^6 - 23\sigma^4 + 13\sigma^2 - 9\right) kh - \sigma^2\left(7\sigma^4 - 38\sigma^2 - 9\right)\Big), \end{aligned} \tag{2.121}$$

where

$$\nu = \left(\sigma^2 - 1\right)^2 k^2 h^2 - 2\sigma\left(\sigma^2 + 1\right) kh + \sigma^2. \tag{2.122}$$

NLSE arises in describing nonlinear waves in various physical contexts, such as nonlinear optics [91], plasma physics [34], nanosized electronics [12], ferromagnetics [7], Bose–Einstein condensates [100], and hydrodynamics [18, 66, 98, 102]. In the general context of weakly nonlinear dispersive waves, this equation was first discussed by Benney and Newell [1]. In the case of gravity waves propagating on the surface of infinite-depth irrotational, inviscid and incompressible fluid, NLSE was first derived by Zakharov [99] using the Hamiltonian formalism and then by Yuen and Lake [98] using the averaged Lagrangian method.

Under certain relationship between the parameters, when

$$-\omega'' a_{0,0,0} < 0, \tag{2.123}$$

NLSE admits exact solutions in the form of *solitons* that exist due to the balance of dispersion and nonlinearity and propagate without changing their shape and keeping their energy [19]. In this case, the uniform carrier wave is unstable with respect to long-wave modulations allowing for the formation of envelope solitons. This type of instability is known as the *modulational* or *Benjamin–Feir instability* [101] (it was discovered for the first time in optics by Bespalov and Talanov [3]). In the case of surface gravity waves, condition (2.123) holds at $kh \gtrsim 1.363$. At the bifurcation point $a_{0,0,0} = 0$ ($kh \approx 1.363$), when the modulational instability changes to stability, NLSE of form (2.120) is not sufficient to describe the wave train evolution since the leading nonlinear term vanishes. In this case, high-order nonlinear and nonlinear dispersive terms should be taken into account.

In the case of infinite depth, such a high-order NLSE (HONLSE) was first derived by Dysthe [23]. It includes the third-order dispersion and cubic nonlinear dispersive terms as well as an additional nonlinear dispersive term describing the input of the wave-induced mean flow. This equation is usually referred to as the fourth-order HONLSE to emphasise the contrast with the third-order NLSE. Janssen [35] rederived Dysthe's equation and corrected the sign at one of the nonlinear dispersive terms. Worthy of mention is also the paper by Lukomsky [52] who derived Dysthe's equation in a different way. Later Trulsen and Dysthe [89] extended the equation derived by Dysthe to broader bandwidth by including the forth- and fifth-order linear dispersion. Debsarma and Das [17] derived a yet more general HONLSE that is one order higher than the equation derived by Trulsen and Dysthe. Original Dysthe's equation was written for the first-harmonic envelope of velocity potential rather than of surface profile. In the case of standard NLSE, this difference is not essential because in that order the first-harmonic amplitudes of the velocity potential and surface displacement differ by a dimensional factor only, which is not true any longer in the HONLSE case, as was discussed by Hogan [33]. Keeping this in mind, Trulsen et al. [90] rewrote Dysthe's equation in terms of the first-harmonic envelope of surface profile while taking into account the linear dispersion to an arbitrary order. Dyachenko and Zakharov [20, 21] made a conformal mapping of the fluid domain into the lower half-plane to derive a counterpart of Dysthe's equation in new canonical variables.

In the case of finite depth, the effect of induced mean flow manifests itself in the third order, so that the NLSE is generally coupled to the equation for the induced mean flow [2]. However, Davey and Stewartson [14] showed that these coupled equations are equivalent to the single NLSE (2.120) derived by Hasimoto and Ono [31] (see also Ref. [77] for more details). On the other hand, such an equivalence is not preserved for high-order equations. The first attempt to derive a HONLSE in the case of finite depth was made by Johnson [36], but only for $kh \approx 1.363$, when the cubic NLSE term vanishes. The similar attempt was made by Kakutani and Michihiro [37] (see also a more formal derivation made later by Parkes [67]). Sedletsky [75, 76] used the multiple-scale technique to derive a single fourth-order HONLSE for the first-harmonic envelope of surface profile by using the additional power expansion (2.119) of the induced mean flow. This equation is the direct counterpart of Dysthe's equation written in terms of the first-harmonic envelope of surface profile [90] but for the case of finite depth. Gandzha et al. [26] reduced this HONLSE to dimensionless form:

$$\begin{aligned} \mathrm{i}u_\tau = -\mathrm{i}a_1 u_\chi + a_2 u_{\chi\chi} - a_{0,0,0}\,|u|^2 u \\ +\mathrm{i}\Big(a_3 u_{\chi\chi\chi} - a_{1,0,0}\,u_\chi |u|^2 - a_{0,0,1}\,u^2 u_\chi^*\Big), \end{aligned} \tag{2.124}$$

where * stands for complex conjugate, the subscripts next to u denote the partial derivatives, and

$$\tau = \frac{\omega}{c}\,t, \quad \chi = kx \tag{2.125}$$

are the dimensionless time and coordinate. The free surface displacement η is related to the amplitude u as follows

$$\begin{aligned} k\eta = \alpha_0 |u|^2 + \alpha_1\,\mathrm{Re}\big(u\exp(\mathrm{i}\theta)\big) + 2\alpha_2\,\mathrm{Re}\big(u^2\exp(2\mathrm{i}\theta)\big), \\ \alpha_0 = \tfrac{2(1-\sigma^2)kh+\sigma}{c\nu}, \quad \alpha_1 = \tfrac{1}{\sqrt{c}}, \quad \alpha_2 = \tfrac{3-\sigma^2}{8c\sigma^3}, \\ \nu = \left(\sigma^2-1\right)^2 k^2h^2 - 2\sigma\left(\sigma^2+1\right)kh + \sigma^2, \end{aligned} \tag{2.126}$$

where $\theta = kx - \omega t = \chi - c\tau$ is the wave phase; c and a_1 are the dimensionless phase and group speeds, respectively,

$$c = -\frac{\omega}{k}\left(k\frac{\partial^2\omega}{\partial k^2}\right)^{-1} = -\frac{4\sigma^2}{\upsilon} > 0, \tag{2.127}$$

$$\begin{aligned} a_1 = \frac{ck}{\omega}c_g = -\frac{2\sigma}{\upsilon}\Big(\left(1-\sigma^2\right)kh + \sigma\Big) > 0, \\ \upsilon = \left(\sigma^2-1\right)\left(3\sigma^2+1\right)k^2h^2 - 2\sigma\left(\sigma^2-1\right)kh - \sigma^2. \end{aligned} \tag{2.128}$$

Equation (2.124) describes the evolution of the first-harmonic envelope of the surface profile with taking into account the third-order dispersion and cubic nonlinear dispersive terms. The contribution of the mean flow that is "hidden" in the values of the coefficients $a_{0,0,0}$, $a_{1,0,0}$, and $a_{0,0,1}$ that depend only on dimensionless depth kh. The notation of coefficients $a_n \ldots$ was selected such that the number of their indices corresponds to the order of nonlinearity in the corresponding term, and the index values correspond to the orders of derivatives present in that term (with complex conjugate derivatives having a separate index). The second- and third-order dispersion coefficients are expressed as follows

$$
\begin{aligned}
a_2 &= \frac{1}{2}, \quad a_3 = \frac{1}{12\sigma\upsilon}\sum_{p=0}^{3} a_3^{(p)}(kh)^p, \\
a_3^{(0)} &= -3\sigma^3, \quad a_3^{(1)} = -3\sigma^2\left(\sigma^2-1\right), \\
a_3^{(2)} &= -3\sigma\left(\sigma^2-1\right)\left(3\sigma^2+1\right), \\
a_3^{(3)} &= \left(\sigma^2-1\right)\left(15\sigma^4-2\sigma^2+3\right).
\end{aligned}
\tag{2.129}
$$

The cubic nonlinear dispersion coefficients are

$$
\begin{aligned}
\begin{pmatrix} a_{1,0,0} \\ a_{0,0,1} \end{pmatrix} &= \frac{1}{32\sigma^5\nu^2}\sum_{p=0}^{5}\begin{pmatrix} 2a_{1,0,0}^{(p)} \\ a_{0,0,1}^{(p)} \end{pmatrix}(kh)^p, \\
a_{1,0,0}^{(0)} &= -\sigma^5\left(17\sigma^4-94\sigma^2-27\right), \\
a_{0,0,1}^{(0)} &= \sigma^5\left(\sigma^4-2\sigma^2+9\right), \\
a_{1,0,0}^{(1)} &= \sigma^4\left(55\sigma^6-269\sigma^4-99\sigma^2-135\right), \\
a_{0,0,1}^{(1)} &= \sigma^4\left(19\sigma^6-137\sigma^4-47\sigma^2-27\right), \\
a_{1,0,0}^{(2)} &= -2\sigma^3\left(19\sigma^8-82\sigma^6-92\sigma^4+98\sigma^2-135\right), \\
a_{0,0,1}^{(2)} &= -2\sigma^3\left(43\sigma^8-296\sigma^6+230\sigma^4-32\sigma^2-9\right), \\
a_{1,0,0}^{(3)} &= 2\sigma^2\left(1-\sigma^2\right)\left(15\sigma^8-14\sigma^6-24\sigma^4+94\sigma^2-135\right), \\
a_{0,0,1}^{(3)} &= 2\sigma^2\left(\sigma^2-1\right)\left(51\sigma^8-272\sigma^6+158\sigma^4+8\sigma^2-9\right), \\
a_{1,0,0}^{(4)} &= \sigma\left(\sigma^2-1\right)^3\left(39\sigma^6+35\sigma^4-3\sigma^2-135\right), \\
a_{0,0,1}^{(4)} &= -\sigma\left(\sigma^2-1\right)^3\left(27\sigma^6-57\sigma^4-7\sigma^2-27\right), \\
a_{1,0,0}^{(5)} &= -\left(\sigma^2-1\right)^5\left(9\sigma^4+10\sigma^2-27\right), \\
a_{0,0,1}^{(5)} &= -\left(\sigma^2-1\right)^5\left(9\sigma^4-10\sigma^2+9\right).
\end{aligned}
\tag{2.130}
$$

The latter coefficients were first derived in Refs. [75, 76] and then rederived in Ref. [78] with keeping one additional term in the multi-scale expansion of the mean flow (this term introduces a small correction to the expressions presented in Ref. [76]). Here, we used the expressions with this additional term taken into account.

Equation (2.124) has exact solutions in the form of solitons [25] and numerical solutions in the form of quasi-solitons [26]. Quasi-solitons are the solutions that have slowly varying amplitude, propagate with nearly constant speed, and possess the unique property of solitons to exist over long periods of time without breaking. Their speed was found to be higher than the speed of the exact NLSE solitons taken as initial conditions in computations [26].

Numerical solutions to Eq. (2.124) can be found by the *split-step Fourier technique*. For the sake of convenience, we proceed to the reference frame moving with speed a_1 (dimensionless group speed):

$$\xi = \chi - a_1\tau, \quad T = \tau. \tag{2.131}$$

The relationship between the derivatives in new and old variables is given by the formulas

$$\frac{\partial}{\partial\chi} = \frac{\partial\xi}{\partial\chi}\frac{\partial}{\partial\xi} + \frac{\partial T}{\partial\chi}\frac{\partial}{\partial T} = \frac{\partial}{\partial\xi},$$
$$\frac{\partial}{\partial\tau} = \frac{\partial\xi}{\partial\tau}\frac{\partial}{\partial\xi} + \frac{\partial T}{\partial\tau}\frac{\partial}{\partial T} = -a_1\frac{\partial}{\partial\xi} + \frac{\partial}{\partial T},$$

so that

$$u_\tau = -\mathrm{i}a_2 u_{\xi\xi} + \mathrm{i}a_{0,0,0}|u|^2 u + \left(a_3 u_{\xi\xi\xi} - a_{1,0,0}u_\xi|u|^2 - a_{0,0,1}u^2 u_\xi^*\right). \tag{2.132}$$

Consider the linear part of HONLSE (2.132):

$$u_\tau = -\mathrm{i}a_2 u_{\xi\xi} + a_3 u_{\xi\xi\xi}, \quad u = u(\xi, \tau). \tag{2.133}$$

Apply the Fourier transform to the function $u(\xi, \tau)$:

$$\hat{u}(\kappa, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\xi, \tau)\exp(-\mathrm{i}\kappa\xi)\mathrm{d}\xi \equiv \mathscr{F}_\kappa[u(\xi, \tau)]. \tag{2.134}$$

The inverse Fourier transform is written as

$$u(\xi, \tau) = \int_{-\infty}^{\infty} \hat{u}(\kappa, \tau)\exp(\mathrm{i}\kappa\xi)\mathrm{d}\kappa \equiv \mathscr{F}_\xi^{-1}[\hat{u}(\kappa, \tau)]. \tag{2.135}$$

The Fourier transforms of the derivatives of function $u(\xi, \tau)$ are expressed as

$$\widehat{(u_\xi)} = \mathrm{i}\kappa\hat{u}, \quad \widehat{(u_{\xi\xi})} = -\kappa^2\hat{u}, \quad \dots, \quad \widehat{(u_{n\xi})} = (\mathrm{i}\kappa)^n\hat{u}. \tag{2.136}$$

Hence, linear equation (2.133) takes the following form in the Fourier space:

$$\hat{u}_\tau = \left(-\mathrm{i}a_2(\mathrm{i}\kappa)^2 + a_3(\mathrm{i}\kappa)^3\right)\hat{u}, \quad \hat{u}(0) \equiv \hat{u}_0. \tag{2.137}$$

This ordinary differential equation can easily be integrated,

$$\widehat{u} = \widehat{u}_0 \exp\big((\mathrm{i}a_2\kappa^2 - \mathrm{i}a_3\kappa^3)\tau\big), \tag{2.138}$$

and the following solution for $u(\xi, \tau)$ is obtained:

$$u_L = \int_{-\infty}^{\infty} \widehat{u}_0 \exp\big((\mathrm{i}a_2\kappa^2 - \mathrm{i}a_3\kappa^3)\tau\big) \exp(\mathrm{i}\kappa\xi)\mathrm{d}\kappa. \tag{2.139}$$

Nonlinear equation (2.132) can be split into the linear and nonlinear parts:

$$u_\tau = -\mathrm{i}a_2 u_{\xi\xi} + \mathrm{i}a_{0,0,0}u|u|^2 + \left(a_3 u_{\xi\xi\xi} - a_{1,0,0}u_\xi |u|^2 - a_{0,0,1}u^2 u_\xi^*\right) \equiv (\mathscr{L} + \mathscr{N})\, u,$$

where

$$\mathscr{L} \equiv -\mathrm{i}a_2\partial_{\xi\xi} + a_3\partial_{\xi\xi\xi}, \tag{2.140}$$

$$\mathscr{N} \equiv \mathrm{i}a_{0,0,0}|u|^2 - a_{1,0,0}u_\xi u^* - a_{0,0,1}uu_\xi^* \tag{2.141}$$

are the linear and nonlinear operators, respectively. The semi-discretisation in time is performed as follows

$$\left.\frac{u(\xi, \tau + \Delta\tau) - u(\xi, \tau)}{\Delta\tau}\right|_{\Delta\tau\to 0} = (\mathscr{L} + \mathscr{N})\, u(\xi, \tau) \quad \Rightarrow$$
$$u(\xi, \tau + \Delta\tau) \approx u(\xi, \tau) + \Delta\tau(\mathscr{L} + \mathscr{N})u(\xi, \tau) \approx \mathrm{e}^{\Delta\tau(\mathscr{L}+\mathscr{N})}u(\xi, \tau),$$

and then the second-order Strang formula for noncommuting operators [83] is used:

$$\mathrm{e}^{\Delta\tau(\mathscr{L}+\mathscr{N})} \equiv S^{(2)}(\Delta\tau)$$
$$= \exp\left(\frac{\Delta\tau}{2}\mathscr{N}\right) \exp\left(\Delta\tau\mathscr{L}\right) \exp\left(\frac{\Delta\tau}{2}\mathscr{N}\right), \tag{2.142}$$
$$= \exp\left(\frac{\Delta\tau}{2}\mathscr{L}\right) \exp\left(\Delta\tau\mathscr{N}\right) \exp\left(\frac{\Delta\tau}{2}\mathscr{L}\right). \tag{2.143}$$

In our computations splitting (2.143) proved to be more accurate than (2.142). The linear part is integrated exactly using relation (2.139)

$$\mathrm{e}^{\Delta\tau\mathscr{L}}u(\xi, \tau) = \mathscr{F}_\xi^{-1}\left[\mathrm{e}^{\Delta\tau\left(-\mathrm{i}a_2(\mathrm{i}\kappa)^2 + a_3(\mathrm{i}\kappa)^3\right)}\mathscr{F}_\kappa[u(\xi, \tau)]\right], \tag{2.144}$$

and the nonlinear part is corrected at each step as follows

$$\mathrm{e}^{\Delta\tau\mathscr{N}}u(\xi, \tau) = \mathrm{e}^{\Delta\tau\left(\mathrm{i}a_{0,0,0}|u|^2 - a_{1,0,0}u_\xi u^* - a_{0,0,1}uu_\xi^*\right)}u(\xi, \tau). \tag{2.145}$$

Following Yoshida [97], a more accurate fourth-order splitting can be introduced as well:

$$S^{(4)}(\Delta\tau) = S^{(2)}\left(p_1\Delta\tau\right) S^{(2)}\left(p_0\Delta\tau\right) S^{(2)}\left(p_1\Delta\tau\right), \quad p_0 = -\frac{2^{1/3}}{2-2^{1/3}} \approx -1.70, \quad p_1 = \frac{1}{2-2^{1/3}} \approx 1.35. \tag{2.146}$$

For a more detailed description of the split-step Fourier technique, the reader can also refer to Ref. [62]. The results of computations of the solutions to Eq. (2.132) are presented in Refs. [25, 26].

2.5 Two-parameter Method for Describing the Nonlinear Evolution of Narrow-Band Wave Trains

In this section, we consider the evolution of narrow-band wave trains of finite amplitude in a nonlinear dispersive system which is described by a (1 + 1) Klein–Gordon equation with arbitrary polynomial nonlinearity:

$$u_{tt} - c^2 u_{xx} + \sum_{p=1}^{P} \alpha_p u^p = 0. \tag{2.147}$$

Here, u is an unknown twice differentiable function of the wave process, $0 < t < \infty$ is time, $-\infty < x < \infty$ is coordinate, c and α_p are arbitrary real constants ($\alpha_1 \neq 0$), and P is an arbitrary positive integer. Let the initial condition at $t = 0$ have the form

$$u(x, 0) = Q(x)\big(\exp(\mathrm{i}kx) + \exp(-\mathrm{i}kx)\big),$$
$$u_t(x, 0) = P(x)\big(\exp(\mathrm{i}kx) + \exp(-\mathrm{i}kx)\big),$$

where k is the carrier wave number.

In the multi-scale method, the unknown function $u(x, t)$ of coordinate and time is looked for in the form of asymptotic expansion in powers of a small nonlinearity parameter ε:

$$u(x, t) = \sum_{n=1}^{\infty} \varepsilon^n u^{(n)}(x, t). \tag{2.148}$$

The wave motion is classified into slow one and fast one by introducing different time scales and different spatial scales:

$$T_n \equiv \mu^n t, \quad X_n \equiv \mu^n x.$$

The derivatives with respect to time and coordinate are expanded into the following series:

$$\frac{\partial}{\partial t} = \sum_{n=0}^{\infty} \mu^n \frac{\partial}{\partial T_n}, \qquad \frac{\partial}{\partial x} = \sum_{n=0}^{\infty} \mu^n \frac{\partial}{\partial X_n}, \tag{2.149}$$

the times T_n and coordinates X_n being assumed to be independent variables. A principal drawback of this method lies in the fact that the parameters ε and μ with different physical meanings (the former characterising the smallness of nonlinearity, and the latter describing the slowness of temporal and spatial variations) are tentatively taken equal: $\varepsilon = \mu$. This admission produces the so-called secular terms in the equations for $u^{(n)}(x, t)$. Such terms, which indefinitely grow with time, are eliminated in each new order of ε by an appropriate choice of free parameters emerging in the solutions of the linear inhomogeneous wave equations derived from the original nonlinear equations for the function $u(x, t)$. This procedure is very awkward, and it is difficult to formulate in algorithmic form.

From here on we follow Ref. [56] to introduce a perturbative technique that was first proposed in Ref. [52] to reduce the original wave equation to a model equation for the wave train envelope (high-order nonlinear Schrödinger equation). The time derivative is expanded into an asymptotic series in two independent parameters that characterise the smallness of amplitudes (ε) and the slowness of their spatial variations (μ). In contrast to the multi-scale method and other perturbative methods in which these two parameters are taken equal, the two-parameter method produces no secular terms.

We look for a solution to Eq. (2.147) in the form of truncated Fourier series with variable coefficients:

$$u(x, t) = \sum_{n=-N_u}^{N_u} u_n(x, t) \mathrm{e}^{\mathrm{i}n(\omega t - kx)}, \quad u_{-n} \equiv u_n^*, \tag{2.150}$$

where ω is the wave train carrier frequency, $N_u + 1$ is the number of harmonics taken into consideration, and * stands for complex conjugate. The same series can be written for all integer powers of the function u:

$$u^p(x, t) = \sum_{n=-pN_u}^{pN_u} (u^p)_n(x, t) \mathrm{e}^{\mathrm{i}n(\omega t - kx)}, \quad p = \overline{2, P}, \tag{2.151}$$

where $(u^p)_{-n} \equiv (u^p)_n^*$. The coefficients $(u^p)_n$ can be expressed recurrently in terms of the coefficients u_n:

$$(u^p)_n = \sum_{n_1=\max(-N_u,\, n-(p-1)N_u)}^{\min(N_u,\, n+(p-1)N_u)} u_{n_1} (u^{p-1})_{n-n_1}.$$

The corresponding expansions of the derivatives are

$$u_{tt}(x,\,t) = \sum_{n=-N_u}^{N_u} \big((u_n)_{tt} + 2\mathrm{i}n\omega(u_n)_t - n^2\omega^2 u_n\big)\mathrm{e}^{\mathrm{i}n(\omega t-kx)}, \tag{2.152}$$

$$u_{xx}(x,\,t) = \sum_{n=-N_u}^{N_u} \big((u_n)_{xx} - 2\mathrm{i}nk(u_n)_x - n^2k^2 u_n\big)\mathrm{e}^{\mathrm{i}n(\omega t-kx)}. \tag{2.153}$$

Substituting (2.150)–(2.153) in (2.147) and equating the coefficients at the like powers of the function $\exp(\mathrm{i}(\omega t - kx))$, we obtain a system of nonlinear differential equations for the coefficients $u_n(x,\,t)$ $(n = \overline{0,\,N_u})$:

$$(u_n)_{tt} - c^2(u_n)_{xx} + 2\mathrm{i}n\big(\omega(u_n)_t + c^2k(u_n)_x\big) + \\ (n^2c^2k^2 - n^2\omega^2 + \alpha_1)u_n + \sum_{p=2}^{P} \alpha_p(u^p)_n = 0. \tag{2.154}$$

Linearisation of these equations at $n = 1$ gives the dispersion relation in the linear approximation:

$$\omega^2 = \alpha_1 + c^2k^2. \tag{2.155}$$

Generally, the system of equations (2.154) is by no means more simple than original equation (2.147). It can be simplified if solutions are looked for in a class of functions with narrow spectrum, $|\Delta k| \ll k$ (*quasi-monochromaticity* condition). In this case, the problem has a formal small parameter $\mu \sim |\Delta k|/k$, and the coefficients $u_n(x,\,t)$ can be regarded as slow functions of x and t. Let us introduce a slow coordinate $\xi = \mu x$ and go over to the variables $u_n = u_n(\mu x,\,t)$.

When there are no resonances between higher harmonics, the amplitudes of Fourier coefficients decrease with increasing number (*quasi-harmonicity* condition):

$$u_n \sim \varepsilon^n A, \quad n \geqslant 1, \quad u_0 \sim \varepsilon^2 A, \quad \varepsilon < 1, \tag{2.156}$$

where $u_1 \equiv \varepsilon A$. The parameter ε can be chosen as the second formal parameter, which is independent of the dispersion parameter μ in the general case. The use of two independent formal parameters is a distinctive feature of this approach as compared to the multi-scale method and other perturbative methods, where these parameters are not distinguished ($\varepsilon = \mu$). When these incomparable parameters are set equal, a perturbative procedure produces nonphysical secular terms.

In contrast to perturbative methods which use the expansions of form (2.148) and (2.149) to reduces Eq. (2.154) to evolution equations of NLSE type, we start from the most general explicit form of such an evolution equation. To this end, the time derivative $(u_1)_t \equiv \varepsilon A_t$ should be expressed in terms of the derivatives $(u_1)_{nx} \equiv \varepsilon \mu^n A_{n\xi}$ with respect to coordinate (designation $A_{n\xi}$ means the n-th derivative with respect to ξ) and all possible combinations of nonlinear terms $\varepsilon^{2n+1}A^{(n+1)}(A^*)^n$. Hence, the derivative A_t can be written as the following asymptotic expansion in terms of parameters ε and μ:

$$A_t = \mathrm{i} \sum_{n_0=0}^{\infty} (\mathrm{i}\mu)^{n_0} \Big(a_{n_0} A_{n_0\xi} + \\ + \varepsilon^2 \sum_{n_1=0}^{n_0} \sum_{n_2=0}^{n_1} a_{n_0-n_1, n_1-n_2, n_2} A_{(n_0-n_1)\xi} A_{(n_1-n_2)\xi} A^*_{n_2\xi} + O(\varepsilon^4) \Big). \tag{2.157}$$

Equation (2.157) is the general form of the evolution equation for the complex first-harmonic amplitude A. The unknown coefficients $a_{n\_}$ can be determined from Eqs. (2.154). To this end, the amplitudes of all other harmonics (u_0, u_2, u_3, …) are expanded in terms of the amplitude of the first harmonic A in the same way as it is done in expansion (2.157):

$$u_0 = \varepsilon^2 \sum_{n_0=0}^{\infty} (\mathrm{i}\mu)^{n_0} \Big(\sum_{n_1=0}^{n_0} b^{(0)}_{n_0-n_1, n_1} A_{(n_0-n_1)\xi} A^*_{n_1\xi} \\ + \varepsilon^2 \sum_{n_1=0}^{n_0} \sum_{n_2=0}^{n_1} \sum_{n_3=0}^{n_2} b^{(0)}_{n_0-n_1, n_1-n_2, n_2-n_3, n_3} \\ \times A_{(n_0-n_1)\xi} A_{(n_1-n_2)\xi} A^*_{(n_2-n_3)\xi} A^*_{n_3\xi} + O(\varepsilon^4) \Big), \tag{2.158}$$

$$u_2 = \varepsilon^2 \sum_{n_0=0}^{\infty} (\mathrm{i}\mu)^{n_0} \Big(\sum_{n_1=0}^{n_0} b^{(2)}_{n_0-n_1, n_1} A_{(n_0-n_1)\xi} A_{n_1\xi} \\ + \varepsilon^2 \sum_{n_1=0}^{n_0} \sum_{n_2=0}^{n_1} \sum_{n_3=0}^{n_2} b^{(2)}_{n_0-n_1, n_1-n_2, n_2-n_3, n_3} \\ \times A_{(n_0-n_1)\xi} A_{(n_1-n_2)\xi} A_{(n_2-n_3)\xi} A^*_{n_3\xi} + O(\varepsilon^4) \Big), \tag{2.159}$$

$$u_3 = \varepsilon^3 \sum_{n_0=0}^{\infty} (\mathrm{i}\mu)^{n_0} \Big(\sum_{n_1=0}^{n_0} \sum_{n_2=0}^{n_1} b^{(3)}_{n_0-n_1, n_1-n_2, n_2} A_{(n_0-n_1)\xi} A_{(n_1-n_2)\xi} A_{n_2\xi} \\ + \varepsilon^2 \sum_{n_1=0}^{n_0} \sum_{n_2=0}^{n_1} \sum_{n_3=0}^{n_2} \sum_{n_4=0}^{n_3} b^{(3)}_{n_0-n_1, n_1-n_2, n_2-n_3\, n_3-n_4, n_4} \\ \times A_{(n_0-n_1)\xi} A_{(n_1-n_2)\xi} A_{(n_2-n_3)\xi} A_{(n_3-n_4)\xi} A^*_{n_4\xi} + O(\varepsilon^4) \Big), \quad \ldots . \tag{2.160}$$

The unknown coefficients $b^{(n)}_{n\_}$ are found along with the coefficients $a_{n\_}$ from the system of equations (2.154) by substituting expansions (2.157)–(2.160) and equating to zero the coefficients at the like powers of the products $\varepsilon^k \mu^m$ in different combinations $(A\_ \ldots A^*\_ \ldots)$. In its essence, this procedure is similar to the method of undetermined coefficients. The coefficient calculation order and the general form of the expansions for A_t and u_n in arbitrary order of ε are given in Ref. [56]. The use of two parameters in anzats (2.157) is of key importance for the coefficient calculation procedure, since the expansions could not be split into linear-independent terms at $\varepsilon = \mu$.

It should be noted that the reduction of Eq. (2.147) with the second time derivative to Eq. (2.157) with the first time derivative puts a constraint on the initial condition for u_t. In this case, $u_t(x, 0)$ is a function of $u(x, 0)$ defined by formula (2.157).

The two-parameter expansions were programmed in symbolic form for an arbitrary order of μ and ε. The evolution equation for the complex amplitude of the first harmonic is

$$\begin{aligned}
A_t = \mathrm{i}\Big(&(\mathrm{i}\mu)a_1A_\xi + (\mathrm{i}\mu)^2a_2A_{\xi\xi} + (\mathrm{i}\mu)^3a_3A_{\xi\xi\xi} + (\mathrm{i}\mu)^4a_4A_{\xi\xi\xi\xi} + O(\mu^5)\\
&+ \varepsilon^2\Big[a_{0,0,0}A|A|^2 + (\mathrm{i}\mu)\ \left(a_{1,0,0}A_\xi|A|^2 + a_{0,0,1}A^2A_\xi^*\right)\\
&+ (\mathrm{i}\mu)^2\left(a_{2,0,0}A_{\xi\xi}|A|^2 + a_{1,1,0}A_\xi^2A^* + a_{1,0,1}|A_\xi|^2A + a_{0,0,2}A^2A_{\xi\xi}^*\right) + O(\mu^3)\Big]\\
&+ \varepsilon^4\Big[a_{0,0,0,0,0}A|A|^4 + (\mathrm{i}\mu)\ \left(a_{1,0,0,0,0}A_\xi|A|^4 + a_{0,0,0,1,0}A^2|A|^2A_\xi^*\right)\\
&+ (\mathrm{i}\mu)^2\left(a_{2,0,0,0,0}A_{\xi\xi}|A|^4 + a_{1,1,0,0,0}A_\xi^2|A|^2A^* + a_{1,0,0,1,0}|A_\xi|^2A|A|^2\right.\\
&\left.+ a_{0,0,0,2,0}A^2|A|^2A_{\xi\xi}^* + a_{0,0,0,1,1}A^3A_\xi^2\right) + O(\mu^3)\Big] + O(\varepsilon^6)\Big).
\end{aligned} \tag{2.161}$$

In each term of this equation, the power of the formal parameter μ points to the overall order of the derivatives with respect to ξ, and the power of the formal parameter ε points to the nonlinearity order. These parameters disappear after going back to the original variables $u_1 = \varepsilon A$ and $x = \xi/\mu$. Taking into account dispersion relation (2.155), the coefficients $a\_$ can be written as (at $P = 5$)

$$a_0 = 0, \quad a_1 = \frac{c^2k}{\omega}, \quad a_2 = \frac{c^2\alpha_1}{2\omega^3}, \quad a_3 = -\frac{c^4k\alpha_1}{2\omega^5},$$

$$a_4 = \frac{\alpha_1c^4(4c^2k^2 - \alpha_1)}{8\omega^7}, \quad a_n = \frac{1}{n!}\frac{\mathrm{d}^n\omega}{\mathrm{d}k^n};$$

$$a_{0,0,0} = \frac{3\alpha_3}{2\omega} - \frac{5\alpha_2^2}{3\omega\alpha_1}, \quad a_{1,0,0} = 2a_{0,0,1} = \frac{c^2k}{\omega^3}\left(\frac{10\alpha_2^2}{3\alpha_1} - 3\alpha_3\right);$$

$$a_{2,0,0} = \frac{c^2}{18\alpha_1\omega^5}\Big(2c^2k^2\left(27\alpha_1\alpha_3 - 14\alpha_2^2\right) - \alpha_1\left(27\alpha_1\alpha_3 - 62\alpha_2^2\right)\Big),$$

$$a_{1,1,0} = \frac{c^2}{36\alpha_1\omega^5}\Big(4c^2k^2\left(27\alpha_1\alpha_3 - 28\alpha_2^2\right) - \alpha_1\left(27\alpha_1\alpha_3 - 38\alpha_2^2\right)\Big),$$

$$a_{1,0,1} = \frac{c^2}{6\alpha_1\omega^5}\Big(4c^2k^2\big(9\alpha_1\alpha_3 - 4\alpha_2^2\big) - \alpha_1\big(9\alpha_1\alpha_3 - 34\alpha_2^2\big)\Big),$$

$$a_{0,0,2} = \frac{c^2}{6\alpha_1\omega^5}\Big(c^2k^2\big(9\alpha_1\alpha_3 + 2\alpha_2^2\big) + 12\alpha_1\alpha_2^2\Big);$$

$$a_{0,0,0,0,0} = \frac{1}{\omega}\Big(-\frac{335\alpha_2^4}{108\alpha_1^3} - \frac{25\alpha_2^4}{18\omega^2\alpha_1^2} + \frac{143\alpha_2^2\alpha_3}{12\alpha_1^2} + \frac{5\alpha_2^2\alpha_3}{2\omega^2\alpha_1} - \frac{9\alpha_3^2}{8\omega^2} + \frac{3\alpha_3^2}{16\alpha_1} - \frac{14\alpha_2\alpha_4}{\alpha_1} + 5\alpha_5\Big),$$

$$a_{1,0,0,0,0} = \frac{c^2k}{\omega^3}\Big(\frac{925\alpha_2^4}{108\alpha_1^3} + \frac{275\alpha_2^4}{18\omega^2\alpha_1^2} - \frac{421\alpha_2^2\alpha_3}{12\alpha_1^2} - \frac{55\alpha_2^2\alpha_3}{2\omega^2\alpha_1} + \frac{99\alpha_3^2}{8\omega^2} - \frac{9\alpha_3^2}{16\alpha_1} + \frac{42\alpha_2\alpha_4}{\alpha_1} - 15\alpha_5\Big),$$

$$a_{0,0,0,1,0} = \frac{c^2k}{\omega^3}\Big(\frac{295\alpha_2^4}{54\alpha_1^3} + \frac{100\alpha_2^4}{9\omega^2\alpha_1^2} - \frac{139\alpha_2^2\alpha_3}{6\alpha_1^2} - \frac{20\alpha_2^2\alpha_3}{\omega^2\alpha_1} + \frac{9\alpha_3^2}{\omega^2} - \frac{3\alpha_3^2}{8\alpha_1} + \frac{28\alpha_2\alpha_4}{\alpha_1} - 10\alpha_5\Big).$$

The expressions for the subsequent coefficients $a\_$ are too cumbersome to be presented in explicit form. The expressions for the complex amplitudes of other harmonics are given in Ref. [56].

Let us illustrate the evolution of a wave train envelope described by the equation of form (2.157). To this end, we rewrite original equation (2.147) in terms of dimensionless variables $\widetilde{x} \equiv kx$ and $\widetilde{t} \equiv ckt$:

$$\widetilde{u}_{\widetilde{t}\widetilde{t}} - \widetilde{u}_{\widetilde{x}\widetilde{x}} + \sum_{p=1}^{P} \widetilde{\alpha}_p \widetilde{u}^p = 0, \quad \widetilde{u} = \frac{u}{u_A}, \quad \widetilde{\alpha}_p = \frac{\alpha_p\, u_A^{p-1}}{(ck)^2}. \tag{2.162}$$

In this case, we have $\widetilde{c} = 1$, $\widetilde{k} = 1$, $\widetilde{\omega}^2 = \widetilde{\alpha}_1 + 1$, and u_A is a typical amplitude of the function u. As an example, let us consider the case $P = 3$ with $\widetilde{\alpha}_1 = 1$, $\widetilde{\alpha}_2 = 0$, and $\widetilde{\alpha}_3 = -1/6$. The values of these parameters correspond to the first two terms in the Taylor expansion of the function $\sin u$. Hereafter, the tildes over the dimensionless variables are omitted. The corresponding coefficients of evolution equation (2.161) are

$$a_1 = \frac{1}{\sqrt{2}}, \quad a_2 = \frac{1}{4\sqrt{2}}, \quad a_3 = -\frac{1}{8\sqrt{2}}, \quad a_4 = \frac{3}{64\sqrt{2}},$$

$$a_{0,0,0} = -\frac{1}{4\sqrt{2}}, \quad a_{1,0,0} = \frac{1}{4\sqrt{2}}, \quad a_{0,0,1} = \frac{1}{8\sqrt{2}},$$

$$a_{2,0,0} = a_{0,0,2} = -\frac{1}{16\sqrt{2}}, \quad a_{1,1,0} = a_{1,0,1} = -\frac{3}{16\sqrt{2}},$$

$$a_{0,0,0,0,0} = -\frac{1}{96\sqrt{2}}, \quad \dots . \tag{2.163}$$

Initially, we retain only those terms in Eq. (2.161) whose overall order of smallness with respect to the parameters ε and μ is no more than two. In this case we obtain a classical NLSE:

$$(u_1)_t = -a_1(u_1)_x - \mathrm{i}a_2(u_1)_{xx} + \mathrm{i}a_{0,0,0}\, u_1|u_1|^2. \tag{2.164}$$

It has an exact one-soliton solution at $a_2\, a_{0,0,0} < 0$:

$$u_1(x,\ t) = U_0\, \mathrm{sech}\big(K(x - x_0 - Vt)\big)\ \mathrm{e}^{\mathrm{i}\kappa x - \mathrm{i}\Omega t}. \tag{2.165}$$

Here, U_0, κ, Ω, and V are the soliton's complex amplitude, wave number, frequency and speed, respectively; x_0 is the soliton's arbitrary initial position, which is usually set equal to zero. By selecting $x_0 = \pm\mathrm{i}\frac{\pi}{2}$, the sech profile can be transformed to csch. Solutions of form (2.165) are called *bright solitons*. They were first derived in Ref. [102]. The following relationships between the soliton parameters can be established in this case:

$$K = |U_0|\sqrt{-\frac{a_{0,0,0}}{2a_2}}, \quad \Omega = \kappa a_1 + \left(K^2 - \kappa^2\right) a_2, \quad V = a_1 - 2\kappa a_2. \tag{2.166}$$

The parameters U_0 and κ are free parameters of the problem. The corresponding approximate solution of Eq. (2.162) is

$$u(x,\ t) = u_1(x,\ t)\exp\big(\mathrm{i}(\sqrt{2}\,t - x)\big) + u_1^*(x,\ t)\exp\big(-\mathrm{i}(\sqrt{2}\,t - x)\big). \tag{2.167}$$

To analyse the effect of high-order dispersive terms on the shape of one-soliton solution (2.165), we consider HONLSE

$$\begin{aligned}(u_1)_t =\ &-\ a_1(u_1)_x - \mathrm{i}a_2(u_1)_{xx} + a_3(u_1)_{xxx} + \mathrm{i}a_4(u_1)_{xxxx} + \mathrm{i}a_{0,0,0}\, u_1|u_1|^2 \\ &- a_{1,0,0}(u_1)_x|u_1|^2 - a_{0,0,1}\, u_1^2(u_1^*)_x - \mathrm{i}a_{2,0,0}(u_1)_{xx}|u_1|^2 - \mathrm{i}a_{1,1,0}(u_1)_x^2\, u_1^* \\ &- \mathrm{i}a_{1,0,1}|(u_1)_x|^2 u_1 - \mathrm{i}a_{0,0,2}\, u_1^2(u_1^*)_{xx} + \mathrm{i}a_{0,0,0,0,0}\, u_1|u_1|^4\end{aligned} \tag{2.168}$$

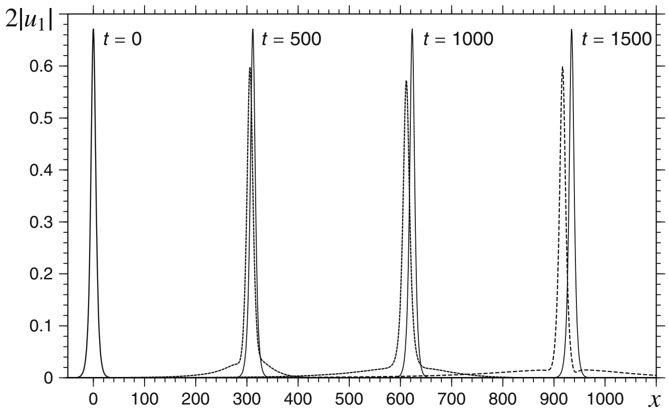

Fig. 2.5 Evolution of the wave train envelope which at the initial moment $t = 0$ is given by function (2.165). (*Solid curve*) exact one-soliton NLSE solution (2.164), (*dashed curve*) numerical solution of HONLSE (2.168)

with the coefficients defined by formulas (2.163). The initial condition is chosen in the form of function (2.165) with

$$U_0 = \frac{1}{10}\sqrt{\frac{2}{|a_{0,0,0}|}}, \quad \kappa = \frac{1}{10\sqrt{|a_2|}},$$

and $x_0 = 0$. Figure 2.5 shows the evolution of such an envelope. To solve Eq. (2.168) numerically, we used the split-step Fourier method described in the previous section. High-order dispersive terms are seen to affect the amplitude, shape and velocity of the soliton solution. This solution shows a typical quasi-soliton behaviour discussed in more detail in Ref. [26].

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