Chapter 2
Vibrations of a Three-Dimensional Body, Plate and Ring

2.1 Equations of Three-Dimensional Body Vibrations

Let us use the Cartesian system of rectilinear orthogonal coordinates $x = (x_i), i = 1, 2, 3$. The equations of three-dimensional body vibrations in the curvilinear coordinate system are given in the Appendix.

The stress state at the point of a solid body is characterized by the tensor

$$\underline{\underline{\tau}} = \tau^{ij}, \quad \underline{\underline{\tau}} = \underline{\underline{\tau}}(x, t), \tag{2.1}$$

where $i, j = 1, 2, 3$.

Displacements at any point of the body are determined by the vector

$$\underline{u} = u_i, \quad \underline{u} = \underline{u}(x, t). \tag{2.2}$$

We denote the vector of body forces by $\underline{q} = q^i, \underline{q} = \underline{q}(x, t)$. They consist of given forces $\underline{p} = p^i, \underline{p} = \underline{p}(x, t)$ and inertial forces $(-\rho\ddot{\underline{u}})$, where $\rho = \rho(x)$ is the specific density of a three-dimensional body. Hence, the dynamic equilibrium equations in the vector-tensor form can be written as [2]

$$-\nabla \cdot \underline{\underline{\tau}} - \underline{q} = 0, \quad \underline{q} = \underline{p} - \rho\ddot{\underline{u}}, \tag{2.3}$$

where $\nabla_i = \frac{\partial}{\partial x_i}$.

Deformations through displacements can be expressed using the operation of differentiation

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla\underline{u} + (\nabla\underline{u})^*), \tag{2.4}$$

where the asterisk denotes the conjugation operation.

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In turn, the total strain tensor $\underline{\underline{\varepsilon}}$ can be represented as a sum of the tensors of the elastic strain and the strain from external causes, which are denoted by $\underline{\underline{\phi}}$ and $\underline{\underline{e}}$, respectively. That is

$$\underline{\underline{\varepsilon}} = \underline{\underline{\phi}} + \underline{\underline{e}}. \tag{2.5}$$

The strain $\underline{\underline{e}}$ can be caused, for example, by the piezo effect, electromagnetic action, variable body heating and others.

An elastic body material will be assumed to be isotropic. We partition the stress tensor $\underline{\underline{\tau}}$ into the spherical and deviatoric parts, which are denoted, respectively, by $\underline{\underline{\sigma}}$ and $\underline{\underline{S}}$

$$\underline{\underline{\tau}} = \underline{\underline{\sigma}} + \underline{\underline{S}}. \tag{2.6}$$

The tensor $\underline{\underline{\sigma}}$ is diagonal

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}, \tag{2.7}$$

where $\sigma = \frac{\tau^{11} + \tau^{22} + \tau^{33}}{3}$ is the mean tensile (compressive) stress.

In which

$$\underline{\underline{S}} = \underline{\underline{\tau}} - \underline{\underline{\sigma}}$$

or in more detail

$$S = \begin{pmatrix} \tau^{11} - \sigma & \tau^{12} & \tau^{13} \\ \tau^{21} & \tau^{22} - \sigma & \tau^{23} \\ \tau^{31} & \tau^{32} & \tau^{33} - \sigma \end{pmatrix}. \tag{2.8}$$

Using formulas (2.7) and (2.8), it is easy to prove that the convolution of the tensors

$$\underline{\underline{\sigma}} \cdot \cdot \underline{\underline{S}} = 0 \tag{2.9}$$

(the convolution of tensors is understood as the sum of the products of their respective components).

We also divide the elastic strain tensor into the spherical and deviatoric parts

$$\underline{\underline{\phi}} = \underline{\underline{\Theta}} + \underline{\underline{\Gamma}}. \tag{2.10}$$

The spherical part of the strain tensor characterizes change of the volume, while the deviatoric part characterizes change of the shape at a given point of an elastic body. According to the law of elasticity

$$\underline{\underline{\Theta}} = k\underline{\underline{\sigma}} \quad \text{and} \quad \underline{\underline{\Gamma}} = \frac{1}{2G}\underline{\underline{S}}, \tag{2.11}$$

where k is the coefficient of volumetric expansion of the material of an elastic body, $k = \frac{1-2\nu}{E}$, E is the tensile (or compressive) modulus of elasticity (Young's modulus); ν is the transverse compression ratio (Poisson's ratio); and G is the shearing modulus of elasticity.

Hence, the total elastic strain will be

$$\underline{\underline{\phi}} = k\underline{\underline{\sigma}} + \frac{1}{2G}\underline{\underline{S}}. \tag{2.12}$$

In view of formulas (2.3)–(2.5) and (2.12) the equations of three-dimensional body vibrations in the vector-tensor form will be recorded as follows

$$\begin{aligned} &-\mathrm{div}\underline{\underline{\tau}} + \rho\ddot{\underline{u}} - \underline{p} = 0, \\ &\mathrm{def}\underline{u} - \left(k\underline{\underline{\sigma}} + \tfrac{1}{2G}\underline{\underline{S}}\right) - \underline{\underline{e}} = 0, \end{aligned} \tag{2.13}$$

where

$$\mathrm{div}\underline{\underline{\tau}} = \nabla \cdot \underline{\underline{\tau}}, \quad \mathrm{def}\underline{u} = \frac{1}{2}(\nabla\underline{u} + (\nabla\underline{u})^*).$$

The objective of this section is to represent the equations of vibrations of a three-dimensional body in the operator form. This can be accomplished by introducing the following designations

$$\begin{aligned} &\xi = \underline{\underline{\tau}}, \quad \eta = \underline{u}, \quad f = p, \quad p = \underline{p}, \quad e = \underline{\underline{e}}, \\ &D\xi = -\mathrm{div}\ \underline{\underline{\tau}}, \quad D^*\eta = \mathrm{def}\underline{u}, \\ &R\ddot{\eta} = \rho\ddot{\underline{u}}, \quad B\xi = k\underline{\underline{\sigma}} + \frac{1}{2G}\underline{\underline{S}}. \end{aligned} \tag{2.14}$$

Hence, Eqs. (2.13) will take the form:

$$\begin{aligned} &D\xi + R\ddot{\eta} - f = 0, \\ &D^*\eta - B\xi - e = 0. \end{aligned} \tag{2.15}$$

The operator equations (2.15) are valid in the whole volume Ω, occupied by the elastic body, excluding its surface Γ.

Let us discover the properties of the operators introduced in the region Ω. At first

$$\begin{aligned}&\int\limits_{\Omega} R\eta_1\eta_2 d\Omega = \int\limits_{\Omega} (\rho\underline{u}_1)\cdot\underline{u}_2 d\Omega = \int\limits_{\Omega}\underline{u}_1\cdot(\rho\underline{u}_2)d\Omega = \int\limits_{\Omega}\eta_1 R\eta_2 d\Omega,\\&\int\limits_{\Omega} R\eta\,\eta d\Omega = \int\limits_{\Omega}\rho\underline{u}\cdot\underline{u}d\Omega \geq \rho_{\min}\int\limits_{\Omega}\underline{u}\cdot\underline{u}d\Omega = \rho_{\min}\int\limits_{\Omega}\eta\,\eta d\Omega > 0,\end{aligned} \tag{2.16}$$

if $\underline{u} = \eta \neq 0$ in the domain Ω.

According to formulas (2.16) the operator of inertia R is self-adjoint (self-conjugate) and positive definite. Note that the first property follows from the second and does not require a separate proof [4], but is simply taken into account.

Let us now discover the properties of the operator B. In view of formulas (2.6), (2.9) and notations (2.14) we have

$$\begin{aligned}\int\limits_{\Omega} B\xi\,\xi\, d\Omega &= \int\limits_{\Omega}\left(k\underline{\underline{\sigma}} + \frac{1}{2G}\underline{\underline{S}}\right)\cdot\cdot(\underline{\underline{\sigma}} + \underline{\underline{S}})d\Omega\\&= \int\limits_{0}^{l}\left(k\underline{\underline{\sigma}}\cdot\cdot\underline{\underline{\sigma}} + \frac{1}{2G}\underline{\underline{S}}\cdot\cdot\underline{\underline{S}}\right)d\Omega \geq k_{\min}\int\limits_{\Omega}(\underline{\underline{\sigma}}\cdot\cdot\underline{\underline{\sigma}} + \underline{\underline{S}}\cdot\cdot\underline{\underline{S}})d\Omega\\&= k_{\min}\int\limits_{\Omega}\underline{\underline{\tau}}\cdot\cdot\underline{\underline{\tau}}\, d\Omega = k_{\min}\int\limits_{\Omega}\xi\,\xi\, d\Omega > 0.\end{aligned} \tag{2.17}$$

Here, the convolution of tensors with themselves is equal to the sum of squares of their elements (i.e. is a positive value). This makes it possible to take the factor $k_{\min}$ out of the integral, since $G = \frac{E}{2(1+\nu)}$ and $k < \frac{1}{2G}$. Formula (2.17) shows that the operator B is positive definite and, hence, self-conjugate (self-adjoint).

Let us proceed to consideration of the properties of the operators D and D^*. We denote the unit vector of the normal to the surface of an elastic body Γ by $\underline{n}$. We also denote distributed forces by $\underline{X} = \underline{n}\cdot\underline{\underline{\tau}}$ and displacement on the surface Γ by $\underline{Y} = \underline{u}$.

Let us write the Clapeyron formula [2]

$$-\int\limits_{\Omega}\mathrm{div}\underline{\underline{\tau}}\cdot\underline{u}d\Omega + \int\limits_{\Gamma}\underline{X}\cdot\underline{Y}d\Gamma = \int\limits_{\Omega}\underline{\underline{\tau}}\cdot\cdot\mathrm{def}\underline{u}d\Omega. \tag{2.18}$$

Using notations (2.14), equality (2.18) can be rewritten as

$$\int_{\Omega} D\xi\,\eta\,\mathrm{d}\Omega = -\int_{\Gamma} XY\mathrm{d}\Gamma + \int_{\Omega} \xi\,D^{*}\eta\,\mathrm{d}\Omega, \tag{2.19}$$

where $X = \underline{X}$ and $Y = \underline{Y}$.

This formula means that the operators D and D^{*} are conjugate in the sense of Lagrange. In the particular case of homogeneous boundary conditions, where there are distributed forces $X = \underline{X} \equiv 0$ on the portion of the surface Γ_1 and displacements $Y = \underline{Y} \equiv 0, \Gamma_1 + \Gamma_2 = \Gamma,$ on the surface portion Γ_2, equality (2.19) can be extended into

$$\int_{\Omega} D\xi\,\eta\mathrm{d}\Omega = \int_{\Omega} \xi\,D^{*}\eta\,\mathrm{d}\Omega,$$

that is, the operators D and D^{*} become merely conjugate.

From the system of equations (2.15) it is not hard to proceed to the single equation with respect to displacement η. Really, from the second equations (2.15) it follows that $\xi = B^{-1}(D^{*}\eta - e)$, and substitution of this expression into the first equations (2.15) gives

$$N\eta + R\ddot{\eta} - f = 0, \tag{2.20}$$

where

$$N = DB^{-1}D^{*}\ \text{and}\ f = p + DB^{-1}e.$$

The appearance and properties of their operators show that Eqs. (2.15) and (2.20) are analogous to the equations of oscillations in the simplest case of a straight rod (1.20) and (1.74).

Introducing oscillation equations into operator equations of the single form is particularly advantageous in that similar algorithms can be applied to solve various problems as long as these algorithms can also be written in the operator form.

2.2 Equations of Plate Vibrations

Let us give another example of drawing up a system of operator equations of the type (2.15). Let us consider the transverse vibrations of a thin rectangular plate. We use the Cartesian system of coordinates $x = x_i, i = 1, 2,$ which are counted along the perpendicular axes located in the middle surface of a plate. Much more complicated oscillation equations of a shell in a curvilinear coordinate system are considered in the Appendix.

We denote the vector of shearing forces by $\underline{Q} = \underline{Q}(x,t), \underline{Q} = Q^i$ and the tensor by $\underline{\underline{M}} = \underline{\underline{M}}(x,t), \underline{\underline{M}} = M^{ij}$. The tensor's elements at $i = j$ are bending moments, while at $i \neq j$ they are torques (t is time).

Further, let $p = p(x,t)$ be the distributed transverse load on the plate, $m = m(x,t)$ the distributed external moment, $\rho = \rho(x)$ the mass per unit area of the plate and $j = j(x)$ the distributed moment of inertia per unit area of the plate.

The vector-tensor equations of equilibrium of the plate considering inertial forces can be given the form [1, 3]

$$\begin{aligned} -\nabla \cdot \underline{Q} &= q\,, \quad q = -\rho\ddot{u} + p, \\ \underline{Q} - \nabla \cdot \underline{\underline{M}} &= \underline{\mu}, \quad \underline{\mu} = -j\ddot{\underline{\vartheta}} + \underline{m}, \end{aligned} \tag{2.21}$$

where $u = u(x,t)$ is displacement perpendicular to the middle surface of the plate; $\underline{\vartheta} = \underline{\vartheta}(x,t), \underline{\vartheta} = \vartheta_i$ is the vector of angles of inclination of the middle surface; and $\nabla = \nabla_i$ is the operator of differentiation along the coordinates $x_i, i = 1, 2$.

The connection between displacements and deformations of shearing and bending is as follows

$$\begin{gathered} \nabla u + \underline{\vartheta} = \underline{\gamma}, \\ \tfrac{1}{2}\left(\nabla\underline{\vartheta} + (\nabla\underline{\vartheta})^*\right) = \underline{\underline{\kappa}}. \end{gathered} \tag{2.22}$$

The total strains of shearing and bending are composed of elastic strains and the strains from external causes

$$\begin{aligned} \underline{\gamma} &= \underline{\varphi} + \underline{g}, \\ \underline{\underline{\kappa}} &= \underline{\underline{\phi}} + \underline{\underline{k}}. \end{aligned} \tag{2.23}$$

The material of the plate will be assumed to be isotropic. We partition the tensor $\underline{\underline{M}}$ into the spherical and deviatoric parts $\underline{\underline{s}}$ and $\underline{\underline{S}}$

$$\underline{\underline{M}} = \underline{\underline{s}} + \underline{\underline{S}}, \tag{2.24}$$

where

$$\underline{\underline{s}} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}, \quad s = \frac{M^{11} + M^{22}}{2}, \quad \underline{\underline{S}} = \underline{\underline{M}} - \underline{\underline{s}}.$$

The law of elasticity can be written as

$$\underline{\varphi} = b\underline{Q}, \quad \underline{\underline{\phi}} = \alpha\underline{\underline{s}} + \beta\underline{\underline{S}}, \tag{2.25}$$

where [3]

$$b = \frac{12}{5}\frac{(1+\nu)}{Eh}, \quad \alpha = \frac{12(1-\nu)}{Eh^3},$$

$$\beta = \frac{6}{Gh^3}, \quad h = h(x)\,\text{is plate thickness}$$

Excluding $q, \underline{\mu}, \gamma$ and $\underline{\underline{\kappa}}$ from Eqs. (2.21), (2.22) and using formulas (2.23), (2.25), we obtain a system of equations describing the bending-shearing vibrations of a thin plate in the vector-tensor form

$$\begin{gathered} -\nabla\cdot\underline{Q} + \rho\ddot{u} - p = 0, \\ \underline{Q} - \nabla\cdot\underline{\underline{M}} + j\ddot{\underline{\vartheta}} - \underline{m} = 0, \end{gathered} \tag{2.26}$$

$$\begin{gathered} \nabla u + \underline{\vartheta} - b\underline{Q} - \underline{g} = 0, \\ \tfrac{1}{2}(\nabla\underline{\vartheta} + (\nabla\underline{\vartheta})^*) - (\alpha\underline{\underline{s}} + \beta\underline{\underline{S}}) - \underline{\underline{k}} = 0. \end{gathered} \tag{2.27}$$

Equations (2.26) and (2.27) are valid in the region Ω, which coincides with the area of the middle surface of the plate but does not include its boundary Γ. Note that these equations have a certain analogy with the equations of the bending-shearing vibrations of a rod (1.46).

Let us introduce the notation

$$\xi = \begin{pmatrix} \underline{Q} \\ \underline{\underline{M}} \end{pmatrix}, \quad \eta = \begin{pmatrix} u \\ \underline{\vartheta} \end{pmatrix}, \quad f = \begin{pmatrix} p \\ \underline{m} \end{pmatrix}, \quad e = \begin{pmatrix} \underline{g} \\ \underline{\underline{k}} \end{pmatrix}, \tag{2.28}$$

$$D\xi = \begin{pmatrix} -\nabla\cdot\underline{Q} \\ \underline{Q} - \nabla\cdot\underline{\underline{M}} \end{pmatrix}, \quad D^*\eta = \begin{pmatrix} \nabla u + \underline{\vartheta} \\ \frac{1}{2}(\nabla\underline{\vartheta} + (\nabla\underline{\vartheta})^*) \end{pmatrix}, \tag{2.29}$$

$$R\ddot{\eta} = \begin{pmatrix} \rho\ddot{u} \\ j\ddot{\underline{\vartheta}} \end{pmatrix}, \quad B\xi = \begin{pmatrix} b\underline{Q} \\ \alpha\underline{\underline{s}} + \beta\underline{\underline{S}} \end{pmatrix} \tag{2.30}$$

In this notation the tensor-vector equations (2.26), (2.27) take the form of operator equations of the type (2.15).

Let us define the properties of the operators entered. To do this, we draw up expressions for the scalar products of $D\xi$ and η, as well as of ξ and $D^*\eta$

$$\begin{aligned} \int_\Omega D\xi\eta\,d\Omega &= \int_\Omega \Big((-\nabla\cdot\underline{Q})u + (\underline{Q} - \nabla\cdot\underline{\underline{M}})\cdot\underline{\vartheta}\Big)d\Omega, \\ \int_\Omega \xi D^*\eta\,d\Omega &= \int_\Omega \Big(\underline{Q}\cdot(\nabla u + \underline{\vartheta}) + \underline{\underline{M}}\cdot\cdot\frac{1}{2}(\nabla\underline{\vartheta} + (\nabla\underline{\vartheta})^*)\Big)d\Omega. \end{aligned} \tag{2.31}$$

At the same time, by virtue of the formula for integration by parts

$$\int_\Omega (-\nabla \cdot \underline{Q}) u \mathrm{d}\Omega = -\int_\Gamma (\underline{n} \cdot \underline{Q}) u \mathrm{d}\Gamma + \int_\Omega \underline{Q} \cdot (\nabla u) \mathrm{d}\Omega$$

and the Clapeyron formula

$$\int_\Omega (-\nabla \cdot \underline{\underline{M}}) \cdot \underline{\vartheta} \mathrm{d}\Omega = \int_\Gamma (-\underline{n} \cdot \underline{\underline{M}}) \cdot \underline{\vartheta} \mathrm{d}\Gamma + \int_\Omega \underline{\underline{M}} \cdot \cdot \frac{1}{2} (\nabla \underline{\vartheta} + (\nabla \underline{\vartheta})^*) \mathrm{d}\Omega$$

we can write

$$\int_\Omega \Big((-\nabla \cdot \underline{Q}) u + (\underline{Q} - \nabla \cdot \underline{\underline{M}}) \cdot \underline{\vartheta} \Big) \mathrm{d}\Omega = -\int_\Gamma \Big((\underline{n} \cdot \underline{Q}) u + (\underline{n} \cdot \underline{\underline{M}}) \cdot \underline{\vartheta} \Big) \mathrm{d}\Gamma$$
$$+ \int_\Omega \left(\underline{Q} \cdot (\nabla u + \underline{\vartheta}) + \underline{\underline{M}} \cdot \cdot \frac{1}{2} (\nabla \underline{\vartheta} + (\nabla \underline{\vartheta})^*) \right) \mathrm{d}\Omega. \tag{2.32}$$

Then we have

$$\int_\Gamma (\underline{n} \cdot \underline{\underline{M}}) \cdot \underline{\vartheta} \mathrm{d}\Gamma = \int_\Gamma \left(M\vartheta - H \frac{du}{ds} \right) \mathrm{d}\Gamma$$
$$= \int_\Gamma \left(M\vartheta + \frac{d\mathrm{d}H}{\mathrm{d}s} u \right) \mathrm{d}\Gamma, \tag{2.33}$$

when the coordinate s counted along the contour of the plate has been introduced, it can also be accepted that

$$\underline{\vartheta} = \begin{pmatrix} \vartheta \\ -\frac{\mathrm{d}u}{\mathrm{d}s} \end{pmatrix} \tag{2.34}$$

and the bending and torsional moments on the contour of the plate can be denoted by M and H, as well as by ϑ and $\left(-\frac{\mathrm{d}u}{\mathrm{d}s} \right)$, the inclination angles along the normal and tangentially to the contour of the plate, correspondingly. In addition, we have used the formula for integration by parts

$$\int_\Gamma H \frac{\mathrm{d}u}{\mathrm{d}s} \mathrm{d}\Gamma = -\int_\Gamma \frac{\mathrm{d}H}{\mathrm{d}s} u \mathrm{d}\Gamma,$$

where the summand outside the integral vanishes because the initial and final points of coordinate s coincide.

Further

$$Q = \underline{n} \cdot \underline{Q}$$

which is the shearing force on the plate contour.

Combining this formula and formulas (2.31)–(2.34), we obtain

$$\int\limits_{\Omega} D\xi\,\eta \mathrm{d}\Omega = -\int\limits_{\Gamma} XY \mathrm{d}\Gamma + \int\limits_{\Omega} \xi\, D^{*}\eta\, \mathrm{d}\Omega,$$

where $X = \underline{X} = \begin{pmatrix} \tilde{Q} \\ M \end{pmatrix}$; $Y = \underline{Y} = \begin{pmatrix} u \\ \vartheta \end{pmatrix}$; and $\tilde{Q} = Q + \frac{\mathrm{d}H}{\mathrm{d}s}$

That is, the operators D and D^{*} introduced are conjugate in the sense of Lagrange and, if the product on the plate contour $\underline{X} \cdot \underline{Y} = XY = 0$, then these operators are simply conjugate.

Finally

$$\begin{aligned}
\int\limits_{\Omega} B\xi\xi \mathrm{d}\Omega &= \int\limits_{\Omega} \Big(b\underline{Q} \cdot \underline{Q} + (\alpha\underline{\underline{s}} + \beta\underline{\underline{S}}) \cdot\cdot (\underline{\underline{s}} + \underline{\underline{S}})\Big) \mathrm{d}\Omega \\
&= \int\limits_{\Omega} (b\underline{Q} \cdot \underline{Q} + \alpha\underline{\underline{s}} \cdot\cdot \underline{\underline{s}} + \beta\underline{\underline{S}} \cdot\cdot \underline{\underline{S}}) \mathrm{d}\Omega \\
&\geq c \int\limits_{\Omega} (\underline{Q} \cdot \underline{Q} + \underline{\underline{s}} \cdot\cdot \underline{\underline{s}} + \underline{\underline{S}} \cdot\cdot \underline{\underline{S}}) \mathrm{d}\Omega = c \int\limits_{\Omega} \xi\xi\, \mathrm{d}\Omega > 0,
\end{aligned}$$

$$\int\limits_{\Omega} R\eta\eta \mathrm{d}\Omega = \int\limits_{\Omega} (\rho\, uu + j\underline{\vartheta})\underline{\vartheta} \mathrm{d}\Omega \geq \mathrm{d} \int\limits_{\Omega} \eta\eta\, \mathrm{d}\Omega > 0,$$

where $c = \min(b, \alpha, \beta)$; and $d = \min(\rho, j)$.

These estimates imply that the operators B and R are positive definite and therefore self-adjoint (self-conjugate).

This is an opportune time to write the equations of the free axially symmetric oscillations of a thin plate, which will be required in what follows. These equations in the polar coordinate system r, ϑ can be recorded on the basis of the formulas in [1] in the form

$$\begin{aligned}
&-\tfrac{1}{r}\tfrac{\partial(rQ_r)}{\partial r} + \rho\ddot{u} = 0, \quad \tfrac{\partial u}{\partial r} + \vartheta = 0, \\
&Q_r - \left(\tfrac{\partial}{\partial r} + \tfrac{1}{r}\right)M_r + \tfrac{1}{r}M_{\vartheta} = 0, \\
&\left(\tfrac{\partial}{\partial r} - \tfrac{1}{r}\right)\vartheta - \beta(M_r - M_{\vartheta}) = 0,
\end{aligned} \tag{2.35}$$

where Q_r, M_r is the shearing force and bending moment on the area element with the normal r; M_{ϑ} is the bending moment on the area element with the normal, which is tangential to the circle with radius r; and u is axial displacement. Moreover,

$\beta = \frac{12(1+\nu)}{Eh^3}$, where $h = h(r)$ is the thickness of the plate. Equations (2.35) can be written in the operator form analogously with what was done for the system of Eq. (1.50). They will be used further in Chap. 10.

2.3 Equations of Ring Vibrations

Let us consider the plane oscillations of a circular ring. We introduce x as the circumferential coordinate, $0 \le x \le l, l = 2\pi R$, and R as the radius of the ring. It is considered that the distributed tangential and radial loads $p^1 = p^1(x, t)$ and $p^2 = p^2(x, t)$ are acting on the ring and that the distributed moment $m = m(x, t)$ is acting on the plane of the ring. Suppose further that $\rho = \rho(x)$ and $j = j(x)$ are the distributed inertial characteristics at translational displacements and angular vibrations of the ring. We denote the coefficients of elasticity of a ring at tensile and shearing strain by $b_1 = b_1(x)$ and $b_2 = b_2(x)$, respectively. Furthermore, it is accepted that external causes such as the piezo effect and thermal heating can generate tensile (compression), shearing and bending strains g_1, g_2 and k.

The unknown quantities are $Q^1 = Q^1(x, t)$ and $Q^2 = Q^2(x, t)$ which are tensile and shearing forces; $M = M(x, t)$ the bending moment in the ring; and $u_1 = u_1(x, t), u_2 = u_2(x, t)$ and $\vartheta = \vartheta(x, t)$ the tangential, radial displacements and rotation of the cross-section of the ring subject to vibrations.

The equations of oscillations under tension (compression) of the ring in the tangential direction are

$$\begin{aligned} &-\frac{\partial Q^1}{\partial x} - \frac{Q^2}{R} = q^1, \quad q^1 = -\rho \ddot{u}_1 + p^1, \\ &\frac{\partial u_1}{\partial x} + \frac{u_2}{R} = \varepsilon_1, \quad \varepsilon_1 = b_1 Q^1 + g_1. \end{aligned} \tag{2.36}$$

The equations of dynamic equilibrium and strain equations under plane vibrations of the ring are

$$\begin{aligned} &-\frac{\partial Q^2}{\partial x} + \frac{Q^1}{R} = q^2, \quad q^2 = -\rho \ddot{u}_2 + p^2, \\ &Q^2 - \frac{\partial M}{\partial x} = \mu, \quad \mu = -j\ddot{\vartheta} + m. \end{aligned} \tag{2.37}$$

$$\begin{aligned} &\frac{\partial u_2}{\partial x} - \frac{u_1}{R} + \vartheta = \varepsilon_2, \quad \varepsilon_2 = b_2 Q^2 + g_2, \\ &\frac{\partial \vartheta}{\partial x} = \kappa, \quad \kappa = \beta M + k. \end{aligned} \tag{2.38}$$

Equations (2.36)–(2.38) of ring oscillations differ from the equations of rectilinear axis rod oscillations (1.6), (1.43)–(1.45) by the former being interconnected with each other due to the presence of summands with the factor $\frac{1}{R}$. The easiest way to see the appearance of such summands is to analyze the example of axisymmetric strain of the ring. In this case there arises the ring tensile strain equal to $\varepsilon = \frac{2\pi(R+u)-2\pi R}{2\pi R} = \frac{u}{R}$, where u is radial displacement.

Let us introduce the following designations

$$\begin{aligned} &p = \underline{p} = p^i, \quad g = \underline{g} = g_i, \quad Q = \underline{Q} = Q^i, \quad u = \underline{u} = u_i, \quad i = 1, 2, \\ &f = (p, m), \quad e = (g, k), \quad \xi = (Q, M), \quad \eta = (u, \vartheta). \end{aligned} \tag{2.39}$$

Let us form the differential operators

$$D = \begin{pmatrix} -\frac{\partial}{\partial x} & \frac{1}{R} & 0 \\ \frac{1}{R} & \frac{\partial}{\partial x} & 0 \\ 0 & 1 & -\frac{\partial}{\partial x} \end{pmatrix} \quad D^* = \begin{pmatrix} -\frac{\partial}{\partial x} & -\frac{1}{R} & 0 \\ \frac{1}{R} & -\frac{\partial}{\partial x} & 0 \\ 0 & 1 & -\frac{\partial}{\partial x} \end{pmatrix}, \tag{2.40}$$

and diagonal algebraic operators of inertia and elasticity

$$R = \operatorname{diag}(\rho, \rho, j) \text{ and } B = \operatorname{diag}(b_1, b_2, \beta). \tag{2.41}$$

Excluding $q^1, q^2, \varepsilon_1, \varepsilon_2, \mu, \kappa$ from Eqs. (2.36)–(2.38) and using the notation introduced for given and unknown quantities (2.39), as well as for the differential and algebraic operators (2.40) and (2.41), we arrive once again at the system of operator equations of the form (1.20). Also confirmed are the properties of the operators contained in these equations

$$\begin{aligned} \int_0^l D\xi\eta \mathrm{d}x &= \int_0^l \left(\begin{pmatrix} -\frac{\partial}{\partial x} & -\frac{1}{R} & 0 \\ \frac{1}{R} & -\frac{\partial}{\partial x} & 0 \\ 0 & 1 & -\frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} Q^1 \\ Q^2 \\ M \end{pmatrix} \right) \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vartheta \end{pmatrix} \mathrm{d}x \\ &= \int_0^l \left(\left(-\frac{\partial Q^1}{\partial x} - \frac{Q^2}{R} \right) u_1 + \left(\frac{Q^1}{R} - \frac{\partial Q^2}{\partial x} \right) u_2 + \left(Q^2 - \frac{\partial M}{\partial x} \right) \vartheta \right) \mathrm{d}x \\ &= \int_0^l \left(Q^1 \frac{\partial u_1}{\partial x} - \frac{Q^2}{R} u_1 + \frac{Q^1}{R} u_2 + Q^2 \frac{\partial u_2}{\partial x} + Q^2 \vartheta + M \frac{\partial \vartheta}{\partial x} \right) \mathrm{d}x \\ &= \int_0^l \left(Q^1 \left(\frac{\partial u_1}{\partial x} + \frac{u_2}{R} \right) + Q^2 \left(-\frac{u_1}{R} + \frac{\partial u_2}{\partial x} + \vartheta \right) + M \frac{\partial \vartheta}{\partial x} \right) \mathrm{d}x \\ &= \int_0^l \begin{pmatrix} Q^1 \\ Q^2 \\ M \end{pmatrix} \cdot \left(\begin{pmatrix} \frac{\partial}{\partial x} & \frac{1}{R} & 0 \\ -\frac{1}{R} & \frac{\partial}{\partial x} & 1 \\ 0 & 0 & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vartheta \end{pmatrix} \right) \mathrm{d}x = \int_0^l \xi D^* \eta \, \mathrm{d}x. \end{aligned}$$

Now that integration by parts has been performed the terms outside the integral vanish because they have equal values at the lower and upper limits of integration at $x = 0, l$ by virtue of the closedness of the ring.

Further

$$\int_0^l B\xi\xi \mathrm{d}x = \int_0^l \left(\begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} Q^1 \\ Q^2 \\ M \end{pmatrix} \right) \cdot \begin{pmatrix} Q^1 \\ Q^2 \\ M \end{pmatrix} \mathrm{d}x \geq b \int_0^l \xi\xi \mathrm{d}x > 0,$$

analogously

$$\int_0^l R\eta\eta \mathrm{d}x = \int_0^l \left(\rho(u_1^2 + u_2^2) + j\vartheta^2 \right) \mathrm{d}x \geq c \int_0^l \eta\eta \mathrm{d}x > 0,$$

where $b = \min(b_1, b_2, \beta)$ and $c = \min(\rho, j)$.

Thus, in the problem of vibrations of a ring the adjointness (conjugacy) of the operators D and D^* has been shown, as have the positive definiteness and, therefore, the self-adjointness (self-conjugacy) of the operators B and R.

The equations of elastic body vibrations in the curvilinear coordinate system are given in the Appendix.

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