Preface

The theory of integrability plays an important role in the study of the dynamics of differential systems. This theory is related to several branches of mathematics, such as algebraic geometry, algebraic topology, the theory of Lie groups and Lie algebras, real and complex analysis, differential geometry, and Riemannian geometry. Here we are mainly concerned with the algebraic and analytic aspects of the integrability of ordinary differential equations and of dynamical systems.

This book summarizes the last two decades of research on the integrability of dynamical systems and related topics obtained by the author and his coauthors, together with relevant results of other specialists. The main emphasis is on ordinary differential equations and singular holomorphic foliations given by the orbits of a projective one-form in the projective real or complex plane.

The theory of integrability was born with Newton's mechanics and developed rapidly. However, until Liouville's work on Riccati equations in 1841, this research generally reduced to the search for different methods of integrating distinct differential equations. The fact that the simple Riccati equations cannot be integrated by quadratures forced researchers to explore new theories and methods to determine if a dynamical system has first integrals, or is integrable, to study the regularity of first integrals (if they exist), and to determine the topology and geometry of their level surfaces.

For higher-dimensional differential systems it is in general difficult to study their dynamics. If a system has a number, say k, of functionally independent first integrals, the study of its dynamics can be reduced to $(n-k)$–dimensional systems. Then it becomes easier to study the dynamics of the reduced lower-dimensional system. Since many mechanical systems have first integrals or invariants, the problems of how to prove their existence and determine their expressions are important and have attracted the interest of many physicists and mathematicians.

This book consists of seven chapters. Chapter 1 introduces some preliminary notions and methods on the theory of integrability of ordinary differential equations, which include the existence and number of functionally independent first integrals surrounding a regular point, and their applications in solving linear and quasilinear partial differential equations. We also characterize the global existence and regularity of the first integrals of planar differential systems in their canonical regions. Section 1.4 contains some fundamental results on the Lax pairs of ordinary differential equations and their application to the search for first integrals. The main tools used in this chapter are the flow box theorem, the C^k (orbital) equivalence of flows, the limit sets of two-dimensional flows, the local topological structure of analytic vector fields around singularities, the finiteness of limit cycles and the Bendixson compactification.

Chapter 2 concentrates on the Jacobian and the inverse Jacobian multipliers, which are fundamental tools in the study of integrability. In Sect. 2.1 we study the existence of Jacobian multipliers and their relation to volume preserving flows. Section 2.2 concerns inverse Jacobian multipliers and their application to the study of the dynamics of differential systems, for instance, the existence and multiplicity of limit cycles and so on. In this subsection we also summarize the results on the existence and regularity of inverse integrating factors in a neighborhood of a singularity or of a limit cycle. In Sect. 2.3 we mainly consider the existence of C^∞ or C^ω local inverse Jacobian multipliers at a singularity of an analytic differential system having a pair of pure imaginary eigenvalues and a two-dimensional center manifold. We also study the Hopf bifurcation using the vanishing multiplicity of inverse Jacobian multipliers. In the last section we introduce the notion of Lie symmetry for a vector field and its application to the construction of inverse Jacobian multipliers. The key tools in this chapter are the theory of normal forms, the symmetry of Lie groups, and the theory of differential forms.

Chapter 3 is on the Darboux theory of integrability and its applications. The Darboux theory of integrability is an important part of the theory of integrability and plays an important role in the study of the dynamics of polynomial differential systems. Section 3.1 focuses on the classical Darboux theory of integrability and on its proof. Section 3.2 presents new developments in the Darboux theory of integrability. In Sect. 3.3 we introduce the Liouvillian and the elementary integrability of polynomial differential systems, and show their relations with the Darboux theory of integrability. Section 3.4 provides a relation between the Liouvillian first integrals and the Darboux polynomials, and its application to Liénard differential systems. The tools used in this chapter are the expression of functionally independent first integrals, the algebraic multiplicity of invariant algebraic hypersurfaces and differential field extension.

A key role in the Darboux theory of integrability is played by the Darboux polynomials, which were first used by Darboux in 1878 to construct first integrals or integrating factors. Chapter 4 is devoted to the study of the degree and the existence of Darboux polynomials of polynomial differential systems. Section 4.1 studies the degree of the Darboux polynomials of planar singular holomorphic foliations. This is one of the classical Poincaré problems. Section 4.2 illustrates methods for classifying the Darboux polynomials of some concrete differential systems from physics and mechanics. This study involves algebraic geometry, weight homogeneous polynomials, and the method of characteristic curves for solving linear partial differential equations.

Chapter 5 studies the equivalence between algebraic integrability and rational integrability. Using this result, we further characterize the polynomial, rational and analytic integrability of some concrete physical models. Section 5.1 focuses on the algebraic integrability of some specific models via the Darboux polynomials. Section 5.2 presents some results on the polynomial and rational integrability of natural Hamiltonian systems. Section 5.3 summarizes the results on the meromorphic integrability of Hamiltonian systems along a given orbit via the differential Galois group. Section 5.4 introduces algorithms to compute the Darboux polynomials and the rational first integrals. In the study of these results we need the Frobenius integrability theorem, the differential Galois group, and so on.

Chapter 6 presents some applications of the Darboux theory of integrability to the center-focus problem, to a weakened version of Hilbert's 16th problem on algebraic limit cycles, and to many other specific models.

Chapter 7 is devoted to the local theory of integrability. The materials are on the existence of local analytic, formal, rational, meromorphic and local Darboux first integrals of analytic (or formal) differential systems in a neighborhood of a singularity. The emphasis is on the equivalent characterization of the analytical dynamical systems which are locally integrable at a singularity, showing that analytic integrability implies the existence of analytic normalization. Furthermore, we consider the existence of embedding flows for integrable diffeomorphisms on a manifold. Finally, we characterize the varieties and the asymptotic expressions of the first integrals of the integrable or partially integrable differential systems.

A preliminary version of this book appeared as a series of lectures for graduate students at Shanghai Jiao Tong University, China. My colleagues, Profs. Jiang Yu and Yilei Tang, attended the lectures and pointed out lots of improvements in the mathematics and its presentation. I really appreciate our interesting discussions and their suggestions. I would also like to thank those students enrolled in my lectures, who corrected many misprints in the mathematics. I must appreciate Prof. Jaume Llibre for his careful reading of the English presentation and his corrections. I am sincerely indebted to several anonymous referees who offered valuable comments and suggestions for improving the structure and presentation of this book. I thank Ramon Peng, the editor of Springer, who has worked a lot during the review process. Finally, I should thank my wife, Niannian Su, for giving me the gift of time during the period in which this book was written.

Shanghai, China Xiang Zhang
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