Chapter 2
Jacobian and Inverse Jacobian Multipliers

This chapter is mainly concerned with the existence and regularity of the Jacobian and the inverse Jacobian multipliers of differential systems near a singularity, a periodic orbit, or a polycycle. We will also use the vanishing multiplicity of inverse Jacobian multipliers to study the multiplicity of a limit cycle, or of a homoclinic loop, and the cyclicity of a singularity.

2.1 Jacobian Multipliers, First Integrals and Integrability

For $n = 2$, system (1.1) is called a *planar differential system*. It can be written in the *Pfaffian form* or as a *differential one-form*

$$\omega := f_2(x)dx_1 - f_1(x)dx_2 = 0. \tag{2.1}$$

If there exists a C^1 function H such that

$$dH = \omega,$$

we say the differential form ω is *exact*, and H is called a *first integral* of the differential form. In this case system (1.1) is *Hamiltonian*, i.e.

$$\frac{dx_1}{dt} = \partial_{x_2} H, \qquad \frac{dx_2}{dt} = -\partial_{x_1} H.$$

Clearly H is a first integral of the Hamiltonian system.

Remark

- In $\mathbb{R}^{2m}$, let $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m$ be the coordinate systems. A canonical Hamiltonian system can be written as

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$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \nabla_{(x,y)} H = \begin{pmatrix} \partial_{y_1} H \\ \vdots \\ \partial_{y_m} H \\ -\partial_{x_1} H \\ \vdots \\ -\partial_{x_m} H \end{pmatrix},$$

where E is the mth order unit matrix, and H is called a *Hamiltonian function*. Obviously the Hamiltonian function is a first integral of the Hamiltonian system. Recall that $\nabla_{(x,y)} H$ is the gradient of H.

- If the differential form ω in (2.1) is exact, the divergence $\operatorname{div}(f) = \partial_{x_1} f_1 + \partial_{x_2} f_2$ of f identically vanishes, where $f = (f_1, f_2)$ is the vector field associated to system (1.1) with $n = 2$. So we have $d\omega = 0$. In this case, by definition the differential form ω is *closed*. Generally, if a differential form in an n-dimensional space is exact, it must be closed; but the inverse is not necessarily correct. We refer to Bott and Tu [41] and Olver [351].

For the planar differential system (1.1), the following results are well known.

Proposition 2.1 *Assume that Ω is a simply connected region of $\mathbb{R}^2$, and that a function $M(x)$ is continuously differentiable on Ω, and $M(x) \neq 0$, $x \in \Omega$. If the divergence of the vector field Mf vanishes identically on Ω, i.e.*

$$\operatorname{div}(Mf) := \partial_{x_1}(Mf_1) + \partial_{x_2}(Mf_2) \equiv 0, \quad x \in \Omega,$$

then system (1.1) *has a continuously differentiable first integral in Ω. Moreover,*

$$H(x) = \int_{x_0}^{x} Mf_2 dx_1 - Mf_1 dx_2, \qquad x \in \Omega,$$

is one of the first integrals of system (1.1)*, where $x_0 \in \Omega$ is an arbitrary fixed point, and the integral is taken over any simple curve connecting x_0 and x which is located in Ω.*

Remark

- The condition that Ω is simply connected ensures that the integral in Proposition 2.1 is uniquely determined by x. In other words, the integral is independent of the choice of the passage from x_0 to x.
- The assumption $M(x) \neq 0$, $x \in \Omega$ implies that system (1.1) and the system $\dot{x} = M(x)f(x)$ have the same first integrals and the same orbits. Their dynamics can be determined by the first integral H.
- The function $M(x)$ in Proposition 2.1 is called the *integrating factor* of system (1.1).
- In some applications the integrating factor $M(x)$ may vanish or may not be well defined on a zero Lebesgue measure subset, say Ω^*, of Ω. If this is the case, the

dynamics of system (1.1) in Ω^* cannot be studied using the first integral, and we need some other tools to investigate it. For instance, the differential equation

$$ydx - xdy = 0$$

has the integrating factor $1/y^2$, which is defined on $\mathbb{R}^2 \setminus \{y = 0\}$.

In higher-dimensional spaces, the corresponding notion of integrating factor is the *Jacobian last multiplier* or simply *Jacobian multiplier*, which by definition is a non-vanishing and continuously differentiable function $M(x)$ defined on Ω, satisfying

$$\operatorname{div}(Mf) := \partial_{x_1}(Mf_1) + \cdots + \partial_{x_n}(Mf_n) \equiv 0, \qquad x \in \Omega,$$

where $f(x) = (f_1(x), \ldots, f_n(x))$ is the vector-valued function appearing in the right-hand side of system (1.1).

Remark Similar to an integrating factor, a Jacobian multiplier may vanish or may not be well defined on a zero Lebesgue measure subset of Ω.

The following proposition is one of Jacobi's classical results, which characterizes the existence of first integrals of system (1.1) via a Jacobian multiplier.

Proposition 2.2 *Assume that M_1 and M_2 are two Jacobian multipliers of system* (1.1)*, and at least one of them is not zero on a full Lebesgue measure subset of Ω, say M_1. If $H(x) = M_2(x)/M_1(x)$ is not a constant, it is a first integral of system* (1.1) *in Ω.*

The proof of Proposition 2.2 follows easily from the definitions of first integral and Jacobian multiplier. The details are left to the reader as an exercise.

Note that if the Jacobian multiplier $M \equiv 1$, the divergence of system (1.1) vanishes, i.e. $\operatorname{div} f \equiv 0$. One of Liouville's classical results shows that any volume in the phase space is invariant under the flow of system (1.1) when its divergence vanishes, see e.g. Arnold [13, Sect. 16, Theorem 1].

Theorem 2.1 (Liouville's theorem) *Let $\varphi_t(x)$ be the flow of system* (1.1) *satisfying $\varphi_0(x) = x \in \Omega$ and let $D \subset \Omega$ be an arbitrary bounded region. Set $\varphi_t(D) = \{\varphi_t(x) |\ x \in D\}$. Assume that the divergence of system* (1.1) *is identically zero. If $\varphi_t(D) \subset \Omega$ then*

$$\text{the volume of } \varphi_t(D) = \text{the volume of } D.$$

In the following we continue to use the notations given in Liouville's theorem. For any continuously differentiable function $M(x)$ defined on Ω, if the integral

$$\int_{\varphi_t(D)} M(x)dx \tag{2.2}$$

is independent of the time t, it is called an *invariant integral* of system (1.1).

Liouville's theorem was extended by Poincaré, taking into account not only the Jacobian multiplier and but also its integral along the phase flow of system (1.1).

Theorem 2.2 (Poincaré's theorem) *Assume that $\varphi_t(x)$ is the flow of system* (1.1) *satisfying $\varphi_0(x) = x \in \Omega$, $D \subset \Omega$ is an arbitrary bounded region, and $M(x) \in C^1(\Omega)$. Then* (2.2) *is an invariant integral of system* (1.1) *if and only if $M(x)$ is a Jacobian multiplier of system* (1.1).

Clearly, Liouville's theorem is a special case of Poincaré's theorem.

Proof of Theorem 2.2. To prove that (2.2) is an invariant integral of system (1.1) is equivalent to proving

$$\frac{d}{dt}\int_{\varphi_t(D)} M(x)dx\bigg|_{t=t_0} \equiv 0, \tag{2.3}$$

where t_0 is an arbitrary point in the interval where the flow φ_t of system (1.1) is defined. Since $D \subset \Omega$ is arbitrary, we only need to prove that (2.3) holds at $t = 0$.

Since $\varphi_t : x \to \varphi_t(x)$ is a diffeomorphism, and

$$\begin{aligned} \varphi_t(x) &= x + f(x)t + O(t^2), \qquad |t| \ll 1, \\ \partial_x\varphi_t(x) &= E + \partial_x f(x)t + O(t^2), \end{aligned}$$

where $\partial_x\varphi_t(x)$ is the Jacobian matrix of $\varphi_t(x)$ with respect to x, we get from linear algebra that

$$\det(\partial_x\varphi_t(x)) = 1 + \mathrm{tr}(\partial_x f(x))t + O(t^2), \qquad |t| \ll 1,$$

where tr denotes the trace of a matrix, that is, the sum of the entries on the diagonal. By calculus one has

$$\int_{\varphi_t(D)} M(y)dy = \int_D M(\varphi_t(x))\det(\partial_x\varphi_t(x))dx.$$

Hence

$$\begin{aligned} &\frac{d}{dt}\int_{\varphi_t(D)} M(y)dy\bigg|_{t=0} \\ &= \int_D \left(\partial_y M(\varphi_t(x))\frac{d\varphi_t(x)}{dt}\det(\partial_x\varphi_t(x)) + M(\phi_t(x))\frac{d\det(\partial_x\varphi_t(x))}{dt}\bigg|_{t=0}\right)dx \\ &= \int_D (\partial_x M(x)f(x) + M(x)\mathrm{tr}(\partial_x f(x)))\,dx = \int_D \mathrm{div}(M(x)f(x))dx. \end{aligned}$$

Since $M(x)$ and $f(x)$ are both continuously differentiable, and D is arbitrary, the equality (2.3) holds if and only if

$$\mathrm{div}(M(x)f(x)) \equiv 0, \qquad x \in \Omega.$$

This proves the theorem. □

In Chap. 1 we introduced the relation between first integrals of equivalent differential systems. Next we discuss the relation between the Jacobian multipliers of equivalent differential systems.

Proposition 2.3 *Assume that $y = G(x)$ is a continuously differentiable and invertible transformation defined on Ω, and that a continuously differentiable function $M(x)$ is a Jacobian multiplier of system (1.1) in Ω. Then*

$$N(y) := M(G^{-1}(y))DG^{-1}(y)$$

is a Jacobian multiplier of the differentiable system

$$\dot{y} = \partial_x G(G^{-1}(y))f(G^{-1}(y)), \tag{2.4}$$

where $DG^{-1}(y)$ is the Jacobian of G^{-1} (i.e. the determinant of the Jacobian matrix of G^{-1}), and $\partial_x G(x)$ is the Jacobian matrix of G.

Proof We could prove this result from the definition, but the calculations are very complicated. Instead, we follow an idea of Berrone and Giacomini [32] and prove it by applying invariant integrals.

Let φ_t and ψ_t respectively be the flows of systems (1.1) and (2.4), which satisfy $\varphi_0(x) = x$ and $\psi_0(y) = y$, respectively, with $y = G(x)$. Since $y = G(x)$ is an invertible transformation sending system (1.1) to system (2.4), it follows that the flows of the systems are conjugate, i.e.

$$G^{-1} \circ \psi_t = \varphi_t \circ G^{-1}.$$

Assume that $V \subset G(\Omega)$ is an arbitrary bounded region. Then we have

$$\begin{aligned}\int_{\psi_t(V)} N(y)dy &= \int_{\psi_t(V)} M(G^{-1}(y))DG^{-1}(y)dy \\ &= \int_{G^{-1}\circ\psi_t(V)} M(x)dx = \int_{\varphi_t\circ G^{-1}(V)} M(x)dx.\end{aligned}$$

Since $M(x)$ is a Jacobian multiplier of system (1.1), it follows from Theorem 2.2 that the integral in the left-hand side of these last equalities is an invariant integral of system (2.4). Hence $N(y)$ is a Jacobian multiplier of system (2.4). □

We now discuss how to use Jacobian multipliers to construct first integrals. For k ($k \le n$) functions $p_1(x), \ldots, p_k(x)$ defined on a domain Ω of an n-dimensional space, we recall that $(c_1, \ldots, c_k) \in \mathbb{C}^n$ are regular values of these functions if $p_1, \ldots, p_k$ are functionally independent at all points $x \in \Omega$ such that $p_1(x) = c_1, \ldots, p_k(x) = c_k$. This kind of point x is a regular point of the functions $p_1(x), \ldots, p_k(x)$. If this is not the case, we call $(c_1, \ldots, c_k)$ critical values of $p_1, \ldots, p_k$, and correspondingly x is a critical point.

Sard's theorem states that critical values form a zero Lebesgue measure subset, see Sard [382], or [410, Theorem II.3.1] and [354].

Theorem 2.3 (Sard's theorem) *Let*

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

be a C^k *function, where* $k \geq \max\{n - m + 1, 1\}$. *If* V *is a critical set of* f, *i.e. a set formed by critical points of* f, *then* $f(V) \subset \mathbb{R}^m$ *is a zero Lebesgue measure subset.*

Theorem 2.4 (Jacobi's theorem) *Assume that the differential system* (1.1) *has* $n-2$ *functionally independent and continuously differentiable first integrals in* Ω

$$H_1(x),\ \ldots,\ H_{n-2}(x)$$

and it has a continuously differentiable Jacobian multiplier $M(x)$. *Then system* (1.1) *has a first integral defined in some neighborhood of each regular point of* $(H_1, \ldots, H_{n-2})$ *which is functionally independent of* $H_1(x),\ \ldots,\ H_{n-2}(x)$.

Proof Assume that $V \subset \Omega$ is an open domain, and consists of regular points of $(H_1, \ldots, H_{n-2})$. Without loss of generality we assume that

$$\begin{aligned} y_i &= H_i(x), \qquad i = 1, \ldots, n-2, \\ y_{n-1} &= x_{n-1}, \\ y_n &= x_n \end{aligned} \tag{2.5}$$

is invertible in V. Denote by $y = G(x)$ the transformation defined in (2.5), and by Δ the Jacobian of this transformation, i.e. $\Delta = \det(\partial_x G(x))$. Then system (1.1) is equivalent to the system

$$\begin{aligned} \dot{y}_i &= 0, \qquad i = 1, \ldots, n-2, \\ \dot{y}_{n-1} &= f_{n-1} \circ G^{-1}(y), \\ \dot{y}_n &= f_n \circ G^{-1}(y), \end{aligned} \tag{2.6}$$

via the transformation (2.5). Clearly system (2.6) has the functionally independent first integrals $I_i(y) = y_i$, $i = 1, \ldots, n-2$. By Proposition 2.3, system (2.6) has the Jacobian multiplier

$$N(y) := M \circ G^{-1}(y) DG^{-1}(y), \qquad y \in G(V).$$

This implies that the two-dimensional system

$$\begin{aligned} \dot{y}_{n-1} &= f_{n-1} \circ G^{-1}(I_1, \ldots, I_{n-2}, y_{n-1}, y_n) =: g_{n-1}(y_{n-1}, y_n), \\ \dot{y}_n &= f_n \circ G^{-1}(I_1, \ldots, I_{n-2}, y_{n-1}, y_n) =: g_n(y_{n-1}, y_n) \end{aligned}$$

has the integrating factor

$$L(y_{n-1}, y_n) = M \circ G^{-1}(y) DG^{-1}(I_1, \ldots, I_{n-2}, y_{n-1}, y_n).$$

Hence it follows from Proposition 2.1 that this last two-dimensional system has the first integral

$$I_{n-1}(y_{n-1}, y_n) = \int L g_n dy_{n-1} - L g_{n-1} dy_n.$$

Obviously I_{n-1} is functionally independent of $I_1, \ldots, I_{n-2}$, because the latter are independent of y_{n-1} and y_n.

By Proposition 1.2, $H_{n-1} := I_{n-1} \circ G(x)$ is a first integral of system (1.1). From the functional independence of $I_1, \ldots, I_{n-2}, I_{n-1}$, we can easily prove that $H_1(x), \ldots, H_{n-2}(x), H_{n-1}(x)$ are functionally independent.

This completes the proof of the theorem. □

Next we discuss the relation between Jacobian multipliers and first integrals of completely integrable autonomous differential systems. In this book the superscript τ will denote the transpose of a matrix, or of a vector.

Proposition 2.4 *Assume that system (1.1) has $n-1$ functionally independent and twice continuously differentiable first integrals $H_1, \ldots, H_{n-1}$ in Ω. Set*

$$M_i = \left(\partial_1 \mathbf{H}, \ldots, \partial_{i-1}\mathbf{H}, \widehat{\partial_i \mathbf{H}}, \partial_{i+1}\mathbf{H}, \ldots, \partial_n \mathbf{H}\right),$$

for $i = 1, \ldots, n$, where the hat denotes the absence of that element in the matrix, $\partial_j \mathbf{H} = (\partial_j H_1, \ldots, \partial_j H_{n-1})^\tau$ and ∂_j is the partial derivative with respect to x_j, $j = 1, \ldots, n$. The following statements hold.

(a) For any smooth function $I(x)$ we have

$$\mathscr{X}(I) = a(x) D_x(I, H_1, \ldots, H_{n-1}),$$

where $\mathscr{X}$ is the vector field associated with system (1.1), $D_x(I, H_1, \ldots, H_{n-1})$ is the Jacobian of $(I, H_1, \ldots, H_{n-1})$ with respect to x, and

$$a(x) = f_i(x) W_i(x)^{-1},$$

for some $i \in \{1, \ldots, n\}$ such that $W_i(x) := (-1)^{1+i}\Delta_i \neq 0$ with $\Delta_i = \det M_i$.

(b) $a(x)$ is an inverse Jacobian multiplier, i.e. $1/a(x)$ is a Jacobian multiplier.

Proof First we claim that if $W_i(x) \neq 0$, then

$$f_j(x) = W_j(x) W_i(x)^{-1} f_i(x), \qquad j = 1, \ldots, n.$$

We now prove this claim. Since $H_1(x), \ldots, H_{n-1}(x)$ are functionally independent and smooth first integrals of $\mathscr{X}$, we have

$$f_1(x)\partial_1\mathbf{H}(x) + \cdots + f_n(x)\partial_n\mathbf{H}(x) = 0.$$

This equation can be equivalently written as

$$M_i \begin{pmatrix} f_1 \\ \vdots \\ f_{i-1} \\ f_{i+1} \\ \vdots \\ f_n \end{pmatrix} = -f_i\partial_i\mathbf{H}.$$

Solving this linear algebraic equation via Cramer's rule gives

$$f_j = -A_j\Delta_i^{-1}f_i, \qquad j = 1,\ldots,i-1,i+1,\ldots,n,$$

where A_j is the determinant of the matrix obtained from M_i replacing its jth column by $\partial_i\mathbf{H}$. Direct calculations show that

$$A_j = (-1)^{i-j-1}\Delta_j = (-1)^{i-1}W_j.$$

This proves the claim.

Next we claim that

$$\sum_{j=1}^{n}\partial_j W_j \equiv 0.$$

In fact, we get from

$$\Delta_i = \det\left(\partial_1\mathbf{H},\ldots,\partial_{i-1}\mathbf{H},\widehat{\partial_i\mathbf{H}},\partial_{i+1}\mathbf{H},\ldots,\partial_n\mathbf{H}\right)$$

that

$$\partial_i W_i = (-1)^{1+i}\sum_{s=1,s\neq i}^{n}\det\left(\partial_1\mathbf{H},\ldots,\partial_i\partial_s\mathbf{H},\ldots,\widehat{\partial_i\mathbf{H}},\ldots,\partial_n\mathbf{H}\right).$$

Note that any element in the summation of $\partial_i W_i$, for example

$$(-1)^{1+i}\det\left(\partial_1\mathbf{H},\ldots,\partial_i\partial_j\mathbf{H},\ldots,\widehat{\partial_i\mathbf{H}},\ldots,\partial_n\mathbf{H}\right), \tag{2.7}$$

with $i > j$, has a counterpart in $\partial_j W_j$, i.e.

$$(-1)^{1+j}\det\left(\partial_1\mathbf{H},\ldots,\widehat{\partial_j\mathbf{H}},\ldots,\partial_i\partial_j\mathbf{H},\ldots,\partial_n\mathbf{H}\right), \tag{2.8}$$

such that their sum identically vanishes. This proves the claim.

Now we can prove the theorem.

(a) The first claim shows that for any smooth function I

$$\mathscr{X}(I) = \sum_{j=1}^{n} f_j \partial_j I = \sum_{j=1}^{n} W_j W_i^{-1} f_i \partial_j I = W_i^{-1} f_i \sum_{j=1}^{n} W_j \partial_j I. \tag{2.9}$$

By the properties of the determinant one can check that

$$D_x(I, H_1, \ldots, H_{n-1}) = \sum_{j=1}^{n} W_j \partial_j I. \tag{2.10}$$

Then the proof of statement (a) follows from (2.9) and (2.10).
(b) This statement follows from

$$\mathscr{X}\left(W_i^{-1} f_i\right) = W_i^{-1} f_i \operatorname{div}(\mathscr{X}). \tag{2.11}$$

We now prove this fact. Replacing I in (2.9) by f_i / W_i yields

$$\mathscr{X}\left(W_i^{-1} f_i\right) = W_i^{-1} f_i \sum_{j=1}^{n} W_j \partial_j \left(W_i^{-1} f_i\right). \tag{2.12}$$

Since $f_j W_i = f_i W_j$, differentiating this equation with respect to x_j gives

$$\partial_j f_j W_i + f_j \partial_j W_i = \partial_j f_i W_j + f_i \partial_j W_j. \tag{2.13}$$

Hence we have

$$\begin{aligned}
\mathscr{X}\left(W_i^{-1} f_i\right) &= W_i^{-3} f_i \sum_{j=1}^{n} \left(\partial_j f_i W_i - f_i \partial_j W_i\right) W_j \\
&= W_i^{-2} f_i \sum_{j=1}^{n} \left(\partial_j f_j W_i + f_j \partial_j W_i - f_i \partial_j W_j\right) - W_i^{-3} f_i \sum_{j=1}^{n} f_i \partial_j W_i W_j \\
&= W_i^{-1} f_i \operatorname{div}(\mathscr{X}) + W_i^{-3} f_i \sum_{j=1}^{n} \left(W_i f_j \partial_j W_i - f_i W_j \partial_j W_i\right) \qquad (2.14) \\
&= W_i^{-1} f_i \operatorname{div}(\mathscr{X}),
\end{aligned}$$

where in the second equality we have used (2.13), and in the third and fourth equalities we have used the second and first claims, respectively. This proves statement (b) and consequently the theorem. □

Proposition 2.4 has been extended to systems of vector fields for matrix Jacobian multipliers, see Weng and Zhang [440, Theorem 1.3].

The last proposition verifies the existence of Jacobian multipliers of a completely integrable differential system. The next result further characterizes the essential property of completely integrable differential systems, and their relation with their Jacobian multipliers.

Theorem 2.5 *Consider the C^k autonomous differential system (1.1) and its associated vector field $\mathscr{X}$ with $k \in (\mathbb{N}\backslash\{1\}) \cup \{\infty, \omega\}$ defined in Ω. Assume that system (1.1) is C^r completely integrable in Ω with $2 \le r \le k$, $\operatorname{div} \mathscr{X} \not\equiv 0$, and that the Lebesgue measure of the set of its singularities is zero. Let $H_1(x), \ldots, H_{n-1}(x)$ be $n-1$ functionally independent C^r first integrals. Then the following statements hold.*

(a) *If $J(x)$ is a smooth Jacobian multiplier of system (1.1), then $J(x)$ is functionally independent of $H_1(x), \ldots, H_{n-1}(x)$.*
(b) *There exists a full Lebesgue measure subset $\Omega_0 \subset \Omega$ in which system (1.1) is C^{r-1} orbitally equivalent to the linear differential system*

$$\dot{y} = y. \tag{2.15}$$

We remark that statement (a) was first proved by Llibre et al. in [292]. Statement (b) was proved for two-dimensional differential systems by Giné and Llibre in [183], and for any finite-dimensional differential systems by Llibre et al. in [292]. We note that Theorem 2.5 is similar to the flow box theorem in some sense.

Proof of Theorem 2.5. Since $H_1(x), \ldots, H_{n-1}(x)$ are functionally independent C^r first integrals, we can assume without loss of generality that

$$W(x) := \det\left(\partial_1 \mathbf{H}, \ldots, \partial_{n-1}\mathbf{H}\right) \neq 0, \qquad x \in \Omega_0 \subset \Omega,$$

where Ω_0 is a full Lebesgue measure subset of Ω, and $\mathbf{H} = (H_1, \ldots, H_{n-1})^\tau$. For $i = 1, \ldots, n-1$, set

$$W_i(x) := \det\left(\partial_1 \mathbf{H}, \ldots, \partial_{i-1}\mathbf{H}, \partial_n \mathbf{H}, \partial_{i+1}\mathbf{H}, \ldots, \partial_{n-1}\mathbf{H}\right).$$

Statement (b) of Proposition 2.4 shows that

$$Q(x) = W(x) f_n(x)^{-1}$$

is a Jacobian multiplier of system (1.1) in Ω_0 except perhaps in a zero Lebesgue measure subset. Note that $f_n(x)$ can vanish only on a zero Lebesgue measure subset of Ω_0, otherwise $f(x) = (f_1(x), \ldots, f_n(x))$ identically vanishes, because $f_i(x) = -W_i(x)W(x)^{-1}f_n(x)$, $i = 1, \ldots, n-1$, which could be obtained from the proof of Proposition 2.4.
(a) First we prove that $Q(x)$ is functionally independent of $H_1(x), \ldots, H_{n-1}(x)$. Set

$$A := \det \begin{pmatrix} \partial_1 Q & \cdots & \partial_{n-1} Q & \partial_n Q \\ \partial_1 \mathbf{H} & \cdots & \partial_{n-1}\mathbf{H} & \partial_n \mathbf{H} \end{pmatrix}.$$

Expanding the determinant A with respect to the first row yields

$$A = \sum_{i=1}^{n} \partial_i Q\, Q_i^*, \tag{2.16}$$

where

$$Q_i^* = (-1)^{1+i} \Delta_i, \tag{2.17}$$

with Δ_i the determinant of the $(n-1) \times (n-1)$-matrix which is obtained from the matrix A by removing the first row and the ith column. Then

$$W(x) = \Delta_n, \quad W_i(x) = (-1)^{n-1-i} \Delta_i, \quad i = 1, \ldots, n-1. \tag{2.18}$$

From (2.16)–(2.18) together with $W_i(x) = -Qf_i$ one gets

$$A = (-1)^{n+1} Q \mathscr{X}(Q) = (-1)^{n+1} Q^2 \mathrm{div} \mathscr{X}.$$

This shows that A can vanish only in a zero Lebesgue measure subset of Ω. Hence, $Q, H_1, \ldots, H_{n-1}$ are functionally independent in Ω.

If $J/Q \equiv$ constant, it is clear that J is functionally independent of $H_1, \ldots, H_{n-1}$. If $J/Q \not\equiv$ constant, then J/Q is a first integral of system (1.1). By Theorem 1.1 J/Q and $H_1, \ldots, H_{n-1}$ are functionally dependent. It follows that J is functionally independent of $H_1, \ldots, H_{n-1}$, because Q is functionally independent of $H_1, \ldots, H_{n-1}$, which ends the proof of the statement.

(b) By Proposition 2.4 and statement (a) it follows that ∇J, $\nabla H_1, \ldots, \nabla H_{n-1}$ are linearly independent at all points of a full Lebesgue measure subset $\widetilde{\Omega}_0 \subset \Omega$. Taking the invertible change of coordinates

$$y_1 = J(x)H_1, \quad \ldots, \quad y_{n-1} = J(x)H_{n-1}, \quad y_n = J(x), \qquad x \in \widetilde{\Omega}_0,$$

one has

$$\dot{y}_i = -y_i\, \mathrm{div} f, \quad \dot{y}_n = -y_n\, \mathrm{div} f.$$

This proves the statement, and consequently the theorem. $\square$

2.2 Inverse Jacobian Multipliers and Their Vanishing Sets

Consider the smooth differential system (1.1) defined in Ω or its associated vector field $\mathscr{X}$, let Ω_0 be an open subset of Ω. A function $V \in C^1(\Omega_0)$ is an *inverse Jacobian multiplier* of system (1.1) if

$$\mathscr{X}(V) = V \operatorname{div} \mathscr{X}.$$

For $n = 2$, the function V is also called an *inverse integrating factor*. Here 'inverse' comes from the fact that if V is an inverse Jacobian multiplier of (1.1), then $M = V^{-1}$ is a Jacobian multiplier of (1.1) in $\Omega_0 \backslash V^{-1}(0)$, where $V^{-1}(0) = \{x \in \Omega_0 | \ V(x) = 0\}$.

Proposition 2.5 *If V is an inverse Jacobian multiplier of the vector field $\mathscr{X}$, then $V^{-1}(0)$ is an invariant set of $\mathscr{X}$.*

Proof By definition of the inverse Jacobian multiplier, it follows that the vector field $\mathscr{X}$ is tangent to $V^{-1}(0)$. So $V^{-1}(0)$ is formed by the orbits of the vector field $\mathscr{X}$. This means that $V^{-1}(0)$ is invariant under the flow of the vector field. □

The above provides a geometric proof of Proposition 2.5. It can also be proved using

$$V(\varphi_t(x)) = V(x) \exp\left(\int_0^t \operatorname{div} \mathscr{X} \circ \varphi_s(x) ds\right), \tag{2.19}$$

which follows from

$$\frac{dV(\varphi_t(x))}{dt} = \mathscr{X}(V) \circ \varphi_t(x) = V \operatorname{div} \mathscr{X} \circ \varphi_t(x),$$

where $\varphi_t(x)$ is the solution of system (1.1) satisfying the initial condition $\varphi_0(x) = x$.

The following result characterizes the location of limit cycles of a planar smooth differential system via the inverse integrating factors.

Theorem 2.6 *Assume that the planar analytic vector field $\mathscr{X}$ defined in Ω has an inverse integrating factor V. If Γ is a limit cycle of $\mathscr{X}$, then $\Gamma \subset V^{-1}(0)$.*

This result was obtained by Giacomini et al. [172, Theorem 9] in 1996, and plays an important role in the study of integrability, the center-focus problem, the limit cycle bifurcation and its relation with Lie symmetry, and so on. In this direction there are plenty of results, see e.g. [31, 64, 66, 68, 70, 72, 99, 115, 116, 139, 149, 150, 160, 161, 163, 164, 167, 168, 171, 178, 185, 186, 209, 226, 257] and the references therein.

In the following we provide a proof of Theorem 2.6, which is different from the original one in [172], where Giacomini et al. applies the first-order de Rham cohomology.

Proof of Theorem 2.6. By Proposition 2.5 it follows that $V^{-1}(0)$ is an invariant set of system (1.1). So if $V^{-1}(0) \cap \Gamma \neq \emptyset$, then $\Gamma \subset V^{-1}(0)$.

On the contrary we assume that $\Gamma \bigcap V^{-1}(0) = \emptyset$. Then there exists a neighborhood Ω_0 of Γ in which V does not vanish. So $M = 1/V$ is an integrating factor of system (1.1) in Ω_0. Obviously $\mathscr{X}$ and $M\mathscr{X}$ have the same orbits in Ω_0. Hence Γ is also a limit cycle of $M\mathscr{X}$. In addition, it follows from Theorem 2.1 (i.e. Liouville's theorem) that the flow of the vector field $M\mathscr{X}$ is area preserving. But this is not possible, because for a domain $\omega_0 \subset \Omega_0$ located in one side of Γ the limit set of

ω_0 under either positive or negative flow of $M\mathscr{X}$ is the limit cycle, and so its area has the limit zero as either $t \to \infty$ or $t \to -\infty$. This contradiction implies that $\Gamma \bigcap V^{-1}(0) \neq \emptyset$ and so $\Gamma \subset V^{-1}(0)$. $\square$

Theorem 2.6 was extended to smooth vector fields on orientable, connected and smooth Riemannian surfaces by Athanassopoulos [16] in 2007.

Remark Theorem 2.6 shows that inverse integrating factors have played a more important role than integrating factors in the study of the dynamics of planar differential systems. We refer the reader to the survey paper [164] by García and Grau.

We now study the multiplicity of limit cycles via inverse integrating factors.

Assume that Γ is a limit cycle of system (1.1). For any $q \in \Gamma$ and a sufficiently small neighborhood Ω_0 of Γ, we define a *Poincaré section* $\Sigma := \{\psi(s) |\ s \in (-\varepsilon, \varepsilon)\} \subset \Omega_0$, which is transversal to the flow of system (1.1), where ψ is an analytic function satisfying $\psi(0) = q$. Let $\phi_t(x)$ be the flow of system (1.1). It induces a map on the Poincaré section Σ, called the *Poincaré map*, and denoted by $\mathfrak{P}$. It is well known that $\mathfrak{P}$ has the same regularity as that of the differential system, see e.g. Ilyashenko and Yakovenko [213], or Zhang [473, Sect. 6.2].

Set $\mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$ and $J = (-\varepsilon, \varepsilon)$. Then there exists a diffeomorphism $\Psi\colon J \times \mathbb{S}^1 \longrightarrow \Omega_0$, a sufficiently small neighborhood of Γ, which provides a local coordinate system (s, θ) in Ω_0 with $s \in J$ and $\theta \in \mathbb{S}^1$, satisfying $\Psi^{-1} \circ \psi(s) = (s, 0)$, see e.g. Fig. 2.1. Clearly the last condition means that $\Sigma = \{\Psi(s, 0),\ s \in J\}$. Ye [451, Sect. 2] presented a concrete such diffeomorphism, i.e.

$$x(s, \theta) = \varphi(\theta) - s\psi'(\theta), \quad y(s, \theta) = \psi(\theta) + s\varphi'(\theta),$$

where $(\varphi(\theta), \psi(\theta))$ is a parameterization of the limit cycle Γ.

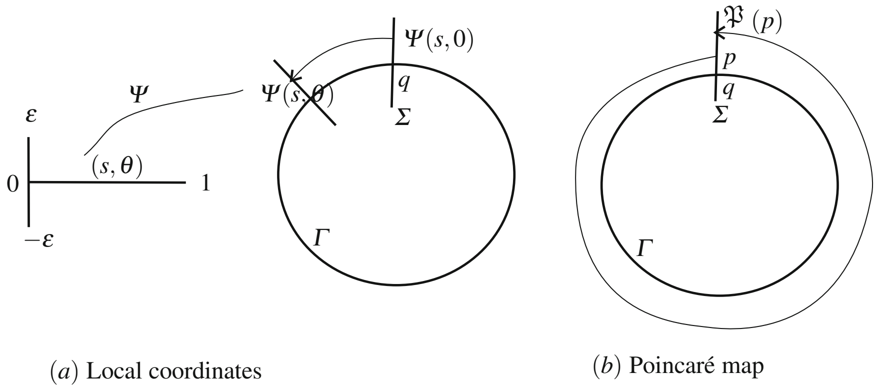

Fig. 2.1 A Poincaré section and Poincaré map

Under the local coordinates $(s,\ \theta)$, system (1.1) can be written as

$$\dot{s} = g_1(s, \theta), \quad \dot{\theta} = g_2(s, \theta).$$

Obviously $g_1(0, \theta) \equiv 0$, and $g_2 \neq 0$ on $J \times \mathbb{S}^1$ for $|\varepsilon|$ sufficiently small. So this last system of equations can be written as a differential equation

$$\frac{ds}{d\theta} = F(s, \theta). \tag{2.20}$$

Denote by

$$\widetilde{X} = \partial_\theta + F(s, \theta)\partial_s$$

the vector field associated to the Eq. (2.20). Note that if system (1.1) is analytic then $\widetilde{X}$ is analytic, because we can choose the diffeomorphism $\Psi(s, \theta)$ to be analytic. Let $\eta_\theta(s)$ be the solution of this last equation with $\eta_0(s) = s$. In the coordinates (s, θ), the Poincaré map $\mathfrak{P}$ can be expressed as $\mathfrak{P}(s) = \eta_1(s)$, because this last equation is periodic of period 1 in θ. It is well known that if system (1.1) is analytic, then the Poincaré map $\mathfrak{P}$ is analytic.

By construction of the Poincaré map $\mathfrak{P}$, it follows that $\mathfrak{P}(0) = 0$ because $s = 0$ corresponds to the limit cycle Γ. We say the limit cycle Γ is

- *hyperbolic*, if $\mathfrak{P}'(0) \neq 1$;
- of *multiplicity m* ($m \geq 2$), if $\mathfrak{P}(s) = s + \beta_m s^m + O(s^{m+1})$, $\beta_m \neq 0$.

Note that if $\mathfrak{P}(s) \equiv s$ then Γ belongs a period annulus, and it is not a limit cycle.

Assume that the planar differential system (1.1) is analytic, and it has the limit cycle Γ. Let V be an inverse integrating factor of system (1.1) in a neighborhood of Γ. By Proposition 2.3 the inverse integrating factor V of system (1.1) is transformed by $(x, y) = \Psi(s, \theta)$ to the inverse integrating factor $\widetilde{V}$ of Eq. (2.20) with

$$\widetilde{V}(s, \theta) = \frac{V(\Psi(s, \theta))}{g_2(s, \theta)D\Psi(s, \theta)}.$$

Recall that $D\Psi(s, \theta)$ is the Jacobian of $\Psi(s, \theta)$. The inverse integrating factor V has a *zero of multiplicity m* over Γ, or *vanishing multiplicity m* over Γ, if

$$\widetilde{V}(s, \theta) = g(\theta)s^m + O(s^{m+1}),$$

with $g(\theta) \not\equiv 0$ on Γ. García et al. [160, Lemma 7] proved that $g(\theta) \neq 0$ for $\theta \in \mathbb{S}^1$, which can be obtained in the following way. By (2.19) we have

$$\widetilde{V}(\eta_\theta(s_0), \theta) = \widetilde{V}(\eta_0(s_0), 0) \exp\left(\int_0^\theta \partial_s F(\eta_\rho(s_0), \rho)d\rho\right), \tag{2.21}$$

where $\eta_\theta(s_0)$ is the solution of equation (2.20) satisfying the initial condition $s(0) = s_0$. Expanding the functions on the two sides of (2.21) as Taylor series of s_0, and comparing the coefficients of s_0^m, one has

$$g(\theta) = g(0)\exp\left(\int_0^\theta \partial_s F(0,\rho)d\rho\right).$$

Since $g(\theta) \not\equiv 0$ on $\mathbb{S}^1$, it follows that $g(\theta) \neq 0$ on $\mathbb{S}^1$. Otherwise $g(\theta) \equiv 0$ on $\mathbb{S}^1$.

The next result, due to García et al. [160, Theorem 4], reveals the relation between the inverse integrating factors and the multiplicity of a limit cycle. Recall that a *period annulus* is a region filled up with periodic orbits.

Theorem 2.7 *Let Γ be a limit cycle of a planar analytic differential system* (1.1)*, and let V be an analytic inverse integrating factor of system* (1.1) *in a neighborhood of Γ. The following statements hold.*

(a) If Γ is a limit cycle of multiplicity m, then V has a zero of multiplicity m over Γ.
(b) If V has a zero of multiplicity m over Γ, then Γ is either a limit cycle of multiplicity m or belongs to a period annulus.

Proof We first claim that the inverse integrating factor $\widetilde{V}$ of Eq. (2.20) and the Poincaré map $\mathfrak{P}(s)$ satisfy

$$\widetilde{V}(\mathfrak{P}(s),1) = \widetilde{V}(s,0)\mathfrak{P}'(s). \tag{2.22}$$

Indeed, it follows from (2.21) that

$$\widetilde{V}(\mathfrak{P}(s),1) = \widetilde{V}(s,0)\exp\left(\int_0^1 \partial_s F(\eta_\theta(s),\theta)d\theta\right). \tag{2.23}$$

Note that $\partial_s\eta_\theta(s)$ satisfies the variational equation

$$\frac{\partial w}{\partial\theta}(s,\theta) = \frac{\partial F}{\partial s}(\eta_\theta(s),\theta)w(s,\theta), \quad w(s,0) = 1.$$

Integrating this equation with respect to θ from 0 to 1 gives

$$w(s,1) = \exp\left(\int_0^1 \partial_s F(\eta_\theta(s),\theta)d\theta\right).$$

By definition, one has $\mathfrak{P}'(s) = \partial_s\eta_1(s) = w(s,1)$. This proves the claim.

The remaining proof can be completed by using (2.22) and the Taylor expansions of $\widetilde{V}(s,1)$ and $\mathfrak{P}(s)$ with respect to s. The details are omitted.

We complete the proof of the theorem. □

Remark Giacomini et al. [173] studied the semistable and convex limit cycles by using the multiplicity of the inverse Jacobian multipliers on the limit cycles.

Theorem 2.7 establishes a relation between the multiplicities of a limit cycle Γ and of the inverse integrating factors over Γ. *What about the existence and the regularity of inverse integrating factors of planar differential systems in a neighborhood of a limit cycle?*

Ensiso and Peralta-Salas [139] in 2009 conducted pioneering work in this direction, and obtained a series of fundamental results in some neighborhood of limit cycles and of singularities.

Theorem 2.8 *Assume that Γ is a limit cycle of a planar analytic differential system* (1.1)*. The following statements hold.*

(a) System (1.1) *has a C^∞ inverse integrating factor V in a neighborhood Ω_0 of Γ, and V vanishes only on Γ.*
(b) The Taylor expansion of V at any point of Γ is not identically zero.
(c) V is analytic in Ω_0 if and only if the Poincaré map defined in a neighborhood of Γ can be embedded in a flow of a one-dimensional analytic vector field.
(d) If the limit cycle Γ is hyperbolic, then the inverse integrating factor V is analytic in Ω_0.

The proof of Theorem 2.8 requires knowledge of the existence of embedding flows of diffeomorphisms and of normal forms of one-dimensional vector fields. The definition of a normal form and the related background will be introduced in the following sections and chapters.

A map $y = F(x),\ x \in \Omega_0 \subset \mathbb{R}^n$, can be *embedded in a flow* $\varphi_t(x)$ (or *embedded in a vector field* $\mathscr{X}$, its solutions are also denoted by $\varphi_t(x)$), if there exists a fixed $T > 0$ such that $\varphi_T(x) = F(x)$. We also say that $\varphi_t(x)$ is an *embedding flow* of the map and $\mathscr{X}$ is an *embedding vector field* of the map. Hereafter we call $\varphi_T(x)$ the time T map of the flow $\varphi_t(x)$.

Proof of Theorem 2.8. The main ideas of the proof follow from Ensiso and Peralta-Salas [139].

(a) Since the planar differential system (1.1) is analytic, the associated Poincaré map $\mathfrak{P}$ of the limit cycle Γ is analytic, and it has the Taylor expansion

$$\mathfrak{P}(s) = s + \beta_0 s^m + o(s^m),$$

with $\beta_0 \neq 0$ and $m \geq 1$. Note that if $m = 1$ then $\beta_0 > -1$ and the limit cycle Γ is hyperbolic; and that if $m > 1$ then the limit cycle Γ is of multiplicity m.

By Takens [420] and Yakovenko [450] the Poincaré map is C^∞ conjugate to the time one map of the flow of the one-dimensional vector field

$$(s^m + \beta_1 s^{2m-1})\partial_s.$$

Moreover, it follows from Yakovenko [450] that there exists a C^∞ diffeomorphism Φ from Ω_0 to $\Theta_0 = (\mathbb{R}, 0) \times \mathbb{S}^1$ under which the vector field $\mathscr{X}$ associated to system (1.1) is transformed to the normal form vector field

$$\overline{\mathscr{X}} = \Phi_* \mathscr{X} = P_{m-1}(s) \left(T^{-1}\partial_\theta + (s^m + \beta_1 s^{2m-1})\partial_s\right),$$

where T is the minimal positive period of Γ, Ω_0 is the neighborhood of Γ defined as before, and P_{m-1} is a polynomial of degree $m-1$ satisfying $P_{m-1}(0) = 1$.

Clearly the vector field $\overline{\mathscr{X}}$ has the inverse integrating factor

$$\overline{V} := P_{m-1}(s) \left(s^m + \beta_1 s^{2m-1}\right).$$

Then

$$V := \overline{V} D\Phi^{-1} \circ \Phi$$

is an inverse integrating factor of $\mathscr{X}$ in a neighborhood of the limit cycle Γ, where $D\Phi^{-1}$ represents the Jacobian of Φ^{-1}. This proves the statement.

(b) The proof follows from the expressions of $\overline{V}$ and V given in the proof of statement (a).

(c) *Necessity*. Here we will use the notations given in the proof of Theorem 2.7. Note that $\widetilde{V}(s, 0) = \widetilde{V}(s, 1)$; we denote them by $\widetilde{W}(s)$. By the assumption that V is analytic, it follows that $\widetilde{V}(s, \theta)$ is analytic for $|s|$ sufficiently small. Hence $\widetilde{W}(s)$ is analytic and consequently it has $s = 0$ as a unique zero point in a suitably small neighborhood of $s = 0$, say $(-\varepsilon_0, \varepsilon_0)$.

By (2.22) one has

$$\frac{\mathfrak{P}'(s)}{\widetilde{W}(\mathfrak{P}(s))} = \frac{1}{\widetilde{W}(s)}, \quad 0 < |s| < \varepsilon_0.$$

Choosing any fixed but sufficiently small $s_0 \neq 0$, and integrating this last equation from $\mathfrak{P}^{-1}(s_0)$ to s_0 gives

$$\int_{s_0}^{\mathfrak{P}(s_0)} \frac{d\mu}{\widetilde{W}(\mu)} = \int_{\mathfrak{P}^{-1}(s_0)}^{s_0} \frac{\mathfrak{P}'(s)}{\widetilde{W}(\mathfrak{P}(s))} ds = \int_{\mathfrak{P}^{-1}(s_0)}^{s_0} \frac{ds}{\widetilde{W}(s)} =: \sigma_0. \tag{2.24}$$

Clearly σ_0 is a finite number and it depends only on s_0.

Consider the one-dimensional analytic differential equation

$$\frac{d\rho}{dt} = \sigma_0 \widetilde{W}(\rho). \tag{2.25}$$

Let $\rho_t(s)$ be the solution of equation (2.25) satisfying the initial condition $\rho_0(s) = s$. Then $\rho_t(0) \equiv 0$ for all $t \in \mathbb{R}$, and $\rho_t(s) \neq 0$ for $s \neq 0$ small. Integrating equation (2.25) along the solution $\rho_t(s_0)$ from 0 to t yields

$$\int_{s_0}^{\rho_t(s_0)} \frac{d\rho}{\widetilde{W}(\rho)} = \sigma_0 t.$$

Comparing this last equality at $t = 1$ with (2.24) we readily obtain $\rho_1(s_0) = \mathfrak{P}(s_0)$. In addition, it is clear that $\rho_1(0) = \mathfrak{P}(0)$. This proves that the Poincaré map $\mathfrak{P}$ is the time one map of the flow of the one-dimensional analytic equation (2.25).

Sufficiency. Assume that $h(\zeta)\partial_\zeta$ is the one-dimensional analytic vector field and $\psi_t(\zeta)$ is its associated flow satisfying $\mathfrak{P}(\zeta) = \psi_1(\zeta)$. Consider the vector field

$$\mathscr{Y} := h(\zeta)\partial_\zeta + \partial_\tau, \quad (\zeta, \tau) \in (\mathbb{R}, 0) \times \mathbb{S}^1.$$

Since the Poincaré maps of $\mathscr{Y}$ and $\widetilde{\mathscr{X}}$ associated to (2.20) are equal, by [233, Lemma 8], which states that two real C^k $(1 \le k \le \omega)$ periodic differential systems of period 1 are C^k equivalent if and only if their Poincaré maps are C^k conjugate, it follows that the vector fields $\mathscr{Y}$ and $\widetilde{\mathscr{X}}$ are analytically equivalent. Note that the vector field $\mathscr{Y}$ has the analytic inverse integrating factor $V^*(\zeta, \tau) = h(\zeta)$. Then the vector field $\widetilde{\mathscr{X}}$ has the analytic inverse integrating factor $\widetilde{V}(s, \theta) := V^* DH^{-1} \circ H(s, \theta)$, where $(\zeta, \tau) = H(s, \theta)$ is the analytic invertible change of coordinates which transforms $\mathscr{Y}$ to $\widetilde{\mathscr{X}}$. Furthermore, it follows from the analytic equivalence of the vector fields $\mathscr{X}$ and $\widetilde{\mathscr{X}}$ that the vector field $\mathscr{X}$ has an analytic inverse integrating factor defined in a neighborhood of the limit cycle Γ.

This proves the sufficiency and consequently the statement.

(d) Since the limit cycle Γ is hyperbolic, it follows that the Poincaré map is hyperbolic, i.e.

$$\mathfrak{P}(s) = v_0 s + o(s),$$

with $v_0 \neq 1$. Because of the analyticity of $\mathfrak{P}(s)$ we get from Belitskii and Tkachenko [30, Theorem 2.5] that $\mathfrak{P}(s)$ is C^ω conjugate to its linear part. Obviously, the linear map $s \to v_0 s$ is the time one map of the flow of the linear vector field $\ln v_0\, s\partial_s + \partial_\tau$. Hence $\mathfrak{P}(s)$ can be embedded in a one-dimensional analytic vector field. Then the proof follows from statement (c).

This completes the proof of the theorem. □

Remark If the limit cycle of an analytic differential system is not hyperbolic, there may not exist an analytic inverse integrating factor in a neighborhood of the limit cycle. Ensiso and Peralta-Salas [139, Example 2.8] constructed a planar analytic differential system which has a semistable limit cycle, and it has no analytic inverse integrating factor in any neighborhood of the limit cycle.

We now turn to the study of the existence and regularity of the inverse integrating factors of planar differential systems in a neighborhood of elementary singularities. Assume that $q \in \Omega$ is a singularity of the planar differential system (1.1), and $\lambda_{1,2}$ are the two eigenvalues of the Jacobian matrix of the system at the singularity q.

- The singularity q is *elementary* if at least one of $\lambda_{1,2}$ is not zero.

- The singularity q is *nondegenerate* if $\lambda_{1,2}$ are both not zero.
- The singularity q is *hyperbolic* if the real parts of $\lambda_{1,2}$ do not vanish.
- The singularity q is *semihyperbolic* if one of the real parts of $\lambda_{1,2}$ is not zero and the other is zero.
- The singularity q is a *hyperbolic saddle* if $\lambda_{1,2}$ have different signs.
- The singularity q is a *weak hyperbolic saddle* if it is a hyperbolic saddle and $\lambda_1 + \lambda_2 = 0$.
- The singularity q is a *strong hyperbolic saddle* if it is a hyperbolic saddle and $\lambda_1 + \lambda_2 \neq 0$.
- The singularity q is a *hyperbolic node* if $\lambda_{1,2}$ have the same sign.
- The singularity q is a *center* if there is a neighborhood of q which is filled up with periodic orbits.
- The singularity q is a *nondegenerate center* if q is a center, and $\lambda_{1,2}$ are a pair of pure imaginary eigenvalues.
- The singularity q is a *focus* if all orbits in some neighborhood of q spirally approach this singularity in either the positive or negative sense.
- The singularity q is a *strong focus* if q is a focus, and the real parts of $\lambda_{1,2}$ are not zeros.
- The singularity q is a *weak focus* if q is a focus, and $\lambda_{1,2}$ are a pair of pure imaginary eigenvalues.
- The singularity q is *nilpotent* if the Jacobian matrix of system (1.1) at q is not zero, but its eigenvalues $\lambda_{1,2}$ are both zero.

We remark that the topological structures of nilpotent singularities have been characterized by Andreev [8], see also Zhang et al. [480].

Theorem 2.9 *If q is an elementary singularity of the planar analytic differential system* (1.1)*, then the system has an inverse integrating factor, say V, in some neighborhood of q. Moreover, the regularity of V can be characterized as follows:*

(a) If q is a hyperbolic saddle or a semihyperbolic singularity, then V is C^∞.
(b) If q is a hyperbolic node, or a nondegenerate center, or a strong focus, then V is C^ω.
(c) If q is a weak focus, then V is C^∞. Furthermore, V is C^ω if and only if the Poincaré map of system (1.1) *can be embedded in a flow of an analytic differential system in a neighborhood of q.*
(d) In all the above cases, the multiplicity of q as a zero of V is finite.

Recall that the inverse integrating factor V at the singularity q has

- *multiplicity m* ($m \geq 1$) if the lowest order term in the Taylor expansion of V at q is of degree m.
- *infinite multiplicity* if V at q is flat, in other words, the Taylor expansion of V at q is identically zero.

Remark Theorem 2.9 and its proof were given in [139, Theorem 1.3] by Ensiso and Peralta-Salas, in which the main tools are the normal forms of system (1.1) at singularities, see Ilyashenko and Yakovenko [213]. By computing the inverse integrating factors of the normal form systems, the authors obtained the inverse integrating factors of system (1.1) at the singularity through the relation between the inverse integrating factors of two equivalent differential systems. We will not prove Theorem 2.9, instead referring the reader to the original paper.

The next result, due to Giné and Peralta-Salas [186], characterizes the existence of a C^∞ inverse integrating factor in a neighborhood of the center.

Theorem 2.10 *The planar analytic differential system* (1.1) *with a* (*possibly degenerate) center at the origin O has a C^∞ inverse integrating factor V defined in a neighborhood B of the origin, which is positive in $B\backslash\{O\}$ and is flat at O (i.e. V and all its derivatives at O vanish).*

García and Maza [167] studied the existence and regularity of inverse integrating factors of planar analytic differential systems in a neighborhood of a simple monodromy singularity. They applied a generalized polar blow up to system (1.1) at the origin, then system (1.1) in a neighborhood of the origin is transformed to a system defined on a cylinder $\{(r, \theta) \in (\mathbb{R}, 0) \times \mathbb{S}^1\}$. Their results were stated in terms of generalized polar coordinates.

It is still an open problem to *characterize the existence and regularity of planar analytic differential systems in a neighborhood of non-elementary singularities.*

The above results have been extended to polycycles. Recall that a *polycycle* of a planar differential system (1.1) consists of finitely many regular orbits and singularities, which form a set Γ, i.e. $\Gamma = \{\psi_i(t)\}_{i=1}^k \cup \{q_i\}_{i=1}^k$ with the ψ_i's regular orbits and the q_i's singularities of the system, which satisfy

- $\lim_{t\to\infty} \psi_i(t) = q_{i+1}$, $\lim_{t\to-\infty} \psi_i(t) = q_i$, $i = 1, \ldots, k$, where $q_{k+1} = q_1$; and
- Γ has an inner (or outer) neighborhood either filled up with periodic orbits or in which any orbit has Γ as its ω or α limit set.

In particular, Γ is called a *homoclinic cycle* if $k = 1$; or a *heteroclinic cycle* if $k = 2$. A polycycle Γ is *compact* if all singularities on Γ are located in the finite plane.

Ensiso and Peralta-Salas [139, Theorem 1.4] extended Theorem 2.6 to polycycles, and obtained the next result.

Theorem 2.11 *Assume that the smooth differential system* (1.1) *has a compact polycycle $\Gamma \subset \Omega$, which is a limit set of some orbits located either in its inner or outer neighborhood. If system* (1.1) *has a C^1 inverse integrating factor V in Ω, then $\Gamma \subset V^{-1}(0)$.*

Ensiso and Peralta-Salas [139] proved the above theorem by using the fact that the universal covering manifold of a manifold is simply connected, and so a closed differential one-form on it is exact. For the definition of the universal covering manifold, see e.g. Sharpe [392].

Remark Berrone and Giacomini [31] proved in 2000 that if a planar smooth differential system (1.1) has an inverse integrating factor V defined in a neighborhood of a hyperbolic saddle p_0 satisfying $V(p_0) = 0$, then V vanishes on all four separatrices of p_0. This result does not hold in general for nonhyperbolic singularities, see e.g. [168].

Next we discuss the regularity of the inverse integrating factors in a neighborhood of a polycycle. Assume that the eigenvalues of the planar differential system (1.1) at a singularity S are λ and μ.

- If $r := -\lambda/\mu = q/p$ is a positive rational number, we call S a $q : p$ *resonant saddle*.

García et al. [160, Proposition 11] in 2010 proved the nonexistence of analytic inverse integrating factors in a neighborhood of a homoclinic cycle.

Proposition 2.6 *Assume that the planar analytic differential system* (1.1) *has a compact homoclinic cycle Γ with the singularity a strong resonant saddle. If system* (1.1) *is not orbitally linearizable* (*formal or analytic*) *in some neighborhood of the singularity, then it cannot have an analytic inverse integrating factor in any neighborhood of Γ.*

García et al. [160, p. 3603] verified that the system

$$\dot{x} = -x + 2y + x^2, \quad \dot{y} = 2x - y - 3x^2 + \frac{3}{2}xy$$

satisfies all conditions of Proposition 2.6, and so it has no analytic inverse integrating factor defined in a neighborhood of the homoclinic loop, which is contained in the invariant algebraic curve $x^2(1-x) - y^2 = 0$, and is homoclinic to the origin.

In the proof of Proposition 2.6 the authors made use of the normal form of system (1.1) at a resonant saddle and the expression of the inverse integrating factors at the saddle. For the reader's convenience, to understand this last proposition, we briefly introduce some fundamental results on normal forms. The details will be given in Chap. 7.

A *formal series* is a series of the form

$$h(x) = Cx + \sum_{k=2}^{\infty} h_k(x), \tag{2.26}$$

where $h_k(x)$ is a vector-valued homogeneous polynomial of degree k. The formal series (2.26) is (*locally*) *invertible* if C is an invertible matrix. A map $y = h(x)$ is an *invertible formal transformation* if $h(x)$ is an invertible formal series. If C is the unit matrix, we call $y = h(x)$ a *near identity formal transformation* or a *formal transformation tangent to the identity*. If the formal series $h(x)$ is convergent, then $y = h(x)$ is an invertible analytic transformation.

The differential systems (1.1) and (1.6) are

- *formally equivalent* if there exists an invertible formal transformation $y = h(x)$ such that

$$\partial_x h(x) f(x) = g \circ h(x).$$

- *formally orbitally equivalent* if system (1.1) is formally equivalent to a system of the form $\dot{y} = a(y)g(y)$, where $a(y)$ is an invertible formal series.

The differential system (1.1) is

- *formally (orbitally) linearizable* if system (1.1) is formally (orbitally) equivalent to a linear system.

Let q be a singularity of system (1.1). Without loss of generality, let q be the origin, and assume system (1.1) has the form

$$\dot{x} = Ax + \sum_{k=2}^{\infty} f_k(x), \tag{2.27}$$

where the f_k's are vector-valued homogeneous polynomials of degree k. We call system (2.27) a *formal differential system*. If the series in (2.27) is convergent, then the system is analytic.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix A.

- The eigenvalues of A satisfy a *resonant relation* if there exists some $j \in \{1, \ldots, n\}$ and $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, $|k| \geq 2$, such that

$$\lambda_j = \langle k, \lambda \rangle,$$

where $|k| = k_1 + \cdots + k_n$. Recall that $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ and $\langle k, \lambda \rangle = k_1\lambda_1 + \cdots + k_n\lambda_n$.
- Otherwise the eigenvalues of A do not satisfy any resonant relation.

Example Let A be a real or complex matrix of order 5 with eigenvalues

$$0, \quad 2, \quad -2, \quad 4 + \pi\mathbf{i}, \quad 4 - \pi\mathbf{i}.$$

Then the resonant relations that the eigenvalues of A satisfy are

$$0 = (k_3 - 4k_4)2 + k_3(-2) + k_4(4 + \pi\mathbf{i}) + k_4(4 - \pi\mathbf{i}),$$

with $k_3 \in \mathbb{N},\ k_4 \in \mathbb{Z}_+,\ k_3 \geq 4k_4$; and

$$2 = (k_3 - 4k_4 + 1)2 + k_3(-2) + k_4(4 + \pi\mathbf{i}) + k_4(4 - \pi\mathbf{i}),$$

with $k_3 \in \mathbb{N},\ k_4 \in \mathbb{Z}_+,\ k_3 \geq 4k_4 - 1$.

Theorem 2.12 (Poincaré linearization theorem) *If the eigenvalues of A in* (2.27) *do not satisfy any resonant relation, then system* (2.27) *is linearizable.*

For $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$, $m \geq 2$, set $x^m = x_1^{m_1} \ldots x_n^{m_n}$. Let e_j be the n-dimensional unit vector with its jth entry equal to 1 and the others all vanishing. We call e_j the *jth unit vector*. A nonlinear monomial $x^m e_j$ in system (2.27) is *resonant* if $\lambda_j = \langle m, \lambda \rangle$. System (2.27) is in the *Poincaré normal form* (or simply *normal form*) if its linear part is in the Jordan normal form, and all nonlinear monomials are resonant.

If the nonlinear term $\psi(x)$ of a near identity transformation $y = x + \psi(x)$ sending system (2.27) to its Poincaré normal form consists of nonresonant monomials, then the transformation is called a *distinguished normalization*. Correspondingly the normal form system is a *distinguished normal form*. A monomial, say y^m, of $\psi(y)$ is *nonresonant* if $\langle m, \lambda \rangle \neq 0$; *resonant* if $\langle m, \lambda \rangle = 0$. Readers should take care of the difference between the resonances of the monomials in the systems and in the transformations.

Example Set $A = \mathrm{diag}(0,\ 2,\ -2,\ 4+\pi\mathbf{i},\ 4-\pi\mathbf{i})$. If an analytic or a formal differential system

$$\dot{x} = Ax + f(x)$$

is in the Poincaré normal form, then its general form is

$$\dot{y} = Ay + \begin{pmatrix} \sum_{k_3 \in \mathbb{N},\, k_1,\, k_4 \in \mathbb{Z}_+,\ k_3 \geq 4k_4} a_{k_1,k_3,k_4} y_1^{k_1} y_2^{k_3-4k_4} y_3^{k_3} y_4^{k_4} y_5^{k_4} \\ \sum_{k_3 \in \mathbb{N},\, k_1,\, k_4 \in \mathbb{Z}_+,\ k_3 \geq 4k_4-1} b_{k_1,k_3,k_4} y_1^{k_1} y_2^{k_3-4k_4+1} y_3^{k_3} y_4^{k_4} y_5^{k_4} \\ \mathbf{0} \end{pmatrix},$$

where $\mathbf{0}$ is the 3-dimensional zero vector.

Theorem 2.13 (Poincaré–Dulac normal form theorem) *An analytic or a formal differential system* (2.27) *is always formally equivalent to its Poincaré normal form via a near identity formal transformation.*

The proof of the Poincaré linearization theorem and the Poincaré–Dulac normal form theorem can be found in [213, 230]. Clearly, the Poincaré linearization theorem is a special case of the Poincaré–Dulac normal form theorem, because if the eigenvalues of the linear part of system (2.27) do not satisfy any resonant relation, then its Poincaré–Dulac normal form is linear.

Problem: *is an analytic differential system* (2.27) *analytically equivalent to its Poincaré normal form*? This is a classical problem, and there are extensive studies on it, see e.g. [30, 33, 42, 83, 91, 92, 97, 132, 134, 138, 192, 195, 197, 198, 213, 230, 232, 339, 344, 367, 390, 400, 401, 408, 409, 412, 413, 420, 444, 466] and the references therein. We will not discuss this problem in the general case. In Chap. 7 we will restrict our study to the analytic integrable (or partial integrable) differential systems.

We go back to the inverse integrating factors. Proposition 2.6 provided a sufficient condition under which an analytic differential system does not have an analytic inverse integrating factor defined in a neighborhood of a homoclinic cycle. The following result provides a link between two analytic inverse integrating factors defined in a neighborhood of a polycycle. See García et al. [160, Theorem 6].

Proposition 2.7 *Assume that system* (1.1) *is analytic and has a compact polycycle* Γ. *If all orbits located in a unilateral neighborhood of* Γ *have* Γ *as either their* α *or* ω *limit set, then in a sufficiently small neighborhood of* Γ *the ratio of any two analytic inverse integrating factors is a constant.*

Similar to Theorem 2.7, García et al. [160, Theorem 6] studied the cyclicity of a homoclinic loop with a hyperbolic saddle on it in terms of the vanishing multiplicity of an inverse integrating factor. Recall that the *cyclicity* of a homoclinic loop Γ of system (1.1) is the maximal number of limit cycles which can bifurcate from it under the prescribed class of perturbations of system (1.1). We note that there are extensive studies on the cyclicity of the homoclinic and the heteroclinic loops of a polynomial differential system under polynomial perturbations, see e.g. [378].

Let Γ be a homoclinic loop consisting of a regular orbit $\varphi(t)$ and a weak hyperbolic saddle p_0. Without loss of generality we can assume that p_0 is at the origin. After an invertible real linear change of variables and a time rescaling (if necessary), by applying the Poincaré–Dulac normal form theory we get a near identity formal change of coordinates which brings system (1.1) to

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \left(1 + \sum_{i=1}^{k-1} c_i x^i y^i\right) \begin{pmatrix} x \\ -y \end{pmatrix} + x^k y^k \begin{pmatrix} a_k x \\ -b_k y \end{pmatrix} + \text{h.o.t.}, \tag{2.28}$$

where h.o.t. denotes the higher-order terms. If $a_k \neq b_k$, we call $d_{k+1} = a_k - b_k$ the *first nonvanishing saddle quantity*. In this case we can take a sufficiently higher-order cut off of the formal transformation under which system (1.1) is analytically equivalent to (2.28). If there does not exist a k such that $a_k \neq b_k$, it follows from Zhang [466] that system (1.1) is analytically orbitally linearizable. Similar to the vanishing multiplicity of an inverse integrating factor over a limit cycle, we can define the vanishing multiplicity of an inverse integrating factor over a homoclinic loop.

Theorem 2.14 *Let* Γ *be a compact homoclinic loop of the planar analytic system* (1.1), *which is homoclinic to a weak hyperbolic saddle, and whose associated Poincaré map is not the identity. Assume that system* (1.1) *has an analytic inverse integrating factor defined on a neighborhood of* Γ, *whose vanishing multiplicity over* Γ *is* m. *The following statements hold.*

(a) $m \geq 1$ *and the first possible nonvanishing saddle quantity is* d_m.
(b) *If* $d_m \neq 0$, *the cyclicity of* Γ *is* $2m - 1$.
(c) *If* $d_m = 0$, *the cyclicity of* Γ *is* $2m$.

We will not prove this theorem. We refer the reader to its original proof given in [160, Theorem 6]. These last results also show the importance of inverse integrating factors in the study of the dynamics of planar differential systems.

Remark Previously we have studied the existence and the regularity of the inverse integrating factors of a planar analytic differential system in a neighborhood of a limit cycle, or of an elementary singularity. But it is still an open problem *to characterize the existence and the regularity of a planar analytic differential system in a neighborhood of a polycycle, or of a homoclinic loop, or of a heteroclinic loop.*

2.3 Inverse Jacobian Multipliers and the Center-Focus Problem

In this section we will study the existence of analytic or C^∞ inverse Jacobian multipliers of an n-dimensional analytic differential system at a singularity having a two-dimensional center manifold with a pair of pure imaginary eigenvalues.

In this direction the first result is the classical Poincaré center theorem, which reveals the essential character of the center-focus problem for planar analytic differential systems. Recall that the *center-focus problem* is to determine whether a singularity of a planar nonlinear differential system with a pair of pure imaginary eigenvalues is a center or a focus.

Theorem 2.15 (Poincaré center theorem) *Assume that a planar analytic differential system has a singularity and its linear part at the singularity has a pair of pure imaginary eigenvalues. Then the singularity is a center if and only if the system has an analytic first integral in a neighborhood of the origin.*

The proof of Theorem 2.15 will be obtained as a consequence of Theorem 7.18, which will be proved later on. So we omit its proof here.

Lyapunov extended the Poincaré center theorem from two-dimensional differential systems to higher-dimensional systems. For the real analytic differential system in $\mathbb{R}^n$

$$\begin{aligned} \dot{x} &= -y + f_1(x, y, z) = F_1(x, y, z), \\ \dot{y} &= \quad x + f_2(x, y, z) = F_2(x, y, z), \\ \dot{z} &= Az + f(x, y, z) = F(x, y, z), \end{aligned} \tag{2.29}$$

where A is an $(n-2) \times (n-2)$ real matrix, and $w = (w_3, \ldots, w_n)^\tau$, $w \in \{z, f, F\}$, with τ the transpose of a matrix, we denote by

$$\mathscr{X} = F_1(x, y, z)\frac{\partial}{\partial x} + F_2(x, y, z)\frac{\partial}{\partial y} + \sum_{j=3}^{n} F_j(x, y, z)\frac{\partial}{\partial z_j}$$

its associated vector field. Set $\mathbf{f} := (f_1, f_2, f) = O(|(x, y, z)|^2)$.

Recall that a set $I \subset \Omega$ is an r-dimensional *invariant manifold* of system (1.1) if it is invariant under the flow of system (1.1) and it is an r-dimensional manifold. For $r \in \{\infty, \omega\}$, it is well known that if a C^r differential system (1.1) has a singularity q, it can be written, after a translation (if necessary), as

$$\dot{x} = Lx + p(x), \tag{2.30}$$

where L is the Jacobian matrix of $f(x)$ at q and $p(x) = O(|x|^2)$ is an n-dimensional C^r vector-valued function.

Let $s,\ u$ and c be the numbers of eigenvalues of L which have negative, positive and zero real parts, respectively. Then system (2.30) can be written, after an invertible linear transformation (if necessary), as

$$\dot{y} = Sy + g_1(y, z, w), \quad \dot{z} = Uz + g_2(y, z, w), \quad \dot{w} = Cw + g_3(y, z, w), \tag{2.31}$$

where $y,\ z$ and w are $s,\ u$ and c-dimensional coordinates respectively, $S,\ U$ and C are square matrices of orders $s,\ u$ and c respectively whose eigenvalues have only negative, positive and zero real parts, and $g_1, g_2, g_3 = O(|(y, z, w)|^2)$ are C^r smooth, $r \in \{\infty, \omega\}$. Clearly the linear differential system

$$\dot{y} = Sy, \qquad \dot{z} = Uz, \qquad \dot{w} = Cw \tag{2.32}$$

respectively has the invariant hyperplanes $H_s := \{z = w = 0\}$, $H_u := \{y = w = 0\}$ and $H_c := \{y = z = 0\}$, which are respectively called the *stable*, *unstable* and *center hyperplanes* of the linear differential system (2.32). Of course, these invariant hyperplanes are $s,\ u$ and c-dimensional manifolds, respectively.

The stable (resp. unstable) manifold theorem shows that for $r \in \{\infty, \omega\}$, the C^r differential system (2.31) has a unique s-dimensional (resp. u-dimensional) C^r invariant manifold W^s (resp. W^u) tangent to the stable (resp. unstable) hyperplane H_s (resp. H_u), which is called the *stable manifold* (resp. *unstable manifold*) of system (2.31) at the origin. The orbits starting on W^s (resp. W^u) will positively (resp. negatively) approach the origin.

The center manifold theorem states that for $r \in \{\infty, \omega\}$, the C^r differential system (2.31) has a c-dimensional C^k invariant manifold W^c tangent to the center hyperplane H_c for any $k \in \mathbb{N}$, which is called the *center manifold* at the origin. It is well known that the center manifold of a C^ω differential system (if it exists) may not be C^ω and even not be C^∞. In general, center manifolds at a singularity may not be unique. For more details, see e.g. Chicone [89, Chap. 4].

For system (2.29), to fix our objective we assume that A has no eigenvalues with vanishing real parts. Under this hypothesis we get from the center manifold theorem that system (2.29) has a two-dimensional center manifold, which is tangent to the (x, y) plane at the origin. Moreover, this center manifold has the expression

$$\mathscr{M}^c = \bigcap_{i=3}^{n} \{z_i = c_i(x, y)\}, \tag{2.33}$$

where the $c_i(x, y)$'s, $i = 3, \ldots, n$, are C^r functions for some $r \in \mathbb{N}$.

Now we can state the Lyapunov center theorem.

Theorem 2.16 (Lyapunov center theorem) *Assume that A has no eigenvalues with vanishing real parts. The following statements hold.*

(a) *System* (2.29) *restricted to* $\mathscr{M}^c$ *has a center at the origin if and only if it admits a real analytic local first integral of the form*

$$\Phi(x, y, z) = x^2 + y^2 + \text{ higher-order terms},$$

which is defined in a neighborhood of the origin in $\mathbb{R}^n$.

(b) *If statement* (*a*) *holds, then the center manifold is unique and analytic.*

For a proof of the Lyapunov center theorem, we refer the reader to Bibikov [33, Theorems 3.1, 3.2 and Sect. 5] and Sijbrand [402].

In the next subsection we will show how to use the inverse integrating factors or the inverse Jacobian multipliers to characterize the center-focus problem of a two or higher-dimensional analytic differential system when restricted to a two-dimensional center manifold.

2.3.1 The Center-Focus Problem via Inverse Integrating Factors or Inverse Jacobian Multipliers

Making use of the inverse integrating factors, this subsection provides a characterization of when a singularity of a two-dimensional differential system with a pair of pure imaginary eigenvalues is a center. For higher-dimensional systems having a singularity with a pair of pure imaginary eigenvalues and a two-dimensional center manifold, we prove the existence and the regularity of the inverse Jacobian multipliers of the systems at the singularity, and we also characterize the singularity to be a center or a focus when restricted to the center manifold via inverse Jacobian multipliers.

In this direction Reeb [370] conducted pioneering work in 1952. He provided a characterization of the center-focus problem via the inverse integrating factors.

Theorem 2.17 (Reeb center theorem) *Assume that a real planar analytic differential system has a singularity with a pair of pure imaginary eigenvalues. Then the singularity is a center if and only if the system has a real analytic and nonvanishing inverse integrating factor in a neighborhood of the singularity.*

One can prove this theorem by making use of the Poincaré center theorem. The details are left to the reader as an exercise.

The Reeb center theorem was extended to three-dimensional differential systems by Buică et al. [49] in 2012. Several new ideas were involved in the proofs of their results.

Theorem 2.18 (Buică, García and Maza center-focus theorem) *Assume that system* (2.29) *is a* 3*-dimensional real analytic differential system, and that* A *is a nonzero real number. The following statements hold.*

(a) *System* (2.29) *restricted to* $\mathscr{M}^c$ *has a center at the origin if and only if it admits an analytic inverse Jacobian multiplier of the form*

$$J(x, y, z) = z + \text{ higher-order terms},$$

which is defined in a neighborhood of the origin. Furthermore, if such an inverse Jacobian multiplier exists, the center manifold is unique and analytic, and it is contained in $J^{-1}(0)$.

(b) *If system* (2.29) *restricted to* $\mathscr{M}^c$ *has a focus at the origin, then there exists a* C^∞ *and nonflat inverse Jacobian multiplier of the form*

$$J(x, y, z) = z(x^2 + y^2)^k + \text{ higher-order terms}, \quad k \geq 2,$$

which is defined in a neighborhood of the origin. Furthermore, there exists a local C^∞ *center manifold* $\mathscr{M}$ *such that* $\mathscr{M} \subset J^{-1}(0)$.

Recall that a C^∞ function is *nonflat* at a point if its Taylor expansion at this point is not identically zero.

We will not prove Theorem 2.18, which can be obtained from the proof of the next Theorems 2.19 and 2.20. These two theorems describe the further extension of Theorem 2.18 from three-dimensional differential systems to any finite-dimensional system (2.29) by Zhang [475], where some ideas and techniques different from those of [49, 50] are involved.

To state the results, we need to introduce some notation. Let $\lambda_3, \ldots, \lambda_n$ be the eigenvalues of the matrix A. Then the linear part of system (2.29) has the eigenvalues $\lambda = (\mathbf{i}, -\mathbf{i}, \lambda_3, \ldots, \lambda_n)$, where $\mathbf{i} = \sqrt{-1}$. For $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n$, we have

$$\langle \ell, \lambda \rangle = \ell_1 \mathbf{i} - \ell_2 \mathbf{i} + \sum_{j=3}^{n} \ell_j \lambda_j.$$

Set

$$\mathscr{R} := \left\{ \ell \in \mathbb{Z}^n | \ \langle \ell, \lambda \rangle = 0, \ \ell + e_j \in \mathbb{Z}_+^n, \, j = 3, \ldots, n \right\},$$

where e_j is the jth unit vector of $\mathbb{R}^n$. Assume that

(H) $\mathscr{R}$ is one-dimensional and A is diagonalizable in $\mathbb{C}$.

We remark that the assumption (H) implies that $\operatorname{Re} \lambda_j \neq 0$ for $j = 3, \ldots, n$. So it follows from the center manifold theorem that system (2.29) has a center manifold tangent to the (x, y) plane at the origin, which can be expressed in the form (2.33). If the eigenvalues of A either all have positive real parts or all have negative real parts, then $\mathscr{R}$ admits a unique $\mathbb{Z}_+$-linearly independent element with $(1, 1, 0)$

as its generator, where 0 is the $(n-2)$-dimensional zero vector. Note that in the three-dimensional case, if A is a nonzero real number, the assumption (H) holds automatically.

If A has complex eigenvalues, since A is real, its complex eigenvalues appear in conjugate pairs. For convenience, when stating our next results we assume without loss of generality that λ_{3+2j} and λ_{3+2j+1}, $j = 0, 1, \ldots, m-1$, are the conjugate complex eigenvalues of A, where $m \in \mathbb{Z}_+$ with $2m \le n-2$. Obviously if A has no complex eigenvalues then $m = 0$.

The next result, due to Zhang [475], characterizes the center of system (2.29) at the origin on the center manifold $\mathscr{M}^c$ via inverse Jacobian multipliers.

Theorem 2.19 *Assume that the differential system* (2.29) *is analytic and satisfies the assumption* (H). *Let* $\lambda_3, \ldots, \lambda_n$ *be the eigenvalues of* A, *which satisfy either* $\mathrm{Re}\lambda_j > 0$ *for all* $j = 3, \ldots, n$ *or* $\mathrm{Re}\lambda_j < 0$ *for all* $j = 3, \ldots, n$, *and* $\lambda_{3+2s+1} = \overline{\lambda_{3+2s}}$, $s = 0, 1, \ldots, m-1$, *with* $m \in \mathbb{Z}_+$ *and* $2m \le n-2$, *where the bar denotes the conjugate of a complex number. The following statements hold.*

(a) *System* (2.29) *restricted to the center manifold* $\mathscr{M}^c$ *has a center at the origin if and only if the system has a local analytic inverse Jacobian multiplier of the form*

$$J(x,y,z) = \prod_{j=0}^{m-1}\left((z_{3+2j} - p_{3+2j}(x,y,z))^2 + (z_{3+2j+1} - p_{3+2j+1}(x,y,z))^2\right) \times \prod_{l=3+2m}^{n}(z_l - p_l(x,y,z))V(x,y,z), \qquad \text{if} \quad m > 0, \tag{2.34}$$

or

$$J(x,y,z) = \prod_{l=3}^{n}(z_l - p_l(x,y,z))V(x,y,z), \qquad \text{if} \quad m = 0,$$

in a neighborhood of the origin, where $p_j = O(|(x,y,z)|^2)$ *for* $j = 3, \ldots, n$, *and* $V(0,0,0) = 1$.

(b) *If system* (2.29) *has the inverse Jacobian multiplier in statement* (a), *then the center manifold* $\mathscr{M}^c$ *is unique and analytic, and* $\mathscr{M}^c \subset J^{-1}(0)$.

We note that the set of matrices satisfying (H) forms a full Lebesgue measure subset of the set of real matrices of order n. The assumption (H) provides only a sufficient condition. If (H) does not hold, it is still an open problem *to determine whether the origin restricted to the center manifold is a center or a focus via inverse Jacobian multipliers*.

When system (2.29) restricted to the center manifold has a focus at the origin, the following result due to Zhang [475] verifies the existence of a C^∞ smooth local inverse Jacobian multiplier around the singularity.

Theorem 2.20 *Assume that the differential system* (2.29) *is analytic and satisfies the assumption* (H)*. The following statements hold.*

(a) *If system* (2.29) *restricted to* $\mathscr{M}^c$ *has a focus at the origin, then the system has a local* C^∞ *inverse Jacobian multiplier of the form*

$$J(x,y,z) = \prod_{j=0}^{m-1}\left[(z_{3+2j} - p_{3+2j}(x,y,z))^2 + (z_{3+2j+1} - p_{3+2j+1}(x,y,z))^2\right]$$
$$\times \prod_{l=3+2m}^{n}(z_l - p_l(x,y,z))\left[(x - q_1(x,y,z))^2 + (y - q_2(x,y,z))^2\right]^l \quad (2.35)$$
$$\times h\left((x - q_1(x,y,z))^2 + (y - q_2(x,y,z))^2\right) V(x,y,z), \qquad \text{if} \quad m > 0,$$

or

$$J(x,y,z) = \prod_{l=3}^{n}(z_l - p_l(x,y,z))\left[(x - q_1(x,y,z))^2 + (y - q_2(x,y,z))^2\right]^l$$
$$\times h\left((x - q_1(x,y,z))^2 + (y - q_2(x,y,z))^2\right) V(x,y,z), \qquad \text{if} \quad m = 0,$$

in a neighborhood of the origin, with $l \geq 2$, $p_j, q_i = O(|(x,y,z)|^2)$*, and* $h(0) = V(0,0,0) = 1$.

(b) *There exists a local* C^∞ *center manifold* $\mathscr{M}$ *such that* $\mathscr{M} \subset J^{-1}(0)$.

The number l in Theorem 2.20 is called the *vanishing multiplicity* of the inverse Jacobian multiplier at the origin.

2.3.1.1 Preparation for the Proof of Theorems 2.19 and 2.20

To prove Theorems 2.19 and 2.20 we need some technical results, which can also be used to study some related problems. To simplify the notation we will replace the real coordinates by the corresponding conjugate complex coordinates. The details are as follows.

For the $(n-2)$-tuple of eigenvalues $\lambda_3, \ldots, \lambda_n$ of A in the Jordan normal form, if $\lambda_j,\ \lambda_{j+1}$ are two complex conjugate eigenvalues, and its associated coordinates are $z_j,\ z_{j+1}$, we choose a pair of conjugate complex coordinates $\zeta_j = z_j + \mathbf{i} z_{j+1}$ and $\zeta_{j+1} = z_j - \mathbf{i} z_{j+1}$ instead of z_j and z_{j+1}, where $\mathbf{i} = \sqrt{-1}$. Moreover, we set $\xi = x + \mathbf{i} y$, $\eta = x - \mathbf{i} y$. By the assumption (H) we can write system (2.29) in these new coordinates as

$$\begin{aligned}
\dot{\xi} &= -\mathbf{i}\xi + p_1(\xi,\eta,\zeta) = P_1(\xi,\eta,\zeta),\\
\dot{\eta} &= \ \ \mathbf{i}\eta + p_2(\xi,\eta,\zeta) = P_2(\xi,\eta,\zeta),\\
\dot{\zeta} &= \ \ \Lambda\zeta + p(\xi,\eta,\zeta) = P(\xi,\eta,\zeta),
\end{aligned} \qquad (2.36)$$

with $\Lambda = \text{diag}(\lambda_3, \ldots, \lambda_n)$, where we have used the fact that A is in the real Jordan normal form, where $\zeta_j = z_j$ for λ_j real, $j \in \{3, \ldots, n\}$. We note that these new coordinates can greatly simplify the notation in the following calculations. Denote by $\widetilde{\mathscr{X}}$ the vector field associated to system (2.36).

Writing Proposition 2.3 according to system (2.29) gives the next result.

Lemma 2.1 *Let J be an inverse Jacobian multiplier of system* (2.29)*. Then an invertible smooth change of coordinates $(x, y, z) = \Psi(u, v, w)$ sends system* (2.29) *to*

$$\dot{\mathbf{w}} = (\partial\Psi(\mathbf{w}))^{-1}\mathbf{F} \circ \Psi(\mathbf{w}),$$

with $\mathbf{F} = (F_1, F_2, F)^\tau$ and $\mathbf{w} = (u, v, w)^\tau$, which has the inverse Jacobian multiplier

$$\widetilde{J}(\mathbf{w}) = J(\Psi(\mathbf{w}))(D\Psi(\mathbf{w}))^{-1},$$

where $\partial\Psi(\mathbf{w})$ and $D\Psi(\mathbf{w})$ denote respectively the Jacobian matrix and the Jacobian of Ψ.

Applying the Poincaré–Dulac normal form theorem to system (2.36), we get the next result.

Lemma 2.2 *Assume that (H) holds. System* (2.36) *is formally equivalent to*

$$\begin{aligned}
\dot{u} &= -u(\mathbf{i} + \rho_1(uv)), \\
\dot{v} &= \quad v(\mathbf{i} + \rho_2(uv)), \\
\dot{w}_j &= \ w_j(\lambda_j + \rho_j(uv)), \qquad j = 3, \ldots, n,
\end{aligned} \tag{2.37}$$

via a distinguished normalization, where the ρ_i's are analytic functions or formal series without constant terms.

Lemma 2.2 assures only the existence of the formal normalization of system (2.36) to its distinguished normal form. What about the regularity of the normalization? The next results provide some sufficient conditions.

Lemma 2.3 *Under the assumption (H), for the distinguished normalization transforming system* (2.36) *to its Poincaré–Dulac normal form* (2.37) *the following statements hold.*

(a) If system (2.36) *restricted to the center manifold $\mathscr{M}^c$ has a focus at the origin, then the distinguished normalization is C^∞.*
(b) If system (2.36) *restricted to $\mathscr{M}^c$ has a center at the origin, and the eigenvalues of A have either all positive real parts or all negative real parts, then the distinguished normalization is analytic.*

Proof (a) In (2.37) the coordinates u and v are conjugate, which forces $\rho_2 = \overline{\rho}_1$. By the assumption that the origin of system (2.37) restricted $w = 0$ is a focus, we must have $\text{Re}\,\rho_1 \neq 0$. Otherwise, the singularity on the two-dimensional manifold will be a center, a contradiction. So by Belitskii [29, Theorem 1] it follows that the distinguished normalization transforming system (2.36) to its normal form (2.37) is C^∞.

(b) According to [33] we write system (2.36) as

$$\begin{aligned} \begin{pmatrix} \dot\xi \\ \dot\eta \end{pmatrix} &= A_1 \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} p_1(\xi,\eta,\zeta) \\ p_2(\xi,\eta,\zeta) \end{pmatrix}, & A_1 &= \begin{pmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix}, \\ \dot\zeta &= A_2\zeta + p(\xi,\eta,\zeta), & A_2 &= \Lambda. \end{aligned}$$

By the assumption (H) it follows that the eigenvalues of A_1 and A_2 are not resonant with each other, i.e.

$$\lambda_j \neq \ell_1(-\mathbf{i}) + \ell_2\mathbf{i} = (\ell_2 - \ell_1)\mathbf{i}, \quad \ell_1,\ \ell_2 \in \mathbb{Z}_+ \quad \text{for } j = 3, \ldots, n.$$

So we get from Bibikov [33, Theorem 10.1] that there exists a distinguished normalization

$$\xi = u + \varphi_1(u,v), \quad \eta = v + \varphi_2(u,v), \quad \zeta = r + \varphi(u,v),$$

with $r = (r_3, \ldots, r_n)$, $\varphi = (\varphi_3, \ldots, \varphi_n)$ and $\varphi_1, \varphi_2, \varphi = O(|(u,v,r)|^2)$, which sends system (2.36) formally to

$$\dot u = -u(\mathbf{i} + \rho_1(uv)), \quad \dot v = v(\mathbf{i} + \rho_2(uv)), \quad \dot r = \Lambda r + h(u,v,r), \tag{2.38}$$

with $\rho_1, \rho_2 = o(1)$, $h = (h_3, \ldots, h_n)^\tau = O(|(u,v,r)|^2)$ and $h(u,v,0) = 0$. According to Bibikov [33], system (2.38) is called a *quasi-normal form* of system (2.36).

Since system (2.38) restricted to the center manifold $r = 0$ has a center at the origin, it has a formal first integral. Recall that a *formal first integral* of an analytic or a formal differential system (1.1) is a formal series which satisfies the Eq. (1.2). Then we get from Zhang [466] that $\rho_1(uv) = \rho_2(uv)$ in (2.38). Applying Bibikov [33, Theorems 10.2, 3.2 and Sect. 5] shows that the distinguished normalization from system (2.36) to (2.38) is convergent. Hence systems (2.36) and (2.38) are analytically equivalent.

Our next aim is to find an analytic change of coordinates which sends system (2.38) to system (2.37). Let

$$u = u, \quad v = v, \quad r = w + \psi(u,v,w)$$

be such a transformation. Some calculations show that

$$\begin{aligned} &\partial_w\psi\,\Lambda w - \mathbf{i}u\partial_u\psi + \mathbf{i}v\partial_v\psi - \Lambda\psi \\ &\quad = \Lambda h(u,v,w+\psi(u,v,w)) - \partial_w\psi\, w\rho + u\rho_1\partial_u\psi - v\rho_2\partial_v\psi, \end{aligned} \tag{2.39}$$

where $w\rho = (w_3\rho_3, \ldots, w_n\rho_n)^\tau$, ψ is a column vector and $\partial_w\psi$ is the Jacobian matrix of ψ with respect to w. In the linear space $\mathscr{H}^{l+p+q}$ formed by $(n-2)$-dimensional vector-valued homogeneous polynomials of degree l in w and of degrees p and q in u and v, respectively, the linear operator

$$L = \frac{\partial}{\partial w} \Lambda w - \mathbf{i} u \frac{\partial}{\partial u} + \mathbf{i} v \frac{\partial}{\partial v} - \Lambda$$

has the spectrum

$$\left\{ \langle k, \lambda \rangle - p\mathbf{i} + q\mathbf{i} - \lambda_j | \; k \in \mathbb{Z}_+^{n-2}, |k| = l, p, q \in \mathbb{Z}_+, j = 3, \ldots, n \right\},$$

see Bibikov [33] for more details.

Expanding ψ, ρ_1, ρ_2, ρ and h as Taylor series and using the assumption (H), we can prove by induction that Eq. (2.39) have a formal series solution ψ consisting of nonresonant monomials. Furthermore, the assumption that the eigenvalues of A have their real parts either all positive or all negative guarantees that there exists a number $\sigma > 0$ such that if $\langle k, \lambda \rangle - p\mathbf{i} + q\mathbf{i} - \lambda_j \neq 0$ for $(k, p, q) \in \mathbb{Z}_+^n$ then

$$\left| \langle k, \lambda \rangle - p\mathbf{i} + q\mathbf{i} - \lambda_j \right| \geq \sigma.$$

Using the techniques in the proof of the classical Poincaré–Dulac normal form theorem (see e.g. Bibikov [33] or Zhang [466]), we can prove the convergence of ψ.

We complete the proof of statement (b), and consequently the lemma. □

The next result verifies the existence of analytic integrating factors of system (2.36) restricted to the center manifold.

Lemma 2.4 *Assume that system* (2.36) *has an analytic inverse Jacobian multiplier of the form*

$$J(\xi, \eta, \zeta) = (\zeta_3 - \psi_3(\xi, \eta, \zeta)) \ldots (\zeta_n - \psi_n(\xi, \eta, \zeta)) V(\xi, \eta, \zeta),$$

where $\psi_s = O(|(\xi, \eta, \zeta)|^2)$ *for* $s \in \{3, \ldots, n\}$, V *is analytic and satisfies* $V(0, 0, 0) \neq 0$. *The following statements hold.*

(a) $\mathscr{M} = \bigcap_{s=3}^{n} \{\zeta_s = \psi_s(\xi, \eta, \zeta)\}$ *is an invariant analytic center manifold of the vector field* $\widetilde{\mathscr{X}}$ *in a neighborhood of the origin.*

(b) $V|_{\mathscr{M}}$ *is an analytic inverse integrating factor of* $\widetilde{\mathscr{X}}|_{\mathscr{M}}$.

Proof (a) Since J is an inverse Jacobian multiplier of $\widetilde{X}$, we have

$$\widetilde{\mathscr{X}}(J) = J \mathrm{div} \widetilde{\mathscr{X}}.$$

This implies that there exist analytic functions

$$K_0(\xi,\eta,\zeta),\quad K_3(\xi,\eta,\zeta),\quad \ldots,\quad K_n(\xi,\eta,\zeta)$$

such that

$$\begin{aligned} \widetilde{\mathscr{X}}(\zeta_s-\psi_s(\xi,\eta,\zeta)) &= K_s(\xi,\eta,\zeta)(\zeta_s-\psi_s(\xi,\eta,\zeta)),\quad s=3,\ldots,n,\\ \widetilde{\mathscr{X}}(V(\xi,\eta,\zeta)) &= K_0(\xi,\eta,\zeta)V(\xi,\eta,\zeta). \end{aligned} \tag{2.40}$$

This shows that each hypersurface $\mathfrak{S}_s := \{\zeta_s = \psi_s(\xi,\eta,\zeta)\}$, $s = 3,\ldots,n$, is invariant under the flow of $\widetilde{\mathscr{X}}$.

In addition, the invariant hypersurface $\mathfrak{S}_s$ can be expressed via the Implicit Function Theorem as

$$\zeta_s = \gamma_s(\xi,\eta),\quad s=3,\ldots,n,$$

in a neighborhood of the origin, where $\gamma_s(\xi,\eta)$ is analytic, and it satisfies $\gamma_s(0,0)=0$ and $\partial_\xi\gamma_s(0,0)=\partial_\eta\gamma_s(0,0)=0$ for $s=3,\ldots,n$. This proves that

$$\mathscr{M} = \bigcap_{s=3}^{n}\{\zeta_s = \gamma_s(\xi,\eta)\},$$

is an analytic center manifold of $\widetilde{\mathscr{X}}$ in a neighborhood of the origin, and is tangent to the (ξ,η) plane at the origin.

(*b*) Set $\gamma(\xi,\eta) = (\gamma_3(\xi,\eta),\ldots,\gamma_n(\xi,\eta))$ with $\gamma_s(\xi,\eta)$ obtained in the proof of (*a*) for $s=3,\ldots,n$. From the first equation of (2.40) we have

$$\widetilde{\mathscr{X}}(\zeta_s-\psi_s(\xi,\eta,\zeta)) = 0 \quad \text{on } \mathscr{M},\quad s=3,\ldots,n.$$

These $n-2$ equations can be written in vector form

$$\left(E-\partial_\zeta\psi\right)P = P_1\partial_\xi\psi + P_2\partial_\eta\psi \qquad \text{on } \mathscr{M}, \tag{2.41}$$

where $\partial_\sigma\psi = (\partial_\sigma\psi_3,\ldots,\partial_\sigma\psi_n)^\tau$, $\sigma\in\{\xi,\eta\}$, and $\partial_\zeta\psi$ is the Jacobian matrix of $\psi=(\psi_3,\ldots,\psi_n)$ with respect to ζ. Moreover, it follows from the construction of $\gamma(\xi,\eta)$ that

$$\gamma(\xi,\eta) = \psi(\xi,\eta,\gamma(\xi,\eta)).$$

This implies

$$\left(E-\partial_\zeta\psi\right)\partial_\xi\gamma = \partial_\xi\psi,\quad \left(E-\partial_\zeta\psi\right)\partial_\eta\gamma = \partial_\eta\psi, \tag{2.42}$$

where $\partial_\sigma\gamma = (\partial_\sigma\gamma_3,\ldots,\partial_\sigma\gamma_n)^\tau$, $\sigma\in\{\xi,\eta\}$.

Set $I(\xi,\eta) = V(\xi,\eta,\gamma(\xi,\eta))$. Some calculations show that

$$
\begin{aligned}
\widetilde{\mathscr{X}}|_{\mathscr{M}}(I(\xi,\eta)) &= P_1(\xi,\eta,\gamma(\xi,\eta))\partial_\xi I + P_2(\xi,\eta,\gamma(\xi,\eta))\partial_\eta I \qquad (2.43)\\
&= P_1[w]\partial_\xi V + P_2[w]\partial_\eta V + \partial_\zeta V P[w]\big|_{\mathscr{M}} = \widetilde{\mathscr{X}}(V)|_{\mathscr{M}} = K_0 V|_{\mathscr{M}} = K_0|_{\mathscr{M}} I,
\end{aligned}
$$

where $[w] = (\xi, \eta, \gamma(\xi,\eta))$, and we have used (2.42) and (2.41) in the proof.

Finally, we only need to prove $K_0|_{\mathscr{M}} = \operatorname{div}(\widetilde{\mathscr{X}}|_{\mathscr{M}})$. First, from (2.40) and the expression of J we have

$$J\operatorname{div}\widetilde{\mathscr{X}} = \widetilde{\mathscr{X}}(J) = (K_0 + K_3 + \cdots + K_n)J.$$

This implies

$$K_0 = \operatorname{div}\widetilde{\mathscr{X}} - K_3 - \cdots - K_n. \qquad (2.44)$$

Second, from (2.40) and the fact that

$$\widetilde{\mathscr{X}}(\zeta_s - \psi_s(\xi,\eta,\zeta)) = P_s - P_1\partial_\xi\psi_s - P_2\partial_\eta\psi_s - \partial_\zeta\psi_s P, \quad s = 3, \ldots, n,$$

we have

$$\operatorname{diag}(K_3,\ldots,K_n)\,(\zeta - \psi(\xi,\eta,\zeta)) = \left(E - \partial_\zeta\psi\right)P - P_1\,\partial_\xi\psi - P_2\,\partial_\eta\psi. \qquad (2.45)$$

Differentiating (2.45) with respect to ζ, and writing the resulting equations in vector form, we get

$$
\begin{aligned}
\operatorname{diag}(K_3,\ldots,K_n) &= \left(E - \partial_\zeta\psi\right)\partial_\zeta P\left(E - \partial_\zeta\psi\right)^{-1}\\
&\quad - \partial_\xi\psi\,\partial_\zeta P_1\left(E - \partial_\zeta\psi\right)^{-1} - \partial_\eta\psi\,\partial_\zeta P_2\left(E - \partial_\zeta\psi\right)^{-1}, \qquad (2.46)
\end{aligned}
$$

where we have used the fact that

$$\partial_\xi\partial_{\zeta_s}\psi_j = \partial_\eta\partial_{\zeta_s}\psi_j = \partial_{\zeta_s}\partial_{\zeta_l}\psi_j = 0 \quad \text{on } \mathscr{M}, \quad \text{for all } 3 \le s, j, l \le n,$$

which are obtained from $\psi_j(\xi,\eta,\zeta) = \zeta_j$ on $\mathscr{M}$, $j = 3, \ldots, n$.

Since similar matrices have the same trace, one has

$$
\begin{aligned}
\operatorname{trace}\left(\left(E - \partial_\zeta\psi\right)\partial_\zeta P\left(E - \partial_\zeta\psi\right)^{-1}\right) &= \operatorname{trace}(\partial_\zeta P) = \sum_{j=3}^{n}\partial_{\zeta_j}P_j,\\
\operatorname{trace}\left(\partial_\xi\psi\,\partial_\zeta P_1\left(E - \partial_\zeta\psi\right)^{-1}\right) &= \operatorname{trace}\left(\left(E - \partial_\zeta\psi\right)^{-1}\partial_\xi\phi\,\partial_\zeta P_1\right)\\
&= \partial_\zeta P_1\left(E - \partial_\zeta\psi\right)^{-1}\partial_\xi\psi,\\
\operatorname{trace}\left(\partial_\eta\psi\,\partial_\zeta P_2\left(E - \partial_\zeta\psi\right)^{-1}\right) &= \operatorname{trace}\left(\left(E - \partial_\zeta\psi\right)^{-1}\partial_\eta\psi\,\partial_\zeta P_2\right)\\
&= \partial_\zeta P_2\left(E - \partial_\zeta\psi\right)^{-1}\partial_\eta\psi.
\end{aligned}
$$

These last three equalities together with (2.46) show that

$$K_3 + \cdots + K_n = \sum_{j=3}^{n} \partial_{\zeta_j} P_j - \partial_\zeta P_1 \left(E - \partial_\zeta \psi\right)^{-1} \partial_\xi \psi - \partial_\zeta P_2 \left(E - \partial_\zeta \psi\right)^{-1} \partial_\eta \psi. \tag{2.47}$$

Combining (2.44), (2.47) and (2.42) yield

$$\begin{aligned} K_0|_{\mathscr{M}} &= \partial_\xi P_1 + \partial_\eta P_2 + \partial_\zeta P_1\, \partial_\xi \gamma + \partial_\zeta P_2\, \partial_\eta \gamma \big|_{\mathscr{M}} \\ &= \partial_\xi P_1(\xi, \eta, \gamma(\xi, \eta)) + \partial_\eta P_2(\xi, \eta, \gamma(\xi, \eta)) = \operatorname{div}(\widetilde{\mathscr{X}}|_{\mathscr{M}}). \end{aligned}$$

This together with (2.43) verifies statement (b).

The lemma is proved. □

The proof of Lemma 2.4 shows that this last lemma also holds for C^∞ smoothness.

We now study the relation between center manifolds and C^∞ inverse Jacobian multipliers.

Lemma 2.5 *Assume that system* (2.29) *satisfies* (H) *and has a* C^∞ *inverse Jacobian multiplier, which is expressed in the complex conjugate coordinates of the form*

$$J(\xi, \eta, \zeta) = \prod_{s=3}^{n} (\zeta_s - \psi_s(\xi, \eta, \zeta)) V(\xi, \eta, \zeta),$$

where $\psi_s = O(|(\xi, \eta, \zeta)|^2)$ *and* $V|_{\zeta_\ell = \psi_\ell(\xi,\eta,\zeta)} \not\equiv 0$ *for any* $\ell \in \{3, \ldots, n\}$. *The following statements hold.*

(a) $\mathscr{M}^* = \bigcap_{s=3}^{n} \{\zeta_s = \psi_s(\xi, \eta, \zeta)\}$ *is a center manifold of system* (2.29) *at the origin.*

(b) *For any smooth center manifold* $\mathscr{M}$ *of system* (2.29) *at the origin, if* $\mathscr{X}|_{\mathscr{M}}$ *has a center at the origin, then* $J|_{\mathscr{M}} = 0$.

Proof (a) The proof is similar to that of Lemma 2.4. The details are omitted.

(b) For any $C_0 \in \mathscr{M}$, which is located in a sufficiently small neighborhood of the origin, let φ_t be the periodic orbit of period T of (2.29) passing through P_0. By the definitions we have

$$\frac{dJ(\varphi_t)}{dt} = \mathscr{X}(J)|_{\varphi_t} = J \operatorname{div} \mathscr{X}|_{\varphi_t}.$$

Integrating this equation from 0 to T yields

$$J(C_0) = J(C_0) \exp\left(\int_0^T \operatorname{div} \mathscr{X}|_{\varphi_s} ds\right). \tag{2.48}$$

Restricting system (2.29) to the center manifold $\mathscr{M}$, and then writing it in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, one gets

$$dt = (1 + O(r))^{-1} d\theta.$$

Integrating this last equation along the periodic orbit yields $T = 2\pi + O(r)$. Hence we have

$$\int_0^T \operatorname{div}\mathscr{X}|_{\varphi_s} ds = \int_0^T (\iota + O(|C_0|))ds = 2\pi\iota + O(|C_0|) \neq 0,$$

in a sufficiently small neighborhood of the origin, where

$$\iota = \operatorname{div}\mathscr{X}|_{\text{at the origin}} = \lambda_3 + \cdots + \lambda_n \neq 0.$$

This together with (2.48) forces

$$J(C_0) = 0.$$

By the arbitrariness of $C_0 \in \mathscr{M}$ one has $J|_{\mathscr{M}} \equiv 0$. This proves statement (b).

This completes the proof of the lemma. □

2.3.1.2 Proof of Theorem 2.19

In the proof of Theorem 2.19 we will also use the notations given in the previous subsection.

(a) *Sufficiency*. As in the last subsection we write system (2.29) in the form (2.36) with complex conjugate coordinates. Lemma 2.4 and its proof show that system (2.36) has an analytic center manifold with the expression

$$\mathscr{M} = \bigcap_{s=3}^{n} \{\zeta_s = \gamma_s(\xi, \eta)\},$$

which is tangent to the (x, y) plane (i.e. (ξ, η) plane), where the γ_s's are analytic functions satisfying $\gamma_s(0, 0) = 0$ and $\partial\gamma_s(0, 0) = 0$. Moreover, system (2.36) restricted to $\mathscr{M}$ has an analytic inverse integrating factor $I(\xi, \eta) = \widetilde{V}(\xi, \eta, \gamma(\xi, \eta))$, where $\widetilde{V}$ is $V(x, y, z)$ written in the complex conjugate coordinates (ξ, η, ζ).

Going back to the real coordinates, the center manifold $\mathscr{M}$ can be expressed in the form

$$\mathscr{M} = \bigcap_{s=3}^{n} \{z_s = h_s(x, y)\}.$$

Set $h(x, y) = (h_3(x, y), \dots, h_n(x, y))$. Since $I(0, 0) = V(0, 0, 0) \neq 0$, and $V(x, y, h(x, y))$ is an inverse integrating factor of system (2.29) when restricted to $\mathscr{M}$, the differential one-form

$$\frac{1}{V(x, y, h(x, y))}\left((x + f_2(x, y, h(x, y)))dx + (y - f_1(x, y, h(x, y)))dy\right)$$

is well defined in a neighborhood of the origin and it is exact. So it has an analytic first integral of the form $H(x, y) = x^2 + y^2 +$ higher-order terms. By the Poincaré center theorem it follows that the origin of the vector field $\mathscr{X}$ when restricted to the center manifold $\mathscr{M}$ is a center. Furthermore, we get from Sijbrand [402, Theorems 6.3 and 7.1] that the center manifold at the origin is unique. So $\mathscr{M}^c = \mathscr{M}$ and consequently system (2.29) restricted to $\mathscr{M}^c$ has a center at the origin. This proves the statement.
Necessity. As in the proof of sufficiency, we still write system (2.29) in the form (2.36). By Lemmas 2.2 and 2.3 there exists a distinguished normalization which transforms system (2.36) to system (2.37).

By the proof of Lemma 2.3, we get that system (2.37) satisfies $g_1 = g_2$, and so it has the inverse Jacobian multiplier $\widetilde{J} = w_3 \dots w_n$. Applying Lemma 2.1 to systems (2.36) and (2.37) and combining the distinguished normalization, we deduce that system (2.36) has the analytic inverse Jacobian multiplier

$$J^* = (\zeta_3 - \psi_3(\xi, \eta, \zeta)) \dots (\zeta_n - \psi_n(\xi, \eta, \zeta))(D(\xi, \eta, \zeta))^{-1},$$

where $\psi_s = O(|(\xi, \eta, \zeta)|^2)$ is analytic, $s = 3, \dots, n$, and $D(\xi, \eta, \zeta)$ is the Jacobian of the transformation from (2.36) to (2.37), and satisfies $D(0, 0, 0) = 1$. Writing J^* in the (x, y, z) coordinates provides an analytic inverse Jacobian multiplier of system (2.29), which is of the form (2.34).
(b) The first statement, i.e. the analyticity and the uniqueness of the center manifolds, follows from the proof of sufficiency of statement (a). The second statement, i.e. $\mathscr{M}^c \subset J^{-1}(0)$, can be obtained from Lemma 2.5 (b).

This completes the proof of the theorem. □

2.3.1.3 Proof of Theorem 2.20

Here we will also use the notation introduced in Sect. 2.3.1.1.
(a) Under the assumption of the theorem, by Lemma 2.3 (a) it follows that system (2.36) is locally C^∞ equivalent to its Poincaré normal form (2.37) with $\rho_1 \neq \rho_2$. Clearly system (2.37) has the C^∞ inverse Jacobian multiplier

$$\widetilde{J}(u, v, w) = w_3 \dots w_n uv(\rho_2(uv) - \rho_1(uv)).$$

Since $\rho_2(s) - \rho_1(s)$ is nonflat at $s = 0$ and $\rho_2(0) - \rho_1(0) = 0$, we deduce that

$$\widetilde{J}(u, v, w) = w_3 \ldots w_n (uv)^{\ell} \delta(uv),$$

with $\ell \geq 2$ and $\delta(0) \neq 0$. Hence system (2.37) has a C^∞ inverse Jacobian multiplier of the form

$$J(\xi, \eta, \zeta) = \prod_{s=3}^{n} (\zeta_s - \psi_s(\xi, \eta, \zeta))((\xi - \psi_1)(\eta - \psi_2))^l \\ \times \delta((\xi - \psi_1)(\eta - \psi_2))(D(\xi, \eta, \zeta))^{-1},$$

where $\psi_s = O(|(\xi, \eta, \zeta)|^2)$ is C^∞ for $s = 1, \ldots, n$. Then the statement follows from the same arguments as in the proof of necessity of statement (a) of Theorem 2.19.
(b) The proof follows from statement (a) and Lemma 2.5.

This completes the proof of the theorem. □

2.3.2 *Hopf Bifurcation via Inverse Jacobian Multipliers*

Using the inverse Jacobian multipliers to study Hopf bifurcation, the first result was due to Buică et al. [50] for a three-dimensional differential system. Roughly speaking, *Hopf bifurcation* is a bifurcation phenomena which provides a mechanism to produce limit cycles from a singularity, i.e. for a smooth differential system with parameters, assume that at given values of the parameters the system has a singularity with a pair of pure imaginary eigenvalues, the others having nonvanishing real parts, if under a sufficiently small perturbation of the parameters the singularity is fixed and its stability changes when restricted to the two-dimensional center manifold, then the system will either produce or lose one limit cycle in a sufficiently small neighborhood of the singularity.

To study the Hopf bifurcation of system (2.29), we consider its analytic perturbation of the form

$$\begin{aligned} \dot{x} &= -y + g_1(x, y, z, \varepsilon) = G_1(x, y, z, \varepsilon), \\ \dot{y} &= \quad x + g_2(x, y, z, \varepsilon) = G_2(x, y, z, \varepsilon), \\ \dot{z} &= Az + g(x, y, z, \varepsilon) = G(x, y, z, \varepsilon), \end{aligned} \tag{2.49}$$

where $\varepsilon \in \mathbb{R}^m$ is an m-dimensional small parameter, g_1 and g_2 are analytic functions, and g is an $(n-2)$-dimensional vector-valued function. Set $\mathbf{g}(x, y, z, \varepsilon) := (g_1, g_2, g)$. We assume that

$$\mathbf{g}(x, y, z, \varepsilon) = O(|(x, y, z)|) \quad \text{and} \quad \mathbf{g}(x, y, z, 0) = \mathbf{f}(x, y, z),$$

where $\mathbf{f}$ is defined in (2.29), and that the Jacobian matrix of $\mathbf{G} = (G_1, G_2, G)$ with respect to (x, y, z) at the origin has the eigenvalues

$$\alpha(\varepsilon) \pm \mathbf{i}, \quad \lambda_s + \beta_s(\varepsilon) \text{ satisfying } \alpha(0) = \beta_s(0) = 0, \quad s = 3, \ldots, n. \tag{2.50}$$

The condition (2.50) together with the assumption (H) ensure that system (2.49) has a two-dimensional center manifold, and its origin is *monodromic* when restricted to this center manifold. By definition, a singularity of a two-dimensional smooth vector field is *monodromic* if all orbits near the singularity spiral around the singularity.

We now study the Hopf bifurcation of system (2.49) at the origin when the parameters ε vary near $0 \in \mathbb{R}^m$. By definition, the *Hopf-cyclicity* (or *cyclicity* for simplicity) of system (2.29) at the origin is the maximal number of limit cycles which can bifurcate from the origin of system (2.49) via Hopf bifurcation. This maximal number is denoted by $\mathrm{Cycl}(\mathscr{X}_\varepsilon, 0)$, where $\mathscr{X}_\varepsilon$ is the vector field associated to (2.49).

The next result, due to Zhang [475], characterizes the Hopf-cyclicity via the vanishing multiplicity of the inverse Jacobian multipliers at the origin, which is an extension of the main result of [50] from three-dimensional differential systems to any finite-dimensional system. Here the main techniques are those of [50].

Theorem 2.21 *Assume that the analytic differential system* (2.29) *satisfies the assumption* (H). *If system* (2.29) *restricted to the center manifold* $\mathscr{M}^c$ *has a focus at the origin, then* $\mathrm{Cycl}(\mathscr{X}_\varepsilon, 0) = \ell - 1$, *where* ℓ *is the vanishing multiplicity of the inverse Jacobian multiplier defined in Theorem 2.20.*

In what follows, for simplicity we assume without loss of generality that the matrix A of system (2.29) is in the real Jordan normal form.

2.3.2.1 Preparation for the Proof of Theorem 2.21

By Theorem 2.20, together with the assumption of Theorem 2.21, we get that system (2.29) has a C^∞ inverse Jacobian multiplier of the form

$$\begin{aligned} J(x, y, z) = \prod_{j=0}^{m-1} \left[(z_{3+2j} - \psi_{3+2j}(x, y, z))^2 + (z_{3+2j+1} - \psi_{3+2j+1}(x, y, z))^2 \right] \\ \times \prod_{s=3+2m}^{n} (z_s - \psi_s(x, y, z))(x^2 + y^2)^\ell V(x, y, z), \end{aligned}$$

where m is the number of pairs of the complex conjugate eigenvalues of A, or

$$J(x, y, z) = \prod_{s=3}^{n} (z_s - \psi_s(x, y, z))(x^2 + y^2)^\ell V(x, y, z),$$

where $\psi_3, \ldots, \psi_n, V \in C^\infty$ satisfy $\psi_s(x, y, z) = O(|(x, y, z)|^2)$ and $V(0, 0, 0) = 1$. Moreover, by Lemmas 2.4 and 2.5 and their proofs system (2.29) has a C^∞ center manifold $\mathscr{M}^c$ at the origin, which is the intersection of the invariant hypersurfaces

$$z_j = \psi_j(x, y, z), \qquad j = 3, \ldots, n. \tag{2.51}$$

As in Sect. 2.3.1, if $m > 0$ we choose the complex conjugate coordinates (w_{3+2s}, w_{3+2s+1}) instead of the real ones (z_{3+2s}, z_{3+2s+1}). In these new coordinates (x, y, w) we can write system (2.29) as

$$\begin{aligned} \dot{x} &= -y + \rho_1(x, y, w), \\ \dot{y} &= x + \rho_2(x, y, w), \\ \dot{w}_s &= w_s(\lambda_s + \rho_s(x, y, w)), \qquad s = 3, \ldots, n, \end{aligned} \tag{2.52}$$

where $\rho_1, \rho_2 = O(|(u, v, w)|^2)$ and $\rho_j = O(|(u, v, w)|)$, $j = 3, \ldots, n$. Obviously system (2.52) has the two-dimensional center manifold $w = 0$, because $\mathrm{Re}\lambda_s \neq 0$ for $s = 3, \ldots, n$. By Lemma 2.1 system (2.52) has the inverse Jacobian multiplier

$$\widetilde{J}(x, y, w) = w_3 \ldots w_n (x^2 + y^2)^\ell \widetilde{V}(x, y, w),$$

with $\widetilde{V} \in C^\infty$ satisfying $\widetilde{V}(0, 0, 0) \neq 0$.

Note that systems (2.29) and (2.52) are C^∞ equivalent near the origin, so we can consider without loss of generality that system (2.49) is a small perturbation of system (2.52). In the cylindrical coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\sigma, \qquad r \geq 0,$$

system (2.49) can be written as

$$\dot{\theta} = 1 + \Theta(\theta, r, \sigma, \varepsilon), \quad \dot{r} = R(\theta, r, \sigma, \varepsilon) \quad \dot{\sigma} = As + \Gamma(\theta, r, \sigma, \varepsilon), \tag{2.53}$$

where Θ, R and Γ satisfy

$$\begin{aligned} &\Theta(\theta, r, \sigma, 0) = O(r), && R(\theta, r, \sigma, 0) = O(r^2), \\ &R(\theta, 0, \sigma, \varepsilon) = 0, && \Gamma(\theta, r, \sigma, 0) = O(r). \end{aligned}$$

Correspondingly, system (2.53) when $\varepsilon = 0$ has the inverse Jacobian multiplier

$$\widetilde{J}(r\cos\theta, r\sin\theta, r\sigma)/r^{n-1} = \sigma_3 \ldots \sigma_n r^{2\ell-1} V^*(\theta, r, \sigma),$$

with $V^*(\theta, 0, 0) =$constant$\neq 0$.

In a sufficiently small neighborhood of the origin, for $|\varepsilon| \ll 1$ we have $1 + \Theta(\theta, r, \sigma, \varepsilon) > 0$. So we can rewrite system (2.53) in an equivalent way as

$$\frac{dr}{d\theta} = a(\theta, r, \sigma, \varepsilon), \quad \frac{d\sigma}{d\theta} = A\sigma + b(\theta, r, \sigma, \varepsilon), \tag{2.54}$$

where $b = (b_3, \ldots, b_n)^\tau$, and b_s has the factor σ_s for $\varepsilon = 0$. Moreover, a and b are defined on the cylinder

$$\mathscr{C} = \left\{(\theta, r, \sigma, \varepsilon) \in \mathbb{R}/(2\pi\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^m : |r|, |\varepsilon| \ll 1\right\}$$

and satisfy

$$a(\theta, 0, \sigma, \varepsilon) = 0, \quad a(\theta, r, \sigma, 0) = O(r^2), \quad b(\theta, r, \sigma, 0) = O(r). \tag{2.55}$$

Let

$$\mathscr{Y}_\varepsilon = \partial_\theta + a(\theta, r, \sigma, \varepsilon)\partial_r + \langle A\sigma + b(\theta, r, \sigma, \varepsilon), \partial_\sigma \rangle$$

be the vector field associated to system (2.54), where $\partial_\sigma = (\partial_{\sigma_3}, \ldots, \partial_{\sigma_n})$. From the inverse Jacobian multiplier of system (2.53) when $\varepsilon = 0$ we get the inverse Jacobian multiplier of the vector field $\mathscr{Y}_0$

$$J_0(\theta, r, \sigma) = \frac{\widetilde{J}(r\cos\theta, r\sin\theta, r\sigma)}{r^{n-1}(1 + \Theta(\theta, r, \sigma, 0))} = \sigma_3 \ldots \sigma_n r^{2\ell-1} V_0(\theta, r, \sigma), \tag{2.56}$$

with $V_0 = 1 + O(r)$ for r sufficiently small.

Now the study of the periodic solutions of system (2.49) is equivalent to the study of the 2π periodic solutions of system (2.54) on the cylinder $\mathscr{C}$. To apply the Poincaré map of system (2.54), we denote by

$$\varphi_\theta(r_0, \sigma_0, \varepsilon) := (r_\theta(r_0, \sigma_0, \varepsilon), \sigma_\theta(r_0, \sigma_0, \varepsilon))$$

the solution of system (2.54) satisfying the initial condition $\varphi_0(r_0, \sigma_0, \varepsilon) = (r_0, \sigma_0) \in \mathscr{C}$. Since the flow of system (2.54) is transversal to the section $\theta = 0$ on the cylinder $\mathscr{C}$, and system (2.54) is 2π periodic in θ, we can define its Poincaré map in a neighborhood of the origin as

$$\mathscr{P}(r_0, \sigma_0; \varepsilon) = \varphi_{2\pi}(r_0, \sigma_0, \varepsilon),$$

which is C^∞ because system (2.54) is C^∞. Set

$$\mathscr{P}_r(r_0, \sigma_0, \varepsilon) = r_{2\pi}(r_0, \sigma_0, \varepsilon) \quad \text{and} \quad \mathscr{P}_\sigma(r_0, \sigma_0, \varepsilon) = \sigma_{2\pi}(r_0, \sigma_0, \varepsilon).$$

We have

$$\mathscr{P}_r(r_0, \sigma_0, \varepsilon) = r_0 + \int_0^{2\pi} p(v, r_v(r_0, \sigma_0, \varepsilon), \sigma_v(r_0, \sigma_0, \varepsilon), \varepsilon)dv,$$
$$\mathscr{P}_\sigma(r_0, \sigma_0, \varepsilon) = e^{A2\pi}\left(E\sigma_0 + \int_0^{2\pi} e^{-Av} q(v, r_v(r_0, \sigma_0, \varepsilon), \sigma_v(r_0, \sigma_0, \varepsilon), \varepsilon)dv\right),$$

where E is the unit matrix of order $n-2$. Now the periodic orbits of system (2.54) are uniquely determined by the zeros of the displacement function

$$\mathscr{D}(r_0, \sigma_0, \varepsilon) := \mathscr{P}(r_0, \sigma_0, \varepsilon) - (r_0, \sigma_0).$$

Setting $\mathscr{D}_r(r_0, \sigma_0, \varepsilon) = \mathscr{P}_r(r_0, \sigma_0, \varepsilon) - r_0$, $\mathscr{D}_\sigma(r_0, \sigma_0, \varepsilon) = \mathscr{P}_\sigma(r_0, \sigma_0, \varepsilon) - \sigma_0$, we have $\mathscr{D} = (\mathscr{D}_r, \mathscr{D}_\sigma)$. By (2.55) we can check that

$$\mathscr{D}_\sigma(0, 0, 0) = 0, \qquad \partial_\sigma \mathscr{D}_\sigma(0, 0, 0) = e^{2\pi A} - E.$$

So by the implicit function theorem it follows that the functional equation

$$\mathscr{D}_\sigma(r_0, \sigma_0, \varepsilon) = 0$$

has a unique C^∞ solution, say $\sigma_0 = \delta(r_0, \varepsilon)$, defined in a neighborhood of $(r_0, \varepsilon) = (0, 0)$. Then the zeros (r_0, σ_0) of the displacement function $\mathscr{D}(r_0, \sigma_0, \varepsilon)$ are uniquely determined by the zeros r_0, in terms of ε, of

$$d(r_0, \varepsilon) := \mathscr{D}_r(r_0, \delta(r_0, \varepsilon), \varepsilon),$$

through $\sigma_0 = \delta(r_0, \varepsilon)$.

Next, to prove Theorem 2.21 we only need to study the zeros r_0 of the reduced function $d(r_0, \varepsilon)$.

2.3.2.2 Proof of Theorem 2.21

The previous subsection reduces the proof of Theorem 2.21 to the study of the zeros of $d(r_0, \varepsilon)$ in r_0.

In [48] Buică and García provided a relation between inverse Jacobian multipliers and the Poincaré map, which in system (2.54) with $\varepsilon = 0$ reads as

$$J_0(0, \mathscr{P}(r_0, \sigma_0, 0)) = J_0(0, r_0, \sigma_0) D\mathscr{P}(r_0, \sigma_0, 0), \tag{2.57}$$

where $D\mathscr{P}$ is the Jacobian of $\mathscr{P}$ with respect to (r_0, s_0). Using the expression of J_0 in (2.56), the equality (2.57) reads

$$\mathscr{P}_{\sigma 3} \ldots \mathscr{P}_{\sigma n} \mathscr{P}_r^{2\ell-1} V_0(0, \mathscr{P}_r, \mathscr{P}_\sigma) = \sigma_{03} \ldots \sigma_{0n} r_0^{2\ell-1} V_0(0, r_0, \sigma_0) D\mathscr{P}(r_0, \sigma_0, 0), \tag{2.58}$$

where $\sigma_0 = (\sigma_{03}, \ldots, \sigma_{0n})$, and

$$\mathscr{P}_\sigma(r_0, \sigma_0, 0) = (\mathscr{P}_{\sigma 3}(r_0, \sigma_0, 0), \ldots, \mathscr{P}_{\sigma n}(r_0, \sigma_0, 0)) = \langle \sigma_0, \mathscr{P}_\sigma^*(r_0, \sigma_0) \rangle,$$

with $\mathscr{P}_\sigma^*$ a suitable $(n-2)$-dimensional C^∞ function, because each hyperplane $\sigma_s = 0$, $s = 3, \ldots, n$, is invariant under the flow of system (2.54) when $\varepsilon = 0$. Some calculations show that

$$D\mathscr{P}(r_0, \sigma_0, 0)|_{\sigma_0 = 0} = \mathscr{P}_{\sigma 3}^* \ldots \mathscr{P}_{\sigma n}^* \partial_r \mathscr{P}_r|_{\sigma_0 = 0}, \quad \delta(r_0, 0) \equiv 0.$$

Via this expression the equality (2.58) simplifies to

$$\mathscr{P}_r^{2\ell-1} V_0(0, \mathscr{P}_r, 0) = r_0^{2\ell-1} V_0(0, r_0, 0) \partial_r \mathscr{P}_r \qquad \text{for} \quad \sigma_0 = 0, \tag{2.59}$$

and

$$d(r_0, 0) = \mathscr{D}(r_0, 0, 0).$$

Set $d(r_0, 0) = \kappa_k r_0^k + O(r_0^{k+1})$ with $\kappa_k \neq 0$. We have

$$\mathscr{P}_r(r_0, 0, 0) = r_0 + \kappa_k r_0^k + O(r_0^{k+1}),$$

and consequently

$$V_0(0, \mathscr{P}_r, 0) = V_0(0, r_0, 0) + O(r_0^k).$$

So the equality (2.59) can be written as

$$V_0(0, r_0, 0) \left((2\ell - 1)\kappa_k r_0^{2\ell-2+k} + O(r^{2\ell-1+k}) \right) + O(r_0^{2\ell-1+k}) = V_0(0, r_0, 0) k \kappa_k r_0^{2\ell-2+k}.$$

This together with $V_0(0, 0, 0) = 1$ shows that

$$k = 2\ell - 1.$$

Moreover $k \geq 3$, because $\ell \geq 2$ by Theorem 2.20.

Using the expression of $d(r_0, 0)$, we get from the Weierstrass preparation theorem that $d(r_0, \varepsilon)$ has at most $2\ell - 1$ zeroes. Since $r_0 = 0$ is a solution of $d(r_0, \varepsilon) = 0$, and system (2.53) is symmetric with respect to $(\theta, r, s) \to (\theta + \pi, -r, -s)$, we see that $d(r_0, \varepsilon) = 0$ has at most $\ell - 1$ positive roots in r_0. This proves that system (2.49) has at most $\ell - 1$ small amplitude limit cycles which bifurcate from the origin of system (2.29) on the two-dimensional center manifold.

The next example

$$\begin{aligned}
\dot{u} &= -v + g_1(u, v, w) + uh(u, v, \varepsilon),\\
\dot{v} &= u + g_2(u, v, w) + vh(u, v, \varepsilon),\\
\dot{w}_j &= w_j(\lambda_j + g_j(u, v)) + w_j h(u, v, \varepsilon), \qquad j = 3, \ldots, n,
\end{aligned} \tag{2.60}$$

shows the existence of $\ell - 1$ limit cycles, where $h(u, v, \varepsilon) = \sum_{s=1}^{\ell-1} \varepsilon^{\ell-s} a_s (u^2 + v^2)^s$, $\mathrm{Re}\lambda_j \neq 0, j = 3, \ldots, n$, and w_j and its conjugation appear in pairs. For more details, see [50, 161].

In short the theorem is proved. □

2.4 Inverse Jacobian Multipliers via Lie Groups

In this section we introduce some fundamental results on the study of inverse integrating factors and of inverse Jacobian multipliers via Lie groups.

An *r-parameter Lie group G* is by definition

- not only an r-dimensional smooth manifold,
- but also a group with its group operator denoted by '$\cdot$', such that
  - the group operation

    $$m : G \times G \to G, \qquad m(g, h) = g \cdot h, \quad g, h \in G,$$

  - and the inverse operation

    $$i : G \to G, \qquad i(g) = g^{-1}, \quad g \in G,$$

    are both smooth on the manifold, where g^{-1} is the inverse element of g under the group action.

Example $G = \mathbb{R}^n$ is an n-parameter Lie group with its group operator defined by the summation of two vectors in $\mathbb{R}^n$ because $\mathbb{R}^n$ is a trivial n-dimensional manifold, and the summation and its inverse operation (i.e. subtraction) are both analytic.

Another simple example is

$$G = \mathrm{SO}(2) := \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \ \theta \in [0,\ 2\pi) \right\},$$

which is a one-parameter Lie group (called the *rotation group on the plane*), where θ denotes the angle rotated counterclockwise and the group operator is the multiplication of two matrices. Indeed, the structure of a one-dimensional manifold of SO(2) can be viewed by identifying G with the unit circle in $\mathbb{R}^2$

$$\mathbb{S}^1 = \{(\cos\theta,\ \sin\theta)|\ \theta \in [0,\ 2\pi)\}.$$

Clearly SO(2) is a group under the action of multiplication of matrices, and the product and inverse operations of matrices in SO(2) are analytic.

Let $V \subset \mathbb{R}^r$ be an open subset containing the origin, and $V_0 \subset V$ be a connected subset having the origin in its interior. An *r-parameter local Lie group* consists of the connected open subset V_0 and a smooth group operator

$$m:\ V \times V \to \mathbb{R}^r,$$

together with its smooth inverse operator

$$i:\ V_0 \to V.$$

Example $V = (-\varepsilon,\ \varepsilon) \subset \mathbb{R}$ is a one-parameter local Lie group with group operation the summation of numbers in V.

Let M be a smooth manifold, and let G be a Lie group with the group operator '$\cdot$'. A *transformation group* acting on M consists of the Lie group G and a smooth map $\varphi:\ G \times M \to M$ satisfying

- $\varphi(g,\ \varphi(h,\ x)) = \varphi(g \cdot h,\ x)$ for all $x \in M$ and all $g,\ h \in G$,
- $\varphi(e,\ x) = x$ for all $x \in M$, where e is the unit element of the group G,
- $\varphi(g^{-1},\ \varphi(g,\ x)) = x$ for all $x \in M$ and all $g \in G$.

A *local transformation group* acting on M consists of the local Lie group $V_0 \subset G$ with the unit element $e \in V_0$ and a smooth map $\varphi:\ V_0 \times M \to M$ satisfying

- $\varphi(g,\ \varphi(h,\ x)) = \varphi(g \cdot h,\ x)$ for all $x \in M$ and all $g,\ h \in V_0$ such that $g \cdot h \in V_0$ and $\varphi(h, x) \in M$,
- $\varphi(e,\ x) = x$ for all $x \in M$,
- $\varphi(g^{-1},\ \varphi(g,\ x)) = x$ for all $x \in M$ and all $g \in V_0$ such that $g^{-1} \in V_0$ and $\varphi(g, x) \in M$.

Examples We now illustrate some transformation groups.

1. Translation groups in $\mathbb{R}^n$. For arbitrary $0 \neq a \in \mathbb{R}^n$ and $G = \mathbb{R}$,

$$\begin{aligned} \varphi:\quad G \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (t, x) &\longrightarrow x + at \end{aligned}$$

   is called a *translation group*. Obviously the group acts on the whole space $\mathbb{R}^n$, and its orbits are all straight lines parallel to the vector a.
2. Rescaling groups in $\mathbb{R}^n$. Let $G = \mathbb{R}^+ := \{t \in \mathbb{R}|\ t > 0\}$ be the multiplication group. Set non-zero $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. Then

$$\begin{aligned} \varphi: \quad & G \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ & (\lambda, x) \longrightarrow (\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_n} x_n), \end{aligned}$$

is called a *rescaling group*. Each orbit generated by this group is a one-dimensional smooth manifold.

3. For the smooth differential system (1.1) defined in Ω, or its associated vector field $\mathscr{X}$, denote by $\varphi_t(x)$ (or $\varphi(t, x)$) its flow satisfying $\varphi(0, x) = x$, $t \in J$, where J is either $\mathbb{R}$ or an open interval centered at the origin. Since J is a single-parameter (local) Lie group, we get from the properties of solutions of the autonomous differential systems that $\varphi_t(x)$ is a one-parameter transformation group acting on Ω. In this sense we call the vector field $\mathscr{X}$ an *infinitesimal generator* of this group action.
   For example, the infinitesimal generator of the one-parameter Lie group SO(2) acting on the plane is
   $$\dot{x} = -y, \qquad \dot{y} = x.$$

The *symmetry group* of system (1.1) is a local transformation group G acting on the subset M of the product space $\mathbb{R} \times \Omega$ of the time and the phase space, such that each integral curve of system (1.1) in M under the action of the group G is still an integral curve of the system provided that its image is still in M.

Example SO(2) is a symmetry group of the differential equation $\dfrac{d^2x}{dt^2} = 0$ because the set of integral curves of the equation is formed by all straight lines in the (t, x) plane, and the action of SO(2) is a rotation in the plane and it transforms each straight line to a straight line.

Next we need the Lie bracket of two vector fields. Let $\mathscr{X}$ and $\mathscr{Y}$ be two smooth vector fields defined on a domain Ω. Their *Lie bracket* is defined as

$$[\mathscr{X}, \mathscr{Y}](f) = \mathscr{X}\mathscr{Y}(f) - \mathscr{Y}\mathscr{X}(f), \quad \text{for all } f \in C^1(\Omega).$$

If the vector fields $\mathscr{X}$ and $\mathscr{Y}$ have local expressions

$$\mathscr{X} = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}, \qquad \mathscr{Y} = \sum_{j=1}^{n} g_j(x) \frac{\partial}{\partial x_j},$$

then

$$[\mathscr{X}, \mathscr{Y}] = \sum_{i=1}^{n} (\mathscr{X}(g_i) - \mathscr{Y}(f_i)) \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(f_j \frac{\partial g_i}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

Direct calculations show that for arbitrary smooth vector fields $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ defined on the domain Ω, their Lie bracket satisfies

- *bilinearity*: for arbitrary constants a, b,

$$[a\mathscr{X} + b\mathscr{Y},\ \mathscr{Z}] = a[\mathscr{X},\ \mathscr{Z}] + b[\mathscr{Y},\ \mathscr{Z}],$$
$$[\mathscr{X},\ a\mathscr{Y} + b\mathscr{Z}] = a[\mathscr{X},\ \mathscr{Y}] + b[\mathscr{X},\ \mathscr{Z}].$$

- *anti-symmetry*: $[\mathscr{X},\ \mathscr{Y}] = -[\mathscr{Y},\ \mathscr{X}]$.
- *Jacobi identity*: $[\mathscr{X},\ [\mathscr{Y},\ \mathscr{Z}]] + [\mathscr{Z},\ [\mathscr{X},\ \mathscr{Y}]] + [\mathscr{Y},\ [\mathscr{Z},\ \mathscr{X}]] = 0$.

The next results provide a relation between the Lie bracket of vector fields and their flows.

Theorem 2.22 *Assume that $\mathscr{X}$ and $\mathscr{Y}$ are two smooth vector fields defined in an open subset $\Omega \subset \mathbb{R}^n$.*

(a) If $\mathscr{Y}$ is an infinitesimal generator of some symmetry group of the vector field $\mathscr{X}$, then the Lie bracket of $\mathscr{X}$ and $\mathscr{Y}$ satisfies

$$[\mathscr{X},\ \mathscr{Y}] = \alpha(x)\mathscr{X},$$

where $\alpha(x)$ is a scalar function, and is a common first integral of $\mathscr{X}$ and $\mathscr{Y}$.

(b) Denote by φ_t and ψ_s the flows of the vector fields $\mathscr{X}$ and $\mathscr{Y}$, respectively. Then the two flows commute, i.e.

$$\varphi_t \circ \psi_s(x) = \psi_s \circ \varphi_t(x),$$

if and only if $[\mathscr{X},\ \mathscr{Y}] = 0$.

Theorem 2.22 (a) can be found in Berrone and Giacomini [32] and Stephani [411], and (b) can be found in Olver [351, Theorem 1.34].

We now discuss the relation between the existence of a symmetry group and an inverse integrating factor (Jacobian multiplier) of a differential system.

Theorem 2.23 *Assume that the planar smooth differential system* (1.1) *has a symmetry group whose infinitesimal generator is $\mathscr{V} = g_1\partial_{x_1} + g_2\partial_{x_2}$. Then system* (1.1) *has an inverse integrating factor of the form $V(x) = f_1 g_2 - f_2 g_1$.*

Proof See Olver [351, Theorem 2.48] and its proof. This result is a special case of the following Theorems 2.24 and 2.25, where we will give a unified proof. □

Example Consider the planar homogeneous differential system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y),$$

where P and Q are homogeneous polynomials of the same degree. We can easily check that this system has a symmetry group whose infinitesimal generator is

$$\dot{x} = x, \qquad \dot{y} = y.$$

So we get from Theorem 2.23 that this last homogeneous differential system has an inverse integrating factor $V(x, y) = xQ(x, y) - yP(x, y)$.

We now extend Theorem 2.23 to higher-dimensional differential systems.

Theorem 2.24 *Assume that the n-dimensional smooth differential system* (1.1) *has* $n-1$ *symmetry groups, and their infinitesimal generators are*

$$\mathscr{Y}_i = g_{i1}\partial_{x_1} + \cdots + g_{in}\partial_{x_n}, \quad i = 1, \ldots, n-1.$$

Then

$$V(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ g_{11} & g_{12} & \cdots & g_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n-1,1} & g_{n-1,2} & \cdots & g_{n-1,n} \end{vmatrix}$$

is an inverse Jacobian multiplier of system (1.1).

Theorem 2.24 is also a special case of the following Theorem 2.25. The proof of Theorem 2.24 will be obtained as an application of Theorem 2.25.

For simplicity of notation we will use differential forms. Here we only introduce some basic notions and fundamental results on differential forms, which will be used later on. For more details, we refer to [1, 13, 351].

Any real smooth function $f(x)$ is a *smooth differential 0-form*. Its differential

$$df = \sum_{i=1}^{n} \partial_{x_i} f \, dx_i$$

is a differential 1-form. Generally, a *smooth differential 1-form* in $\mathbb{R}^n$ is

$$\omega = a_1(x)dx_1 + \cdots + a_n(x)dx_n,$$

where the $a_i(x)$'s are real smooth functions. The differential form dx_i acting on the vector ∂_{x_j} becomes a real function, i.e.

$$\langle dx_i;\ \partial_{x_j} \rangle = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

The differential 1-form ω acting on the vector field $\mathscr{X}$ associated with system (1.1) becomes a real function, i.e.

$$\langle \omega;\ \mathscr{X} \rangle = \sum_{i=1}^{n} a_i(x) f_i(x).$$

For differential 1-forms $\omega_1, \ldots, \omega_k$, their *wedge product* $\omega_1 \wedge \cdots \wedge \omega_k$ is by definition

$$\langle \omega_1 \wedge \cdots \wedge \omega_k;\ \mathscr{X}_1, \ldots, \mathscr{X}_k \rangle = \det(\langle \omega_i;\ \mathscr{X}_j \rangle),$$

for arbitrary vectors $\mathscr{X}_1, \ldots, \mathscr{X}_k$. From the properties of determinants we easily see that the wedge product satisfies

- *multi-linearity*: for an arbitrary differential 1-form $\omega_1, \ldots, \omega_i, \omega_i', \ldots, \omega_k'$ and arbitrary constants a, b, we have

$$\begin{aligned}\omega_1 \wedge \cdots \wedge (a\,\omega_i + b\,\omega_i') \wedge \cdots \wedge \omega_k = {} & a\,\omega_1 \wedge \cdots \wedge \omega_i \wedge \cdots \wedge \omega_k \\ & + b\,\omega_1 \wedge \cdots \wedge \omega_i' \wedge \cdots \wedge \omega_k,\end{aligned}$$

- *anti-symmetry*: $\omega_i \wedge \omega_i = 0$, $\omega_i \wedge \omega_j = -\omega_j \wedge \omega_i$ for $i \neq j$.

In the coordinates $x = (x_1, \ldots, x_n)$ of $\mathbb{R}^n$, a *smooth differential k-form* is by definition

$$\omega = \sum_I a_I(x) dx^I,$$

where $I := \{(i_1, \ldots, i_k)\}$ is a set of strictly monotone and increasing indices of multiplicity k, i.e. $1 \le i_1 < i_2 < \cdots < i_k \le n$, $dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and the $a_I(x)$'s are real smooth functions.

Generally, the *wedge product* of differential forms is defined by satisfying the following properties:

- *bilinearity*: for arbitrary differential forms $\omega,\ \omega'$ and $\theta,\ \theta'$, and arbitrary constants a, b,

$$(a\,\omega + b\,\omega') \wedge \theta = a\,\omega \wedge \theta + b\,\omega' \wedge \theta, \quad \omega \wedge (a\,\theta + b\,\theta') = a\,\omega \wedge \theta + b\,\omega \wedge \theta'.$$

- *associative law*: for arbitrary differential forms $\omega,\ \theta,\ \zeta$,

$$\omega \wedge (\theta \wedge \zeta) = (\omega \wedge \theta) \wedge \zeta.$$

- *anti-commutative law*: for an arbitrary differential k-form ω and differential l-form θ,

$$\omega \wedge \theta = (-1)^{kl}\, \theta \wedge \omega.$$

Clearly any differential form of order larger than n in an n-dimensional space is zero.

The *differential* or *exterior derivative* of a differential k-form $\omega = \sum\limits_I a_I(x) dx^I$ is defined as

$$d\omega = \sum_I da_I(x) dx^I = \sum_I \sum_{j=1}^n \partial_{x_j} a_I(x) dx_j \wedge dx^I,$$

and it is a differential $(k+1)$-form.

The *Lie derivative* of a differential form with respect to a vector field $\mathscr{X}$, denoted by $L_{\mathscr{X}}$, is defined by satisfying the following properties:

- *linearity*: for arbitrary differential forms $\omega,\ \theta$ and arbitrary constants a, b,

$$L_{\mathscr{X}}(a\,\omega + b\,\theta) = a\,L_{\mathscr{X}}(\omega) + b\,L_{\mathscr{X}}(\theta).$$

- *Leibniz identity*: for arbitrary differential forms $\omega,\ \theta$,

$$L_{\mathscr{X}}(\omega\wedge\theta) = L_{\mathscr{X}}(\omega)\wedge\theta + \omega\wedge L_{\mathscr{X}}(\theta).$$

- *commutativity with differential*: $L_{\mathscr{X}}(d\omega) = dL_{\mathscr{X}}(\omega)$.

From the definition we easily see that the Lie derivative of a differential k-form with respect to the vector field $\mathscr{X}$ satisfies

$$L_{\mathscr{X}}\left(\sum_I a_I(x)dx^I\right) = \sum_I\left(L_{\mathscr{X}}(a_I)dx^I + \sum_{j=1}^k a_I dx_{i_1}\wedge\cdots\wedge L_{\mathscr{X}}(dx_{i_j})\wedge\cdots\wedge dx_{i_k}\right).$$

For the vector field $\mathscr{X} = \sum_{i=1}^n f_i\partial_{x_i}$, the Lie derivative of a smooth function $a(x)$ with respect to $\mathscr{X}$ is by definition

$$L_{\mathscr{X}}(a(x)) = f_1\partial_{x_1}a(x) + \cdots + f_n\partial_{x_n}a(x).$$

In particular,

$$L_{\mathscr{X}}(dx_i) = dL_{\mathscr{X}}(x_i) = df_i.$$

The Lie derivative of a vector field $\mathscr{Y}$ with respect to $\mathscr{X}$ is by definition

$$L_{\mathscr{X}}(\mathscr{Y}) = [\mathscr{X}, \mathscr{Y}].$$

The Lie derivative of vector fields and of differential forms with a vector field can be defined in a unified way, see e.g. Olver [351, Definition 1.63].

Theorem 2.25 *Let $\mathscr{X}$ be the vector field associated to system* (1.1)*. Assume that the n vector fields $\mathscr{X}_1, \ldots, \mathscr{X}_n$ defined in Ω satisfy*

$$L_{\mathscr{X}}(\mathscr{X}_i) = [\mathscr{X},\ \mathscr{X}_i] = \sum_{j=1}^n a_{ij}(x)\mathscr{X}_j, \qquad a_{ij}(x)\in C^1(\Omega), \quad 1\le i, j\le n,$$

and the trace of the matrix $(a_{ij}(x))$ formed by the coefficients of these last expressions vanishes, i.e. $\mathrm{tr}(a_{ij}(x))\equiv 0$*. Then*

$$V(x) = \det(\langle dx_i;\ \mathscr{X}_j\rangle)$$

is an inverse Jacobian multiplier of system (1.1)*.*

Proof The proof follows from Berrone and Giacomini [31]. By the properties of determinants, it follows that the Lie derivative of a differential k-form ω acting on vector fields $\mathscr{X}_1, \ldots, \mathscr{X}_k$ with respect to $\mathscr{X}$ is

$$\begin{aligned} L_{\mathscr{X}}(\langle \omega;\ \mathscr{X}_1, \ldots, \mathscr{X}_k\rangle) =&\langle L_{\mathscr{X}}(\omega);\ \mathscr{X}_1, \ldots, \mathscr{X}_k\rangle \\ &+ \sum_{i=1}^{k}(\langle \omega;\ \mathscr{X}_1, \ldots, L_{\mathscr{X}}(\mathscr{X}_i), \ldots, \mathscr{X}_k\rangle). \end{aligned}$$

Next we compute the three parts by using the volume form $\sigma = dx_1 \wedge \cdots \wedge dx_n$ to replace ω. From the wedge product of differential 1-forms we get

$$L_{\mathscr{X}}(\langle \sigma;\ \mathscr{X}_1, \ldots, \mathscr{X}_n\rangle) = L_{\mathscr{X}} \det(\langle dx_i, \mathscr{X}_j\rangle) = L_{\mathscr{X}} V = \mathscr{X}(V).$$

Then it follows from the properties of the Lie derivative and the linearity of the wedge product of differential forms that

$$\begin{aligned} \langle L_{\mathscr{X}}(\sigma);\ \mathscr{X}_1, \ldots, \mathscr{X}_n\rangle &= \sum_{i=1}^{n}\langle dx_1 \wedge \cdots \wedge L_{\mathscr{X}}(dx_i) \wedge \cdots \wedge dx_n;\ \mathscr{X}_1, \ldots, \mathscr{X}_n\rangle \\ &= \sum_{i=1}^{n}\langle dx_1 \wedge \cdots \wedge df_i \wedge \cdots \wedge dx_n;\ \mathscr{X}_1, \ldots, \mathscr{X}_n\rangle \\ &= \sum_{i=1}^{n}\langle dx_1 \wedge \cdots \wedge \partial_{x_i} f_i dx_i \wedge \cdots \wedge dx_n;\ \mathscr{X}_1, \ldots, \mathscr{X}_n\rangle \\ &= \sum_{i=1}^{n}\partial_{x_i} f_i \langle dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_n;\ \mathscr{X}_1, \ldots, \mathscr{X}_n\rangle = V \operatorname{div} f, \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{n}(\langle \sigma;\ \mathscr{X}_1, \ldots, L_{\mathscr{X}}(\mathscr{X}_i), \ldots, \mathscr{X}_n\rangle) &= \sum_{i=1}^{n}\left(\left\langle \sigma;\ \mathscr{X}_1, \ldots, \sum_{j=1}^{n} a_{ij}(x)\mathscr{X}_j, \ldots, \mathscr{X}_n\right\rangle\right) \\ &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}(x)\langle \sigma;\ \mathscr{X}_1, \ldots, \mathscr{X}_j, \ldots, \mathscr{X}_n\rangle\right) \\ &= \sum_{i=1}^{n}(a_{ii}(x)\langle \sigma;\ \mathscr{X}_1, \ldots, \mathscr{X}_i, \ldots, \mathscr{X}_n\rangle) \\ &= \operatorname{tr}(a_{ij})V = 0. \end{aligned}$$

These verify $\mathscr{X}(V) = V \operatorname{div} f = V \operatorname{div} \mathscr{X}$, and consequently V is an inverse Jacobian multiplier of system (1.1). This proves the theorem. $\square$

Proof of Theorem 2.24. Set $\mathscr{X}_1 = \mathscr{X}, \ \mathscr{X}_i = \mathscr{Y}_{i-1}, \ i = 2, \ldots, n$. Then we have

$$V(x) = \det(\langle dx_i; \ \mathscr{X}_j \rangle).$$

By the assumption and Theorem 2.22 (a) we get that there exist functions $a_i(x), \ i = 2, \ldots, n$, such that

$$\begin{aligned} L_{\mathscr{X}}(\mathscr{X}_1) &= [\mathscr{X}, \ \mathscr{X}_1] = [\mathscr{X}, \ \mathscr{X}] = 0, \\ L_{\mathscr{X}}(\mathscr{X}_i) &= [\mathscr{X}, \ \mathscr{X}_i] = [\mathscr{X}, \ \mathscr{Y}_{i-1}] = a_i(x)\mathscr{X} = a_i(x)\mathscr{X}_1, \quad i = 2, \ldots, n. \end{aligned}$$

Obviously the trace of the matrix formed by the coefficients of the expressions of these Lie derivatives is zero. It follows from Theorem 2.25 that $V(x)$ is an inverse Jacobian multiplier of system (1.1). This proves the theorem. □

Some of the results on the relation between inverse integrating factors and limit cycles for planar differential systems can be extended to inverse Jacobian multipliers for higher-dimensional differential systems. Roughly speaking, a limit cycle of a higher-dimensional differential system is an isolated periodic orbit.

Theorem 2.26 *Assume that $V(x)$ is a smooth inverse Jacobian multiplier of the smooth differential system* (1.1) *in an open set $\Omega_0 \subset \Omega$, and that $\Gamma \subset \Omega_0$ is a limit cycle of system* (1.1)*, which is an ω (resp. α) limit set of all nearby orbits. Then $\Gamma \subset V^{-1}(0)$.*

Proof The main idea of the proof comes from [31]. Let φ_t be the flow of system (1.1). Since the inverse Jacobian multiplier V satisfies

$$\mathscr{X}(V) = V \mathrm{div} f,$$

if $\varphi_t(x) \in \Omega_0$ we have

$$V(\varphi_t(x)) = V(x) \exp\left(\int_0^t \mathrm{div} f \circ \varphi_s(x) ds\right).$$

This shows that for any orbit $\gamma \subset \Omega_0$, we have either $\gamma \subset V^{-1}(0)$ or $\gamma \bigcap V^{-1}(0) = \emptyset$. Without loss of generality we assume that Γ is the ω limit set of all its nearby orbits, and so it is asymptotically stable.

On the contrary, we assume that $\Gamma \not\subseteq V^{-1}(0)$. Then there exists a closed tubular neighborhood $B_\varepsilon(\Gamma)$ centered at Γ of radius $\varepsilon > 0$, such that

$$0 < m \le V(x) \le M, \qquad \text{for } x \in B_\varepsilon(\Gamma),$$

where $m, M \in \mathbb{R}$. Since Γ is asymptotically stable, we assume without loss of generality that

$$\varphi_t(B_\varepsilon(\Gamma)) \subset B_\varepsilon(\Gamma), \qquad t > 0.$$

For $D_0 \subset \mathbb{R}^n$, denote by $\|D_0\|$ its volume. It follows from Theorem 2.2 (i.e. Poincaré's theorem) that the integral

$$\int_{\varphi_t(B_\varepsilon(\Gamma))} \frac{dx}{V(x)}$$

is invariant with respect to the time t. Hence we have

$$\frac{\|\varphi_t(B_\varepsilon(\Gamma))\|}{m} \geq \int_{\varphi_t(B_\varepsilon(\Gamma))} \frac{dx}{V(x)} = \int_{B_\varepsilon(\Gamma)} \frac{dx}{V(x)} \geq \frac{\|B_\varepsilon(\Gamma)\|}{M}, \quad t > 0.$$

On the other hand, since

$$\lim_{t\to\infty} \varphi_t(B_\varepsilon(\Gamma)) \subset \Gamma,$$

we have

$$\lim_{t\to\infty} \|\varphi_t(B_\varepsilon(\Gamma))\| = 0.$$

This contradiction implies that $\Gamma \subset V^{-1}(0)$. $\square$

For higher-dimensional differential systems it is still an open problem to *characterize the existence and regularity of the inverse Jacobian multipliers near a singularity or a limit cycle, except in the cases studied in Sect.* 2.3.

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