PART I MODELING

1 INTRODUCTION ............................................. 3
   1.1 Vibration Tests ........................................ 5
      1.1.1 Free Vibration Test ............................ 6
      1.1.2 Forced Vibration Test .......................... 7
      1.1.3 Ambient Vibration Test ......................... 8
   1.2 Uncertainties ........................................ 9
      1.2.1 Variability and Identification Uncertainty .......... 9
      1.2.2 Sources of Identification Uncertainty .............. 10
   1.3 OMA Methods ...................................... 10
   1.4 Non-Bayesian Approach ............................... 11
      1.4.1 Eliminating Random Response ................... 12
      1.4.2 Exploiting Statistics ............................ 12
      1.4.3 Identification Uncertainty ........................ 14
   1.5 Bayesian Approach ................................... 15
      1.5.1 Philosophy ................................... 15
      1.5.2 Posterior Distribution and Statistics .............. 16
      1.5.3 Computing Posterior Statistics .................... 17
      1.5.4 Formulations and Algorithms ..................... 17
      1.5.5 Maximum Likelihood Estimation ................... 18
      1.5.6 Drawbacks and Limitations ..................... 18
   1.6 Overview of This Book ............................... 19
      1.6.1 Modeling .................................... 20
      1.6.2 Inference .................................... 20
      1.6.3 Algorithms ................................... 21
      1.6.4 Uncertainty Laws ................................ 21
   1.7 How to Use This Book ............................... 22
      1.7.1 Student ..................................... 22
      1.7.2 Researcher ................................... 24
2 Spectral Analysis of Deterministic Process ........................................ 29
  2.1 Periodic Process (Fourier Series) ......................................... 30
     2.1.1 Complex Exponential Form .................................. 32
     2.1.2 Parseval Equality ........................................ 34
  2.2 Non-periodic Process (Fourier Transform) .................................. 35
     2.2.1 From Fourier Series to Fourier Transform ....................... 35
     2.2.2 Properties of Fourier Transform ............................... 37
     2.2.3 Dirac Delta Function ........................................ 38
     2.2.4 Parseval Equality ........................................... 38
  2.3 Discrete-Time Approximation with FFT .................................... 39
     2.3.1 Fast Fourier Transform ..................................... 40
     2.3.2 Approximating Fourier Transform and Fourier Series ............ 42
     2.3.3 Parseval Equality ........................................... 43
  2.4 Distortions in Fourier Series .............................................. 43
     2.4.1 Nyquist Frequency .......................................... 44
     2.4.2 Aliasing .................................................. 44
     2.4.3 Leakage .................................................. 46
  2.5 Distortions in Fourier Transform .......................................... 49
  2.6 Summary of FFT Approximations ............................................ 50
  2.7 Summary of Fourier Formulas, Units and Conventions ...................... 50
     2.7.1 Multiplier in Fourier Transform ............................... 50
  2.8 Connecting Theory with Matlab ............................................ 53
  2.9 FFT Algorithm ........................................................ 54
     2.9.1 Basic Idea ................................................ 55
     2.9.2 Computational Effort ........................................ 56
  References .............................................................................. 57

3 Structural Dynamics and Modal Testing .......................................... 59
  3.1 SDOF Dynamics ............................................................. 60
     3.1.1 Natural Frequency ............................................. 61
     3.1.2 Damping Ratio ............................................... 63
     3.1.3 Damped Free Vibration ....................................... 63
     3.1.4 Logarithmic Decrement Method ................................. 67
     3.1.5 Harmonic Excitation ........................................... 68
     3.1.6 Simplifying Algebra with Complex Number ...................... 71
     3.1.7 Dynamic Amplification ....................................... 72
     3.1.8 Half-Power Bandwidth Method ................................. 74
     3.1.9 Principle of Superposition ................................... 77
     3.1.10 Periodic Excitation .......................................... 78
## 6 Measurement Basics

6.1 Data Acquisition Process ........................................... 205  
6.2 Channel Noise ..................................................... 206  
6.3 Sensor/Hardware Noise ........................................... 207  
6.4 Sensor Principle .................................................. 209  
6.5 Aliasing ............................................................. 212  
6.6 Quantization Error .................................................. 213  
   6.6.1 Statistical Properties ....................................... 215  
   6.6.2 Power Spectral Density ..................................... 215  
6.7 Synchronization ................................................... 216  
6.8 Channel Noise Calibration ......................................... 218  
   6.8.1 Base Isolation ................................................. 219  
   6.8.2 Huddle Test ................................................... 220  
   6.8.3 Three Channel Analysis ..................................... 222  

References .......................................................... 224

## 7 Ambient Data Modeling and Analysis

7.1 Resonance Band Characteristics ................................. 226  
   7.1.1 Single Mode .................................................. 227  
   7.1.2 Multi-mode ................................................... 228  
7.2 PSD Spectrum ..................................................... 228  
   7.2.1 Procedure .................................................... 229  
7.3 Singular Value Spectrum .......................................... 231  
   7.3.1 Single Mode .................................................. 232  
   7.3.2 Multi-mode ................................................... 234  
7.4 Illustration with Field Data ..................................... 237  
   7.4.1 Time Histories ................................................. 238  
   7.4.2 Sample PSD (No Averaging) ................................. 238  
   7.4.3 Sample PSD (Averaged) ..................................... 239  
   7.4.4 Singular Value Spectrum ................................... 240  
7.5 Asynchronous Data ................................................ 241  
   7.5.1 Two Measurement Groups ................................... 242  
   7.5.2 Multiple Measurement Groups ............................... 247  
7.6 Microtremor Data ................................................ 249  
   7.6.1 Background Seismic Noise .................................. 249  
   7.6.2 Site Amplification and H/V Spectrum ....................... 252  
7.7 Simulation of Ambient Data ....................................... 255  
   7.7.1 Gaussian Scalar Process .................................... 255  
   7.7.2 Gaussian Vector Process .................................... 258  
   7.7.3 Quantifying Noise Level .................................... 260  

References .......................................................... 261
# Part II Inference

## 8 Bayesian Inference

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.1</td>
<td>Bayes’ Theorem</td>
<td>266</td>
</tr>
<tr>
<td>8.2</td>
<td>Updating Knowledge Using Data</td>
<td>267</td>
</tr>
<tr>
<td>8.3</td>
<td>System Identification Framework</td>
<td>268</td>
</tr>
<tr>
<td>8.4</td>
<td>Identifiability</td>
<td>268</td>
</tr>
<tr>
<td>8.5</td>
<td>Globally Identifiable Problems</td>
<td>274</td>
</tr>
<tr>
<td>8.5.1</td>
<td>Quality of Gaussian Approximation</td>
<td>275</td>
</tr>
<tr>
<td>8.6</td>
<td>Locally Identifiable Problems</td>
<td>283</td>
</tr>
<tr>
<td>8.7</td>
<td>Unidentifiable Problems</td>
<td>284</td>
</tr>
<tr>
<td>8.8</td>
<td>Model Class Selection</td>
<td>285</td>
</tr>
<tr>
<td>8.8.1</td>
<td>Comparing Model Classes with Evidence</td>
<td>285</td>
</tr>
<tr>
<td>8.8.2</td>
<td>Model Trade-off</td>
<td>286</td>
</tr>
</tbody>
</table>

## 9 Classical Statistical Inference

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.1</td>
<td>Statistical Estimators</td>
<td>293</td>
</tr>
<tr>
<td>9.1.1</td>
<td>Quality Statistics</td>
<td>293</td>
</tr>
<tr>
<td>9.1.2</td>
<td>Bias and Convergence</td>
<td>294</td>
</tr>
<tr>
<td>9.1.3</td>
<td>Empirical Statistics</td>
<td>294</td>
</tr>
<tr>
<td>9.2</td>
<td>Maximum Likelihood Estimator</td>
<td>295</td>
</tr>
<tr>
<td>9.3</td>
<td>Cramér-Rao Bound</td>
<td>300</td>
</tr>
<tr>
<td>9.3.1</td>
<td>Easier but Looser Bounds</td>
<td>307</td>
</tr>
<tr>
<td>9.3.2</td>
<td>General Form</td>
<td>310</td>
</tr>
<tr>
<td>9.3.3</td>
<td>Derivation</td>
<td>310</td>
</tr>
<tr>
<td>9.4</td>
<td>Fisher Information Matrix for Gaussian Data</td>
<td>312</td>
</tr>
<tr>
<td>9.4.1</td>
<td>Real Gaussian</td>
<td>312</td>
</tr>
<tr>
<td>9.4.2</td>
<td>Complex Gaussian</td>
<td>315</td>
</tr>
<tr>
<td>9.5</td>
<td>Asymptotic Properties of ML Estimator</td>
<td>316</td>
</tr>
<tr>
<td>9.6</td>
<td>Comparison with Bayesian Inference</td>
<td>319</td>
</tr>
<tr>
<td>9.6.1</td>
<td>Philosophical Perspectives</td>
<td>319</td>
</tr>
<tr>
<td>9.6.2</td>
<td>Maximum Likelihood Estimator</td>
<td>320</td>
</tr>
<tr>
<td>9.6.3</td>
<td>Cramér-Rao Bound and Uncertainty Law</td>
<td>321</td>
</tr>
</tbody>
</table>

## 10 Bayesian OMA Formulation

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.1</td>
<td>Single Setup Data</td>
<td>325</td>
</tr>
<tr>
<td>10.1.1</td>
<td>Likelihood Function</td>
<td>326</td>
</tr>
<tr>
<td>10.1.2</td>
<td>Single Mode</td>
<td>329</td>
</tr>
<tr>
<td>10.2</td>
<td>Remarks to Formulation</td>
<td>329</td>
</tr>
<tr>
<td>10.2.1</td>
<td>Complex Gaussian FFT</td>
<td>330</td>
</tr>
<tr>
<td>10.2.2</td>
<td>Selected Frequency Band</td>
<td>330</td>
</tr>
<tr>
<td>10.2.3</td>
<td>Prediction Error Model</td>
<td>331</td>
</tr>
<tr>
<td>10.2.4</td>
<td>Measurement Type</td>
<td>332</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>10.2.5</td>
<td>Mode Shape Scaling</td>
<td>333</td>
</tr>
<tr>
<td>10.2.6</td>
<td>Leakage</td>
<td>335</td>
</tr>
<tr>
<td>10.3</td>
<td>Multi-setup Data</td>
<td>336</td>
</tr>
<tr>
<td>10.3.1</td>
<td>Global and Local Mode Shape</td>
<td>337</td>
</tr>
<tr>
<td>10.3.2</td>
<td>Reference DOFs</td>
<td>337</td>
</tr>
<tr>
<td>10.3.3</td>
<td>Parameters in Different Setups</td>
<td>338</td>
</tr>
<tr>
<td>10.3.4</td>
<td>Likelihood Function</td>
<td>339</td>
</tr>
<tr>
<td>10.3.5</td>
<td>Single Mode</td>
<td>340</td>
</tr>
<tr>
<td>10.4</td>
<td>Asynchronous Data</td>
<td>341</td>
</tr>
<tr>
<td>10.4.1</td>
<td>PSD Matrix</td>
<td>341</td>
</tr>
<tr>
<td>10.4.2</td>
<td>Single Mode</td>
<td>343</td>
</tr>
</tbody>
</table>

11 Bayesian OMA Computation | 345 |
| 11.1 | Posterior Most Probable Value | 346 |
| 11.2 | Posterior Covariance Matrix | 348 |
| 11.2.1 | Mapping with Free Parameters | 348 |
| 11.2.2 | Transformation of Covariance Matrix | 349 |
| 11.2.3 | Hessian of Composite Function | 349 |
| 11.2.4 | Transformation Invariance | 351 |
| 11.2.5 | Constraint Singularity | 352 |
| 11.2.6 | Pseudo-inverse | 353 |
| 11.2.7 | Singular Vector Formula | 355 |
| 11.2.8 | Dimensionless Hessian | 356 |
| 11.3 | Mode Shape Uncertainty | 359 |
| 11.3.1 | Norm Constraint Singularity | 360 |
| 11.3.2 | Stochastic Representation | 360 |
| 11.3.3 | Expected MAC and Mode Shape c.o.v | 361 |

Part III Algorithms | 365 |
| 12 Single Mode Problem | 366 |
| 12.1 | Alternative Form of NLLF | 368 |
| 12.2 | Algorithm for MPV | 368 |
| 12.3 | High s/n Asymptotics of MPV | 368 |
| 12.3.1 | Initial Guess of MPV | 370 |
| 12.4 | Posterior Covariance Matrix | 370 |
| 12.4.1 | General Expressions | 371 |
| 12.4.2 | Condensed Expressions | 372 |
| 12.5 | Synthetic Data Examples | 374 |
| 12.6 | Laboratory/Field Data Examples | 381 |

References | 390 |
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>Multi-mode Problem</td>
<td>391</td>
</tr>
<tr>
<td>13.1</td>
<td>Mode Shape Subspace</td>
<td>393</td>
</tr>
<tr>
<td>13.1.1</td>
<td>Orthonormal Basis Representation</td>
<td>393</td>
</tr>
<tr>
<td>13.2</td>
<td>Alternative Form of NLLF</td>
<td>394</td>
</tr>
<tr>
<td>13.3</td>
<td>Most Probable Mode Shape Basis</td>
<td>396</td>
</tr>
<tr>
<td>13.3.1</td>
<td>Hyper Angle Representation</td>
<td>396</td>
</tr>
<tr>
<td>13.3.2</td>
<td>Rotation Matrix</td>
<td>397</td>
</tr>
<tr>
<td>13.3.3</td>
<td>Newton Iteration</td>
<td>398</td>
</tr>
<tr>
<td>13.4</td>
<td>Most Probable Spectral Parameters</td>
<td>402</td>
</tr>
<tr>
<td>13.4.1</td>
<td>Parameterizing Structured Matrices</td>
<td>402</td>
</tr>
<tr>
<td>13.5</td>
<td>Algorithm for MPV</td>
<td>403</td>
</tr>
<tr>
<td>13.6</td>
<td>High s/n Asymptotics of MPV</td>
<td>404</td>
</tr>
<tr>
<td>13.6.1</td>
<td>Initial Guess of MPV</td>
<td>405</td>
</tr>
<tr>
<td>13.7</td>
<td>Posterior Covariance Matrix</td>
<td>405</td>
</tr>
<tr>
<td>13.7.1</td>
<td>General Expressions</td>
<td>407</td>
</tr>
<tr>
<td>13.7.2</td>
<td>Condensed Expressions</td>
<td>408</td>
</tr>
<tr>
<td>13.8</td>
<td>Illustrative Examples</td>
<td>413</td>
</tr>
<tr>
<td>References</td>
<td></td>
<td>418</td>
</tr>
<tr>
<td>14</td>
<td>Multi-setup Problem</td>
<td>419</td>
</tr>
<tr>
<td>14.1</td>
<td>Local Least Squares</td>
<td>420</td>
</tr>
<tr>
<td>14.2</td>
<td>Global Least Squares</td>
<td>422</td>
</tr>
<tr>
<td>14.2.1</td>
<td>Partial Solutions</td>
<td>423</td>
</tr>
<tr>
<td>14.2.2</td>
<td>Limiting Behavior of Solution</td>
<td>424</td>
</tr>
<tr>
<td>14.2.3</td>
<td>Iterative Algorithm</td>
<td>425</td>
</tr>
<tr>
<td>14.2.4</td>
<td>Reference Condensation</td>
<td>426</td>
</tr>
<tr>
<td>14.3</td>
<td>Bayesian Method</td>
<td>427</td>
</tr>
<tr>
<td>14.3.1</td>
<td>Alternative Form of NLLF</td>
<td>428</td>
</tr>
<tr>
<td>14.3.2</td>
<td>Partial MPV of Global Mode Shape</td>
<td>430</td>
</tr>
<tr>
<td>14.3.3</td>
<td>Algorithm for MPV</td>
<td>431</td>
</tr>
<tr>
<td>14.3.4</td>
<td>High s/n Asymptotic MPV</td>
<td>431</td>
</tr>
<tr>
<td>14.3.5</td>
<td>Initial Guess</td>
<td>433</td>
</tr>
<tr>
<td>14.3.6</td>
<td>Asymptotic Weight for Global Least Squares</td>
<td>433</td>
</tr>
<tr>
<td>14.3.7</td>
<td>Posterior Covariance Matrix</td>
<td>434</td>
</tr>
<tr>
<td>14.4</td>
<td>Representative Statistics</td>
<td>437</td>
</tr>
<tr>
<td>14.5</td>
<td>Field Applications</td>
<td>438</td>
</tr>
<tr>
<td>References</td>
<td></td>
<td>450</td>
</tr>
</tbody>
</table>

Part IV Uncertainty Laws

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>Managing Identification Uncertainties</td>
<td>455</td>
</tr>
<tr>
<td>15.1</td>
<td>Context and Key Formulas</td>
<td>456</td>
</tr>
<tr>
<td>15.2</td>
<td>Understanding Uncertainty Laws</td>
<td>460</td>
</tr>
<tr>
<td>15.2.1</td>
<td>Data Length and Usable Bandwidth</td>
<td>461</td>
</tr>
</tbody>
</table>
Operational Modal Analysis
Modeling, Bayesian Inference, Uncertainty Laws
Au, S.-K.
2017, XXIII, 542 p. 158 illus., 28 illus. in color., Hardcover
ISBN: 978-981-10-4117-4