Chapter 2
Spectral Analysis of Deterministic Process

Abstract This chapter analyzes a deterministic process in the frequency domain via Fourier series (for period process) and Fourier Transform (for non-period process). Estimators based on a sample time history of the process are introduced and their possible distortions arising from finite sampling rate (aliasing) and duration (leakage) are discussed. The Fast Fourier Transform provides an efficient computational tool for digital implementation. The chapter contains a section that connects the presented mathematical theory to implementation in Matlab, which is a convenient platform for scientific computing.

Keywords Fourier series · Fourier Transform · Fast Fourier Transform · Matlab · Aliasing · Leakage

A time series, or 'process', describes how a quantity varies with time. Viewing as a function of time allows one to see, e.g., when it is zero, how fast it changes, the minimum and maximum values. This 'time domain' view is not the only perspective. One useful alternative is the 'frequency domain' view, as a sum of 'harmonics' (sines and cosines) at different frequencies and studying how their amplitude varies with frequency. This view is especially relevant for a variety of processes that exhibit variations at different time scales, revealing the characteristics of contributing activities. It also offers a powerful means for studying the oscillatory response of systems with resonance behavior.

In this chapter we introduce the theory for analyzing a process in the frequency domain, namely, 'Fourier analysis'. The basic result is that a periodic process can be written as a 'Fourier series' (FS), which is a sum of harmonics at discrete frequencies. An analogous result holds for a non-periodic process with finite energy, where the sum becomes an integral over a continuum of frequencies and is called 'Fourier Transform' (FT). In digital computations, the integrals involved in Fourier analysis can be approximated in discrete time and computed efficiently via the 'Fast Fourier Transform' (FFT) algorithm. This approximation leads to distortions, namely, limited scope by 'Nyquist frequency', 'aliasing' and 'leakage'. These must be in check so that the calculated results reflect well their targets and are

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correctly interpreted. Fourier theory is often introduced in undergraduate texts of differential equations, e.g., Boyce and DiPrima (2005). See Champeney (1987) for Fourier theorems and Sundararajan (2001) for FFT. Digital signal processing is a closely related subject, see, e.g., Lathi (2000). Smith (2002) gives a non-technical coverage.

As the title tells, this chapter is about 'deterministic process', where the subject time series is taken as 'fixed' or 'given'. No probability concept is involved. While all data (including the ones in operational modal analysis) are by definition deterministic when they are obtained, their downstream effects can be understood much better when they are modeled using probabilistic concepts. This will be taken up in Chap. 4, where Fourier analysis is applied in a probabilistic context.

2.1 Periodic Process (Fourier Series)

A periodic process repeats itself at a fixed time interval. We say that a function $x(t)$ of time t (s) is 'periodic with period T' if T is the smallest value such that

$$x(t+T) = x(t) \quad \text{for any } t \tag{2.1}$$

The proviso 'the smallest value' is necessary to remove ambiguity, because if (2.1) holds then for any integer m:

$$x(t+mT) = x(t+(m-1)T+T) = x(t+(m-1)T) = \cdots = x(t+T) = x(t) \tag{2.2}$$

According to the Fourier theorem, a periodic process $x(t)$ can be written as a 'Fourier series' (FS):

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos \bar{\omega}_k t + \sum_{k=1}^{\infty} b_k \sin \bar{\omega}_k t \qquad \bar{\omega}_k = \frac{2\pi k}{T} \tag{2.3}$$

Here, a_k and b_k are called the (real) 'Fourier series coefficients', associated with harmonic oscillations with period T/k (s), i.e., frequency $\bar{\omega}_k = 2\pi k/T$ (rad/s). This can be seen by noting that as t goes from 0 to T/k the argument $\bar{\omega}_k t$ in the cosine and sine terms goes from 0 to 2π, hence completing one cycle. The term a_0 accounts for the constant 'static' level of the process. Generally, a periodic process need not be just a finite sum of cosines and sines, but the Fourier theorem says that by including an *infinite* number of them with systematically increasing frequencies it is possible to represent *any* periodic process.

Amplitude and Phase
By writing

$$a_k \cos \bar{\omega}_k t + b_k \sin \bar{\omega}_k t = \sqrt{a_k^2 + b_k^2}\left[\underbrace{\frac{a_k}{\sqrt{a_k^2 + b_k^2}}}_{\cos \phi_k} \cos \bar{\omega}_k t + \underbrace{\frac{b_k}{\sqrt{a_k^2 + b_k^2}}}_{\sin \phi_k} \sin \bar{\omega}_k t\right] \tag{2.4}$$

and using the compound angle formula $\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$, (2.3) can be written as

$$x(t) = a_0 + \sum_{k=1}^{\infty} \underbrace{\sqrt{a_k^2 + b_k^2}}_{\text{amplitude}} \cos(\underbrace{\bar{\omega}_k}_{\text{frequency}} t - \underbrace{\phi_k}_{\text{phase}}) \qquad \tan \phi_k = \frac{b_k}{a_k} \tag{2.5}$$

Expressions of Fourier Series Coefficients
Clearly, a_k and b_k depend on $x(t)$. They are given by

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt \tag{2.6}$$

$$\begin{aligned} a_k &= \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos \bar{\omega}_k t\, dt \\ b_k &= \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin \bar{\omega}_k t\, dt \end{aligned} \qquad k = 1, 2, \ldots \tag{2.7}$$

In these expressions, the integration domain can be any continuous interval of length T because $x(t)$ has period T.

Proof of (2.6) and (2.7) (Fourier Series Coefficients)
The expression of a_0 in (2.6) can be shown by integrating both sides of (2.3) w.r.t. t from $-T/2$ to $T/2$ and noting that the integrals of sines and cosines on the RHS are all equal to zero. The derivation of a_k and b_k $(k \geq 1)$ makes use of the following results (j and k are non-zero integers):

$$\int_{-T/2}^{T/2} \cos \bar{\omega}_j t \, \cos \bar{\omega}_k t \, dt = \begin{cases} 0 & j \neq k \\ T/2 & j = k \end{cases} \tag{2.8}$$

$$\int_{-T/2}^{T/2} \sin \bar{\omega}_j t \ \sin \bar{\omega}_k t \, dt = \begin{cases} 0 & j \neq k \\ T/2 & j = k \end{cases} \tag{2.9}$$

$$\int_{-T/2}^{T/2} \cos \bar{\omega}_j t \ \sin \bar{\omega}_k t \, dt = 0 \qquad \text{any } j, k \tag{2.10}$$

In particular, multiplying both sides of $x(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos \bar{\omega}_j t + \sum_{j=1}^{\infty} b_j \sin \bar{\omega}_j t$ with $\cos \bar{\omega}_k t$ and integrating from $-T/2$ to $T/2$,

$$\begin{aligned} \int_{-T/2}^{T/2} x(t) \cos \bar{\omega}_k t \, dt &= a_0 \underbrace{\int_{-T/2}^{T/2} \cos \bar{\omega}_k t \, dt}_{0} + \overbrace{\sum_{j=1}^{\infty} a_j \underbrace{\int_{-T/2}^{T/2} \cos \bar{\omega}_j t \cos \bar{\omega}_k t \, dt}_{T/2 \text{ if } j=k;\ \text{else } 0}}^{a_k T/2} \\ &+ \sum_{j=1}^{\infty} b_j \underbrace{\int_{-T/2}^{T/2} \sin \bar{\omega}_j t \cos \bar{\omega}_k t \, dt}_{0} = \frac{a_k T}{2} \end{aligned} \tag{2.11}$$

Rearranging gives the expression of a_k in (2.7). Multiplying $x(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos \bar{\omega}_j t + \sum_{j=1}^{\infty} b_j \sin \bar{\omega}_j t$ with $\sin \bar{\omega}_k t$ and following similar steps gives the expression of b_k in (2.7). ■

2.1.1 Complex Exponential Form

The FS in (2.3) contains for each frequency a sine and cosine term. It is possible to combine the two terms into a single term using the 'Euler formula':

$$e^{\mathbf{i}\theta} = \cos \theta + \mathbf{i} \sin \theta \qquad \mathbf{i}^2 = -1 \tag{2.12}$$

for any real number θ. Conversely, cosine and sine can be written in terms of $e^{\pm\mathbf{i}\theta}$ as

$$\cos\theta = \frac{1}{2}(\mathrm{e}^{\mathbf{i}\theta} + \mathrm{e}^{-\mathbf{i}\theta}) \qquad \sin\theta = \frac{1}{2\mathbf{i}}(\mathrm{e}^{\mathbf{i}\theta} - \mathrm{e}^{-\mathbf{i}\theta}) \tag{2.13}$$

Applying these identities to (2.3) and writing $1/2\mathbf{i} = -\mathbf{i}/2$ gives

$$\begin{aligned} x(t) &= a_0 + \sum_{k=1}^{\infty} \frac{a_k}{2}(\mathrm{e}^{\mathbf{i}\bar{\omega}_k t} + \mathrm{e}^{-\mathbf{i}\bar{\omega}_k t}) + \sum_{k=1}^{\infty} -\frac{\mathbf{i}b_k}{2}(\mathrm{e}^{\mathbf{i}\bar{\omega}_k t} - \mathrm{e}^{-\mathbf{i}\bar{\omega}_k t}) \\ &= \underbrace{a_0}_{c_0} + \sum_{k=1}^{\infty} \underbrace{\frac{1}{2}(a_k - \mathbf{i}b_k)}_{c_k} \mathrm{e}^{\mathbf{i}\bar{\omega}_k t} + \sum_{k=1}^{\infty} \underbrace{\frac{1}{2}(a_k + \mathbf{i}b_k)}_{c_{-k}} \mathrm{e}^{-\mathbf{i}\bar{\omega}_k t} \end{aligned} \tag{2.14}$$

The FS can then be written in a compact manner as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathbf{i}\bar{\omega}_k t} \qquad \bar{\omega}_k = \frac{2\pi k}{T} \tag{2.15}$$

where $\{c_k\}_{k=-\infty}^{\infty}$ are the 'complex Fourier series coefficients', related to the real ones by

$$c_0 = a_0 \qquad \begin{aligned} c_k &= \tfrac{1}{2}(a_k - \mathbf{i}b_k) \\ c_{-k} &= \tfrac{1}{2}(a_k + \mathbf{i}b_k) \end{aligned} \qquad k = 1, 2, \ldots \tag{2.16}$$

Substituting (2.7) into (2.16) gives c_k in terms of $x(t)$:

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\mathrm{e}^{-\mathbf{i}\bar{\omega}_k t}\, dt \qquad \bar{\omega}_k = \frac{2\pi k}{T},\ k = 0, \pm 1, \pm 2, \ldots \tag{2.17}$$

It can be verified that c_k has the 'conjugate mirror property':

$$c_{-k} = \overline{c_k} \tag{2.18}$$

where a bar on top denotes the complex conjugate, i.e., $\overline{a + b\mathbf{i}} = a - b\mathbf{i}$ for real numbers a and b.

The complex form of FS significantly simplifies algebra in Fourier analysis and is widely used. It takes some time to master algebraic skills with complex numbers but is worthwhile to do so.

2.1.2 Parseval Equality

The 'energy' of a process when viewed in the time or frequency domain can be equated via the 'Parseval equality', which appears in different forms depending on the context. For a periodic process with finite energy $\int_{-T/2}^{T/2} x(t)^2\, dt < \infty$,

$$\underbrace{\frac{1}{T}\int_{-T/2}^{T/2} x(t)^2 dt}_{\text{Time domain}} = \underbrace{a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2)}_{\text{Freq. domain real coeff.}} = \underbrace{\sum_{k=-\infty}^{\infty} |c_k|^2}_{\text{Freq. domain complex coeff.}} \tag{2.19}$$

The leftmost expression may be interpreted as the average energy per unit time. The Parseval equality says that this energy can be viewed as the sum of contributions from harmonics at different frequencies. If one substitutes the FS of $x(t)$ in (2.3) into the leftmost expression, after squaring, one will obtain an integral of a double sum of cosine and sine products. The non-trivial (and beautiful) result is that only the integrals of cos $\times$ cos and sin $\times$ sin terms with the same frequency are non-zero, giving the neat frequency domain expressions in the middle and rightmost that contain no cross terms.

Proof of (2.19) (Parseval Equality, Fourier Series)
We prove the Parseval equality (2.19) using the complex FS in (2.15). Writing $x^2 = x\bar{x}$,

$$\begin{aligned}
\frac{1}{T}\int_{-T/2}^{T/2} x(t)^2\, dt &= \frac{1}{T}\int_{-T/2}^{T/2} \overbrace{\left(\sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathbf{i}\varpi_k t}\right)}^{x(t)} \overbrace{\left(\sum_{j=-\infty}^{\infty} \overline{c_j} \mathrm{e}^{-\mathbf{i}\varpi_j t}\right)}^{\overline{x(t)}} dt \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} c_k \overline{c_j} \underbrace{\int_{-T/2}^{T/2} \mathrm{e}^{\mathbf{i}(\varpi_k - \varpi_j)t}\, dt}_{T \text{ if } j=k;\ \text{else } 0} \\
&= \sum_{k=-\infty}^{\infty} |c_k|^2
\end{aligned} \tag{2.20}$$

In the second equality, the order of infinite sum and integration has been swapped, which is legitimate when the process has finite energy. The result of the integral can be reasoned as follow. Clearly, it is equal to T when $j = k$. Otherwise $(j \neq k)$,

$$\int_{-T/2}^{T/2} \mathrm{e}^{\mathbf{i}(\bar{\omega}_k-\bar{\omega}_j)t}\,dt = \frac{\mathrm{e}^{\mathbf{i}\overbrace{(\bar{\omega}_k-\bar{\omega}_j)T/2}^{\pi(k-j)}} - \mathrm{e}^{-\mathbf{i}\overbrace{(\bar{\omega}_k-\bar{\omega}_j)T/2}^{\pi(k-j)}}}{\mathbf{i}\underbrace{(\bar{\omega}_k-\bar{\omega}_j)}_{2\pi(k-j)/T}} = \frac{2\mathbf{i}\overbrace{\sin\pi(k-j)}^{0}}{2\pi\mathbf{i}(k-j)/T} = 0 \tag{2.21}$$

■

2.2 Non-periodic Process (Fourier Transform)

If a process is not periodic then it cannot be written as a FS. In this case, if it has finite energy then it can still be represented as a sum of harmonics, although now there is a continuum of contributing frequencies, each with infinitesimal contribution. Specifically, a process $x(t)$ defined for $-\infty < t < \infty$ with finite energy $\int_{-\infty}^{\infty} x(t)^2\,dt < \infty$ can be written as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\mathrm{e}^{\mathbf{i}\omega t}\,d\omega \tag{2.22}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathbf{i}\omega t}\,dt \tag{2.23}$$

Here, $X(\omega)$ is called the 'Fourier Transform' (FT) of $x(t)$; and $x(t)$ is the 'inverse Fourier Transform' of $X(\omega)$. In FT, the frequency ω is continuous-valued. This is in contrast to FS, where the frequencies $\{\bar{\omega}_k\}$ are discrete-valued. The factor $1/2\pi$ in (2.22) may appear peculiar but it can be explained by consideration of units; see Sect. 2.7.1 later.

2.2.1 From Fourier Series to Fourier Transform

FT can be reasoned from FS as follow. Consider approximating $x(t)$ by a periodic function $x_p(t)$ with period T, where $x_p(t) = x(t)$ for $-T/2 < t < T/2$ but simply repeats itself elsewhere. This is illustrated in Fig. 2.1.

Intuitively, $x_p(t)$ (as a function) converges to $x(t)$ as $T \to \infty$. For a given T, let c_k be the complex FS coefficients of $x_p(t)$ at frequency $\bar{\omega}_k = 2\pi k/T$. Then

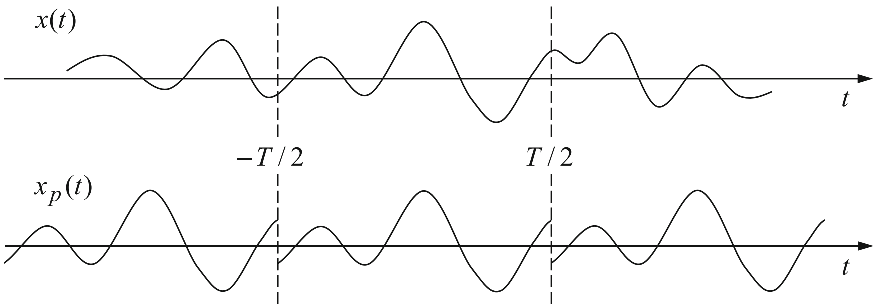


Fig. 2.1 Original process $x(t)$ and periodic proxy $x_p(t)$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x_p(t)\mathrm{e}^{-\mathbf{i}\bar{\omega}_k t}\,dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\mathrm{e}^{-\mathbf{i}\bar{\omega}_k t}\,dt \tag{2.24}$$

since $x_p(t) = x(t)$ for $-T/2 < t < T/2$. Using Cauchy-Schwartz inequality (Sect. C.5.6),

$$|c_k|^2 \le \frac{1}{T^2}\underbrace{\int_{-T/2}^{T/2} x(t)^2\,dt}_{\le \int_{-\infty}^{\infty} x(t)^2\,dt} \times \underbrace{\int_{-T/2}^{T/2} \underbrace{|\mathrm{e}^{-\mathbf{i}\bar{\omega}_k t}|^2}_{1}\,dt}_{T} \le \frac{1}{T}\int_{-\infty}^{\infty} x(t)^2\,dt \tag{2.25}$$

Since $\int_{-\infty}^{\infty} x(t)^2\,dt < \infty$, the above implies $|c_k| \to 0$ as $T \to \infty$. This indicates that FS coefficients are not legitimate quantities for studying the frequency characteristics of a non-period process, because they all diminish trivially as $T \to \infty$, no matter what the process is. The factor $1/T$ is the source of diminishing magnitude.

The following function is motivated by taking out the factor $1/T$ in (2.24) and replacing $\bar{\omega}_k$ by the continuous-valued frequency variable ω:

$$X_p(\omega) = \int_{-T/2}^{T/2} x(t)\mathrm{e}^{-\mathbf{i}\omega t}\,dt \tag{2.26}$$

By construction, $c_k = X_p(\bar{\omega}_k)/T$ and so

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathbf{i}\bar{\omega}_k t} = \sum_{k=-\infty}^{\infty} X_p(\bar{\omega}_k)\mathrm{e}^{\mathbf{i}\bar{\omega}_k t}\frac{1}{T} \tag{2.27}$$

As $T \to \infty$, $x_p(t) \to x(t)$ as a function. Also, the frequency interval $\Delta\omega = \bar{\omega}_{k+1} - \bar{\omega}_k = 2\pi/T$ diminishes. The infinite sum on the RHS of (2.27) tends to an integral. Thus,

$$x(t) = \lim_{T\to\infty} x_p(t) = \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} X_p(\bar{\omega}_k)\mathrm{e}^{\mathbf{i}\bar{\omega}_k t}\frac{\Delta\omega}{2\pi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\mathrm{e}^{\mathbf{i}\omega t}\,d\omega \tag{2.28}$$

where

$$X(\omega) = \lim_{T\to\infty} X_p(\omega) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathbf{i}\omega t}\,dt \tag{2.29}$$

In the above reasoning, we have swapped the order of limit and integration. This is legitimate when the process has finite energy.

2.2.2 *Properties of Fourier Transform*

Some properties of FT are listed in Table 2.1. They can be shown directly from definition. The symbol $\mathcal{F}\{x\}$ denotes the FT of $x(t)$. It is a function of frequency ω but this is omitted for simplicity.

Table 2.1 Some properties of Fourier Transform

Property	Description
Conjugate mirror	Let $X(\omega)$ be the FT of $x(t)$. Then $X(-\omega) = \overline{X(\omega)}$
Linearity	For any scalars a and b, $\mathcal{F}\{ax + by\} = a\mathcal{F}\{x\} + b\mathcal{F}\{y\}$
Differentiation	$\mathcal{F}\{\dot{x}\} = \mathbf{i}\omega\mathcal{F}\{x\}$
Time shift	For any s, let $y(t) = x(t+s)$. Then $\mathcal{F}\{y\} = \mathrm{e}^{\mathbf{i}\omega s}\mathcal{F}\{x\}$
Convolution	Let $z(t)$ be the 'convolution' between $x(t)$ and $y(t)$, defined as $z(t) = \int_{-\infty}^{\infty} x(t-\tau)y(\tau)d\tau$ The FT of convolution is equal to the product of FTs, i.e., $\mathcal{F}\{z\} = \mathcal{F}\{x\}\mathcal{F}\{y\}$

2.2.3 Dirac Delta Function

The ‘Dirac Delta function’ $\delta(t)$, or Delta function in short, is an idealized unit impulse of arbitrarily short duration centered at $t = 0$. It does not exist physically but is frequently used in analysis and modeling. It has the property that

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)x(t)\,dt = x(0) \tag{2.30}$$

for any $\varepsilon > 0$ and function $x(t)$. The FT of the Delta function is simply the constant 1 because

$$\int_{-\infty}^{\infty} \delta(t)\mathrm{e}^{-\mathbf{i}\omega t}dt = \mathrm{e}^{-\mathbf{i}\omega(0)} = 1 \quad \text{for any } \omega \tag{2.31}$$

The inverse FT of the constant 1 gives the FT representation of the Delta function:

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathbf{i}\omega t}d\omega \tag{2.32}$$

This is frequently used in Fourier analysis.

2.2.4 Parseval Equality

For a non-period process $x(t)$ that has FT $X(\omega)$, the Parseval equality reads

$$\int_{-\infty}^{\infty} x(t)^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \tag{2.33}$$

Proof of (2.33) (Parseval Equality, Fourier Transform)
The proof for (2.33) has a similar structure as the one in (2.20) for FS, except that the infinite sums now become integrals:

$$
\begin{aligned}
\int_{-\infty}^{\infty} x(t)^2\, dt &= \int_{-\infty}^{\infty} \overbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\mathrm{e}^{\mathbf{i}\omega t} d\omega\right]}^{x(t)} \overbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega')}\mathrm{e}^{-\mathbf{i}\omega' t}\, d\omega'\right]}^{\overline{x(t)}} dt \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\omega)\overline{X(\omega')}\mathrm{e}^{\mathbf{i}(\omega-\omega')t}\, d\omega'\, d\omega dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\omega)\overline{X(\omega')}\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathbf{i}(\omega-\omega')t}\, dt}_{\delta(\omega-\omega')}\, d\omega'\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \underbrace{\int_{-\infty}^{\infty} \overline{X(\omega')}\delta(\omega-\omega')\, d\omega'}_{\overline{X(\omega)}}\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega
\end{aligned}
\tag{2.34}
$$

In arriving at the third equality, the order of integration w.r.t. t and (ω, ω') has been swapped. This is legitimate when the process has finite energy. ■

2.3 Discrete-Time Approximation with FFT

In digital computations, a process is sampled at discrete time instants. The integrals in FS and FT can be approximated by a Riemann sum on the grid of sampled time instants. Let $\{x_j = x(j\Delta t)\}_{j=0}^{N-1}$ be the N sample values of $x(t)$ at equal time interval Δt (s). A discrete-time approximation to the FT of $x(t)$ is constructed by replacing the integral in (2.23) with a Riemann sum:

$$
X(\omega) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathbf{i}\omega t}\, dt \approx \hat{X}(\omega) = \sum_{j=0}^{N-1} x_j \mathrm{e}^{-\mathbf{i}\omega j\Delta t}\Delta t \tag{2.35}
$$

Here, $\hat{X}(\omega)$ is called the 'discrete-time Fourier Transform' (DTFT) of $x(t)$. Evaluating $\hat{X}(\omega)$ for a given value of ω involves a summation in the time domain. In practice it is only calculated at the following uniformly spaced frequencies:

$$\omega_k = \frac{2\pi k}{N\Delta t} \text{ (rad/s)} \quad k = 0, \ldots, N-1 \text{ (FFT frequencies)} \tag{2.36}$$

It is because the values of $\hat{X}(\omega)$ at these frequencies can be evaluated very efficiently. This is discussed next.

2.3.1 Fast Fourier Transform

The 'Fast Fourier Transform' (FFT) algorithm (Cooley and Tukey 1965) provides an efficient means for calculating the values of DTFT at a specific set of frequencies as in (2.36). It is commonly coded in commercial software or programming packages; see Sect. 2.9 later for an introduction. Here we focus on the definition and properties of FFT.

The FFT of $\{x_j\}_{j=0}^{N-1}$ is the sequence $\{y_k\}_{k=0}^{N-1}$ defined by

$$y_k = \sum_{j=0}^{N-1} x_j e^{-2\pi \mathbf{i} jk/N} \qquad k = 0, \ldots, N-1 \quad \text{(FFT)} \tag{2.37}$$

The 'inverse FFT' of $\{y_k\}_{k=0}^{N-1}$ is defined as the sequence $\{z_j\}_{j=0}^{N-1}$ where

$$z_j = \frac{1}{N}\sum_{k=0}^{N-1} y_k e^{2\pi \mathbf{i} jk/N} \qquad j = 0, \ldots, N-1 \quad \text{(inverse FFT)} \tag{2.38}$$

Note that $\{y_k\}_{k=0}^{N-1}$ is generally complex-valued, even though $\{x_j\}_{j=0}^{N-1}$ is real-valued. In the literature, $\{y_k\}_{k=0}^{N-1}$ is referred as the 'discrete Fourier Transform' (DFT, not to be confused with DTFT) of $\{x_j\}_{j=0}^{N-1}$. In this book, we simply refer it as FFT, because DFT is almost always evaluated via FFT and there is little distinction between the two terms.

Inverse FFT Recovers the Original Sequence
The following shows that inverse FFT indeed recovers the original sequence that produces the FFT, i.e., $x_j = z_j, j = 0, \ldots, N-1$. Substituting $y_k = \sum_{r=0}^{N-1} x_r e^{-2\pi \mathbf{i} rk/N}$ into (2.38),

$$z_j = \frac{1}{N}\sum_{k=0}^{N-1}\underbrace{\sum_{r=0}^{N-1} x_r \mathrm{e}^{-2\pi\mathbf{i}rk/N}}_{y_k}\,\mathrm{e}^{2\pi\mathbf{i}jk/N} = \frac{1}{N}\sum_{r=0}^{N-1} x_r \underbrace{\sum_{k=0}^{N-1}\mathrm{e}^{2\pi\mathbf{i}(j-r)k/N}}_{N\text{ if } r=j;\text{else } 0} = x_j \tag{2.39}$$

The result of the sum over k used above is discussed next.

Exponential Sum Formula
Sums of exponentials are frequently encountered in Fourier analysis and it is worth to get familiarized with their analytical formulas. One basic result is that for any integer k,

$$\sum_{j=0}^{N-1}\mathrm{e}^{2\pi\mathbf{i}jk/N} = \begin{cases} N & k/N = 0, \pm 1, \pm 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{2.40}$$

The first case when k is an integer multiple of N is trivial because $\mathrm{e}^{2\pi\mathbf{i}p} = 1$ for any integer p. Otherwise, using the geometric series summation formula $\sum_{j=0}^{N-1} a^j = (1-a^N)/(1-a)$ with $a = \mathrm{e}^{2\pi\mathbf{i}k/N}$,

$$\sum_{j=0}^{N-1}\mathrm{e}^{2\pi\mathbf{i}jk/N} = \frac{1-\overbrace{\mathrm{e}^{2\pi\mathbf{i}k}}^{1}}{1-\underbrace{\mathrm{e}^{2\pi\mathbf{i}k/N}}_{\neq 1}} = 0 \qquad (\frac{k}{N}\text{ not integer}) \tag{2.41}$$

Taking complex conjugate on (2.40) shows that the result is the same if $2\pi\mathbf{i}jk/N$ is replaced by $-2\pi\mathbf{i}jk/N$.

Conjugate Mirror Property
The FFT $\{y_k\}_{k=0}^{N-1}$ of a real-valued sequence $\{x_j\}_{j=0}^{N-1}$ has the conjugate mirror property that

$$y_{N-k} = \overline{y_k} \tag{2.42}$$

This is because

$$y_{N-k} = \sum_{j=0}^{N-1} x_j \mathrm{e}^{-2\pi\mathbf{i}j(N-k)/N} = \sum_{j=0}^{N-1} x_j \underbrace{\mathrm{e}^{-2\pi\mathbf{i}j}}_{1}\mathrm{e}^{2\pi\mathbf{i}jk/N} = \overline{\sum_{j=0}^{N-1} x_j \mathrm{e}^{-2\pi\mathbf{i}jk/N}} = \overline{y_k} \tag{2.43}$$

The conjugate mirror property is illustrated in Fig. 2.2. About half of the FFT sequence carries redundant information, in the sense that it can be produced as the complex conjugate of the other half.

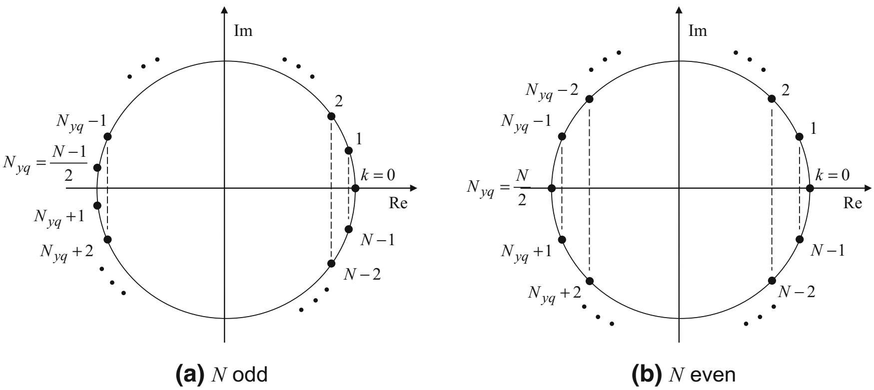


Fig. 2.2 Conjugate mirror property of the FFT of a real-valued sequence. **a** N odd; **b** N even. N_{yq} = integer part of $N/2$, is the index at or just below the 'Nyquist frequency' (Sect. 2.4.1)

2.3.2 Approximating Fourier Transform and Fourier Series

Back to the problem of approximating the FT $X(\omega_k)$ with the FFT of $\{x_j = x(j\Delta t)\}_{j=0}^{N-1}$ as in (2.35). Denoting $\hat{X}_k = \hat{X}(\omega_k)$ and noting $\omega_k j\Delta t = 2\pi jk/N$,

$$\hat{X}_k = \Delta t \underbrace{\sum_{j=0}^{N-1} x_j e^{-2\pi \mathbf{i} jk/N}}_{y_k(\mathrm{FFT})} \tag{2.44}$$

Thus,

$$X(\omega_k) \approx \hat{X}_k = y_k \Delta t \tag{2.45}$$

For FS coefficients, let $\{x_j = x(j\Delta t)\}_{j=0}^{N-1}$ be the sampled sequence of a periodic process $x(t)$ with period T. Assume that $N\Delta t = T$ so that the FFT frequency $\omega_k = 2\pi k/N\Delta t$ coincides with the FS frequency $\bar{\omega}_k = 2\pi k/T$. Approximating the integral in (2.17) by a Riemann sum,

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-\mathbf{i}\bar{\omega}_k t}\, dt \approx \hat{c}_k = \frac{1}{N\Delta t} \underbrace{\sum_{j=0}^{N-1} x_j e^{-2\pi \mathbf{i} jk/N}}_{y_k} \Delta t \tag{2.46}$$

Thus

$$c_k \approx \hat{c}_k = \frac{y_k}{N} \tag{2.47}$$

The use of $\hat{c}_k$ in (2.46) can be generalized to allow $N\Delta t \neq T$, which may provide convenience in practice, e.g., when the period is not known, data duration is not equal to the period, or it is desirable to estimate the FS of measured data using more than one period to average out noise.

2.3.3 *Parseval Equality*

For a real-valued sequence $\{x_j\}_{j=0}^{N-1}$ with FFT $\{y_k\}_{k=0}^{N-1}$, Parseval equality reads

$$\sum_{j=0}^{N-1} x_j^2 = \frac{1}{N}\sum_{k=0}^{N-1} |y_k|^2 \tag{2.48}$$

Proof of (2.48) (Parseval Equality, FFT)
The structure of the proof is similar to that for FS in (2.20) or FT in (2.34):

$$\begin{aligned}
\sum_{j=0}^{N-1} x_j^2 &= \sum_{j=0}^{N-1} \overbrace{\left[\frac{1}{N}\sum_{k=0}^{N-1} y_k e^{2\pi \mathbf{i} jk/N}\right]}^{x_j} \overbrace{\left[\frac{1}{N}\sum_{r=0}^{N-1} \overline{y_r} e^{-2\pi \mathbf{i} jr/N}\right]}^{\overline{x_j}} \\
&= \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\sum_{r=0}^{N-1} y_k \overline{y_r} e^{2\pi \mathbf{i} j(k-r)/N} \\
&= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{r=0}^{N-1} y_k \overline{y_r} \underbrace{\sum_{j=0}^{N-1} e^{2\pi \mathbf{i} j(k-r)/N}}_{N \text{ if } r=k;\text{else } 0} = \frac{1}{N}\sum_{k=0}^{N-1} |y_k|^2
\end{aligned} \tag{2.49}$$

■

2.4 Distortions in Fourier Series

The discrete-time approximation of FS and FT leads to errors of a characteristic nature. In the first place, the approximation is valid only up to the 'Nyquist frequency' ($1/2\Delta t$ Hz). It is also contaminated with harmonics in the original process beyond the Nyquist frequency. This is known as 'aliasing', where high frequency variations are mistaken as low frequency ones. It is the same mechanism that our eyes see the rotor blade of a flying helicopter as slowly rotating (sometimes reversing direction). There is further contamination from harmonics that do not have an integer multiple of cycles within the measured time span, in a $O(1/N\Delta t)$ neighborhood of the subject frequency (and aliased counterparts). This is known as 'leakage'. Aliasing is not repairable after discrete-time sampling and so harmonics beyond the Nyquist frequency should be filtered out beforehand. Leakage can be suppressed by increasing data duration. We first discuss the distortions in the FFT approximation of FS. The discussion for FT follows in the next section.

Recall the context in (2.46), where $x(t)$ is a periodic process with complex FS coefficients $\{c_k\}_{k=-\infty}^{\infty}$; $\{x_j = x(j\Delta t)\}_{j=0}^{N-1}$ is a discrete-time sample sequence of $x(t)$; $\{y_k\}_{k=0}^{N-1}$ is the FFT of $\{x_j\}_{j=0}^{N-1}$; and $\hat{c}_k = y_k/N$ is a discrete-time approximation of c_k.

2.4.1 Nyquist Frequency

Although $\{\hat{c}_k\}_{k=0}^{N-1}$ is a sequence with N terms, only the first half is informative. This stems from the conjugate mirror property:

$$\hat{c}_{N-k} = \overline{\hat{c}_k} \tag{2.50}$$

It implies that $\hat{c}_k$ is conjugate symmetric about the index $N/2$, i.e., frequency $f_{N_{yq}} = (N/2)/N\Delta t = 1/2\Delta t$ (Hz), which is called the 'Nyquist frequency'. As a result, $\hat{c}_k$ can only give a proper estimation of c_k up to the Nyquist frequency.

2.4.2 Aliasing

Aliasing occurs when the original process $x(t)$ contains harmonics at frequencies beyond the Nyquist frequency. Suppose the data duration is equal to the period of $x(t)$, i.e., $N\Delta t = T$. Then it can be shown that (see the end)

$$\hat{c}_k = \sum_{m=-\infty}^{\infty} c_{mN+k} \tag{2.51}$$

In addition to c_k (the $m = 0$ term), $\hat{c}_k$ also contains other terms, $c_{k\pm N}$, $c_{k\pm 2N}$ and so on. To see the contributing (positive) frequencies, separate the sum into positive and negative m, and use the conjugate mirror property of c_k:

$$\begin{aligned}\hat{c}_k &= c_k + \sum_{m=1}^{\infty} c_{mN+k} + \sum_{m=-\infty}^{-1} c_{mN+k} \\ &= c_k + (c_{N+k} + c_{2N+k} + \cdots) + (c_{-N+k} + c_{-2N+k} + \cdots) \\ &= c_k + (c_{N+k} + c_{2N+k} + \cdots) + (\overline{c_{N-k}} + \overline{c_{2N-k}} + \cdots)\end{aligned} \tag{2.52}$$

Combining the two infinite sums,

$$\underbrace{\hat{c}_k}_{\text{freq.}\mathrm{f}_k} = \underbrace{c_k}_{\text{freq.}\mathrm{f}_k} + \underbrace{\sum_{m=1}^{\infty} (\underbrace{c_{mN+k}}_{\text{freq.}mf_s+\mathrm{f}_k} + \underbrace{\overline{c_{mN-k}}}_{\text{freq.}mf_s-\mathrm{f}_k})}_{\text{aliasing}} \tag{2.53}$$

That is, $\hat{c}_k$ is contaminated with contributions from frequencies (in Hz) $f_s \pm \mathrm{f}_k$, $2f_s \pm \mathrm{f}_k$, …, where $f_s = 1/\Delta t$ is the sampling frequency and $\mathrm{f}_k = k/N\Delta t$ is the subject FFT frequency. Aliasing occurs by the same mechanism when the data duration is an integer multiple of the period. When the data duration is not even an integer multiple of the period, there will also be 'leakage'; see the next section.

Example 2.1 (**Aliasing with a single harmonic**) Consider $x(t) = 2\cos 2\pi ft$, which is a single harmonic with frequency f (Hz). Its real FS coefficients $\{a_k, b_k\}$ are zero except $a_1 = 2$. Its complex FS coefficients $\{c_k\}$ are zero except $c_1 = 1$ and $c_{-1} = 1$. Suppose we obtain the samples $\{x_j = x(j\Delta t)\}_{j=0}^{N-1}$ at $\Delta t = 0.1$ s and $N = 10$, i.e., for a duration of $N\Delta t = 1$ s. Using $\{x_j\}_{j=0}^{N-1}$, we estimate the complex FS coefficients by the FFT approximation $\hat{c}_k = N^{-1}\sum_{j=0}^{N-1} x_j e^{-2\pi \mathbf{i} jk/N}$ in (2.46).

Figure 2.3 illustrates the possible distortions in $\hat{c}_k$, depending on the source frequency f. The plots on the left column show $x(t)$ (dashed line) and the sample points $\{x_j\}_{j=0}^{N-1}$ (dots). The plots on the right column show $|\hat{c}_k|$ (dot with stick) versus the FFT frequency $\mathrm{f}_k = k/N\Delta t$ (Hz) for $k = 0, \ldots, N-1$. The shaded part from 5 Hz (Nyquist frequency) to 10 Hz (sampling frequency) is just the mirror image of that from 0 to 5 Hz. It is usually not plotted but is shown here for illustration. When $f = 1$ or 4 Hz, $|\hat{c}_k|$ is correctly estimated up to the Nyquist frequency. When $f = 6$ Hz or 11 Hz, which are beyond the Nyquist frequency, the harmonic is mistaken (aliased) to be 4 Hz and 1 Hz, respectively. ■

Proof of (2.51) (Aliasing, Fourier Series)

Since $N\Delta t = T$, $\bar{\omega}_r = 2\pi r/T = 2\pi r/N\Delta t$ and so $\bar{\omega}_r j\Delta t = 2\pi jr/N$. Substituting $x_j = \sum_{r=-\infty}^{\infty} c_r e^{2\pi \mathbf{i} jr/N}$ into $\hat{c}_k$ in (2.46),

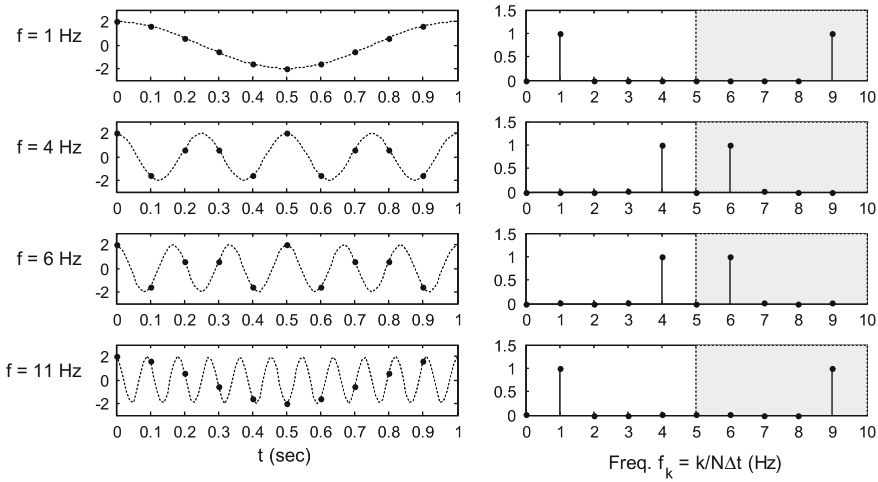


Fig. 2.3 FS amplitude of $x(t) = 2\cos 2\pi ft$ estimated via the FFT of $\{x_j = x(j\Delta t)\}_{j=0}^{N-1}$ with $\Delta t = 0.1$ s and $N = 10$, for $f = 1, 4, 6$ and 11 Hz. *Left column*: *dashed line* = $x(t)$, dot = x_j. *Right column*: *dot with stick* = $|\hat{c}_k|$

$$\begin{aligned}\hat{c}_k &= \frac{1}{N}\sum_{j=0}^{N-1}\overbrace{\sum_{r=-\infty}^{\infty} c_r e^{2\pi \mathbf{i} jr/N}}^{x_j}\, e^{-2\pi \mathbf{i} jk/N} \\ &= \frac{1}{N}\sum_{r=-\infty}^{\infty} c_r \times \underbrace{\sum_{j=0}^{N-1} e^{2\pi \mathbf{i} j(r-k)/N}}_{N \text{ if } (r-k)/N=0,\pm 1,\pm 2,\ldots;\text{else } 0} = \sum_{m=-\infty}^{\infty} c_{mN+k}\end{aligned} \tag{2.54}$$

■

2.4.3 Leakage

Leakage occurs in the FFT approximation of FS when the data duration $N\Delta t$ is not an integer multiple of the period T. In this case, a given FS frequency $\bar{\omega}_r = 2\pi r/T$ need not be matched by a FFT frequency $\omega_k = 2\pi k/N\Delta t$. When this occurs, the FS harmonic will 'leak out' to other FFT frequencies. Leakage (and aliasing) can be explained by the following general formula that expresses $\hat{c}_k$ as the convolution of $\{c_r\}_{r=-\infty}^{\infty}$ with a 'kernel function' $K_1(\omega)$:

$$\hat{c}_k = \sum_{r=-\infty}^{\infty} c_r K_1(\omega_k - \bar{\omega}_r) \tag{2.55}$$

$$K_1(\omega) = \frac{1}{N}\sum_{j=0}^{N-1} \mathrm{e}^{-\mathbf{i}\omega j\Delta t} = \frac{\sin(N\omega\Delta t/2)}{N\,\sin(\omega\Delta t/2)}\mathrm{e}^{-\mathbf{i}(N-1)\omega\Delta t/2} \tag{2.56}$$

See the end for proof. In terms of the dimensionless variable $u = \omega\Delta t/2\pi$,

$$K_1(\omega) = D_N(u)\mathrm{e}^{-\mathbf{i}\pi(N-1)u} \qquad u = \frac{\omega\Delta t}{2\pi} \tag{2.57}$$

where

$$D_N(u) = \frac{\sin N\pi u}{N\sin \pi u} \tag{2.58}$$

is known as the 'Dirichlet kernel', which plays an important role in Fourier theory.

Equation (2.55) indicates that $\hat{c}_k$ contains contributions from all FS frequencies. The sum need not even contain a term at the subject frequency ω_k. The contribution from frequency $\bar{\omega}_r$ is not directly c_r, but is 'attenuated' by $K_1(\omega_k - \bar{\omega}_r)$, which depends on how far $\bar{\omega}_r$ is from ω_k.

Figure 2.4 shows a schematic plot of $|D_N(u)|$. It has a period of 1, with a symmetric basic branch on $(-1/2, 1/2)$. In this branch it has a global maximum of 1 at $u = 0$; and a series of zeros at $u = \pm 1/N, \pm 2/N, \ldots$, up to $\pm 1/2$. For u ranging between $\pm 1/2$, $\omega = 2\pi u/\Delta t$ ranges between $\pm\pi/\Delta t$ (rad/s), i.e., ±Nyquist frequency. Due to convolution effect, $\hat{c}_k$ comprises the harmonics in $x(t)$ at frequencies

(1) near the subject frequency $\mathrm{f}_k = k/N\Delta t$ Hz;
(2) in a $O(1/N\Delta t)$ (Hz) neighborhood around f_k (leakage); and
(3) in the $O(1/N\Delta t)$ (Hz) neighborhoods around $f_s \pm \mathrm{f}_k$, $2f_s \pm \mathrm{f}_k$, … (aliased counterparts of leakage), where $f_s = 1/\Delta t$ Hz is the sampling frequency.

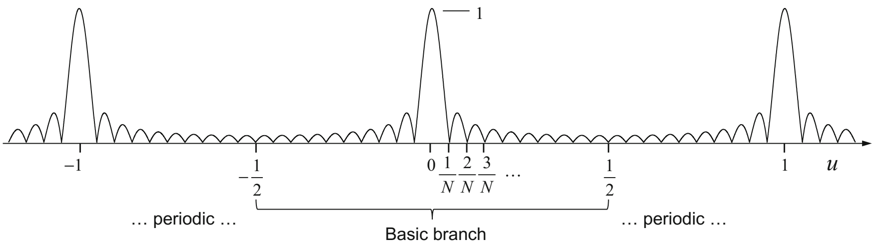


Fig. 2.4 Schematic plot of $|D_N(u)|$, the modulus of the Dirichlet kernel

Example 2.2 (**Leakage from a single harmonic**) Revisit Example 2.1. Everything else being the same, the sampling duration is now slightly extended to $N\Delta t = 1.2$ s, i.e., with $N = 1.2/0.1 = 12$ points ($\Delta t = 0.1$ s). Results analogous to Fig. 2.3 are shown in Fig. 2.5. For $f = 1, 4, 6$ and 11 Hz, the number of cycles within the data duration is $fN\Delta t = 1.2$, 4.8, 7.2 and 13.2. None of these are integer and so leakage occurs, in addition to aliasing (when $f = 6$ and 11 Hz). The FFT frequencies are now $\mathrm{f}_k = k/N\Delta t = 0, 0.833, 1.667, \ldots, 9.167$ Hz, instead of 0, 1, …, 9 Hz in Fig. 2.3. ■

Proof of (2.55) (Leakage, Fourier Series)
Substituting the FS $x_j = \sum_{r=-\infty}^{\infty} c_r \mathrm{e}^{\mathbf{i}\bar{\omega}_r j\Delta t}$ into $\hat{c}_k$ in (2.46),

$$\hat{c}_k = \frac{1}{N}\sum_{j=0}^{N-1} \underbrace{\sum_{r=-\infty}^{\infty} c_r \mathrm{e}^{\mathbf{i}\bar{\omega}_r j\Delta t}}_{x_j} \mathrm{e}^{-\mathbf{i}\omega_k j\Delta t} = \sum_{r=-\infty}^{\infty} c_r \underbrace{\frac{1}{N}\sum_{j=0}^{N-1} \mathrm{e}^{-\mathbf{i}(\omega_k-\bar{\omega}_r)j\Delta t}}_{K_1(\omega_k-\bar{\omega}_r)} \tag{2.59}$$

where

$$K_1(\omega) = \frac{1}{N}\sum_{j=0}^{N-1} \mathrm{e}^{-\mathbf{i}\omega j\Delta t} \tag{2.60}$$

as defined in (2.56). Using $\sum_{j=0}^{N-1} a^j = (1-a^N)/(1-a)$ with $a = \mathrm{e}^{-\mathbf{i}\omega\Delta t}$,

$$K_1(\omega) = \frac{1-\mathrm{e}^{-\mathbf{i}N\omega\Delta t}}{N(1-\mathrm{e}^{-\mathbf{i}\omega\Delta t})} \tag{2.61}$$

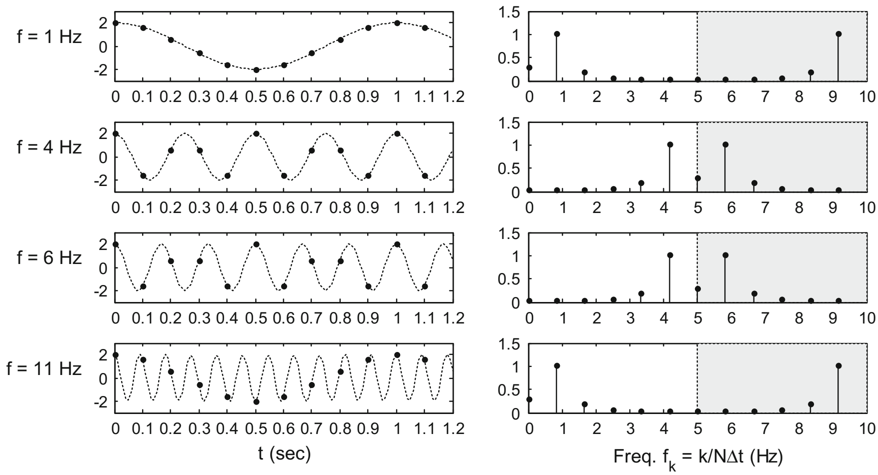


Fig. 2.5 FS amplitude of $x(t) = 2\cos 2\pi ft$ estimated via the FFT of $\{x_j = x(j\Delta t)\}_{j=0}^{N-1}$ with $\Delta t = 0.1$ s and $N = 12$, for $f = 1, 4, 6$ and 11 Hz. Same legend as in Fig. 2.3

Note that for any real θ,

$$1 - \mathrm{e}^{-\mathbf{i}\theta} = \mathrm{e}^{-\mathbf{i}\theta/2}\underbrace{(\mathrm{e}^{\mathbf{i}\theta/2} - \mathrm{e}^{-\mathbf{i}\theta/2})}_{2\mathbf{i}\sin(\theta/2)} = 2\mathbf{i}\mathrm{e}^{-\mathbf{i}\theta/2}\sin(\theta/2) \tag{2.62}$$

Using this identity,

$$K_1(\omega) = \frac{2\mathbf{i}\mathrm{e}^{-\mathbf{i}N\omega\Delta t/2}\sin(N\omega\Delta t/2)}{2\mathbf{i}\mathrm{e}^{-\mathbf{i}\omega\Delta t/2}N\sin(\omega\Delta t/2)} = \frac{\sin(N\omega\Delta t/2)}{N\sin(\omega\Delta t/2)}\mathrm{e}^{-\mathbf{i}(N-1)\omega\Delta t/2} \tag{2.63}$$

which is the rightmost expression in (2.56). ■

2.5 Distortions in Fourier Transform

Nyquist frequency limit, aliasing and leakage in the FFT approximation of FT occurs by a similar mechanism as in FS. Analogous to (2.55), the DTFT approximation $\hat{X}(\omega)$ in (2.35) is related to the target FT $X(\omega)$ by a convolution integral:

$$\hat{X}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega')K_2(\omega - \omega')d\omega' \tag{2.64}$$

$$K_2(\omega) = N\Delta t K_1(\omega) = N\Delta t D_N(u)\mathrm{e}^{-\mathbf{i}\pi(N-1)u} \qquad u = \frac{\omega\Delta t}{2\pi} \tag{2.65}$$

See the end for proof.

Aliasing and leakage can be explained based on (2.64). To see aliasing, note that $|D_N(u)|$ has local maxima 1 at $u = 0, \pm 1, \pm 2, \ldots$; see Fig. 2.4. Correspondingly, $|K_2(\omega - \omega')|$ as a function of ω' has local maxima at $\omega' = \omega \pm 2\pi r/\Delta t$ for $r = 0, 1, 2, \ldots$. Using the conjugate mirror property of FT, $X(\omega - 2\pi r/\Delta t) = \overline{X(2\pi r/\Delta t - \omega)}$, and so $\hat{X}(\omega)$ receives significant contributions from the frequencies $2\pi r/\Delta t \pm \omega$ ($r = 0, 1, 2, \ldots$). This is aliasing.

To see leakage, note that $\hat{X}(\omega)$ at frequency ω receives contribution from the FT $X(\omega')$ of the original process at frequency ω'. The contribution is attenuated by $K_2(\omega - \omega')$, which depends on how far ω' is from ω. For the contribution to be non-zero, ω' need not be the subject frequency ω or aliased counterparts. This is leakage. Since a non-periodic process generally contains harmonics at a continuum of frequencies, leakage exists regardless of the data duration. Nevertheless, the effect diminishes as the data duration increases, because $|K_2(\omega - \omega')|$ is negligible for $|\omega - \omega'| \gg 2\pi/N\Delta t$.

Proof of (2.64) (Distortion, Fourier Transform)
Substituting the inverse FT $x_j = (2\pi)^{-1} \int_{-\infty}^{\infty} X(\omega')\mathrm{e}^{\mathbf{i}\omega' j\Delta t} d\omega'$ into $\hat{X}(\omega)$ in (2.35),

$$\begin{aligned}\hat{X}(\omega) &= \sum_{j=0}^{N-1} \overbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega')\mathrm{e}^{\mathbf{i}\omega' j\Delta t} d\omega'\right]}^{x_j} \mathrm{e}^{-\mathbf{i}\omega j\Delta t}\Delta t \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega') \underbrace{N\Delta t \times \overbrace{\frac{1}{N}\sum_{j=0}^{N-1} \mathrm{e}^{-\mathbf{i}(\omega-\omega')j\Delta t}}^{K_1(\omega-\omega')}}_{K_2(\omega-\omega')} d\omega'\end{aligned} \tag{2.66}$$

which gives (2.64). ■

2.6 Summary of FFT Approximations

Table 2.2 summarizes the Fourier formulas and their FFT approximations. They are generally related by a convolution (sum or integral). Relevant sections are indicated. The last row for power spectral density applies to a stationary stochastic process, which is discussed in Chap. 4.

2.7 Summary of Fourier Formulas, Units and Conventions

Table 2.3 summarizes the Fourier formulas and their Parseval equalities. Unit matters, and is indicated. The table assumes that $x(t)$ has a unit of volt (V) and time is measured in second (s). Relevant sections are indicated.

2.7.1 Multiplier in Fourier Transform

One common confusion in Fourier analysis is the definition of the inverse FT, in particular, the multiplier $1/2\pi$ in $x(t) = (2\pi)^{-1}\int_{-\infty}^{\infty} X(\omega)\mathrm{e}^{\mathbf{i}\omega t} d\omega$. Different authors may use a different multiplier. It may appear arbitrary but in fact has a direct implication on the unit of the FT $X(\omega)$. As seen in Table 2.3, if $x(t)$ has unit V,

Table 2.2 Summary of FFT approximations to Fourier series, Fourier Transform and power spectral density; $y_k = \sum_{j=0}^{N-1} x_j \mathrm{e}^{-2\pi \mathbf{i} jk/N}$ is the FFT of $\{x_j = x(j\Delta t)\}_{j=0}^{N-1}$; $D_N(u) = \sin(N\pi u)/N \sin \pi u$ is the Dirichlet kernel

Process	Theoretical		FFT approximation at $\omega_k = 2\pi k/N\Delta t$ (rad/s)	Relationship between theory and approximation
	Time domain	Frequency domain		
Periodic (period T)	$x(t) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathbf{i}\bar{\omega}_k t}$ $\bar{\omega}_k = 2\pi k/T$ (rad/s) (Sect. 2.1.1)	$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\mathrm{e}^{-\mathbf{i}\bar{\omega}_k t}dt$ (Sect. 2.1.1)	$\hat{c}_k = \frac{1}{N} y_k$ (Sect. 2.3.2)	$\hat{c}_k = \sum_{r=-\infty}^{\infty} c_r K_1(\omega_k - \bar{\omega}_r)$ $K_1(\omega) = D_N(u)\mathrm{e}^{-\mathbf{i}\pi(N-1)u}$, $u = \omega\Delta t/2\pi$ (Sect. 2.4.3)
Non-periodic	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\mathrm{e}^{\mathbf{i}\omega t}d\omega$ (Sect. 2.2)	$X(\omega) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathbf{i}\omega t}dt$ (Sect. 2.2)	$\hat{X}_k = \Delta t y_k$ (Sect. 2.3.2)	$\hat{X}_k = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)K_2(\omega_k - \omega)d\omega$ $K_2(\omega) = N\Delta t D_N(u)\mathrm{e}^{-\mathbf{i}\pi(N-1)u}$, $u = \omega\Delta t/2\pi$ (Sect. 2.5)
Stochastic stationary	$\mathbf{x}(t)$ = vector process	$\mathbf{X}_T(\omega) = \frac{1}{\sqrt{T}}\int_0^T \mathbf{x}(t)\mathrm{e}^{-\mathbf{i}\omega t}dt$	$\hat{\mathbf{X}}_k = \sqrt{\Delta t/N}\,\mathbf{y}_k$ $\{\mathbf{y}_k\}_{k=0}^{N-1}$ is the FFT of $\{\mathbf{x}(j\Delta t)\}_{j=0}^{N-1}$	
	$\mathbf{R}(\tau) = E[\mathbf{x}(t+\tau)\mathbf{x}(t)^T]$ (Sect. 4.1)	$\mathbf{S}(\omega) = \lim_{T\to\infty} E[\mathbf{X}_T(\omega)\mathbf{X}_T(\omega)^*]$ (Sect. 4.2)	$\hat{\mathbf{S}}_k = \hat{\mathbf{X}}_k\hat{\mathbf{X}}_k^*$ (Sect. 4.5.2)	$E[\hat{\mathbf{S}}_k] = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{S}(\omega)F_N(\omega_k - \omega)d\omega$ $F_N(\omega) = N\Delta t D_N^2(u)$, $u = \omega\Delta t/2\pi$ (Sect. 4.7.2)
	$\mathbf{R}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{S}(\omega)\mathrm{e}^{\mathbf{i}\omega\tau}d\omega$ (Sect. 4.4.3)	$\mathbf{S}(\omega) = \int_{-\infty}^{\infty} \mathbf{R}(\tau)\mathrm{e}^{-\mathbf{i}\omega\tau}d\tau$ (Sect. 4.4.3)		

Table 2.3 Summary of Fourier formulas and units

Quantity	Time domain	Frequency domain	Parseval equality
Fourier series (period T)	$\underbrace{x(t)}_{\mathrm{V}} = \sum_{k=-\infty}^{\infty} \underbrace{c_k}_{\mathrm{V}} \mathrm{e}^{\mathbf{i}\bar{\omega}_k t}$	$\underbrace{c_k}_{\mathrm{V}} = \underbrace{\frac{1}{T}}_{1/\mathrm{s}} \int_{-T/2}^{T/2} \underbrace{x(t)}_{\mathrm{V}} \mathrm{e}^{-\mathbf{i}\bar{\omega}_k t} \underbrace{dt}_{\mathrm{s}}$	$\int_{-T/2}^{T/2} \underbrace{x(t)^2}_{\mathrm{V}^2} \underbrace{dt}_{\mathrm{s}} = \underbrace{T}_{\mathrm{s}} \sum_{k=-\infty}^{\infty} \underbrace{\lvert c_k \rvert^2}_{\mathrm{V}^2}$ (Sect. 2.1.2)
Fourier Transform	$\underbrace{x(t)}_{\mathrm{V}} = \underbrace{\frac{1}{2\pi}}_{1/\mathrm{rad}} \int_{-\infty}^{\infty} \underbrace{X(\omega)}_{\mathrm{V/Hz}} \mathrm{e}^{\mathbf{i}\omega t} \underbrace{d\omega}_{\mathrm{rad/s}}$	$\underbrace{X(\omega)}_{\mathrm{V/Hz}} = \int_{-\infty}^{\infty} \underbrace{x(t)}_{\mathrm{V}} \mathrm{e}^{-\mathbf{i}\omega t} \underbrace{dt}_{\mathrm{s}}$	$\int_{-\infty}^{\infty} \underbrace{x(t)^2}_{\mathrm{V}^2} \underbrace{dt}_{\mathrm{s}} = \underbrace{\frac{1}{2\pi}}_{1/\mathrm{rad}} \int_{-\infty}^{\infty} \underbrace{\lvert X(\omega) \rvert^2}_{\mathrm{V}^2/\mathrm{Hz}^2} \underbrace{d\omega}_{\mathrm{rad/s}}$ (Sect. 2.2.4)
Fast Fourier Transform	$\underbrace{x_j}_{\mathrm{V}} = \frac{1}{N} \sum_{k=0}^{N-1} \underbrace{y_k}_{\mathrm{V}} \mathrm{e}^{2\pi \mathbf{i} jk/N}$	$\underbrace{y_k}_{\mathrm{V}} = \sum_{j=0}^{N-1} \underbrace{x_j}_{\mathrm{V}} \mathrm{e}^{-2\pi \mathbf{i} jk/N}$	$\sum_{j=0}^{N-1} \underbrace{x_j^2}_{\mathrm{V}^2} = \frac{1}{N} \sum_{k=0}^{N-1} \underbrace{\lvert y_k \rvert^2}_{\mathrm{V}^2}$ (Sect. 2.3.3)

$X(\omega)$ has unit V/Hz rather than V/(rad/s), despite the fact that the integral in $x(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} X(\omega)\mathrm{e}^{\mathbf{i}\omega t} d\omega$ is w.r.t. ω (rad/s). The factor $1/2\pi$ (with unit 1/rad) makes up for this; $\omega/2\pi$ is the frequency in Hz. If one omits 2π and writes $x(t) = \int_{-\infty}^{\infty} X(\omega)\mathrm{e}^{\mathbf{i}\omega t} d\omega$, then $X(\omega)$ must have a unit of V/(rad/s), and vice versa. The same rule applies to the FFT approximation $\hat{X}(\omega)$. This issue is not relevant to the FS coefficient c_k because the multiplier $e^{\mathbf{i}\bar{\omega}_k t}$ in $x(t) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathbf{i}\bar{\omega}_k t}$ is dimensionless.

2.8 Connecting Theory with Matlab

Matlab provides a convenient platform for signal processing and scientific/engineering computing in general. This section presents the connection of some theoretical results with the functions in Matlab. The focus is on FFT and related functions. In the following, quantities in Matlab are in `typewriter` font.

In Matlab, the index of an array starts from 1. In this book, the index of a discrete-time sequence starts from 0, which is found to simplify presentation. Thus, $\{x_j\}_{j=0}^{N-1}$ in this book is an array `x` of length `N` in Matlab with

$$\mathtt{x(1)} = x_0, \quad \mathtt{x(2)} = x_1, \ldots, \mathtt{x(N)} = x_{N-1}.$$

Let $\{y_k\}_{k=0}^{N-1}$ be the FFT of $\{x_j\}_{j=0}^{N-1}$. In Matlab, FFT can be performed by the built-in function `fft`. The call `y = fft(x)` returns an array `y` of length `N`. By definition,

$$y_k = \sum_{j=0}^{N-1} x_j \mathrm{e}^{-2\pi \mathbf{i} jk/N} \qquad k = 0, \ldots, N-1 \tag{2.67}$$

$$\mathtt{y(k)} = \sum_{\mathtt{j=1}}^{\mathtt{N}} \mathtt{x(j)}\, \mathtt{e}^{-2\pi \mathbf{i} \mathtt{(j-1)(k-1)/N}} \qquad \mathtt{k} = \mathtt{1}, \ldots, \mathtt{N} \tag{2.68}$$

Correspondingly,

$$\begin{aligned} &\mathtt{y(1)} = y_0 && \text{(zero frequency)} \\ &\mathtt{y(2)} = y_1 && (\text{frequency } \mathrm{f}_1 = 1/N\Delta t \text{ Hz}) \\ &\mathtt{y(3)} = y_2 && (\text{frequency } \mathrm{f}_2 = 2/N\Delta t \text{ Hz}) \end{aligned}$$

and so on, up to

$$\mathtt{y(Nq+1)} = y_{N_{yq}}$$

where `Nq` = `floor(N/2)` (integer part of $N/2$). The remaining entries in `y` are just the conjugate mirror image of those below the Nyquist frequency.

Table 2.4 shows the connection of some theoretical properties with Matlab. The symbol '$\equiv$' denotes that the quantities on both sides are numerically the same when evaluated in Matlab. Unless otherwise stated, `x` and `y` are assumed to be real `N` by 1 array. See Chap. 4 for the last three properties regarding correlation function and power spectral density.

2.9 FFT Algorithm

To supplement Sect. 2.3.1, the FFT algorithm is briefly introduced here. Originally, evaluating the DTFT in (2.35) at N frequencies requires a computational effort of the order of N^2, i.e., $O(N^2)$. Using the FFT algorithm, it is reduced to $O(N \log_2 N)$, although the values are evaluated at a specific set of equally spaced frequencies. The key lies in the discovery that, for N being some power of 2, i.e., $N = 2^m$ for some integer m, a FFT sequence of length N can be obtained from two FFT sequences of length $N/2$; and similarly each FFT sequence of length $N/2$ can be obtained from another two FFT sequences of length $N/4$; and so on. The general algorithm has provisions for other cases of N, but here we confine our discussion to $N = 2^m$.

Table 2.4 Connection between theory and Matlab

Property	Matlab
First entry of FFT	`y = fft(x);` `y(1)` ≡ `sum(x)`
Conjugate mirror	`y = fft(x);` `y(2:end)` ≡ `conj(y(end:-1:2))`
Inverse FFT	`ifft(fft(x))` ≡ `x`
Parseval equality	`sum(abs(fft(x)).^2)` ≡ `N*sum(x.^2)`
Symmetry of convolution	`conv(x,y)` ≡ `conv(y,x)`
Convolution theorem	`fft(conv(x,y))`≡ `fft([x;zeros(N-1,1)]).*fft([y;zeros(N-1,1)])`
Wiener-Khinchin theorem	`s = fft([0;xcorr(x,y)]); % (2N,1) array` `s(1:2:end-1)` ≡ `fft(x).*conj(fft(y))`
Asymmetry of `xcorr`	`xcorr(x,y)` ≡ `xcorr(y(end:-1:1),x(end:-1:1))`
`xcorr` and `conv`	`xcorr(x,y)` ≡ `conv(x,y(end:-1:1))`

2.9.1 Basic Idea

Given $\{x_j\}_{j=0}^{N-1}$, let $\{y_k\}_{k=0}^{N-1}$ be the FFT sequence to be computed, i.e.,

$$y_k = \sum_{j=0}^{N-1} x_j e^{-2\pi \mathbf{i} jk/N} \quad k = 0, \ldots, N-1 \tag{2.69}$$

Let

$$w_N = e^{-2\pi \mathbf{i}/N} \tag{2.70}$$

so that $(w_N)^N = 1$. Then y_k can be written in terms of w_N:

$$y_k = \sum_{j=0}^{N-1} x_j w_N^{jk} \tag{2.71}$$

Separating the sum into even and odd terms of j,

$$\begin{aligned} y_k = &\quad x_0 w_N^{0\cdot k} + x_2 w_N^{2\cdot k} + \cdots + x_{N-2} w_N^{(N-2)\cdot k} && (j \text{ even}) \\ &+ x_1 w_N^{1\cdot k} + x_3 w_N^{3\cdot k} + \cdots + x_{N-1} w_N^{(N-1)\cdot k} && (j \text{ odd}) \\ = &\sum_{r=0}^{(N/2)-1} x_{2r} w_N^{2rk} + \sum_{r=0}^{(N/2)-1} x_{2r+1} w_N^{(2r+1)k} \end{aligned} \tag{2.72}$$

Since $w_N^2 = w_{N/2}$, we can write $w_N^{2rk} = w_{N/2}^{rk}$ and $w_N^{(2r+1)k} = w_{N/2}^{rk} w_N^k$. Then

$$y_k = \sum_{r=0}^{(N/2)-1} x_{2r} w_{N/2}^{rk} + w_N^k \sum_{r=0}^{(N/2)-1} x_{2r+1} w_{N/2}^{rk} \tag{2.73}$$

On the other hand, suppose we separate $\{x_j\}_{j=0}^{N-1}$ into two sequences of length $N/2$, one containing the even j terms and the other containing the odd j terms, i.e., $\{x_{2r}\}_{r=0}^{(N/2)-1}$ and $\{x_{2r+1}\}_{r=0}^{(N/2)-1}$, respectively. Their FFTs are respectively given by

$$\begin{aligned} y'_k &= \sum_{r=0}^{(N/2)-1} x_{2r} e^{-2\pi \mathbf{i} rk/(N/2)} = \sum_{r=0}^{(N/2)-1} x_{2r} w_{N/2}^{rk} \\ y''_k &= \sum_{r=0}^{(N/2)-1} x_{2r+1} e^{-2\pi \mathbf{i} rk/(N/2)} = \sum_{r=0}^{(N/2)-1} x_{2r+1} w_{N/2}^{rk} \end{aligned} \qquad k = 0, \ldots, \frac{N}{2} - 1 \tag{2.74}$$

Comparing (2.73) and (2.74), we see that

$$y_k = y'_k + w_N^k y''_k \qquad k = 0, \ldots, \frac{N}{2} - 1 \tag{2.75}$$

The first half of $\{y_k\}_{k=0}^{N-1}$ can thus be obtained from $\{y'_k\}_{k=0}^{(N/2)-1}$ and $\{y''_k\}_{k=0}^{(N/2)-1}$. The remaining half can be produced using the conjugate mirror property of FFT, i.e., $y_k = \overline{y_{N-k}}$.

The above shows that the FFT of a length N sequence can be obtained from the FFTs of two length $N/2$ sequences. The FFT of each length $N/2$ sequence can be further obtained from the FFTs of another two length $N/4$ sequences, and so on. Carrying this on recursively, eventually it involves the FFT of a length 1 sequence, i.e., a number, which is the sequence itself.

2.9.2 Computational Effort

To assess the computational effort of the FFT algorithm, let C_N denote the number of multiplications to produce a FFT sequence of length N. The effort for additions, subtractions and complex conjugate can be considered negligible. Also, assume that $\{w_N^k\}_{k=0}^{N-1}$ has been computed upfront. Its computational effort is not counted in C_N. Based on (2.75),

$$\underbrace{C_N}_{\text{FFT of length}\, N} = 2 \underbrace{C_{N/2}}_{\text{FFT of length}\, N/2} + \underbrace{N/2}_{\text{Multiply}\, w_N^k \,\text{for}\, N/2\, \text{times}} \tag{2.76}$$

By sequential substitution or induction, and recalling $N = 2^m$, it can be shown that

$$C_N = C_1 N + \frac{N}{2} \log_2 N \tag{2.77}$$

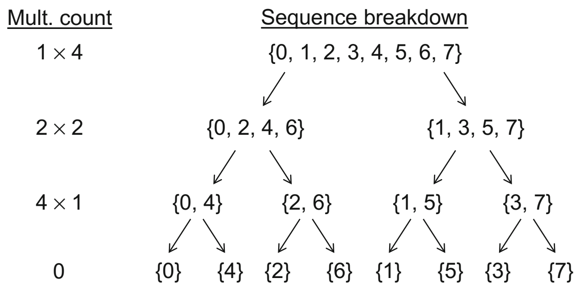
Since the FFT of a length 1 sequence is just the sequence itself, $C_1 = 0$. Consequently,

$$C_N = \frac{N}{2} \log_2 N \tag{2.78}$$

The computational effort for the FFT of a length N sequence is therefore $O(N \log_2 N)$ instead of $O(N^2)$.

*Example 2.3 (**FFT algorithm**)* Figure 2.6 illustrates the recursive breakdown of the calculations for a FFT sequence of length $N = 2^3 = 8$. On the right, each brace

Fig. 2.6 Illustration of FFT algorithm for a sequence of length $N = 8$



contains a sequence. The FFT of the original sequence {0, 1, …, 7} is obtained from the FFTs of two shorter sequences {0, 2, 4, 6} and {1, 3, 5, 7}. The same applies to other sequences.

The left side of the figure counts the number of multiplications involved in producing the longer sequences from the shorter ones. Starting from the bottom, no multiplication is needed to obtain the FFTs of the sequences of length 1. To produce the FFT of {0, 4} from the FFTs of {0} and {4}, it requires 1 multiplication because the first FFT entry $(k = 0)$ involves a multiplication with w_2^k; the second can be produced as the complex conjugate of the first. The same applies to other sequences {2, 6}, {1, 5} and {3, 7}. The number of multiplications to produce the four sequences of length 2 from 8 sequences of length 1 is therefore 4 × 1. Similarly, to produce the FFT of {0, 2, 4, 6} from the FFTs of {0, 4} and {2, 6}, it involves 2 multiplications for the first two FFT entries $(k = 0, 1)$; the other two entries $(k = 2, 3)$ do not involve any multiplication as they are produced from the complex conjugate of the first two. The number of multiplications to produce the FFT of two sequences of length 4 from four sequences of length 2 is therefore 2 × 2. Finally, it involves 1 × 4 multiplications to obtain the FFT of {0, 1, …, 7} from the FFTs of {0, 2, 4, 6} and {1, 3, 5, 7}. The total number of multiplications is 4 × 1 + 2 × 2 + 1 × 4 = 12. This checks with (2.78), which gives $C_8 = (8/2)\log_2 8 = 12$. ■

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