2.1 1-D Quasilinear Wave Equations

Consider the following 1-D quasilinear wave equation

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left( K \left( u, \frac{\partial u}{\partial x} \right) \right) = F \left( u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t} \right),
\]

(2.1)

where \( t \) is the time variable, \( x \) is the spatial variable, \( u \) is a scalar unknown function of \((t, x)\), and \( K = K(u, v) \) is a given \( C^2 \) function with

\[
K_v(u, v) > 0,
\]

(2.2)

and, without loss of generality, we may assume that

\[
K(0, 0) = 0,
\]

(2.3)

moreover, \( F = F(u, v, w) \) is a given \( C^1 \) function, satisfying

\[
F(0, 0, 0) = 0.
\]

(2.4)

By (2.4), \( u = 0 \) is an equilibrium of Eq. (2.1).

The initial condition is given as

\[
t = 0 : \quad u = \varphi(x), \quad u_t = \psi(x), \quad 0 \leq x \leq L,
\]

(2.5)

where \( L \) is the length of the spatial interval, \( \varphi(x) \) and \( \psi(x) \) are \( C^2 \) and \( C^1 \) functions, respectively, on the interval \( 0 \leq x \leq L \).
On one end \(x = 0\), we prescribe any one of the following boundary conditions:

\[
\begin{align*}
    x = 0 : \quad & u = h(t) \quad \text{(Dirichlet type),} \\
    x = 0 : \quad & u_x = h(t) \quad \text{(Neumann type),} \\
    x = 0 : \quad & u_x - \alpha u = h(t) \quad \text{(Third type),} \\
    x = 0 : \quad & u_x - \beta u_t = h(t) \quad \text{(Dissipative type),}
\end{align*}
\]

where \(\alpha\) and \(\beta\) are given positive constants, \(h(t) \in C^2\) (for (2.6a)) or \(C^1\) (for (2.6b)-(2.6d)) is a given function or a control function to be determined.

Similarly, on another end \(x = L\), we prescribe any one of the following boundary conditions:

\[
\begin{align*}
    x = L : \quad & u = \tilde{h}(t) \quad \text{(Dirichlet type),} \\
    x = L : \quad & u_x = \tilde{h}(t) \quad \text{(Neumann type),} \\
    x = L : \quad & u_x + \tilde{\alpha} u = \tilde{h}(t) \quad \text{(Third type),} \\
    x = L : \quad & u_x + \tilde{\beta} u_t = \tilde{h}(t) \quad \text{(Dissipative type).}
\end{align*}
\]

where \(\tilde{\alpha}\) and \(\tilde{\beta}\) are given positive constants, \(\tilde{h}(t) \in C^2\) (for (2.7a)) or \(C^1\) (for (2.7b)-(2.7d)) is a given function or a control function to be determined.

For the convenience of statement, in what follows we denote that

\[
\delta = \begin{cases} 
    2 & \text{in case (2.6a)} \\
    1 & \text{in cases (2.6b)-(2.6d)}
\end{cases}, \quad \tilde{\delta} = \begin{cases} 
    2 & \text{in case (2.7a)} \\
    1 & \text{in cases (2.7b)-(2.7d)}
\end{cases}
\]

### 2.2 Semi-global \(C^2\) Solutions to the Mixed Initial-Boundary Value Problem

Similarly to Chap. 1, in order to get the exact boundary controllability of nodal profile for the quasilinear wave equation (2.1), we should first prove the existence and uniqueness of the semi-global \(C^2\) solution to the forward mixed initial-boundary value problem of the quasilinear wave equation (2.1) with the initial condition (2.5) and with the boundary conditions (2.6)-(2.7), and the corresponding result can be presented in the following lemma (cf. [12, 19]).

**Lemma 2.1**  Under the assumptions given at the beginning of Sect. 2.1, suppose furthermore that the conditions of \(C^2\) compatibility (see Remark 2.1) are satisfied at the points \((t, x) = (0, 0)\) and \((0, L)\), respectively. Then, for any given \(T_0 > 0\), the forward mixed initial-boundary value problem (2.1), (2.5) and (2.6)-(2.7) admits a unique semi-global \(C^2\) solution \(u = u(t, x)\) with small \(C^2\) norm on the domain

\[
R(T_0) = \{(t, x) | 0 \leq t \leq T_0,\ 0 \leq x \leq L\}, \quad (2.8)
\]
provided that the norms \( \| (\varphi, \psi) \|_{C^2[0, L] \times C^1[0, L]}, \| h \|_{C^2[0, T_0]} \) and \( \| \tilde{h} \|_{C^2[0, T_0]} \) are small enough (depending on \( T_0 \)).

**Remark 2.1** It follows from the initial condition (2.5) that

\[
\begin{align*}
    u(0, 0) &= \varphi(0), \quad u_x(0, 0) = \varphi'(0), \quad u_{xx}(0, 0) = \varphi''(0), \\
    u_t(0, 0) &= \psi(0), \quad u_{tx}(0, 0) = \psi'(0),
\end{align*}
\]

then from the Eq. (2.1) we have

\[
u_{tt}(0, 0) = K_u(\varphi(0), \varphi'(0))\varphi'(0) + K_v(\varphi(0), \varphi'(0))\varphi''(0) + F(\varphi(0), \varphi'(0), \psi(0)).
\]

Putting these values to the boundary condition (2.6) on \( \psi = 0 \) with respect to \( t \), respectively, we get the **conditions of \( C^2 \) compatibility** at the point \((t, x) = (0, 0)\).

More precisely, the conditions of \( C^2 \) compatibility at the point \((t, x) = (0, 0)\) are respectively as follows:

\[
\begin{align*}
    &\begin{cases}
    \varphi(0) = h(0), \\
    \psi(0) = h'(0), \\
    K_u(\varphi(0), \varphi'(0))\varphi'(0) + K_v(\varphi(0), \varphi'(0))\varphi''(0) + F(\varphi(0), \varphi'(0), \psi(0)) = h''(0); \\
    \end{cases} & (2.9a) \\
    &\begin{cases}
    \varphi'(0) = h(0), \\
    \psi'(0) = h'(0); \\
    \end{cases} & (2.9b) \\
    &\begin{cases}
    \varphi'(0) - \alpha \varphi(0) = h(0), \\
    \psi'(0) - \alpha \psi(0) = h'(0); \\
    \end{cases} & (2.9c) \\
    &\begin{cases}
    \varphi'(0) - \beta \varphi(0) = h(0), \\
    \psi'(0) - \beta (K_u(\varphi(0), \varphi'(0))\varphi'(0) + K_v(\varphi(0), \varphi'(0))\varphi''(0) \\
    + F(\varphi(0), \varphi'(0), \psi(0))) = h'(0). \\
    \end{cases} & (2.9d)
\end{align*}
\]

The conditions of \( C^2 \) compatibility at the point \((t, x) = (0, L)\) can be given in a similar way.

**Proof of Lemma 2.1** Setting

\[
v = \frac{\partial u}{\partial x}, \quad w = \frac{\partial u}{\partial t},
\]

the Eq. (2.1) can be reduced to the following first order quasilinear system:

\[
\begin{align*}
    &\begin{cases}
    \frac{\partial u}{\partial t} = w, \\
    \frac{\partial v}{\partial t} - \frac{\partial w}{\partial x} = 0, \\
    \frac{\partial w}{\partial t} - K_v(u, v) \frac{\partial v}{\partial x} = F(u, v, w) + K_u(u, v) v \overset{\text{def}}{=} \tilde{F}(u, v, w),
    \end{cases} & (2.11)
\end{align*}
\]
where $\tilde{F}(u, v, w)$ is still a $C^1$ function of $u$, $v$ and $w$, satisfying

$$\tilde{F}(0, 0, 0) = 0. \quad (2.12)$$

It is easy to see that, (2.11) is a strictly hyperbolic system with three distinct real eigenvalues $\lambda_i$ ($i = 1, 2, 3$):

$$\lambda_1 = -\sqrt{K_v(u, v)} < \lambda_2 = 0 < \lambda_3 = \sqrt{K_v(u, v)}. \quad (2.13)$$

Thus, the characteristics for the system (2.11) are given by

$$\frac{dx_i}{dt} = \lambda_i \quad (i = 1, 2, 3). \quad (2.14)$$

Moreover, the corresponding left eigenvectors can be taken as

$$l_1 = (0, \sqrt{K_v}, 1), \quad l_2 = (1, 0, 0), \quad l_3 = (0, -\sqrt{K_v}, 1). \quad (2.15)$$

Let

$$U = (u, v, w)^T. \quad (2.16)$$

The original initial condition (2.5) can be reduced to

$$t = 0: \quad U = (\varphi(x), \varphi'(x), \psi(x))^T, \quad 0 \leq x \leq L, \quad (2.17)$$

and the original boundary conditions (2.6) on $x = 0$ and (2.7) on $x = L$ can be correspondingly replaced by

$$x = 0: \quad w = h'(t), \quad (2.18a)$$

$$x = 0: \quad v = h(t), \quad (2.18b)$$

$$x = 0: \quad v - \alpha u = h(t), \quad (2.18c)$$

$$x = 0: \quad v - \beta w = h(t) \quad (2.18d)$$

and

$$x = L: \quad w = \tilde{h}'(t), \quad (2.19a)$$

$$x = L: \quad v = \tilde{h}(t), \quad (2.19b)$$

$$x = L: \quad v + \tilde{\alpha} u = \tilde{h}(t), \quad (2.19c)$$

$$x = L: \quad v + \tilde{\beta} w = \tilde{h}(t), \quad (2.19d)$$

respectively.
2.2 Semi-global $C^2$ Solutions to the Mixed Initial-Boundary Value Problem

Let

\[ v_i = l_i(U)U \quad (i = 1, 2, 3) \]  \hspace{1cm} (2.20)

be the diagonal variables corresponding to $\lambda_i(U) \ (i = 1, 2, 3)$, respectively. It is easy to see that, at least in a neighbourhood of $U = 0$, the boundary conditions (2.18) and (2.19) can be rewritten as

\begin{align*}
  x = 0 : & \quad v_3 = -v_1 + 2h'(t), \quad (2.21a) \\
  x = 0 : & \quad v_3 = v_1 - 2\sqrt{K_v(v_2, h(t))}h(t), \quad (2.21b) \\
  x = 0 : & \quad v_3 = v_1 - 2\sqrt{K_v(v_2, \alpha v_2 + h(t))}(\alpha v_2 + h(t)), \quad (2.21c) \\
  x = 0 : & \quad v_3 = p_4(h(t), v_1, v_2) + \tilde{p}_4(t) \quad (2.21d)
\end{align*}

and

\begin{align*}
  x = L : & \quad v_1 = -v_3 + 2\tilde{h}'(t), \quad (2.22a) \\
  x = L : & \quad v_1 = v_3 + 2\sqrt{K_v(v_2, \tilde{h}(t))}\tilde{h}(t), \quad (2.22b) \\
  x = L : & \quad v_1 = v_3 + 2\sqrt{K_v(v_2, \tilde{h}(t) - \alpha v_2)}(\tilde{h}(t) - \alpha v_2), \quad (2.22c) \\
  x = L : & \quad v_1 = q_4(\tilde{h}(t), v_1, v_2) + \tilde{q}_4(t), \quad (2.22d)
\end{align*}

respectively.

It is easy to see that the boundary conditions (2.21) on $x = 0$ and (2.22) on $x = L$ are of the form

\[ x = 0 : \quad v_3 = G_3(t, v_1, v_2) + H_3(t) \]  \hspace{1cm} (2.23)

and

\[ x = L : \quad v_1 = G_1(t, v_2, v_3) + H_1(t) \]  \hspace{1cm} (2.24)

respectively, with

\[ G_1(t, 0, 0) \equiv G_3(t, 0, 0) \equiv 0. \]  \hspace{1cm} (2.25)

Thus, Lemma 2.1 can be immediately obtained by Lemma 1.2 in the case $n = 3$.

2.3 Uniqueness of $C^2$ Solution to the One-Sided Mixed Initial-Boundary Value Problem

To study the exact boundary controllability of nodal profile for quasilinear wave equations, we need the uniqueness of $C^2$ solution to the one-sided mixed initial-boundary value problem.
For this purpose, we consider the one-sided forward mixed initial-boundary value problem for the equation (2.1) with the initial data (2.5) and the boundary conditions (2.6) on \( x = 0 \). By Lemma 1.4, it is easy to prove the following lemma.

**Lemma 2.2** The \( C^2 \) solution \( u = u(t, x) \) to the one-sided forward mixed initial-boundary value problem (2.1) and (2.5)–(2.6) is unique on its maximum determinate domain

\[
\{ (t, x) | t \geq 0, \ 0 \leq x \leq x(t) \}, \tag{2.26}
\]

where \( x = x(t) \) denotes the leftmost characteristic passing through the point \( (t, x) = (0, L) \), i.e.,

\[
\begin{cases}
  x'(t) = -\sqrt{K_v(u(t, x(t)), v(t, x(t)))}, \\
  x(0) = L.
\end{cases} \tag{2.27}
\]

Noting (2.2), we now exchange the role of \( t \) and \( x \), and consider the following one-sided rightward mixed initial-boundary value problem for the equation (2.1) with the initial data given at the \( t \)-axis:

\[
x = 0: \quad u = a(t), \ ux = b(t), \quad 0 \leq t \leq T_1, \tag{2.28}
\]

and the boundary condition coming from the original initial data (2.5):

\[
t = 0: \quad u = \varphi(x), \quad 0 \leq x \leq L. \tag{2.29}
\]

By Lemma 2.2, the \( C^2 \) solution \( u = u(t, x) \) to the one-sided rightward mixed initial-boundary value problem (2.1) and (2.28)–(2.29) is unique on its maximum determinate domain

\[
\{ (t, x) | 0 \leq t \leq t(x), \ x \geq 0 \}, \tag{2.30}
\]

where \( t = t(x) \) denotes the downmost characteristic passing through the point \( (t, x) = (T_1, 0) \):

\[
\begin{cases}
  t'(x) = -\frac{1}{\sqrt{K_v(u(t(x), x), v(t(x), x))}}, \\
  t(0) = T_1.
\end{cases} \tag{2.31}
\]

For the solution \( u = u(t, x) \) with \( |u(t, x)| + |u_x(t, x)| \leq \epsilon_0, \ \epsilon_0 > 0 \) being small enough, we have always

\[
t(x) \geq t(x), \tag{2.32}
\]
2.3 Uniqueness of $C^2$ Solution to the One-Sided Mixed Initial-Boundary Value Problem

where $t = t(x)$ is defined by

$$
\begin{cases}
    t'(x) = \inf_{|u|+|v| \leq \epsilon_0} \frac{1}{\sqrt{K_v(u, v)}}, \\
    t(0) = T_1.
\end{cases}
$$

in which

$$
\beta \overset{\text{def.}}{=} \inf_{|u|+|v| \leq \epsilon_0} \frac{1}{\sqrt{K_v(u, v)}}
$$

is a constant, then $t = t(x)$ is the straight line

$$
t = T_1 + \beta x.
$$

Hence, the $x$ coordinate of the intersection point of $t = t(x)$ with the $x$-axis is equal to

$$
-\frac{T_1}{\beta} = \frac{T_1}{\sup_{|u|+|v| \leq \epsilon_0} \frac{1}{\sqrt{K_v(u, v)}}}.
$$

Thus, in order that the maximum determinate domain (2.30) contains the interval $[0, L]$ on the $x$-axis, it is sufficient to take $T_1$ so large that

$$
T_1 \geq L \sup_{|u|+|v| \leq \epsilon_0} \frac{1}{\sqrt{K_v(u, v)}}.
$$

In fact, in this situation, the triangular domain

$$
\left\{(t, x) \mid 0 \leq t \leq \frac{T_1}{L} (L - x), \ 0 \leq x \leq L \right\}
$$

must be included in the maximum determinate domain (2.30), then the $C^2$ solution $u = u(t, x)$ to the one-sided rightward mixed initial-boundary value problem (2.1) and (2.28)–(2.29) should be unique on it.

Thus, we have

**Lemma 2.3** Under assumption (2.2), if

$$
T_1 > \frac{L}{\sqrt{K_v(0, 0)}},
$$

then for the one-sided rightward mixed initial-boundary value problem (2.1) and (2.28)–(2.29), the maximum determinate domain of the small $C^2$ solution $u = u(t, x)$ is...
with \(|u(t, x)| + |u_x(t, x)| \leq \epsilon_0 \) (\(\epsilon_0 > 0\) being small enough) must contain the whole interval \([0, L]\) on the \(x\)-axis.

For the one-sided leftward mixed initial-boundary value problem of the Eq. (2.1) with the initial condition

\[
x = L : \quad u = \bar{a}(t), \quad u_x = \bar{b}(t), \quad 0 \leq t \leq T_1
\]  

(2.39)

and the boundary condition (2.29), similar results hold.
Exact Boundary Controllability of Nodal Profile for Quasilinear Hyperbolic Systems
Li, T.; Wang, K.; Gu, Q.
2016, IX, 108 p. 27 illus., Softcover
ISBN: 978-981-10-2841-0