Chapter 2
Universal Fuzzy Models and Universal Fuzzy Controllers for Non-affine Nonlinear Systems

2.1 Introduction

It is known that mathematical modeling of many real engineering plants often leads to complex nonlinear systems, which brings severe difficulties to stability analysis and controller synthesis [1–6]. In recent decades, researchers have been seeking some effective methods to handling complex nonlinear systems and the T–S fuzzy-model-based approach is one of most important approaches. T–S fuzzy models describe a nonlinear system by a group of fuzzy IF-THEN rules in the form of local linear models which are smoothly connected by fuzzy membership functions. This relatively simple structure provides systematic stability analysis and controller design of T–S fuzzy control systems by exploiting conventional control theory, see several books and survey papers [7–11] and the references therein for details.

T–S fuzzy models are shown to be universal function approximators in the sense that they are able to approximate any smooth nonlinear functions to arbitrary degree of accuracy in any convex compact region [12–24]. However, it has been argued in [20] that the commonly used T–S fuzzy models, where the control variables are not included in the premise variables, are only able to approximate affine nonlinear systems to any degree of accuracy on any compact set. In other words, only the stabilization problem of affine nonlinear systems can be solved based on the commonly used T–S fuzzy models. Control design of more general non-affine nonlinear systems via T–S fuzzy models is still a challenge.

In addition, several critical questions still remain open. For example, how to construct a T–S fuzzy model such that it is an universal function approximator to non-affine nonlinear systems? Are these T–S fuzzy models universal fuzzy models in the sense that the approximation error between states of the T–S fuzzy approximation model and the original non-affine nonlinear system can be arbitrarily small [9]? How can controller design of the obtained T–S fuzzy models be facilitated? Given a non-affine nonlinear systems which can be stabilized by an appropriately defined controller, does there exist a fuzzy controller to stabilize it? The last problem is the so-called universal fuzzy controller problem [25–33]. Furthermore, how can one design the universal fuzzy controller if it exists?

Recently, some results on control design of non-affine nonlinear systems based on modified T–S fuzzy models have appeared in literatures [34–38]. However, these
approaches are difficult to implement due to its high complexity. In addition, some preliminary results on universal fuzzy controller problem have been reported in [30, 31]. However, it is found that when applying the approaches proposed in [30, 31] to the case of non-affine nonlinear systems, controller design has to be accomplished by solving a set of nonlinear equations which might be a very difficult task in general.

In this chapter, we investigate the universal fuzzy model problem and universal fuzzy controller problem in the context of non-affine nonlinear systems based on a class of generalized T–S fuzzy models. The generalized T–S fuzzy models is proved to be universal fuzzy models for non-affine nonlinear systems under some sufficient conditions. An approach to robust controller design for non-affine nonlinear systems based on this kind of generalized T–S fuzzy models is then developed. The results of universal fuzzy controllers for two classes of non-affine nonlinear systems are then given, and constructive procedures to obtain the universal fuzzy controllers are also provided. An example is finally presented to show the effectiveness of the proposed approach.

2.2 Universal Fuzzy Models for Non-affine Nonlinear Systems

Consider a nonlinear system described by the following equation,

\[ \dot{x}(t) = f(x(t), u(t)), \]

(2.1)

where \( x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathcal{X} \subseteq \mathbb{R}^n \) and \( u(t) = [u_1(t), \ldots, u_m(t)]^T \in \mathcal{U} \subseteq \mathbb{R}^m \). Throughout the book, it is always assumed that the origin is the equilibrium of the system, that is, \( f(0, 0) = 0 \), and \( f(x, u) \) is a continuously differentiable function on \( \mathcal{X} \times \mathcal{U} \). Further it is assumed that \( \mathcal{X} \times \mathcal{U} \) is a compact region in \( \mathbb{R}^n \times \mathbb{R}^m \).

In order to develop an approach to control of such general nonlinear systems via a way of Takagi–Sugeno (T–S) fuzzy modeling, we consider to approximate the nonlinear system (2.1) by the following class of generalized T–S fuzzy models,

**Plant rule** \( R_l : I F x_1(t) \) is \( \mu_{l1} \) AND \ldots AND \( x_n(t) \) is \( \mu_{ln} \); \( u_1(t) \) is \( \nu_{l1} \) AND \ldots AND \( u_m(t) \) is \( \nu_{lm} \), THEN

\[ \dot{x}(t) = A_l x(t) + B_l u(t); l \in \mathcal{L} := \{1, 2, \ldots, r\}, \]

(2.2)

where \( \mathcal{L} \) denotes the \( l \)th rule, \( r \) is the total number of rules, \( \mu_{li} \) and \( \nu_{li} \) are the fuzzy sets, \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) is the input vector, and \([A_l, B_l]\) are the matrices of the \( l \)th local model.

Via the commonly used fuzzy inference method, that is, the center-average defuzzifier, product inference and singleton fuzzifier, the T–S fuzzy system (2.2) can be expressed as follows,
\[
\dot{x}(t) = \hat{f}(x(t), u(t)),
\]
with
\[
\begin{align*}
\hat{f}(x(t), u(t)) &= \sum_{i=1}^{r} \mu_i(x, u)[A_i x(t) + B_i u(t)], \\
\mu_i(x, u) &= \frac{\prod_{j=1}^{n} \mu_i^j(x_j) \prod_{j=1}^{m} \nu_j^i(u_j)}{\sum_{i=1}^{r} \prod_{j=1}^{n} \mu_i^j(x_j) \prod_{j=1}^{m} \nu_j^i(u_j)},
\end{align*}
\]
where \(\mu_i(x, u)\) are the so-called normalized fuzzy membership functions satisfying
\(\mu_i(x, u) \geq 0\) and \(\sum_{i=1}^{r} \mu_i(x, u) = 1\).

**Remark 2.1** It is noted that the T–S fuzzy model (2.2) differs from the commonly used T–S fuzzy models [8] where the control variables are not included in the premise part of the rules. It has been shown in [20] that the commonly used T–S fuzzy models are only able to approximate affine nonlinear systems. Thus, to represent a more general nonlinear systems given as in (2.1), a more general T–S fuzzy model, such as that given in (2.2) or (2.3) is needed. This kind of models are referred to be the generalized T–S fuzzy models in this book.

Before proceeding further, we introduce a lemma first.

**Lemma 2.1** ([20]) If vector value function \(f(z) = [f_1(z_1, \ldots, z_N), \ldots, f_n(z_1, \ldots, z_N)]^T\) is continuously differentiable on \(Z = \prod_{i=1}^{N} [\alpha_i, \beta_i]\) with \(0 \in Z\) and \(f(0) = 0\), then for \(i = 1, \ldots, N\) the vector value function
\[
G_i(z) = g_i(z_i, \ldots, z_N) = \begin{cases} f(0, \ldots, \hat{z}_i, \ldots, 0, \ldots, \hat{z}_N) - f(0, \ldots, 0, \ldots, \hat{z}_N), & \hat{z}_i \neq 0 \\ \frac{\partial f(0, \ldots, 0, \ldots, \hat{z}_i, \ldots, \hat{z}_N)}{\partial z_i}, & \hat{z}_i = 0 \end{cases}
\]
is continuous on \(Z\) and
\[
f(z) = \sum_{i=1}^{N} G_i(z) z_i = \sum_{i=1}^{N} g_i(z_i, \ldots, z_N) z_i.
\]

Then one has the following result on the universal function approximation capability of the generalized T–S fuzzy models.

**Theorem 2.1** For any given function \(f(x, u) \in C^1\) on the compact set \(\mathcal{X} \times \mathcal{U}\) with \(f(0, 0) = 0\) and any positive constant \(\varepsilon_f\), there exists a T–S fuzzy model \(\hat{f}(x, u) = \sum_{i=1}^{r} \mu_i(x, u)[A_i x(t) + B_i u(t)]\) given in (2.3) such that, for any \((x, u) \in \mathcal{X} \times \mathcal{U}\),
\[
\hat{f}(x, u) = f(x, u) + \varepsilon(x, u),
\]
and
\[
\|\varepsilon(x, u)\| = \|\Delta E(x, u)\| \leq \varepsilon_f \|\tilde{x}\|,
\]
where \(\tilde{x} = [x_1, \ldots, x_n, u_1, \ldots, u_m]^T \in \mathcal{R}^{m+n}\).
Proof} Let \( f(x, u) = [f_1(x, u), \ldots, f_n(x, u)]^T \), then
\[
\begin{align*}
\frac{\partial f(0, \ldots, 0, x_i, u_j, \ldots, u_m) - f(0, \ldots, 0, x_{i+1}, \ldots, u_m)}{u_j}, & \quad x_i \neq 0 \\
\frac{\partial f(0, \ldots, 0, x_{i+1}, \ldots, u_m)}{\partial x_i}, & \quad x_i = 0
\end{align*}
\]
\[
\begin{align*}
g_A(x, u) &= \sum_{l=1}^{r} \mu_l(x, u)A_l \\
g_B(x, u) &= \sum_{l=1}^{r} \mu_l(x, u)B_l.
\end{align*}
\]

Based on the construction schemes given, for example, [15, 16], we have that for any \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \), there always exists two fuzzy systems \( g_A(x, u) \) and \( g_B(x, u) \) such that,
\[
\begin{align*}
g_A(x, u) &= g_1(x, u) + \Delta A(x, u), \quad \|\Delta A(x, u)\|_\infty \leq \varepsilon_1, \\
g_B(x, u) &= g_2(x, u) + \Delta B(x, u), \quad \|\Delta B(x, u)\|_\infty \leq \varepsilon_2,
\end{align*}
\]
where \( \Delta A(x, u) \in \mathbb{R}^{m \times n} \) and \( \Delta B(x, u) \in \mathbb{R}^{m \times n} \) are the approximation errors.

Then it follows from (2.12), (2.13) and \( \sum_{l=1}^{r} \mu_l(x, u) = 1 \) that
\[
\begin{align*}
\sum_{l=1}^{r} \mu_l(x, u)\left\{ A_l x + B_l u \right\} &= f(x, u) + \varepsilon(x, u),
\end{align*}
\]
where \( \varepsilon(x, u) = \Delta A(x, u)x + \Delta B(x, u)u. \)

Based on (2.12) and (2.13), one has
\[
\|\varepsilon(x,u)\| \leq \|\Delta A(x,u)\| \|x\| + \|\Delta B(x,u)\| \|u\|
\]
\[
\leq \varepsilon_1 \|x\| + \varepsilon_2 \|u\| \leq \varepsilon_f \|\bar{x}\|, \quad (2.15)
\]

where \(\varepsilon_f = \sqrt{2} \max(\varepsilon_1, \varepsilon_2)\).

Thus the proof is completed.

In fact, the proof of Theorem 2.1 provides a constructive procedure to obtain the
generalized T–S model with the aid of the following fact.

**Fact 2.1** The matrix function \(Q(x) = [Q^{ij}(x)] \in \mathbb{R}^{m \times n}\) where \(Q^{ij}(x) = \sum_{l=1}^{r_0} \mu_l^{ij}(x)q_l^{ij}\), \(\mu_l^{ij}(x) \geq 0\) and \(\sum_{l=1}^{r_0} \mu_l^{ij}(x) = 1\), can be rewritten as \(Q(x) = \sum_{l=1}^{r_0} \nu_l(Q)\)
\(Q_l\) where \(Q_l \in \mathbb{R}^{m \times n}\), \(\nu_l(x) \geq 0\) and \(\sum_{l=1}^{r_0} \nu_l(x) = 1\).

**Proof** We first define \(L_0 = \{1, \ldots, r_0\}\), \(\mathcal{I} = \{1, \ldots, m\}\), \(\mathcal{J} = \{1, \ldots, n\}\) and \(\mathcal{L}_0 = \{1, \ldots, r_0^{m \times n}\}\). Let \(Q_l = [q_{k(i,j)}^l]\), where \(i \in \mathcal{I}\), \(j \in \mathcal{J}\), \(\hat{l} \in \mathcal{L}_0\) and \(k(i,j) : \mathcal{I} \times \mathcal{J} \rightarrow \mathcal{L}_0\). For any \((i_0, j_0, l_0) \in \mathcal{I} \times \mathcal{J} \times L_0\), denote the set
\(\{\hat{l}|k(i_0, j_0) = l_0, \hat{l} \in \mathcal{L}_0\}\) as \(\mathcal{L}_0(i_0, j_0, l_0)\).

Here we prove that if \(\nu_l(x) = \prod_{(i,j)\in \mathcal{I} \times \mathcal{J}} \mu_l^{ij}\), then
\[
Q(x) = \sum_{l=1}^{r_0^{m \times n}} \nu_l(x) Q_l. \quad (2.16)
\]

For any \((i_0, j_0, l_0) \in \mathcal{I} \times \mathcal{J} \times \mathcal{L}_0\), if the coefficients of \(q_{l_0}^{i_0,j_0}\) in both sides of (2.16) are equal, then (2.16) holds.

From the left-hand side of (2.16), the coefficient of \(q_{l_0}^{i_0,j_0}\) is \(c_{l_0}^{i_0,j_0} = \mu_{l_0}^{i_0,j_0}\). From the right-hand side of (2.16), the coefficient of \(q_{l_0}^{i_0,j_0}\) is
\[
c_{l_0}^{i_0,j_0} = \sum_{(i,j)\in \mathcal{I} \times \mathcal{J}} \prod_{\hat{l} \in \mathcal{L}_0(i_0,j_0,l_0)} \mu_{k(i,j)}^{ij}
= \mu_{l_0}^{i_0,j_0} \left\{ \sum_{\hat{l} \in \mathcal{L}_0(i_0,j_0,l_0)} \prod_{(i,j)\in \mathcal{I} \times \mathcal{J} - \{(i_0,j_0)\}} \mu_{k(i,j)}^{ij} \right\}
= \mu_{l_0}^{i_0,j_0} \left\{ \prod_{(i,j)\in \mathcal{I} \times \mathcal{J} - \{(i_0,j_0)\}} \sum_{k(i,j)\in \mathcal{L}_0} \mu_{k(i,j)}^{ij} \right\} = \mu_{l_0}^{i_0,j_0}. \quad (2.17)
\]

Then one can easily conclude that \(\nu_l(x) \geq 0\) and \(\sum_{l=1}^{r_0^{m \times n}} \nu_l(x) = 1\). Thus the proof is completed.
Then one has the following constructive algorithm.

**Algorithm 2.1** Given a non-affine nonlinear system \( \dot{x} = F(x, u) \), one can construct a generalized T–S fuzzy model \( \dot{\hat{x}}(t) = \hat{F}(x(t), u(t)) = \sum_{l=1}^{r_1} \mu_l(x, u)[C_l x(t) + D_l u(t)] \) to approximate \( F(x, u) \) by the following steps.

**Step 1.** By using Lemma 2.1, transform \( F(x, u) \) into the product of \( F'(\bar{x}) \) and \( \bar{x} \), where \( F'(\bar{x}) = [F'_{ij}(\bar{x})] \in \mathbb{R}^{n \times (m+n)} \), \( i \in \mathcal{I} = \{1, \ldots, n\} \), \( j \in \mathcal{J} = \{1, \ldots, m+n\} \) is a continuous function and \( \bar{x} = [x_1, \ldots, x_n, u_1, \ldots, u_m]^T \in \mathbb{R}^{m+n} \).

**Step 2.** By using the approximation schemes given in [15] or [16], construct the Type II fuzzy model [9], that is, \( \hat{F}'_{ij}(\bar{x}) \) of each element \( F'_i(\bar{x}) \). Then the corresponding fuzzy model for \( F'(\bar{x}) \) is denoted by \( \hat{F}'(\bar{x}) = [\hat{F}_{ij}(\bar{x})] \). Without loss of generality, it is assumed that each fuzzy model \( \hat{F}'_{ij}(\bar{x}) \) has \( r_0 \) rules. Suppose \( \hat{F}'_{ij}(\bar{x}) = \sum_{l=1}^{r_0} \mu_l^j(\bar{x}) q_{ij}^l \), where \( \mu_l^j(\bar{x}) \) are normalized membership functions and \( q_{ij} \) are positive constants.

**Step 3.** By using Fact 2.1, rewrite \( \hat{F}'(\bar{x}) \) as \( \hat{F}'(\bar{x}) = \sum_{l=1}^{r_0} \nu_l(\bar{x}) Q_l \).

In this way, one can construct the generalized T–S fuzzy model for \( \dot{\bar{x}} = F(x, u) \), that is, \( \dot{\bar{x}}(t) = \hat{F}(x(t), u(t)) = \hat{F}'(\bar{x}) \bar{x} = \sum_{l=1}^{r_1} \mu_l(x, u)[C_l x(t) + D_l u(t)] \), where \( r_1 = r_0^{(m+n) \times n} \), \( \mu_l(x, u) = \nu_l(\bar{x}) \) and \( Q_l = [C_l, D_l] \).

**Remark 2.2** Based on the construction scheme shown in the Algorithm 2.1, the error bound \( \varepsilon_f \) can be made arbitrarily small by choosing large enough number of fuzzy rules. Noticing that \( \mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^m \) are bounded, thus the norm bound of the approximation error \( \|\varepsilon(t)\| \) can be arbitrarily small by choosing \( \varepsilon_f > 0 \) accordingly. In other words, the fuzzy dynamic model given in (2.2) can approximate the general nonlinear system in (2.1) to any degree of accuracy.

**Remark 2.3** From Theorem 2.1, one can conclude that the stabilization of a non-affine nonlinear system (2.1) can be solved as a robust stabilization problem of its corresponding generalized T–S fuzzy model with the approximation error as the uncertainty term. It is also noted that with the control variables included in the premise part of fuzzy rules, the two commonly used control schemes, that is, the parallel distributed compensation and the local compensation [9], can not be applied directly.

**Remark 2.4** It should be noted that the result in Theorem 2.1 only answers the approximation problems between two static nonlinear functions, that is, \( \hat{f}(x, u) \) and \( f(x, u) \). However, the approximation errors between the states of two dynamic systems, that is, systems described in (2.1) and (2.3), might grow as time goes. Much care should be taken in dealing with the approximation errors between two dynamic systems, instead of static functions.

For convenience of analysis, we rewrite the generalized T–S fuzzy system in (2.3) as

\[
\dot{\hat{x}}(t) = \hat{f}(\hat{x}(t), u(t)).
\]

(2.18)
Let $GF$ be the set of all dynamic fuzzy models of the form (2.2). Before proceeding further we introduce following definition.

**Definition 2.1** $GF$ are said to be universal fuzzy models for non-affine nonlinear systems as in (2.1), if the approximation error between (2.1) and (2.18) can be made arbitrary small, that is, for any given positive constant $\varepsilon > 0$, there exists a T–S fuzzy system in (2.3) such that for the two dynamic systems (2.1) and (2.18) under the same initial condition and control inputs,

$$\sup_{t \geq 0} \| \hat{x}(t) - x(t) \|^2 < \varepsilon.$$  

(2.19)

We first introduce the following useful lemma.

**Lemma 2.2** (Generalized Gronwall inequality) For any continuous and nonnegative function $\beta(t)$ and $\gamma(t)$, if

$$\varphi(t) \leq \alpha(t) + \gamma(t) \int_0^t \beta(s) \varphi(s) ds,$$  

(2.20)

then one has that

$$\varphi(t) \leq \alpha(t) + \gamma(t) \int_0^t \alpha(s) \beta(s) \frac{\dot{\phi}(s)}{\phi(t)} ds,$$  

(2.21)

where $\phi(t) = e^{-\int_0^t \gamma(s) \beta(s) ds}$.

**Proof** From (2.20), one has that

$$\frac{d}{dt} \{ \phi(t) \int_0^t \beta(s) \varphi(s) ds \} = \beta(t) \phi(t) \varphi(t) - \beta(t) \phi(t) \gamma(t) \int_0^t \beta(s) \varphi(s) ds < \beta(t) \phi(t) \alpha(t).$$  

(2.22)

Therefore,

$$\int_0^t \beta(s) \varphi(s) ds \leq \int_0^t \alpha(s) \beta(s) \frac{\dot{\phi}(s)}{\phi(t)} ds.$$  

(2.23)

Then,

$$\varphi(t) \leq \alpha(t) + \gamma(t) \int_0^t \beta(s) \varphi(s) ds \leq \alpha(t) + \gamma(t) \int_0^t \alpha(s) \beta(s) \frac{\dot{\phi}(s)}{\phi(t)} ds,$$  

(2.24)

which completes the proof.
For the non-affine nonlinear system given in (2.1), denote the Jacobian matrix of the function $f(x, u)$ at the origin as

$$J_f |_{[x,u]=[0,0]} = \frac{\partial f(x, u)}{\partial [x^T, u^T]^T} |_{[x,u]=[0,0]} = [A, B],$$

(2.25)

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$.

Denote $b(x, u) = f(x, u) - Ax$ and $\hat{b}(x, u) = \hat{f}(x, u) - Ax$. Then it is known that $\hat{b}(\cdot)$ satisfies the Lipschitz condition, i.e., there exists a positive constant $\beta_b$ such that

$$\|\hat{b}(x_1, u) - \hat{b}(x_2, u)\| \leq \beta_b \|x_1 - x_2\|, \quad x_1, x_2 \in \mathcal{X}.$$  

(2.26)

Then we are ready to give the following result.

**Theorem 2.2** GF are universal fuzzy models for non-affine nonlinear systems as in (2.1), if $A$ is a Hurwitz matrix satisfying $\|e^{At}\| \leq ce^{-\alpha t}$, for two constants $c$ and $\alpha > 0$, and

$$\beta_b < \frac{\alpha}{2c},$$

(2.27)

where $A$, and $\beta_b$ are defined in (2.25), and (2.26), respectively.

**Proof** The non-affine nonlinear system (2.1) can be rewritten as

$$\dot{x}(t) = Ax(t) + b(x(t), u(t)).$$

(2.28)

Then by using Theorem 2.1, given any positive constants $\varepsilon_b > 0$, one can obtain the following generalized T–S fuzzy model,

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \hat{b}(\hat{x}(t), u(t)),$$

(2.29)

such that,

$$\sup_{(x,u) \in \mathcal{X} \times \mathcal{U}} \|\hat{b}(x(t), u(t)) - b(x(t), u(t))\| < \varepsilon_b \|\hat{x}(t)\|.$$  

(2.30)

Denote $e(t) = \hat{x}(t) - x(t)$. Then from (2.28) and (2.29), we have,

$$\dot{e}(t) = Ae(t) + F(t),$$

(2.31)

where

$$F(t) = \hat{b}(\hat{x}(t), u(t)) - b(x(t), u(t)),$$

(2.32)
and

\[ \| F(t) \| \leq \| \hat{b}(\hat{x}(t), u(t)) - \hat{b}(x(t), u(t)) \| + \| \hat{b}(x(t), u(t)) - b(x(t), u(t)) \| \leq \beta_b \| e(t) \| + \varepsilon_b \| \hat{x}(t) \|. \] (2.33)

The solutions of Eq. (2.31) is

\[ e(t) = \int_0^t e^{A(t-s)} F(s) ds. \] (2.34)

Then one has

\[ \| e(t) \| \leq \int_0^t \| e^{A(t-s)} \| F(s) \| ds \leq \frac{M \varepsilon_b c}{\alpha} (1 - e^{-\alpha t}) + c \beta_b e^{-\alpha t} \int_0^t e^{\alpha s} \| e(s) \| ds \leq \frac{M \varepsilon_b c}{\alpha} + c \beta_b e^{-\alpha t} \int_0^t e^{\alpha s} \| e(s) \| ds. \] (2.35)

By using Lemma 2.1 under condition (2.27), one has

\[ \| e(t) \| \leq \frac{M \varepsilon_b c}{\alpha} \left[ 1 + c \beta_b \int_0^t e^{(\alpha - c \beta_b)(s-t)} ds \right] \leq \frac{M c}{\alpha - c \beta_b} \varepsilon_b. \] (2.36)

From the proof procedure of Theorem 2.1, \( \varepsilon_b \) can be made arbitrarily small, which implies that \( \| e(t) \| \) can be also arbitrarily small. Thus the proof is completed.

2.3 Robust Stabilization Controller Design

It has been shown in Theorem 2.1 that a non-affine nonlinear system described by (2.1) can be exactly expressed in a compact set by a generalized T–S fuzzy model in (2.3) with some uncertainties as follows,

\[ \dot{x}(t) = \sum_{l=1}^{r} \mu_l(x, u) \left\{ A_l x(t) + B_l u(t) + \varepsilon(x(t), u(t)) \right\}, \] (2.37)

where

\[ \varepsilon(x(t), u(t)) = \Delta E(x, u) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \]
In this section, we will develop a stabilization controller design approach to the non-affine nonlinear systems (2.1) via the way of T–S fuzzy models in terms of system (2.3). The following dynamic feedback controller will be employed.

**Plant rule** $R^l$ : IF $x_1(t)$ is $\mu_1^l$ AND ... AND $x_n(t)$ is $\mu_n^l$; $u_1(t)$ is $\nu_1^l$ AND ... AND $u_m(t)$ is $\nu_m^l$, THEN

$$
\dot{u}(t) = F_l x(t) + G_l u(t); \quad l \in \mathcal{L} := \{1, 2, \ldots, r\},
$$

(2.38)

which can be rewritten via the standard fuzzy blending as

$$
\dot{u}(t) = \sum_{l=1}^{r} \mu_l(x, u)[F_l x(t) + G_l u(t)].
$$

(2.39)

Then the closed-loop control system can be described by

$$
\dot{\tilde{x}}(t) = \left\{ \sum_{l=1}^{r} \mu_l(\tilde{x}(t)) \mathcal{A}_l + R \Delta E(\tilde{x}(t)) \right\} \tilde{x}(t),
$$

(2.40)

where $\mathcal{A}_l = \bar{A}_l + \bar{B}_l \bar{K}_l$ and

$$
\begin{align*}
\bar{x}(t) &= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, & \bar{A}_l &= \begin{bmatrix} A_l & B_l \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix}, & \bar{B}_l &= \begin{bmatrix} 0_{n \times m} \\ I_m \end{bmatrix}, \\
R &= \begin{bmatrix} I_n \\ 0_{m \times n} \end{bmatrix}, & \bar{K}_l &= \begin{bmatrix} F_l & G_l \end{bmatrix}.
\end{align*}
$$

(2.41)

**Remark 2.5** The basic idea of the dynamic feedback fuzzy controller in (2.38) is to treat the input variables as states of the closed-loop system consisting of (2.37) and (2.38), and design the controller gains $F_l$ and $G_l$ such that the closed-loop control system (2.40) is stable.

Two useful Lemmas are introduced first.

**Lemma 2.3** ([39]) Let $M$ and $N$ be two matrices with appropriate dimensions, then for any positive constant $\varepsilon > 0$, one has that

$$
M^T N + N^T M \leq \frac{1}{\varepsilon} M^T M + \varepsilon N^T N.
$$

(2.42)

**Lemma 2.4** ([39]) The closed-loop fuzzy control system (2.40) is globally asymptotically stable if there exists a positive definite matrix $P$ such that the following matrix inequalities are satisfied,

$$
(\mathcal{A}_l + R \Delta E)^T P + P (\mathcal{A}_l + R \Delta E) < 0, \quad l \in \mathcal{L}.
$$

(2.43)
Before proceeding, we introduce the following definition first.

**Definition 2.2** The close-loop control system (2.40) is said to be semi-globally uniformly exponentially stable on a compact set \( \mathcal{X} \times \mathcal{U} \subseteq \mathbb{R}^n \times \mathbb{R}^m \) which contains the equilibrium, if there exist positive constants \( C \) and \( \lambda > 0 \) and a region \( \mathcal{X}_0 \times \mathcal{U}_0 \subseteq \mathcal{X} \times \mathcal{U} \), such that given any initial states \( \bar{x}_0 \in \mathcal{X}_0 \times \mathcal{U}_0 \), the solution \( \bar{x}(t) \) of (2.40) exists for all \( t \geq 0 \) and \( \| \bar{x}(t) \| \leq C \| \bar{x}_0 \| e^{-\lambda t} \).

Suppose the upper bound of the uncertainty \( \Delta E \) is given by

\[
\Delta E^T \Delta E \leq \bar{\varepsilon}^2 I_{(m+n)},
\]

(2.44)

Then one has the following result,

**Theorem 2.3** The nonlinear system (2.1) can be semi-globally asymptotically stabilized by the fuzzy controller (2.38), if there exist a positive definite matrix \( X \) and a set of matrices \( Q_l, l \in \mathcal{L} \), such that the following LMIs are satisfied,

\[
\begin{bmatrix}
X \bar{A}_l^T + \bar{A}_l X + \bar{B}_l Q_l + Q_l^T \bar{B}_l^T + RR^T \bar{\varepsilon} X & -I \\
\bar{\varepsilon} X & -I
\end{bmatrix} < 0, \quad l \in \mathcal{L}.
\]

(2.45)

Moreover, the controller gains are given by

\[
\bar{K}_l = Q_l X^{-1}, \quad l \in \mathcal{L}.
\]

(2.46)

**Proof** It is noted that the non-affine nonlinear system (2.1) can be exactly expressed by the generalized T–S fuzzy model with some uncertainty as described in (2.37) in any compact set. Thus if the system (2.37) can be asymptotically stabilized by the controller (2.39), with the bounded initial condition on the state \( x(0) \) and the control \( u(0) \), the original general nonlinear systems can be shown to be semi-globally asymptotically stabilized.

It follows from Lemma 2.3 that for any given positive constant \( \varepsilon \),

\[
(\mathcal{A}_l + R \Delta E)^T P + P (\mathcal{A}_l + R \Delta E) = \mathcal{A}_l^T P + P \mathcal{A}_l + \Delta E^T R^T P + PR \Delta E \\
\leq \mathcal{A}_l^T P + P \mathcal{A}_l + \frac{1}{\varepsilon} PRR^T P + \varepsilon \bar{\varepsilon}^2 I.
\]

(2.47)

Using Schur’s Complement with \( X^T = X \) and \( Q_l = \bar{K}_l X \), (2.45) implies

\[
X \mathcal{A}_l^T + \mathcal{A}_l X + R R^T + \varepsilon^2 XX < 0.
\]

(2.48)

Multiplying \( X^{-1} \) from both sides to (2.48) with the fact \( X^{-1} = \frac{1}{\varepsilon} P \), we have

\[
\mathcal{A}_l^T P + P \mathcal{A}_l + \frac{1}{\varepsilon} PRR^T P + \varepsilon \bar{\varepsilon}^2 I < 0.
\]

(2.49)
Thus the matrix inequalities (2.43) hold if the LMIs in (2.45) are satisfied. Then based on Lemma 2.4, the closed-loop control system (2.40) is asymptotically stable if (2.45) holds. Thus one has shown that the original non-affine nonlinear system (2.1) can be semi-globally asymptotically stabilized by the controller in (2.39). Then the proof is completed.

**Remark 2.6** The matrix inequalities in (2.48) are difficult to be satisfied if the approximation error bound $\bar{\varepsilon}$ is too large. To achieve better approximation, that is, smaller approximation error, more fuzzy rules are normally needed to construct the T–S fuzzy models, which would increase the number of LMIs in (2.45) significantly and thus its complexity. In practice, it is a trade-off problem to decide how many rules should be used in T–S fuzzy models.

**Remark 2.7** It is noted that Theorem 2.3 is based on a common quadratic Lyapunov function. Less conservative but more complex stabilization results based on piecewise Lyapunov functions can also be also obtained. Readers can refer to [40] for the case of discrete-time systems.

### 2.4 Universal Fuzzy Controllers for a Class of Non-affine Nonlinear Systems

In Sect. 2.3, we have proposed an approach to stabilization of the non-affine nonlinear system (2.1) via its generalized T–S fuzzy model (2.3), by using a class of dynamic feedback controllers as in (2.38). In this section, we will study the so-called universal fuzzy controller problem [25, 26, 28–32], that is, if the nonlinear system (2.1) can be stabilized by a dynamic feedback controller described by $\dot{u}(t) = g(x(t), u(t))$, does there exist a fuzzy controller given by (2.38) to stabilize the nonlinear system?

Let $GFC$ be the set of all fuzzy controllers of the form (2.38) and $NNS$ the set of all nonlinear systems of the form (2.1). First we introduce the following definitions.

**Definition 2.3** ([41]) Any $f \in NNS$ is said to be globally uniformly exponentially stabilizable, if there exists a dynamic feedback control law $\dot{u}(t) = g(x(t), u(t))$ such that the closed-loop control system

\[
\begin{cases}
\dot{x}(t) = f(x(t), u(t)) \\
\dot{u}(t) = g(x(t), u(t))
\end{cases}
\]  

(2.50)

is globally uniformly exponentially stable, that is, there exist positive constants $C$ and $\lambda > 0$, such that given any initial states $(x(0), u(0))$, the solution $(x(t), u(t))$ of (2.50) exists for all $t \geq 0$ and satisfies

\[
\|[(x(t)^T, u(t)^T)^T] \leq C\|[(x(0)^T, u(0)^T)^T]e^{-\lambda t}.
\]  

(2.51)
Definition 2.4 ([41]) GFC are said to be universal fuzzy controllers, if for any \( f \in NNS \) which is globally uniformly exponentially stabilizable there exists a dynamic feedback fuzzy control law \( \hat{g}(x, u) \in GFC \) such that the closed-loop control system

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) \\
\dot{u}(t) &= \hat{g}(x(t), u(t))
\end{align*}
\]

(2.52)

is semi-globally uniformly exponentially stable on a compact set \( \mathcal{R} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^m \).

Then we are ready to present the main results of this section.

Theorem 2.4 GFC are universal fuzzy controllers for a class of nonlinear systems which belong to NNS and are globally uniformly exponentially stabilizable.

Proof Since \( f \in NNS \) is globally uniformly exponentially stabilizable, there exists a control law \( \dot{u}(t) = g(x(t), u(t)) \in C^1 \) such that the closed-loop control system (2.50) is globally uniformly exponentially stable.

Denote \( \bar{x} = [x_1, \ldots, x_n, u_1, \ldots, u_m]^T \subset \mathbb{R}^{m+n} \).

According to Theorem 2.1, for a given small enough constant \( \varepsilon_g > 0 \), one can find a fuzzy control law \( \dot{u}(t) = \hat{g}(x, u) \in GFC \) such that

\[
\hat{g}(x, u) = g(x, u) + \varepsilon(x, u),
\]

(2.53)

where

\[
\|\varepsilon(x, u)\| \leq \varepsilon_g \|\bar{x}\|.
\]

(2.54)

Thus the closed-loop control system (2.52) can be rewritten as

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)), \\
\dot{u}(t) &= g(x(t), u(t)) + \varepsilon(x, u).
\end{align*}
\]

(2.55)

For sake of simplicity, rewrite (2.50) as

\[
\dot{x}(t) = G(\bar{x}(t)),
\]

(2.56)

and (2.52) as

\[
\dot{x}(t) = \hat{G}(\bar{x}(t)) = G(\bar{x}(t)) + \bar{\varepsilon}(\bar{x}),
\]

(2.57)

where

\[
\begin{align*}
G(\bar{x}(t)) &= [f(x(t), u(t))^T, g(x(t), u(t))^T]^T, \\
\hat{G}(\bar{x}(t)) &= [f(x(t), u(t))^T, \hat{g}(x(t), u(t))^T]^T, \\
\bar{\varepsilon}(\bar{x}) &= [0, \varepsilon(x, u)^T]^T.
\end{align*}
\]

(2.58)

It follows from Lyapunov converse theorem [6] that the globally uniformly exponential stability of the system (2.56) is equivalent to the existence of a Lyapunov function \( V(\bar{x}(t)) \), and some positive constants \( c_1, c_2, c_3 \) and \( c_4 \) such that
\[ c_1\|\ddot{x}(t)\|^2 \leq V(\ddot{x}(t)) \leq c_2\|\ddot{x}(t)\|^2, \quad (2.59) \]
\[ \frac{\partial V(\ddot{x}(t))}{\partial \ddot{x}} G(\ddot{x}(t)) \leq -c_3\|\ddot{x}(t)\|^2, \quad (2.60) \]
\[ \|\frac{\partial V(\ddot{x}(t))}{\partial \ddot{x}}\| \leq c_4\|\ddot{x}(t)\|. \quad (2.61) \]

The derivative of this Lyapunov function along the trajectories of the system (2.57) satisfies
\[ \frac{\partial V(\ddot{x}(t))}{\partial \ddot{x}} \hat{G}(\ddot{x}(t)) = \frac{\partial V(\ddot{x}(t))}{\partial \ddot{x}} G(\ddot{x}(t)) + \frac{\partial V(\ddot{x}(t))}{\partial \ddot{x}} \epsilon(\ddot{x}) \]
\[ \leq -c_3\|\ddot{x}(t)\|^2 + c_4\epsilon_g\|\ddot{x}(t)\|^2. \quad (2.62) \]

Thus if one chooses a fuzzy control law such that \( \epsilon_g < c_3/c_4 \), it follows from (2.62) that
\[ \dot{V}(\ddot{x}(t)) < -\tilde{c}\|\ddot{x}(t)\|^2, \text{ where } \tilde{c} = c_3 - c_4\epsilon_g. \quad (2.63) \]

Then one can conclude that (2.57), or equivalently, (2.52) is semi-globally uniformly exponentially stable on the compact set \( \mathcal{X} \times \mathcal{U} \). Thus via Definition 2.4 GFC are universal fuzzy controllers.

If a reference model \( \dot{x}(t) = G_m(\ddot{x}(t)) = [f(x(t), u(t))^T, g_m(x(t), u(t))^T]^T \) is given, one can apply Algorithm 2.1 to obtain the model reference fuzzy controller. That is, one can construct a fuzzy control law \( \hat{g}(x, u) \in GFC \) such that for any given \( \epsilon_m > 0 \),
\[ \hat{g}(x, u) = g_m(x, u) + \epsilon(x, u), \quad (2.64) \]
where
\[ \|\epsilon(x, u)\| \leq \epsilon_m \|\ddot{x}\|, \quad (2.65) \]
and the closed-loop control system
\[ \dot{x}(t) = \hat{G}(\ddot{x}(t)) = G_m(\ddot{x}(t)) + \tilde{\epsilon}(\ddot{x}), \quad (2.66) \]
where \( \hat{G}(\ddot{x}(t)) = [f(x(t), u(t))^T, \hat{g}(x(t), u(t))^T]^T \) and \( \tilde{\epsilon}(\ddot{x}) = [0, \epsilon(x, u)^T]^T \), is semi-globally uniformly exponentially stable on the compact set \( \mathcal{X} \times \mathcal{U} \).

Remark 2.8 In [30, 31], the design of the model reference fuzzy control law is equivalent to solving a set of nonlinear equations, which might be very difficult, or even without explicit solutions in some cases. These difficulties do not exist in our approach due to better approximation capability of the generalized T–S fuzzy model and the dynamic feedback fuzzy controller adopted.
2.5 Universal Fuzzy Controllers for More General Non-affine Nonlinear Systems

In Sect. 2.4, we have shown that the fuzzy controllers defined in (2.38) are universal fuzzy controllers for nonlinear systems which are uniformly exponentially stabilizable. In this section, we will consider more general nonlinear systems which are only globally asymptotically stabilizable.

For sake of simplicity, we use the simplified forms as in (2.56) and (2.57) to represent the closed-loop control systems given as in (2.50) and (2.52), respectively.

We first introduce the following definitions.

**Definition 2.5** ([6]) A function \( \alpha : \mathbb{R}^+ \to \mathbb{R}^+ \) is said to belong to class \( \mathcal{K} \) if it is continuous, strictly increasing and \( \alpha(0) = 0 \). It is said to belong to class \( \mathcal{K} \infty \) if it belongs to class \( \mathcal{K} \) and \( \alpha(s) \to \infty \) as \( s \to \infty \). A function \( \beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) is said to belong to class \( \mathcal{KL} \) if for each fixed \( t \geq 0 \), the function \( \beta(\cdot, t) \) is a \( \mathcal{K} \) function and for each fixed \( s \geq 0 \), \( \beta(s, t) \to 0 \) as \( t \to \infty \).

A useful property of class \( \mathcal{K} \) functions, which will be needed subsequently, is given in the following Lemma.

**Lemma 2.5** ([6]) Let \( \alpha_1 \) be a \( \mathcal{K} \) function and \( \alpha_2 \) be a \( \mathcal{K} \infty \) function. Denote the inverse function of \( \alpha_i \) as \( \alpha_i^{-1} \). Then, \( \alpha_1 \circ \alpha_2^{-1} \) belongs to class \( \mathcal{K} \).

**Definition 2.6** ([41]) Any \( f \in NNS \) is said to be semi-globally uniformly asymptotically stabilizable on a compact set \( \mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^m \) which contains the equilibrium, if there exists a control law \( \dot{u}(t) = g(x(t), u(t)) \) such that the closed-loop control system (2.56) is semi-globally uniformly asymptotically stable on a compact set \( \mathcal{X} \times \mathcal{U} \), that is, there exist \( \mathcal{KL} \) function \( \beta \) and a region \( \mathcal{X}_0 \times \mathcal{U}_0 \subset \mathcal{X} \times \mathcal{U} \), such that given any initial states \( \bar{x}(0) \in \mathcal{X}_0 \times \mathcal{U}_0 \) the solution \( \bar{x}(t) \) of (2.56) exists for all \( t \geq 0 \) and satisfies

\[
\|\bar{x}(t)\| \leq \beta(\|\bar{x}(0)\|, t). \tag{2.67}
\]

**Definition 2.7** ([41]) Any \( f \in NNS \) is said to be globally uniformly asymptotically stabilizable if all the conditions in Definition 2.6 hold globally.

**Definition 2.8** ([41]) System (2.57) is said to be semi-globally input-to-state stable (SISS) on a compact set \( \mathcal{X} \times \mathcal{U} \) if there exist a \( \mathcal{KL} \) function \( \beta \) and a \( \mathcal{K} \) function \( \gamma \), such that given any initial states \( \bar{x}(0) \in \mathcal{X}_0 \times \mathcal{U}_0 \subset \mathcal{X} \times \mathcal{U} \) the solution \( \bar{x}(t) \) of (2.57) exists for all \( t \geq 0 \) and satisfies

\[
\|\bar{x}(t)\| \leq \beta(\|\bar{x}(0)\|, t) + \gamma(\sup_{0 \leq \tau \leq t} \|\bar{x}(\tau)\|). \tag{2.68}
\]

**Definition 2.9** ([41]) GFC are said to be universal practical fuzzy controllers, if for any \( f \in NNS \) which is globally uniformly asymptotically stabilizable there exists a dynamic feedback fuzzy control law \( \hat{g}(x, u) \in GFC \) such that the closed-loop control system (2.57) is semi-globally input-to-state stable on a compact set \( \mathcal{X} \times \mathcal{U} \).
**Definition 2.10** ([41]) \(GFC\) are said to be universal asymptotical fuzzy controllers, if for any \(f \in NNS\) which is globally uniformly asymptotically stabilizable there exists a dynamic feedback fuzzy control law \(\hat{g}(x, u) \in GFC\) such that the closed-loop system (2.57) is semi-globally uniformly asymptotically stable on a compact set \(\mathcal{X} \times \mathcal{U}\).

**Theorem 2.5** \(GFC\) are universal practical fuzzy controllers for the non-affine non-linear systems as in (2.1) which are globally uniformly asymptotically stabilizable.

**Proof** Since \(f \in NNS\) is globally uniformly asymptotically stabilizable, there exists a control law \(\hat{u}(t) = g(x(t), u(t)) \in C^1\) such that the closed-loop control system

\[
\dot{x}(t) = G(\bar{x}(t)),
\]

where \(G(\bar{x}(t)) = [f(x(t), u(t))^T, g(x(t), u(t))^T]^T\), is globally uniformly asymptotically stable.

Based on Theorem 2.1, for a given small enough constant \(\varepsilon_\delta > 0\), one can find a dynamic feedback fuzzy control law \(\hat{u}(t) = \hat{g}(x, u) \in GFC\) such that

\[
\hat{g}(x, u) = g(x, u) + \varepsilon(x, u),
\]

where

\[
\|\varepsilon(x, u)\| \leq \varepsilon_\delta \|\bar{x}\|.
\]

And the closed-loop control system consisting of this fuzzy control law and system (2.1) can be rewritten as

\[
\dot{x}(t) = \hat{G}(\bar{x}(t)) = G(\bar{x}(t)) + \varepsilon(\bar{x}),
\]

where \(\hat{G}(\bar{x}(t)) = [f(x(t), u(t))^T, \hat{g}(x(t), u(t))^T]^T\) and \(\varepsilon(\bar{x}) = [0, \varepsilon(x, u)^T]^T\).

By the Lyapunov converse theorem [6], if \(G(\bar{x}(t))\) is globally uniformly asymptotically stable then there exist a Lyapunov function \(V(\bar{x}(t))\), a \(\mathcal{K}_\infty\) function \(\alpha_1(\cdot)\), a \(\mathcal{K}_\infty\) function \(\alpha_2(\cdot)\), a \(\mathcal{K}\) function \(\alpha_3(\cdot)\) and a positive constant \(c\) such that

\[
\alpha_1(\|\bar{x}(t)\|) \leq V(\bar{x}(t)) \leq \alpha_2(\|\bar{x}(t)\|),
\]

\[
\frac{\partial V(\bar{x}(t))}{\partial \bar{x}} G(\bar{x}(t)) \leq -\alpha_3(\|\bar{x}(t)\|),
\]

\[
\|\frac{\partial V(\bar{x}(t))}{\partial \bar{x}}\| \leq c.
\]

The derivative of this Lyapunov function along the trajectories of the system (2.57) satisfies

\[
\frac{\partial V(\bar{x}(t))}{\partial \bar{x}} \hat{G}(\bar{x}(t)) = \frac{\partial V(\bar{x}(t))}{\partial \bar{x}} G(\bar{x}(t)) + \frac{\partial V(\bar{x}(t))}{\partial \bar{x}} \varepsilon(\bar{x})
\]

\[
\leq -\alpha_3(\|\bar{x}(t)\|) + c\varepsilon_\delta \|\bar{x}(t)\|.
\]
Since \( \bar{x}(t) \in \mathcal{X} \times \mathcal{U} \) and \( \mathcal{X} \times \mathcal{U} \) is a compact set, there exists a positive constant \( \sigma \) such that \( \|\bar{x}(t)\| < \sigma \) for all \( \bar{x}(t) \in \mathcal{X} \times \mathcal{U} \). Thus there exists a \( \varepsilon_g > 0 \) such that if \( \|\bar{x}(t)\| \geq \alpha_3^{-1}(c\varepsilon_g\sigma) \), \( \dot{V}(\bar{x}(t)) \leq 0 \). Thus \( V(\bar{x}(t)) \) is an ISS-Lyapunov function for system (2.57). And thus the system (2.57) is semi-globally input-to-state stable on the compact set \( \mathcal{X} \times \mathcal{U} \). Thus via Definition 2.9 GFC are universal practical fuzzy controllers.

Before presenting the other main result of this section, we first introduce a lemma.

**Lemma 2.6** ([42]) For each continuous and positive definite function \( \alpha \), there exists a KL function \( \beta_\alpha(s, t) \) with the following property: if \( y(\cdot) \) is any (locally) absolutely continuous function defined for \( t \geq 0 \) and with \( y(t) \geq 0 \) for all \( t \), \( y(\cdot) \) satisfies the differential inequality

\[
\dot{y}(t) \leq -\alpha(y(t)) \quad (2.77)
\]

with \( y(0) = y_0 \geq 0 \), then one has

\[
y(t) \leq \beta_\alpha(y_0, t). \quad (2.78)
\]

**Theorem 2.6** GFC are universal asymptotical fuzzy controllers for the non-affine nonlinear systems as in (2.1) which are globally uniformly asymptotically stabilizable, if for the \( \mathcal{K} \) function \( \alpha_3(\cdot) \) given in (2.74) there exist a \( \mathcal{K} \) function \( \alpha_4(\cdot) \) and a positive constant \( \gamma \) such that

\[
\inf_{\|\bar{x}(t)\| > 0, \bar{x}(t) \in \mathcal{X} \times \mathcal{U}} \frac{\alpha_3(\|\bar{x}(t)\|) - \alpha_4(\|\bar{x}(t)\|)}{\|\bar{x}(t)\|} \geq \gamma. \quad (2.79)
\]

**Proof** Based on Theorem 2.5, if we choose a fuzzy control law \( \hat{u}(t) = \hat{g}(x, u) \in GFC \) such that (2.70) and (2.71) hold, and \( \varepsilon_g \leq \frac{\gamma}{\varepsilon_g} \), then

\[
\dot{V}(\bar{x}(t)) \leq -\alpha_3(\|\bar{x}(t)\|) + c\varepsilon_g\|\bar{x}(t)\| \leq -\alpha_4(\|\bar{x}(t)\|)
\]

\[
\leq -\alpha_4 \circ \alpha_2^{-1}(V(\bar{x}(t))) = -\tilde{\alpha}_4(V(\bar{x}(t))), \quad (2.80)
\]

where \( \tilde{\alpha}_4 = \alpha_4 \circ \alpha_2^{-1} \).

From Lemma 2.5, \( \tilde{\alpha}_4 \) is a \( \mathcal{K} \) function. Based on Lemma 2.6, we know that there exists a \( \mathcal{K} \mathcal{L} \) function \( \tilde{\beta}(\cdot) \), such that for every solution \( \bar{x}(t) \) of (2.72),

\[
V(\bar{x}(t)) \leq \tilde{\beta}(V(\bar{x}(0)), t). \quad (2.81)
\]

And one can conclude that the closed-loop control system (2.72) is semi-globally uniformly asymptotically stable on the compact set \( \mathcal{X} \times \mathcal{U} \). Thus via Definition 2.10 GFC are universal asymptotical fuzzy controllers.

**Corollary 2.1** GFC are universal asymptotical fuzzy controllers for the nonlinear systems described in (2.1) which are globally uniformly asymptotically stabilizable,
if for the $\mathcal{K}$ function $\alpha_3(\cdot)$ given in (2.74), there exists a positive constant $\gamma > 0$ such that

$$
\inf_{\|\hat{x}(t)\| > 0, \hat{x}(t) \in \mathcal{X} \times \mathcal{Y}} \frac{\alpha_3(\|\hat{x}(t)\|)}{\|\hat{x}(t)\|} \geq \gamma. \tag{2.82}
$$

**Remark 2.9** It is easily observed that, for the class of nonlinear systems discussed in Sect. 2.4, the condition (2.79) or (2.82) always holds. In other words, universal fuzzy controllers always implies universal practical/asymptotical controllers.

If a reference model $\dot{\bar{x}}(t) = G_m(\bar{x}(t)) = [f(x(t), u(t))]^T, g_m(x(t), u(t))^T$ is given, one can apply the Algorithm 2.1 to obtain the model reference fuzzy controller. That is, one can construct a fuzzy control law $\hat{g}(x, u) \in FC$ such that for any given $\varepsilon_m > 0$,

$$
\hat{g}(x, u) = g_m(x, u) + \varepsilon(x, u), \tag{2.83}
$$

where

$$
\|\varepsilon(x, u)\| \leq \varepsilon_m \|\bar{x}\|, \tag{2.84}
$$

and the closed-loop control system

$$
\dot{\bar{x}}(t) = \hat{G}(\bar{x}(t)) = G_m(\bar{x}(t)) + \bar{\varepsilon}(\bar{x}), \tag{2.85}
$$

where $\hat{G}(\bar{x}(t)) = [f(x(t), u(t))]^T, \hat{g}(x(t), u(t))^T]^T$ and $\bar{\varepsilon}(\bar{x}) = [0, \varepsilon(x, u)^T]^T$, is semi-globally input-to-state stable on the compact set $\mathcal{X} \times \mathcal{Y}$.

**Remark 2.10** In Theorem 2.6 and Corollary 2.1, we have provided some sufficient conditions for the fuzzy controllers as in (2.38) to be universal asymptotical fuzzy controllers for non-affine nonlinear systems which are globally uniformly asymptotically stabilizable. In Theorem 4.2 in [31], the authors also presented a result that Mamdani-type fuzzy controllers are universal fuzzy controllers for nonlinear systems which are globally uniformly asymptotically stabilizable. However, that theorem needs some improvement as follows.

**Theorem 2.7** Mamdani-type fuzzy controllers described in (2.3) in [31] are universal asymptotical fuzzy controllers for the systems described in (2.1) in [31] which are globally uniformly asymptotically stabilizable, if for the $\mathcal{K}$ function $\alpha_3(\cdot)$ given in (4.7) in [31], there exist a $\mathcal{K}$ function $\alpha_4(\cdot)$ and a positive constant $\gamma > 0$ such that

$$
\inf_{\|x(t)\| > 0, x(t) \in \mathcal{X}} \frac{\alpha_3(\|x(t)\|) - \alpha_4(\|x(t)\|)}{\|x(t)\|} \geq \gamma. \tag{2.86}
$$
2.6 An Illustrative Example

Example 2.1 Control of the inverted pendulum is a benchmark example to demonstrate novel nonlinear control approaches. It is also noted that actuator models in the inverted pendulum are often given by the hyperbolic tangent functions of control input $u$ [43]. To show the performance of our controller design results, we consider the balancing problem of an inverted pendulum on a cart. The equations of motion for the pendulum are given as

$$
\dot{x}_1 = x_2 \\
\dot{x}_2 = \frac{g \sin(x_1) - amlx_2^2 \sin(2x_1)/2 - a \cos(x_1) \arctan(u) + 0.55u \times 500}{4l/3 - aml \cos^2(x_1)},
$$

where $x_1$ denotes the angle of pendulum from the vertical, $x_2$ is the angular velocity, $g = 9.8 \text{ m/s}^2$ is the gravity constant, $m$ is the mass of pendulum, $M$ is the mass of the cart, $a = \frac{1}{M+m}$, and $2l$ is the length of the pendulum. Note the input is given by $\arctan(u) + 0.55u$ with an amplifier of gain 500 connected. In this study, we choose $m = 2.0 \text{ kg}$, $M = 8.0 \text{ kg}$, $2l = 1.0 \text{ m}$.

By using Algorithm 2.1, one can obtain the corresponding generalized T–S fuzzy approximator of the nonlinear plant. The Gaussian-type membership functions are employed as the fuzzy basis functions [15, 16]. And the interpolation points set are chosen as follows,

$$\mathcal{O} = \{(0, 0, 0), (0, 0, \pm 2), (0, 0, \pm 4.5), (\pm 60^\circ, 0, 0), (\pm 60^\circ, 0, \pm 2), (\pm 60^\circ, 0, \pm 4.5),
\quad (\pm 88^\circ, 0, 0), (\pm 88^\circ, 0, \pm 2), (\pm 88^\circ, 0, \pm 4.5)\}.$$

Then the following generalized T–S fuzzy model can be obtained,

**Plant rule** $R_l$: IF $|x_1|$ is $\mu_l^l$ And $|u|$ is $\nu^l$ THEN

$$\dot{x}(t) = A_l x(t) + B_l u(t) + \Delta E(x, u), l \in \mathcal{L} := \{1, 2, \ldots, 9\},$$

where

$$
A_1 = A_2 = A_3 = \begin{bmatrix} 0 & 1 \\ 17.2941 & 0 \end{bmatrix},
A_4 = A_5 = A_6 = \begin{bmatrix} 0 & 1 \\ 5.8512 & 0 \end{bmatrix},
A_7 = A_8 = A_9 = \begin{bmatrix} 0 & 1 \\ 0.3593 & 0 \end{bmatrix},
B_1 = \begin{bmatrix} 0 \\ -27.36 \end{bmatrix},
B_2 = \begin{bmatrix} 0 \\ -19.48 \end{bmatrix},
B_3 = \begin{bmatrix} 0 \\ -15.56 \end{bmatrix}.
$$
Fig. 2.1 Membership functions of $x_1$ for Example 2.1

$$B_4 = \begin{bmatrix} 0 \\ -12.07 \end{bmatrix}, B_5 = \begin{bmatrix} 0 \\ -8.6 \end{bmatrix}, B_6 = \begin{bmatrix} 0 \\ -6.87 \end{bmatrix},$$

$$B_7 = \begin{bmatrix} 0 \\ -0.81 \end{bmatrix}, B_8 = \begin{bmatrix} 0 \\ -0.57 \end{bmatrix}, B_9 = \begin{bmatrix} 0 \\ -0.46 \end{bmatrix},$$

and the membership functions are illustrated in Figs. 2.1 and 2.2.

In practice, it might be difficult to determine the upper bounds of approximation errors, that is, $\tilde{\varepsilon}$. In this case study, we choose to calculate $\gamma(x, u) = \|\varepsilon(x, u)\|$, where $\varepsilon(x, u)$ is defined in (2.7), at a number of vertex points $(x, u) \in [-\pi/2, \pi/2] \times [-3, 3] \times [-5, 5]$, within the operating range of the pendulum. It is found that $\gamma(x, u) < 0.3$. It is noted that only finite tests can be conducted. However, one can choose to test more points within the operating region thus to improve the precision of the obtained upper bounds.

Fig. 2.2 Membership functions of $u$ for Example 2.1
Then by using Theorem 2.3 with \( \bar{\epsilon} \) being chosen as 0.3, the fuzzy controller in (2.39) can be obtained with the corresponding controller gains given by

\[
\begin{align*}
\bar{K}_1 &= \begin{bmatrix} 745.5975 & 808.7737 & -163.7714 \end{bmatrix}, \\
\bar{K}_2 &= \begin{bmatrix} 529.8384 & 574.5674 & -117.8914 \end{bmatrix}, \\
\bar{K}_3 &= \begin{bmatrix} 422.6347 & 458.1939 & -95.0947 \end{bmatrix}, \\
\bar{K}_4 &= \begin{bmatrix} 332.6444 & 360.4870 & -75.9555 \end{bmatrix}, \\
\bar{K}_5 &= \begin{bmatrix} 237.4104 & 257.1409 & -55.7087 \end{bmatrix}, \\
\bar{K}_6 &= \begin{bmatrix} 190.0873 & 205.7692 & -45.6454 \end{bmatrix}, \\
\bar{K}_7 &= \begin{bmatrix} 26.8120 & 28.5248 & -10.9249 \end{bmatrix}, \\
\bar{K}_8 &= \begin{bmatrix} 20.2189 & 21.3705 & -9.5230 \end{bmatrix}, \\
\bar{K}_9 &= \begin{bmatrix} 17.2702 & 18.1598 & -8.8948 \end{bmatrix}.
\end{align*}
\]

The state trajectories of the closed-loop control system under initial condition \( x(0) = (80^\circ, 0) \) are shown in Fig. 2.3. The initial condition for the dynamic fuzzy controller is \( u_0 = 0 \). It can be observed that pendulum can be stabilized.

It is noted that commonly used T–S fuzzy models in [44] cannot be easily used to describe the non-affine inverted pendulum. However, to make a reasonable comparison, commonly used T–S fuzzy models can be recognized as a special case of the generalized T–S fuzzy models, in which the control input is dealt with only around zero. And one can obtain the following T–S fuzzy model,

**Plant rule \( R^1 \):** IF \( |x_1| \) is about 0° And \( |u| \) is about 0

\[
\text{THEN } \dot{x}(t) = \hat{A}_1 x(t) + \hat{B}_1 u(t) + \Delta \hat{E}(x, u).
\]

**Plant rule \( R^2 \):** IF \( |x_1| \) is about 60° And \( |u| \) is about 0

\[
\text{THEN } \dot{x}(t) = \hat{A}_2 x(t) + \hat{B}_2 u(t) + \Delta \hat{E}(x, u).
\]

**Fig. 2.3** State trajectories for Example 2.1
Table 2.1 Comparison of control design performance for Example 2.1

<table>
<thead>
<tr>
<th>Models</th>
<th>Interpolation points set</th>
<th>Stabilizable interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commonly used T–S fuzzy model</td>
<td>$</td>
<td>x_1</td>
</tr>
<tr>
<td>Generalized T–S fuzzy model</td>
<td>$</td>
<td>x_1</td>
</tr>
<tr>
<td>Generalized T–S fuzzy model</td>
<td>$</td>
<td>x_1</td>
</tr>
</tbody>
</table>

**Plant rule $R^3$ :** If $|x_1|$ is about $88^\circ$ and $|u|$ is about $0$

**THEN** $\dot{x}(t) = \hat{A}_3 x(t) + \hat{B}_3 u(t) + \Delta \hat{E}(x, u).$

where

$$\hat{A}_1 = \begin{bmatrix} 0 & 1 \\ 17.2941 & 0 \end{bmatrix}, \hat{A}_2 = \begin{bmatrix} 0 & 1 \\ 5.8512 & 0 \end{bmatrix}, \hat{A}_3 = \begin{bmatrix} 0 & 1 \\ 0.3593 & 0 \end{bmatrix},$$

$$\hat{B}_1 = \begin{bmatrix} 0 \\ -27.36 \end{bmatrix}, \hat{B}_2 = \begin{bmatrix} 0 \\ -12.07 \end{bmatrix}, \hat{B}_3 = \begin{bmatrix} 0 \\ -0.81 \end{bmatrix}.$$

Via the standard fuzzy blending method in Sect. 2.2, one can easily see that the above T–S fuzzy model is actually in the form of the commonly used T–S fuzzy models.

To illustrate the advantages of the proposed approaches, a number of simulations have been conducted to compare them with those based on commonly used T–S fuzzy models, and the results are summarized in the Table 2.1, where the stabilizable interval indicates the pendulum can be stabilized by fuzzy controllers under initial conditions $(x_1(0), 0, 0)$ with the maximum $\|x_1(0)\|$. One can observe that the proposed control design approach performs much better than the approach based on commonly used T–S fuzzy model. It can also be seen that the more fuzzy rules leads to better control performance, nevertheless with higher computation cost at the same time. This is a tradeoff problem in applications.

2.7 Conclusions

In this chapter, some new results on the universal fuzzy model problem and universal controller problem for non-affine nonlinear systems are provided. A class of generalized T–S fuzzy models are shown to be universal function approximators to non-affine nonlinear systems and they are also shown to be universal fuzzy models for non-affine nonlinear systems under some sufficient conditions. Detailed construction procedure of such generalized T–S fuzzy models are also provided.
An approach to semi-globally stabilization of non-affine nonlinear systems are then developed by using a class of dynamic fuzzy controllers. It is shown that this kind of dynamic fuzzy controllers are universal (or practical/asymptotic) fuzzy controllers for non-affine nonlinear systems which are globally uniformly exponentially stabilizable or globally uniformly asymptotically stabilizable under some sufficient conditions. Constructive procedures to obtain universal fuzzy controllers are also provided. Simulation studies are finally presented to show the advantages of the proposed approaches.

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