# Chapter 2
# Preferences and Operators

**Abstract** This chapter focuses on how operations over binary relations work. We provide a series of basic observations on operations over binary relations and extend the classical results of Graham et al. (Complements and transitive closures. Discret Math 2(1):17–29, 1972) and Fishburn (Operations on binary relations. Discret Math 21(1):7–22, 1978). Moreover, we introduce various closure operators and clarify their implications.

**Keywords** Binary relation · Preference · Unary operator · Multi-arity operator · Transitive closure · Choice

## 2.1 Introduction

This chapter explains a basic framework for a theory of binary relations. Our principal interpretation of a binary relation $R$ is a weak preference relation of some individual or some organization. When $(x, y)$ is an element of the binary relation, $x$ "is at least as good as" (or "is weakly preferred to") $y$ for the individual/organization.

We introduce various types of operators that we use throughout this book. Two types of operators must be distinguished. The first is a class of *unary operators*. Each operator in this class maps a binary relation to a unique binary relation. The dual and complement operators are included in this class. A given preference relation can possibly change to a different relation using an operator in this class.

The second type of operator is a class of *multi-arity operators*. Each multi-arity operator maps a collection of binary relations to a unique binary relation. In this class, the union and intersection are included. A given collection of preference relations is aggregated into some collective preference using an operator in this class.

We provide four examples of operators for the weak preference interpretation. Under the "dual" of a weak preference relation, $x$ is at least as good as $y$ if $y$ is at least as good as $x$ under the original preference. The ranking is reversed under the dual. Thus, the dual generates a preference that is opposite to the original. Under the "complement" of a weak preference relation, $x$ is at least as good as $y$ if $x$ is not at least as good as $y$ under the original preference. This is the negation of the



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original. Consider the situation in which two individuals have preferences, $R_1$ and $R_2$, respectively. The intersection and union are $R_1 \cap R_2$ and $R_1 \cup R_2$, respectively. The former represents agreement between the two individuals, while the latter represents support by some individual.

Dual, complement, union, and intersection are *simple operators* in the sense that they can be used to express other operators.[1] For example, the indifference relation is defined as follows: $x$ is indifferent to $y$ if $x$ is at least as good as $y$ and $y$ is at least as good as $x$. The procedure generating the indifference relation is an operator for binary relations. In particular, it is equal to the intersection of the given relation and its dual. The indifference operator is a compound operator in that it consists of a combination of simple operators.

In this chapter, we clarify how a combination of operators works based on Graham et al. (1972), Fishburn (1978), and Cato (2012). The main focus of these works is the number of binary relations generated by sequential applications of five unary operators. However, we do not examine this problem closely, because the main aim of this section is to provide a basic tool for economic analysis.

The rest of this chapter is organized as follows. Section 2.2 introduces definitions of binary relations and operators. The dual and complement operators are basic operators. By combining them with the union and intersection, we construct various operators over binary relations In Sect. 2.3, we examine the logical implications of combinations of operators. Section 2.4 focuses on a class of closure operators, which are particularly important in our analysis. This section includes the results of Graham et al. (1972) and Cato (2012). Section 2.5 provides concluding remarks.

## 2.2 Binary Relations

Let $X$ be the set of alternatives and $\mathscr{X}$ be the collection of non-empty subsets of $X$. A binary relation on $X$ is a subset of $X \times X$. We use the letters $R, R', Q, Q' \ldots$ to denote binary relations. Let $\mathscr{B}$ be the set of binary relations on $X$. Given a binary relation $R$ and $A \subseteq X$ the *restriction to* $A$ is denoted by $R|_A$, i.e., $R|_A = R \cap (A \times A)$.

The diagonal relation is defined as follows:

$$\Delta = \{(x, y) \in X \times X : x = y\}.$$

A unary operator $\rho$ is a mapping from $\mathscr{B}$ to $\mathscr{B}$. Thus, given a binary relation $R$, $\rho(R) \in \mathscr{B}$ is a binary relation. We first define the basic unary operators for binary relations. Given a binary relation $R$, the *dual* and *complement* of $R$ are defined as follows:

$$d(R) = \{(x, y) \in X \times X : (y, x) \in R\},$$

and

[1] The composition is also simple.

$$c(R) = \{(x, y) \in X \times X : (x, y) \notin R\}.$$

We often write $d(R)$ and $c(R)$ as $(R)^d$ and $(R)^c$, respectively. We also express them as $R^d$ and $R^c$ respectively if it causes no confusion.

The *symmetric* and *asymmetric* parts of $R$ are respectively denoted by $I(R)$ and $P(R)$: given a binary relation $R$, $I(R)$ and $P(R)$ are binary relations defined by

$$I(R) = \{(x, y) \in X \times X : (x, y) \in R \text{ and } (y, x) \in R\},$$

and

$$P(R) = \{(x, y) \in X \times X : (x, y) \in R \text{ and } (y, x) \notin R\}.$$

When $R$ is interpreted as a preference, $P(R)$ corresponds to a strict preference and $I(R)$ corresponds to an indifference. That is, $(x, y) \in P(R)$ means that "$x$ is strictly preferred to $y$"; $(x, y) \in I(R)$ means that "$x$ is indifferent to $y$."

Next, we define the set of non-comparable factors as follows:

$$N(R) = \{(x, y) \in X \times X : (x, y) \notin R \text{ and } (y, x) \notin R\}.$$

If $(x, y) \in N(R)$, preference $R$ cannot judge the ranking between $x$ and $y$. Then, $N(R)$ represents indecisiveness.

Moreover, the following operator is of interest:

$$J(R) = I(R) \cup N(R).$$

That is, $(x, y) \in J(R)$ means that $x$ and $y$ are indifferent or non-comparable.

Note that

$$I(R) = R \cap d(R),\ P(R) = R \cap c(d(R)), \text{ and } N(R) = c(R) \cap c(d(R)).$$

These expressions are useful to understanding two points. First, $P$, $I$, $N$, and $J$ are unary operators for binary relations. Second, they are not basic, because they are compositions of basic operators. Note that the union and the intersection are basic operators, but they are not unary. They are mappings from the product of $\mathscr{B}$ to $\mathscr{B}$: they are said to be binary operators.

For simplicity, we often write $c(d(R))$ as $(R)^{cd}$ (or $R^{cd}$). Then, $P(R) = R \cap R^{cd}$ and $N(R) = R^c \cap R^{cd}$. As we will see later, $c(d(R)) = d(c(R))$ holds for any $R \in \mathscr{B}$. Therefore, we can use $R^{cd}$ and $R^{dc}$ interchangeably.

## 2.3 Basic Results on Operators

In this section, we provide basic results on operators. The following result is useful to understanding the role of operators. The first and second results are particularly important, because $c$ and $d$ are basic operators. Note that (iii)–(v) are derived from (i) and (ii).

**Lemma 2.1** *Let $R$ be a binary relation on $X$. Then,*

*(i)* $c(R \cap R') = c(R) \cup c(R')$ *and* $c(R \cup R') = c(R) \cap c(R')$ *(De Morgan's laws);*
*(ii)* $d(R \cap R') = d(R) \cap d(R')$ *and* $d(R \cup R') = d(R) \cup d(R')$*;*
*(iii)* $I(R \cap R') = I(R) \cap I(R')$ *and* $I(R \cup R') \supseteq I(R) \cup I(R')$*;*
*(iv)* $P(R \cap R') \supseteq P(R) \cap P(R')$ *and* $P(R \cup R') \subseteq P(R) \cup P(R')$*;*
*(v)* $N(R \cap R') \supseteq N(R) \cup N(R')$ *and* $N(R \cup R') = N(R) \cap N(R')$*.*

*Proof* (i) We first show that $c(R \cap R') = c(R) \cup c(R')$. We can prove the claim as follows:

$$\begin{aligned}(x, y) \in c(R \cap R') &\Leftrightarrow (x, y) \notin R \cap R' \\ &\Leftrightarrow [(x, y) \notin R \text{ or } (x, y) \notin R'] \\ &\Leftrightarrow (x, y) \in c(R) \cup c(R').\end{aligned}$$

We next show that $c(R \cup R') = c(R) \cap c(R')$. We can prove this claim as follows:

$$\begin{aligned}(x, y) \in c(R \cup R') &\Leftrightarrow (x, y) \notin R \cup R' \\ &\Leftrightarrow [(x, y) \notin R \text{ and } (x, y) \notin R'] \\ &\Leftrightarrow (x, y) \in c(R) \cap c(R').\end{aligned}$$

(ii) We can prove the first claim as follows:

$$\begin{aligned}d(R \cap R') &= d(\{(x, y) \in X \times X : (x, y) \in R \text{ and } (x, y) \in R'\}) \\ &= \{(y, x) \in X \times X : (x, y) \in R \text{ and } (x, y) \in R'\} \\ &= \{(x, y) \in X \times X : (x, y) \in d(R) \text{ and } (x, y) \in d(R')\} \\ &= d(R) \cap d(R').\end{aligned}$$

We can prove the second claim as follows:

$$\begin{aligned}d(R \cup R') &= d(\{(x, y) \in X \times X : (x, y) \in R \text{ or } (x, y) \in R'\}) \\ &= \{(y, x) \in X \times X : (x, y) \in R \text{ or } (x, y) \in R'\} \\ &= \{(x, y) \in X \times X : (x, y) \in d(R) \text{ or } (x, y) \in d(R')\} \\ &= d(R) \cup d(R').\end{aligned}$$

(iii) Recall that $I(R) = R \cap d(R)$. The first claim can be proved as follows:

$$\begin{aligned} I(R\cap R') &= (R\cap R')\cap d(R\cap R') && \text{(definition of } I)\\ &= (R\cap R')\cap (d(R)\cap d(R')) && \text{(Lemma 2.1 (ii))}\\ &= (R\cap d(R))\cap (R'\cap d(R'))\\ &= I(R)\cap I(R'). && \text{(definition of } I) \end{aligned}$$

The second claim can be proved as follows:

$$\begin{aligned} I(R\cup R') &= (R\cup R')\cap d(R\cup R') && \text{(definition of } I)\\ &= (R\cup R')\cap (d(R)\cup d(R')) && \text{(Lemma 2.1 (ii))}\\ &= ((R\cup R')\cap d(R))\cup ((R\cup R')\cap d(R'))\\ &= ((R\cap d(R))\cup (R'\cap d(R))\cup ((R\cap d(R'))\cup (R'\cap d(R'))\\ &= I(R)\cup I(R')\cup (R'\cap d(R))\cup ((R\cap d(R')). && \text{(definition of } I) \end{aligned}$$

Thus, $I(R\cup R')\supseteq I(R)\cup I(R')$.

(iv) Recall that $P(R) = R\cap c(d(R))$. The first claim can be proved as follows:

$$\begin{aligned} P(R\cap R') &= (R\cap R')\cap c(d(R\cap R')) && \text{(definition of } P)\\ &= (R\cap R')\cap ((R)^{cd}\cup (R')^{cd}) && \text{(Lemma 2.1 (i) and (ii))}\\ &= (R\cap ((R)^{cd}\cup (R')^{cd}))\cap (R'\cap ((R)^{cd}\cup (R')^{cd}))\\ &= ((R\cap (R)^{cd})\cup (R\cap (R')^{cd}))\cap ((R'\cap (R)^{cd})\cup (R'\cap (R')^{cd}))\\ &= (P(R)\cup (R\cap (R')^{cd}))\cap (P(R')\cup (R'\cap (R')^{cd})). && \text{(definition of } P) \end{aligned}$$

Thus, $P(R\cap R')\supseteq P(R)\cap P(R')$.

The second claim can be proved as follows:

$$\begin{aligned} P(R\cup R') &= (R\cup R')\cap c(d(R\cup R')) && \text{(definition of } P)\\ &= (R\cup R')\cap ((R)^{cd}\cap (R')^{cd}) && \text{(Lemma 2.1 (i) and (ii))}\\ &= (R\cap (R)^{cd}\cap (R')^{cd})\cup (R'\cap (R)^{cd}\cap (R')^{cd})\\ &= (P(R)\cap (R')^{cd})\cup (P(R')\cap (R)^{cd}). && \text{(definition of } P) \end{aligned}$$

Thus, $P(R\cup R')\subseteq P(R)\cup P(R')$.

(v) Recall that $N(R) = c(R)\cap c(d(R))$. The first claim can be proved as follows:

$$\begin{aligned} N(R\cap R') &= c(R\cap R')\cap c(d(R\cap R')) && \text{(definition of } N)\\ &= ((R)^{c}\cup (R')^{c})\cap ((R)^{cd}\cup (R')^{cd}) && \text{(Lemma 2.1(i) and (ii))}\\ &= ((R)^{c}\cap ((R)^{cd}\cup (R')^{cd}))\cup ((R')^{c}\cap ((R)^{cd}\cup (R')^{cd}))\\ &= (((R)^{c}\cap (R)^{cd})\cup ((R)^{c}\cap (R')^{cd}))\cup (((R')^{c}\cap (R)^{cd})\cup ((R')^{c}\cap (R')^{cd}))\\ &= (N(R)\cup ((R)^{c}\cap (R')^{cd}))\cup (((R')^{c}\cap (R)^{cd})\cup N(R')). && \text{(definition of } N) \end{aligned}$$

Thus, $N(R\cap R')\supseteq N(R)\cup N(R')$.

The second claim can be proved as follows:

$$\begin{aligned}
N(R \cup R') &= c(R \cup R') \cap c(d(R \cup R')) && \text{(definition of } N\text{)}\\
&= ((R)^c \cap (R')^c) \cap ((R)^{cd} \cap (R')^{cd}) && \text{(Lemma 2.1 (i) and (ii))}\\
&= ((R)^c \cap (R)^{cd}) \cap ((R')^c \cap (R')^{cd}) &&\\
&= N(R) \cap N(R'). && \text{(definition of } N\text{)}
\end{aligned}$$

■

Next, we examine what happens when each operator is applied twice.[2]

**Lemma 2.2** *Let $R$ be a binary relation on $X$. Then,*

*(i)* $c(c(R)) = R$;
*(ii)* $d(d(R)) = R$;
*(iii)* $P(P(R)) = P(R)$;
*(iv)* $I(I(R)) = I(R)$;
*(v)* $N(N(R)) = R \cup d(R)$.

*Proof* (i) This claim is obvious by the definition of $c$.
(ii) This claim is obvious by the definition of $d$.
(iii) Since $P(R) = R \cap c(d(R))$, we can prove the claim as follows:

$$\begin{aligned}
P(P(R)) &= P(R \cap R^{cd}) && \text{(definition of } P\text{)}\\
&= (R \cap R^{cd}) \cap (R \cap R^{cd})^{cd} && \text{(definition of } P\text{)}\\
&= (R \cap R^{cd}) \cap (R^{cd} \cup R) && \text{(Lemmas 2.1 and 2.2 (i) and (ii))}\\
&= P(R). && \text{(definition of } P\text{)}
\end{aligned}$$

(iv) Since $I(R) = R \cap d(R)$, we can prove the claim as follows:

$$\begin{aligned}
I(I(R)) &= I(R \cap R^d) && \text{(definition of } I\text{)}\\
&= (R \cap R^d) \cap (R \cap R^d)^d && \text{(definition of } I\text{)}\\
&= (R \cap R^d) \cap ((R)^d \cup R^{dd}) && \text{(Lemmas 2.1 and 2.2 (ii))}\\
&= I(R). && \text{(definition of } I\text{)}
\end{aligned}$$

(v) Since $N(R) = c(R) \cap c(d(R))$, we can prove the claim as follows:

$$\begin{aligned}
N(N(R)) &= N(R^c \cap R^{cd}) && \text{(definition of } N\text{)}\\
&= (R^c \cap R^{cd})^c \cap (R^c \cap R^{cd})^{cd} && \text{(definition of } N\text{)}\\
&= (R \cup R^d) \cap (R^d \cup R) && \text{(Lemmas 2.1 and 2.2 (i) and (ii))}\\
&= R \cup R^d.
\end{aligned}$$

■

[2](i)–(iv) of Lemma 2.2 are found in Lemma 2.1 of Fishburn (1978). He does not provide proofs.

Each of $P$, $I$, and $N$ has a relevant economic meaning: strict preference, indifference, and indecisiveness, respectively. The asymmetric and symmetric parts satisfy the idempotence property, while the non-comparable factor does not. Moreover, it can be the case that $J(J(R)) \neq J(R)$. Assume that $X = \{x, y\}$ and $R = \{(x, y)\}$. Then, $J(R) = \Delta$ and $J(J(R)) = \{(x, y), (y, x)\}$.

Consider two binary relations $R$ and $R'$, such that $R \subseteq R'$. What happens if we take some operator over them. The following result clarifies that this relationship is preserved or reversed for operators other than $P$.

**Lemma 2.3** *Let $R$, $R'$ be binary relations on $X$ such that $R \subseteq R'$. Then,*

*(i)* $c(R') \subseteq c(R)$;
*(ii)* $d(R) \subseteq d(R')$;
*(iii)* $c(d(R')) \subseteq c(d(R))$;
*(vi)* $I(R) \subseteq I(R')$;
*(v)* $N(R') \subseteq N(R)$.

*Proof* (i) The claim can be proved as follows:

$$\begin{aligned}(x, y) \in c(R') &\Leftrightarrow (x, y) \notin R' && \text{(definition of } c)\\ &\Rightarrow (x, y) \notin R && (R \subseteq R')\\ &\Leftrightarrow (x, y) \in c(R). && \text{(definition of } c)\end{aligned}$$

(ii) The claim can be proved as follows:

$$\begin{aligned}(x, y) \in d(R) &\Leftrightarrow (y, x) \in R && \text{(definition of } d)\\ &\Rightarrow (y, x) \in R' && (R \subseteq R' \text{and (ii))}\\ &\Leftrightarrow (y, x) \in d(R). && \text{(definition of } d)\end{aligned}$$

(iii) By (ii), we have $d(R) \subseteq d(R')$. Then, (i) implies $c(d(R') \subseteq c(d(R))$.

(iv) The claim can be proved as follows:

$$\begin{aligned}I(R) &= R \cap d(R) && \text{(definition of } I)\\ &\subseteq R' \cap d(R') && (R \subseteq R' \text{and (ii))}\\ &= I(R'). && \text{(definition of } I)\end{aligned}$$

(v) The claim can be proved as follows:

$$\begin{aligned}N(R') &= c(R') \cap c(d(R')) && \text{(definition of } N)\\ &\subseteq c(R) \cap c(d(R)) && (R \subseteq R' \text{and (ii)(iii))}\\ &= N(R). && \text{(definition of } N)\end{aligned}$$

■

Assume that $X = \{x, y, z\}$ and let

$$R_1 = \{(x, y)\} \text{ and } R_2 = \{(x, y), (y, z), (y, x)\}.$$

Note that $R_1 \subseteq R_2$, but

$$P(R_1) = \{(x, y)\} \text{ and } P(R_2) = \{(y, z)\}.$$

Therefore, neither $P(R_1) \subseteq P(R_2)$ nor $P(R_2) \subseteq P(R_1)$ is true.

We have the set inclusion result for $P$ under a more restrictive assumption.

**Lemma 2.4** *Let $R$, $R'$ be binary relations on $X$. If $R \subseteq P(R')$, then $P(R) \subseteq P(R')$.*

*Proof* Note that $P(R) \subseteq R$. Since $R \subseteq P(R')$, we have $P(R) \subseteq P(R')$. ■

The following result states that the dual operator has an order-invariant property.[3]

**Lemma 2.5** *Let $R$ be a binary relation on $X$. Then,*

*(i)* $d(c(R)) = c(d(R))$;
*(ii)* $d(P(R)) = P(d(R))$;
*(iii)* $d(I(R)) = I(d((R)) = I(R)$;
*(iv)* $d(N(R)) = N(d(R)) = N(R)$.

*Proof* (i) We can prove this claim as follows:

$$\begin{aligned}(x, y) \in d(c(R)) &\Leftrightarrow (y, x) \in c(R)\\ &\Leftrightarrow (y, x) \notin R\\ &\Leftrightarrow (x, y) \notin d(R)\\ &\Leftrightarrow (x, y) \in c(d(R)).\end{aligned}$$

(ii) We can prove this claim as follows:

$$\begin{aligned}d(P(R)) &= d(R \cap c(d(R))) && \text{(definition of } P)\\ &= d(R) \cap d(c(d(R))) && \text{(Lemma 2.1)}\\ &= d(R) \cap c(d(d(R))) && \text{(Lemma 2.5 (i))}\\ &= P(d(R)). && \text{(definition of } P)\end{aligned}$$

(iii) We can prove this claim as follows:

$$\begin{aligned}d(I(R)) &= d(R \cap d(R)) && \text{(definition of } I)\\ &= d(R) \cap d(d(R)) && \text{(Lemma 2.1)}\\ &= R \cap d(R) && \text{(Lemma 2.2)}\\ &= I(R). && \text{(definition of } I)\end{aligned}$$

[3] (i)–(iii) of Lemma 2.5 are found in Lemma 2.3 of Fishburn (1978). He does not provide proofs.

We can prove $I(d(R)) = I(R)$ in a similar manner.
(iv) We can prove $d(N(R)) = N(R)$ as follows:

$$\begin{aligned} d(N(R)) &= d(c(R) \cap c(d(R))) && \text{(definition of } N\text{)} \\ &= d(c(R)) \cap d(c(d(R))) && \text{(Lemma 2.1)} \\ &= c(d(R)) \cap c(d(d(R))) && \text{(Lemma 2.5 (i))} \\ &= c(d(R)) \cap c(R) && \text{(Lemma 2.2 (ii))} \\ &= N(R). && \text{(definition of } N\text{)} \end{aligned}$$

■

From Lemma 2.2 (i) (ii) and Lemma 2.5 (ii), it is easy to see that

$$d(c(d(c(R)))) = R.$$

Note that $d(P(R))$ is not identical to $P(R)$, in general. Let $R$ on be a binary relation on $\{x, y\}$ such that

$$R = \{(x, y)\}.$$

Then, $P(R) = \{(x, y)\}$ and $d(P(R)) = \{(y, x)\}$.
By Lemma 2.2 (ii) and Lemma 2.5 (i), it is generally true that

$$d(P(d(R))) = P(R).$$

Lemma 2.5 implies that the non-comparable factor $N$ and the symmetric part $I$ are invariant for the dual operator. The two operators have the same property, in this respect. Then, indecisiveness has a similar implication to indifference.
Moreover, Lemma 2.5 implies that

$$d(J(R)) = J(R).$$

The following result states that $J(R)$ corresponds to the non-comparable factor of the asymmetric part. It provides an alternative expression of $J$.

**Lemma 2.6** *Let $R$ be a binary relation on $X$. Then, $J(R) = N(P(R))$.*

*Proof* By definition, $J(R) = I(R) \cup N(R)$. We can prove the claim as follows:

$$\begin{aligned} N(P(R)) &= N(R \cap (R)^{cd}) && \text{(definition of } P\text{)} \\ &= (R \cap (R)^{cd})^c \cap (R \cap (R)^{cd})^{cd} && \text{(definition of } N\text{)} \\ &= (R^c \cup (R)^d) \cap (R^{cd} \cup R) && \text{(Lemmas 2.1, 2.2, and 2.5)} \\ &= (R^c \cap (R^{cd} \cup R)) \cup ((R)^d \cap (R^{cd} \cup R)) \\ &= ((R^c \cap R^{cd}) \cup (R^c \cap R)) \cup (((R)^d \cap R^{cd}) \cup (R^d \cap R)) \end{aligned}$$

$$= (R^c \cap R^{cd}) \cup (R^d \cap R)$$
$$= N(R) \cup I(R). \qquad \text{(definitions of } N \text{ and } I)$$

■

The complement of the dual of the asymmetric part is important to our analysis. Hereafter, we often refer to it as the *co-dual*. By definition,

$$(x, y) \in (P(R))^{cd} \Leftrightarrow (y, x) \notin P(R).$$

Given a preference $R$, $x$ is at least as good as $y$ with respect to $(P(R))^{cd}$ if $y$ is not strictly better than $x$ with respect to $R$. The following is a basic result on the co-dual.

**Lemma 2.7** *Let $R$ be a binary relation on $X$. Then,*

*(i)* $R \cup \Delta \subseteq (P(R))^{cd}$;
*(ii)* $P((P(R))^{cd}) = P(R)$.

*Proof* (i) Note that $P(R) = R \cap R^{cd}$. By Lemma 2.1,

$$(P(R))^{cd} = (R \cap R^{cd})^{cd} = R^{cd} \cup R.$$

Then, it follows that $R \subseteq (P(R))^{cd}$.

Note that $\Delta \setminus R \subseteq R^c$. Since $d(\Delta) = \Delta$, we have $\Delta \setminus R \subseteq R^{cd}$. Since $(P(R))^{cd} = R^{cd} \cup R$, we have $\Delta \subseteq (P(R))^{cd}$. Thus, we conclude that $R \cup \Delta \subseteq (P(R))^{cd}$.

(ii) Note that $P(R))^{cd} = R^{cd} \cup R$. Then, $P((P(R))^{cd}) = P(R^{cd} \cup R)$.

By the definition of $P$, $P(R^{cd} \cup R) = (R^{cd} \cup R) \cap (R^{cd} \cup R)^{cd}$. By Lemmas 2.1, 2.2, and 2.5,

$$(R^{cd} \cup R) \cap (R^{cd} \cup R)^{cd} = (R^{cd} \cup R) \cap (R^{cd} \cap R) = R^{cd} \cap R.$$

Then, $P((P(R))^{cd}) = P(R)$. ■

Now, we check the working of the complement of the dual of the symmetric part. By definition,

$$(x, y) \in (I(R))^{cd} \Leftrightarrow (y, x) \notin I(R).$$

Since $(I(R))^{cd} = (I(R))^{dc}$, Lemma 2.5 implies that

$$(I(R))^{cd} = (I(R))^c.$$

The following result is a counterpart of Lemma 2.7.

**Lemma 2.8** *Let $R$ be a binary relation on $X$. Then,*

*(i)* $(I(R))^{cd} = R^c \cup R^{cd}$;
*(ii)* $I((I(R))^{cd}) = (I(R))^c$.

*Proof* (i) Note that $I(R) = R \cap R^d$. The claim can be proved as follows:

$$\begin{aligned} (I(R))^{cd} &= (R \cap R^d)^{cd} && \text{(definition of } I) \\ &= R^c \cup R^{cd}. && \text{(Lemmas 2.1, 2.2, and 2.5)} \end{aligned}$$

(ii) Note that $(I(R))^{cd} = (I(R))^c$. The claim can be proved as follows:

$$\begin{aligned} I((I(R))^{cd}) &= c(I(R)) \cap d((I(R))^c) && \text{(definition of } I) \\ &= c(I(R)) \cap c(d(I(R))) && \text{(Lemma 2.5 (i))} \\ &= c(I(R)) \cap c(I(R)) && \text{(Lemma 2.5 (iii))} \\ &= c(I(R)). && \text{(definition of } I) \end{aligned}$$

■

## 2.4 Closure Operators

In this section, we introduce a special class of operators. A *closure operator* is a unary operator $\varphi$ from $\mathscr{B}$ to $\mathscr{B}$ that satisfies the following three properties[4]: for all $R, R' \in \mathscr{B}$,

(i) $R \subseteq \varphi(R)$ (extensiveness);
(ii) $R \subseteq R' \Rightarrow \varphi(R) \subseteq \varphi(R')$ (monotonicity);
(iii) $\varphi(\varphi(R)) = \varphi(R)$ (idempotence).

A trivial example of a closure operator is an identity mapping: $\varphi(R) = R$ for all $R \in \mathscr{B}$. The results of the previous section show that (i) $I$, $P$, $d$, and $c$ satisfy idempotence, and (ii) $I$ and $d$ satisfy monotonicity. Extensiveness is not satisfied by any of $I$, $P$, $d$, and $c$. The co-dual satisfies extensiveness and idempotence, but does not satisfy monotonicity.

The following is a fundamental observation on a closure operator. We say that $R$ is *$\varphi$-closed* if $\varphi(R) = R$.

**Proposition 2.1** *Let $\varphi$ be a closure operator. For each $R \in \mathscr{B}$, $\varphi(R)$ is the smallest $\varphi$-closed binary relation containing $R$, i.e., $\varphi(R) \subseteq R'$ whenever $R \subseteq R'$ and $\varphi(R') = R'$.*

*Proof* Take any $R \in \mathscr{B}$. By the idempotence of $\varphi$, $\varphi(R)$ is $\varphi$-closed. It suffices to show that $\varphi(R)$ is the smallest. Assume that $R \subseteq R'$ and $\varphi(R') = R'$. By monotonicity, $R \subseteq R'$ implies that $\varphi(R) \subseteq \varphi(R')$. Then, $\varphi(R) \subseteq R'$. ■

[4]See Berge (1963) for a detailed discussion of closure operators.

As an auxiliary step, we introduce the *composition* of $R$ and $R'$, which is defined as follows:

$$R \circ R' = \{(x, y) \in X : (x, z) \in R \text{ and } (z, y) \in R' \text{ for some } z \in X\}.$$

The composition is a *binary operator*. By employing the composition, we can induce the following unary operator, which is important to our analysis:

$$\rho(R) = R \circ R.$$

Note that given a weak preference $R$, $x$ is at least as good as $z$ with respect to $R \circ R$ if $x$ is at least as good as $y$ and $y$ is at least as good as $z$ with respect to the original preference. Note that $\rho(R)$ is not a closure operator.

Next, we provide some basic results on the composition operator.

**Lemma 2.9** *Let $R$, $R'$ be binary relations on $X$.*

*(i) If $\Delta \subseteq R'$, then $R \subseteq R \circ R'$.*
*(ii) If $\Delta \subseteq R$, then $R' \subseteq R \circ R'$.*

The proof of Lemma 2.9 is straightforward. Thus, we omit it here.

**Lemma 2.10** *Let $R$, $R'$ be binary relations on $X$. Then, $d(R \circ R') = d(R') \circ d(R)$.*

*Proof* The claim can be proved as follows:

$$\begin{aligned}(x, y) \in d(R \circ R') &\Leftrightarrow (y, x) \in R \circ R' \\ &\Leftrightarrow (y, z) \in R \text{ and } (z, x) \in R' \text{ for some } z \in X \\ &\Leftrightarrow (x, z) \in d(R') \text{ and } (z, y) \in d(R) \text{ for some } z \in X \\ &\Leftrightarrow (x, y) \in d(R') \circ d(R)\end{aligned}$$

■

Define

$$R^{(0)} = R \text{ and } R^{(\kappa)} = R^{(\kappa-1)} \circ R \text{ for } \kappa \in \mathbb{N}.$$

**Lemma 2.11** *If $\Delta \subseteq R$, then $R \subseteq R^{(\kappa)}$ for all $\kappa \in \mathbb{N}$.*

*Proof* We prove the claim by mathematical induction. First, Lemma 2.9 implies that $R \subseteq R \circ R$. Thus, we have $R \subseteq R^{(1)}$. Let $n \in \mathbb{N}$. Now, suppose that $R \subseteq R^{(n)}$. Since $\Delta \subseteq R$, Lemma 2.9 implies that $R \subseteq R^{(n)} \circ R$. Therefore, $R \subseteq R^{(n+1)}$. The claim is proved. ■

Now, we can provide examples of closure operators. First, we introduce the *transitive closure* of $R$:

$$tc(R) = \bigcup_{\kappa=0}^{\infty} R^{(\kappa)}.$$

The *reflexive closure* is defined as follows:

$$rc(R) = R \cup \Delta.$$

The *symmetric closure* is defined as follows:

$$sc(R) = R \cup d(R).$$

The *consistent closure* is defined as follows:

$$kc(R) = R \cup (tc(R) \cap d(R)).$$

The consistent closure was recently proposed by Bossert et al. (2005) in order to capture a special property of binary relations. It is clear that $kc(R) \subseteq tc(R)$. An important property of the consistent closure is that $kc(tc(R)) = tc(R)$. To clarify the working of $kc$, we provide the following example. Assume that $X = \{x, y, z\}$. Let us consider

$$R_1 = \{(x, y), (y, z)\} \text{ and } R_2 = \{(x, y), (y, z), (z, x), (x, z)\}.$$

Note that

$$tc(R_1) = R_1 \cup \{(x, z)\} \text{ and } tc(R_2) = R_2 \cup \{(y, x), (z, y)\}.$$

Since $(z, x) \notin R_1$, we have $kc(R_1) = R_1$. Since $(x, y) \in R_2$ and $(y, z) \in R_2$, we have $(y, x) \in kc(R_2)$ and $(z, y) \in kc(R_2)$, and thus, $kc(R_2) = tc(R_2)$.

The above operators all satisfy the three properties.[5]

**Proposition 2.2** *tc, rc, sc, and kc are closure operators.*

*Proof* It is clear that $rc$ and $sc$ are closure operators.

We show that $tc$ is a closure operator. By definition, $R \subseteq tc(R)$. Now, we check that $R \subseteq R' \Rightarrow tc(R) \subseteq tc(R')$. Suppose that $R \subseteq R'$ and $(x, y) \in tc(R)$. Then, there exist $K \in \mathbb{N}$ and $x^0, x^1, \ldots, x^K \in X$ such that

$$x^0 = x, x^K = y, \text{ and } (x^{k-1}, x^k) \in R \text{ for all } k \in \{1, \ldots, K\}.$$

Since $R \subseteq R'$,

$$x^0 = x, x^K = y, \text{ and } (x^{k-1}, x^k) \in R' \text{ for all } k \in \{1, \ldots, K\}.$$

[5] See Bossert and Suzumura (2010) for discussions on $tc$ and $kc$.

Then, we have $(x, y) \in tc(R')$.

We now prove that $tc(tc(R)) = tc(R)$. Since $tc(R) \subseteq tc(tc(R))$, it suffices to show that $tc(tc(R)) \subseteq tc(R)$. If $(x, y) \in tc(tc(R))$, then

$$x^0 = x, x^K = y, \text{ and } (x^{k-1}, x^k) \in tc(R) \text{ for all } k \in \{1, \dots, K\}.$$

Since $(x^{k-1}, x^k) \in tc(R)$, for each $k \in \{1, \dots, K\}$, there exist $L_k \in \mathbb{N}$ and $x_k^0, x_k^1, \dots, x_k^{L_k} \in X$ such that

$$x_k^0 = x^{k-1}, x_k^{L_k} = x^k, \text{ and } (x^{\ell-1}, x^\ell) \in R \text{ for all } \ell \in \{1, \dots, L_k\}.$$

Let $M = \sum_{k=1}^K L_k$. This implies that there exist $x^0, x^1, \dots, x^M \in X$ such that

$$x^0 = x, x^M = y, \text{ and } (x^{m-1}, x^m) \in R \text{ for all } m \in \{1, \dots, M\}.$$

Then, we have $(x, y) \in tc(R)$. Therefore, $tc$ is a closure operator.

Next, we show that $kc$ is a closure operator. By definition, $R \subseteq kc(R)$. We check that $R \subseteq R' \Rightarrow kc(R) \subseteq kc(R')$. Suppose that $R \subseteq R'$. Then, $d(R) \subseteq d(R')$ (by Lemma 2.3) and $tc(R) \subseteq tc(R')$. Thus, we have $tc(R) \cap d(R) \subseteq tc(R') \cap d(R')$. This implies that $kc(R) \subseteq kc(R')$. Then, we prove that $kc(kc(R)) = kc(R)$. Since $kc(R) \subseteq kc(kc(R))$, it suffices to show that $kc(kc(R)) \subseteq kc(R)$. Note that

$$kc(kc(R)) = kc(R) \cup (tc(kc(R)) \cap d(kc(R))).$$

We need to show that $tc(kc(R)) \cap d(kc(R)) \subseteq kc(R)$. Since $kc(R) \subseteq tc(R)$, we have $tc(kc(R)) \subseteq tc(tc(R)) = tc(R)$. Since $R \subseteq kc(R)$, we have $tc(R) \subseteq tc(kc(R))$. Then, $tc(kc(R)) = tc(R)$. Thus,

$$tc(kc(R)) \cap d(kc(R)) = tc(R) \cap d(kc(R)).$$

Since $kc(R) = R \cup (tc(R) \cap d(R))$, Lemma 2.1 (i) and the definition of $rc$ imply that

$$d(R \cup (tc(R) \cap d(R))) = d(R) \cup d(tc(R) \cap d(R)) = d(R) \cup \big(d(tc(R)) \cap R\big) = d(tc(R)) \cup \big(d(R) \cap R\big).$$

Thus,

$$\begin{aligned} tc(kc(R)) \cap d(kc(R)) &= tc(R) \cap \Big(d(tc(R)) \cup \big(d(R) \cap R\big)\Big) \\ &= \Big(tc(R) \cap d(tc(R))\Big) \cup \Big(tc(R) \cap \big(d(R) \cap R\big)\Big). \end{aligned}$$

Note that

$$\Big(tc(R) \cap \big(d(R) \cap R\big)\Big) \subseteq \Big(tc(R) \cap d(tc(R))\Big),$$

because $(d(R) \cap R) \subseteq d(R) \subseteq d(tc(R))$. Thus, $tc(kc(R)) \cap d(kc(R)) = tc(R) \cap d(tc(R))$. By construction, $tc(R) \cap d(tc(R)) = tc(R) \cap d(R)$. Thus, we have

$$tc(kc(R)) \cap d(kc(R)) = tc(R) \cap d(R).$$

Thus, we conclude that $tc(kc(R)) \cap d(kc(R)) \subseteq kc(R)$. Thus, $kc$ is a closure operator. ■

The following result states the relationship between the dual and each closure operator.[6]

**Lemma 2.12** *Let $R$ be a binary relation on $X$. Then,*

*(i)* $d(tc(R)) = tc(d(R))$;
*(ii)* $d(rc(R)) = rc(d(R))$;
*(iii)* $d(sc(R)) = sc(d(R)) = sc(R)$;
*(iv)* $d(kc(R)) = kc(d(R))$.

*Proof* (i) It suffices to show that $d(R^{(\kappa)}) = (d(R))^{(\kappa)}$ for all $\kappa = 0, 1, 2, 3 \ldots$. By definition, we have $d(R^{(0)}) = (d(R))^{(0)}$. Take a natural number $n \in \mathbb{N}$. Suppose that $d(R^{(n)}) = (d(R))^{(n)}$. Then, we have the following:

$$\begin{aligned} d(R^{(n+1)}) &= d(R^{(n)} \circ R) && \text{(Definition of } R^{(n+1)}\text{)} \\ &= d(R) \circ d(R^{(n)}) && \text{(Lemma 2.10)} \\ &= d(R) \circ (d(R))^{(n)} && \text{(Our Supposition)} \\ &= (d(R))^{(n+1)}. \end{aligned}$$

Thus, we have $d(tc(R)) = tc(d(R))$.

(ii) Since $rc(R) = R \cup \Delta$,

$$d(rc(R)) = d(R \cup \Delta) = d(R) \cup d(\Delta) = d(R) \cup \Delta = rc(d(R)),$$

from Lemma 2.1.

(iii) Since $sc(R) = R \cup d(R)$,

$$d(sc(R)) = d(R \cup d(R)) = d(R) \cup R = sc(R),$$

and

$$d(sc(R)) = d(R \cup d(R)) = d(R) \cup d(d(R)) = sc(d(R)),$$

from Lemmas 2.1 and 2.2.

---

[6]Lemma 2.12 (i) is found in Lemma 2.1 of Fishburn (1978).

(iv) Since $kc(R) = R \cup (tc(R) \cap d(R))$, we have the following:

$$\begin{aligned} d(kc(R)) &= d(R \cup (tc(R) \cap d(R))) && \text{(Definition of } kc\text{)} \\ &= d(R) \cup d(tc(R) \cap d(R)) && \text{(Lemma 2.1)} \\ &= d(R) \cup (tc(d(R)) \cap d(d(R))) && \text{(Lemmas 2.1 and 2.12 (i))} \\ &= kc(d(R)). && \text{(Definition of } kc\text{)} \end{aligned}$$

■

The following result provides a characterization of $sc$.

**Lemma 2.13** *Let $R$ be a binary relation on $X$. Then,*

*(i)* $sc(R) = N(N(R))$*;*
*(i)* $sc(R^c) = (I(R))^{cd}$.

*Proof* (i) The claim follows directly from Lemma 2.2 (iv) and the definition of $sc$.
(ii) The claim follows directly from Lemma 2.8 (i) and the definition of $sc$. ■

The following result shows that $I$-closedness is equivalent to $sc$-closedness.

**Lemma 2.14** *Let $R$ be a binary relation on $X$. Then, $sc(R) = R$ if and only if $I(R) = R$.*

*Proof* "If." Suppose that $I(R) = R$. Then, $R \cap d(R) = R$, which implies that $R \subseteq d(R)$. By taking the dual, $R \cap d(R) = d(R)$. Then, we have $d(R) \subseteq R$. Thus, $R = d(R)$. This implies that $sc(R) = R$.

"Only if." If $sc(R) = R$, then $R \cup d(R) = R$, which implies that $d(R) \subseteq R$. If $R \cup d(R) = R$, then $d(R) \cup R = d(R)$, which implies that $R \subseteq d(R)$. Thus, $R = d(R)$. This implies that $I(R) = R$. ■

The following result follows from the extensiveness of closure operators.

**Lemma 2.15** *Let $R$ be a binary relation on $X$. Then,*

*(i)* $c(tc(c(R))) \subseteq R$*;*
*(ii)* $c(sc(c(R)) = I(R)$*;*
*(iii)* $c(rc(c(R)) = R \setminus \Delta$*;*
*(iv)* $c(kc(c(R))) \subseteq R$.

*Proof* (i) Because the transitive closure is extensive, we have $c(R) \subseteq tc(c(R))$. This implies that

$$c(tc(c(R))) \subseteq c(c(R)) = R,$$

from Lemma 2.2 (i).

(ii) The result follows from Lemma 2.5 (iii) and Lemma 2.13 (ii).

(iii) By the definition of $rc$, $rc(c(R)) = c(R) \cup \Delta$. Thus,

$$\begin{aligned} c(rc(c(R))) &= c(c(R) \cup \Delta) \\ &= R \cap (\Delta)^c \\ &= R \setminus \Delta, \end{aligned}$$

from Lemmas 2.1 (i) and 2.2 (i) and the definition of $rc$.

(iv) The proof is the same as that of (i). ■

It is straightforward to check that $c(\varphi(c(R))) \subseteq R$ for all closure operators $\varphi$, because of extensiveness. Two operators, $rc$ and $sc$, show more involving results.

The next lemma clarifies an interesting property of $kc$, in which the consistent closure works as an identity for the symmetric relation.

**Lemma 2.16** *Let $R$ be a binary relation on $X$. Then, $kc(I(R)) = I(R)$.*

*Proof* We can show the claim as follows:

$$\begin{aligned} kc(I(R)) &= I(R) \cup \big(tc(I(R)) \cap d(I(R))\big) && \text{(definition of } kc) \\ &= I(R) \cup \big(tc(I(R)) \cap I(R)\big) && \text{(Lemma 2.5)} \\ &= \big(I(R) \cup tc(I(R))\big) \cap I(R)) \\ &= I(R). && (I(R) \subseteq tc(I(R))) \end{aligned}$$

■

The following result states how each closure operator works with the symmetric operator.[7]

**Lemma 2.17** *Let $R$ be a binary relation on $X$. Then,*

*(i)* $I(tc(I(R))) = tc(I(R))$;
*(ii)* $I(rc(I(R))) = rc(I(R))$;
*(iii)* $I(sc(I(R))) = sc(I(R)) = I(R)$;
*(iv)* $I(kc(I(R))) = kc(I(R))$.

*Proof* (i) We can show the claim as follows:

$$\begin{aligned} I(tc(I(R))) &= tc(I(R)) \cap d(tc(I(R))) && \text{(definition of } I) \\ &= tc(I(R)) \cap (tc((I(R))^d)) && \text{(Lemma 2.12 (i))} \\ &= tc(I(R)) \cap tc(I(R)) && \text{(Lemma 2.5 (iii))} \\ &= tc(I(R)). \end{aligned}$$

[7] Lemma 2.17 (i) is found in Lemma 2.3 of Fishburn (1978).

(ii) We can show the claim as follows:

$$
\begin{aligned}
I(rc(I(R))) &= I(I(R) \cup \Delta) && \text{(definition of } rc\text{)}\\
&= (I(R) \cup \Delta) \cap (I(R) \cup \Delta)^d && \text{(definition of } I\text{)}\\
&= I(R) \cup \Delta && \text{(Lemmas 2.1 (ii) and 2.5 (iii))}\\
&= rc(I(R)). && \text{(definition of } rc\text{)}
\end{aligned}
$$

(iii) We can show the claim as follows:

$$
\begin{aligned}
I(sc(I(R))) &= sc(I(R)) \cup d(sc(I(R))) && \text{(definition of } I\text{)}\\
&= sc(I(R)) \cup (sc(d(I(R)))) && \text{(Lemma 2.12 (iii))}\\
&= sc(I(R)) && \text{(Lemma 2.5 (iii))}\\
&= I(R) \cup d(I(R)) && \text{(definition of } sc\text{)}\\
&= I(R). && \text{(Lemma 2.5 (iii))}
\end{aligned}
$$

(iv) We can show the claim as follows:

$$
\begin{aligned}
I(kc(I(R))) &= I(I(R)) && \text{(Lemma 2.16)}\\
&= I(R) && \text{(Lemma 2.2 (iv))}\\
&= kc(I(R)). && \text{(Lemma 2.16)}
\end{aligned}
$$

■

Graham et al. (1972) clarify the set of binary relations generated by taking the transitive closure and the complement sequentially. They show that only a small number of relations can be generated. Cato (2012) extends their result to the case of the consistent closure. We present the results on the transitive closure and the consistent closure in a parallel manner. We first show an auxiliary result.[8]

**Lemma 2.18** *Let $R$ be a binary relation on $X$. Then,*

*(i)* $tc(R) \circ c(tc(c(tc(R)))) \subseteq c(tc(c(tc(R))))$;
*(ii) For* $\kappa \in \mathbb{N}$, *if* $(x, y) \in (kc(R))^{(\kappa)} \circ c(kc(c(kc(R))))$, *then* $(y, x) \notin P(c(kc(c(kc(R)))))$.

*Proof* (i) By way of contradiction, suppose that $(x, y) \in tc(R)$, $(y, z) \in c(tc(c(tc(R))))$, and $(x, z) \in tc(c(tc(R)))$ for some $x, y, z \in X$. By Lemma 2.15 (i), we have $(y, z) \in tc(R)$. Since $(x, y) \in tc(R)$ and $(y, z) \in tc(R)$, it follows that $(x, z) \in tc(R)$. Note that $(x, z) \in tc(c(tc(R)))$, but that $(x, z) \notin c(tc(R))$. Since $(x, z) \in tc(c(tc(R)))$, $(x, w) \in c(tc(R))$ and $(w, z) \in tc(c(tc(R)))$ for some $w \in X$.

[8] The first claim of Lemma 2.18 is provided by Graham et al. (1972), and the second claim of Lemma 2.18 is provided by Cato (2012).

Either $(y, w) \in tc(R)$ or $(y, w) \notin tc(R)$ must be true. In the former case, we have $(x, y) \in tc(R)$ and $(y, w) \in tc(R)$. This implies that $(x, w) \in tc(R)$. This contradicts the fact that $(x, w) \in c(tc(R))$. In the latter case, we have $(y, w) \notin tc(R)$ and $(w, z) \in tc(c(tc(R)))$. Since $(y, w) \in c(tc(R))$ and $c(tc(R)) \subseteq tc(c(tc(R)))$, we have $(y, w) \in tc(c(tc(R)))$. Since $(y, w) \in tc(c(tc(R)))$ and $(w, z) \in tc(c(tc(R)))$, it follows that $(y, z) \in tc(c(tc(R)))$. This contradicts the supposition that $(y, z) \in c(tc(c(tc(R))))$. This completes the proof.

(ii) By way of contradiction, suppose that there exists $\kappa \in \mathbb{N}$ such that $(x, y) \in (kc(R))^{(\kappa)} \circ c(kc(c(kc(R))))$ and $(y, x) \in P(c(kc(c(kc(R)))))$. By Lemma 2.15 (iv), we have $c(kc(c(kc(R)))) \subseteq kc(R)$. This implies that $(x, y) \in (kc(R))^{(\kappa)} \circ kc(R)$. If $(y, x) \in P(kc(R))$, then $(x, y) \in tc(kc(R)) \cap d(kc(R))$ because $(kc(R))^{(\kappa)} \circ kc(R) \subseteq tc(kc(R))$. Then, $(x, y) \in kc(kc(R))$ by construction of $kc$. Since $kc(R)$ is a closure operator, we have $(x, y) \in kc(R)$. This contradicts the fact that $(y, x) \in P(kc(R))$. Thus, $(y, x) \notin P(kc(R))$.

By the definition of $P$, $(y, x) \in P(c(kc(c(kc(R)))))$ if and only if

$$(y, x) \in c(kc(c(kc(R)))), \tag{2.1}$$

and

$$(x, y) \in kc(c(kc(R))). \tag{2.2}$$

By Lemma 2.15 (iv), we have $c(kc(c(kc(R)))) \subseteq kc(R)$. Thus, (2.1) implies that

$$(y, x) \in kc(R). \tag{2.3}$$

Since $(y, x) \in kc(R)$ and $(y, x) \notin P(kc(R))$, it follows that $(x, y) \in kc(R)$. Thus, we have

$$(x, y) \notin c(kc(R)). \tag{2.4}$$

From (2.2) and (2.4), we obtain $(x, y) \in kc(c(kc(R))) \setminus c(kc(R))$. By the definition of $kc$, $(y, x) \in c(kc(R))$, and thus, $(y, x) \notin kc(R)$. This contradicts (2.3). ■

Now, we show the main results of Graham et al. (1972) and Cato (2012).[9]

**Proposition 2.3** (Graham et al. 1972; Cato 2012) *Let $R$ be a binary relation on $X$. Then,*

*(i)* $tc(c(tc(c(tc(R))))) = c(tc(c(tc(R))))$*;*
*(ii)* $kc(c(kc(c(kc(R))))) = c(kc(c(kc(R))))$.

[9]The first claim of Lemma 2.18 is provided by Graham et al. (1972), and the second claim of Lemma 2.18 is provided by Cato (2012).

*Proof* (i) Since $c(tc(c(tc(R)))) \subseteq tc(c(tc(c(tc(R)))))$ by extensiveness, it suffices to show that

$$tc(c(tc(c(tc(R))))) \subseteq c(tc(c(tc(R)))).$$

Suppose that $(x, y) \in tc(c(tc(c(tc(R)))))$. Then, there exist $K \in \mathbb{N}$ and $x^0, x^1, \ldots, x^K \in X$ such that

$$x^0 = x, (x^{k-1}, x^k) \in c(tc(c(tc(R)))) \text{ for all } k \in \{1, \ldots, K\}, \text{ and } x^K = y.$$

If $c(tc(c(tc(R)))) \circ c(tc(c(tc(R)))) \subseteq c(tc(c(tc(R))))$, then $(x, y) \in tc(c(tc(c(tc(R)))))$. Thus, we have to prove that $c(tc(c(tc(R))))\circ c(tc(c(tc(R)))) \subseteq c(tc(c(tc(R))))$. Suppose that $(x, y) \in c(tc(c(tc(R)))) \circ c(tc(c(tc(R))))$ for some $x, y \in X$. Then, there exists $z \in X$ such that $(x, z) \in c(tc(c(tc(R))))$ and $(z, y) \in c(tc(c(tc(R))))$. Since $c(tc(c(tc(R)))) \subseteq tc(R)$ by Lemma 2.15 (i), it follows that $(x, y) \in tc(R)$. Then, we have $(x, y) \in tc(R) \circ c(tc(c(tc(R))))$. Hence, Lemma 2.18 (i) implies that $(x, y) \in c(tc(c(tc(R))))$. This completes the proof.

(ii) Since $c(kc(c(kc(R)))) \subseteq kc(c(kc(c(kc(R)))))$, we have to show that

$$kc(c(kc(c(kc(R))))) \subseteq c(kc(c(kc(R)))).$$

By the definition of $kc$, this is true if and only if

$$tc(c(kc(c(kc(R))))) \cap d(c(kc(c(kc(R))))) \subseteq c(kc(c(kc(R)))).$$

Then, it suffices to show that

$$(x, y) \in tc(c(kc(c(kc(R))))) \Rightarrow (y, x) \notin P(c(kc(c(kc(R))))).$$

Suppose that $(x, y) \in (c(kc(c(kc(R)))))^{(\kappa)}$ for some $\kappa \in \mathbb{N} \setminus \{1\}$. Since $c(kc(c(kc(R)))) \subseteq kc(R)$ by Lemma 2.15 (iv), it follows that $(x, y) \in (kc(R))^{(\kappa-1)} \circ c(kc(c(kc(R))))$. Hence, Lemma 2.18 (ii) implies that $(y, x) \notin P(c(kc(c(kc(R)))))$. This completes the proof. ■

Proposition 2.3 implies that we can induce at most 10 binary relations from $c$ and $tc$ (or $kc$). In other words, we have the following relations from $c$ and $tc$:

$$R, tc(R), c(R), c(tc(R)), tc(c(R)), tc(c(tc(R))), c(tc(c(R))),$$
$$c(tc(c(tc(R)))), tc(c(tc(c(R)))), c(tc(c(tc(c(R))))).$$

We have the following relations from $c$ and $kc$:

$$R, kc(R), c(R), c(kc(R)), kc(c(R)), kc(c(kc(R))), c(kc(c(R))),$$
$$c(kc(c(kc(R)))), kc(c(kc(c(R)))), c(kc(c(kc(c(R))))).$$

In the preceding analysis, we focused on four fundamental closure operators. Next, we introduce variants of these closure operators. Given $A \subseteq X \times X$, the *A-closure* is defined as follows:

$$\varphi_A(R) = R \cup A.$$

The *indifference-transitive closure* is defined as follows:

$$\varphi_I(R) = R \cup I(tc(R)).$$

The *transitive-indifference closure* is defined as follows:

$$\varphi_I^*(R) = R \cup tc(I(R)).$$

The *reflexive-transitive closure* is defined as follows:

$$\varphi_{rtc}(R) = rc(tc(R)).$$

The *symmetric-reflexive closure* is defined as follows:

$$\varphi_{src}(R) = sc(rc(R)).$$

The following proposition shows that the abovementioned operators are closure operators.

**Proposition 2.4** *$\varphi_A$, $\varphi_I$, $\varphi_I^*$, $\varphi_{rtc}$, and $\varphi_{rsc}$ are closure operators.*

*Proof* It is clear that $\varphi_A$ is a closure operator.

Now, we show that $\varphi_I$ is a closure operator. By construction, $R \subseteq \varphi_I(R)$. Moreover, if $R \subseteq R'$, then the monotonicity of $tc$ implies that $tc(R) \subseteq tc(R')$. By Lemma 2.3 (iv), $I(tc(R)) \subseteq I(tc(R'))$. Thus, $\varphi_I(R) \subseteq \varphi_I(R')$. We prove that $\varphi_I(\varphi_I(R)) = \varphi_I(R)$. Note that

$$\varphi_I(\varphi_I(R)) = \varphi_I(R) \cup I(tc(\varphi_I(R))).$$

Since $R \subseteq tc(\varphi_I(R)) \subseteq tc(R)$, monotonicity implies that $tc(R) \subseteq tc(tc(\varphi_I(R))) \subseteq tc(tc(R))$. By the idempotence of $tc$, $tc(R) = tc(\varphi_I(R)))$. Thus,

$$\begin{aligned}\varphi_I(\varphi_I(R)) &= \varphi_I(R) \cup I(tc(R)) \\ &= (R \cup I(tc(R))) \cup I(tc(R)) \\ &= R \cup I(tc(R))).\end{aligned}$$

Therefore, $\varphi_I$ is a closure operator.

Now, we show that $\varphi_I^*$ is a closure operator. By construction, $R \subseteq \varphi_I^*(R)$. Moreover, if $R \subseteq R'$, then Lemma 2.3 (iv) implies that $I(R) \subseteq I(R')$. By the monotonicity of $tc$, $tc(I(R)) \subseteq tc(I(R'))$. Thus, $\varphi_I^*(R) \subseteq \varphi_I^*(R')$. We prove that

$\varphi_I^*(\varphi_I^*(R)) = \varphi_I^*(R)$. By extensiveness, it suffices to show that $\varphi_I^*(\varphi_I^*(R)) \subseteq \varphi_I^*(R)$. Note that

$$\varphi_I^*(\varphi_I^*(R)) = \varphi_I^*(R) \cup tc(I(\varphi_I^*(R))).$$

We have to show that

$$tc(I(\varphi_I^*(R))) \subseteq \varphi_I^*(R). \tag{2.5}$$

We have the following:

$$\begin{aligned}
I(\varphi_I^*(R)) &= \Big(R \cup tc(I(R))\Big) \cap d\Big(R \cup tc(I(R))\Big) && \text{(definition of } I\text{)}\\
&= \Big(R \cup tc(I(R))\Big) \cap \Big(R^d \cup tc(d(I(R)))\Big) && \text{(Lemmas 2.1 and 2.12)}\\
&= \Big(R \cup tc(I(R))\Big) \cap \Big(R^d \cup tc(I(R))\Big) && \text{(Lemma 2.5)}\\
&= \Big(R \cap R^d\Big) \cup \Big(R \cap tc(I(R))\Big) \cup \Big(R^d \cap tc(I(R))\Big) \cup tc(I(R))\\
&= tc(I(R)).
\end{aligned}$$

Therefore, $tc(I(\varphi_I^*(R))) = tc(I(R))$ by the idempotence of $tc$. We conclude that $tc(I(\varphi_I^*(R))) \subseteq \varphi_I^*(R)$, and thus, $\varphi^*$ is a closure operator.

Next, we show that $\varphi_{rtc}$ is a closure operator. Note that $R \subseteq tc(R)$ by the extensiveness of $tc$. Since $rc$ is also extensive, $tc(R) \subseteq rc(tc(R))$. Then, $R \subseteq \varphi_{rtc}(R)$. If $R \subseteq R'$, then $tc(R) \subseteq tc(R')$, which implies that $\varphi_{rtc}(R) = rc(tc(R)) \subseteq rc(tc(R')) = \varphi_{rtc}(R')$. It suffices to show that $\varphi_{rtc}(\varphi_{rtc}(R)) = \varphi_{rtc}(R)$. Note that $\Delta \subseteq rc(tc(R))$. Then, monotonicity implies that $tc(\Delta) \subseteq tc(rc(tc(R)))$. By extensiveness, we have $\Delta \subseteq tc(R)$, and thus,

$$\begin{aligned}
\varphi_{rtc}(\varphi_{rtc}(R)) &= tc(rc(tc(R)))\\
&= tc(\Delta \cup tc(R)).
\end{aligned}$$

Because $\Delta \cup tc(R) = tc(\Delta \cup tc(R))$, $\varphi_{rtc}(\varphi_{rtc}(R)) = \varphi_{rtc}(R)$.

Finally, we show that $\varphi_{rsc}$ is a closure operator. Note that $R \subseteq \varphi_{rsc}(R)$ by the extensiveness of $sc$ and $rc$. If $R \subseteq R'$, then $sc(R) \subseteq sc(R')$, which implies that $\varphi_{rsc}(R) = rc(sc(R)) \subseteq rc(sc(R')) = \varphi_{rsc}(R')$. We can prove that $\varphi_{rsc}(\varphi_{rsc}(R)) = \varphi_{rsc}(R)$ as follows:

$$\begin{aligned}
rc(sc(\varphi_{rsc}(R))) &= \Delta \cup sc(\varphi_{rsc} R) && \text{(definition of } rc\text{)}\\
&= sc(rc(sc(R))) && (\Delta \subseteq sc(\varphi_{rsc} R))\\
&= sc(\Delta \cup sc(R)) && \text{(definition of } rc\text{)}\\
&= (\Delta \cup sc(R)) \cup d(\Delta \cup sc(R)) && \text{(definition of } sc\text{)}\\
&= \Delta \cup sc(R) && \text{(Lemmas 2.1 (ii) and 2.12 (iii))}\\
&= rc(sc(R)). && \text{(definition of } rc\text{)}
\end{aligned}$$

Therefore, $\varphi_{rsc}$ is a closure operator. ■

Note that $\varphi_I$ is different from $\varphi_I^*$. To see this, assume that $X = \{x, y, z\}$ and $R = \{(x, y), (y, z), (z, x)\}$. Note that

$$I(R) = \emptyset \text{ and } tc(R) = X \times X.$$

Therefore, $\varphi_I(R) = X \times X$ and $\varphi_I^*(R) = R$.

The *symmetric-transitive operator* is defined as follows:

$$\varphi_{stc}(R) = sc(tc(R)).$$

$\varphi_{stc}$ is not a closure operator because it is extensive and monotonic but not idempotent. To see this, assume that $X = \{x, y, z\}$ and $R = \{(x, y), (x, z)\}$. Then,

$$\begin{aligned}
&tc(R) = R, \\
&sc(tc(R)) = \{(x, y), (y, x), (x, z), (z, x)\}, \\
&tc(sc(tc(R))) = \{(x, y), (y, x), (x, z), (z, x), (y, z), (z, y)\} \cup \Delta, \\
&sc(tc(sc(tc(R)))) = tc(sc(tc(R))).
\end{aligned}$$

Thus, we have $\varphi_{stc}(\varphi_{stc}(R)) \neq \varphi_{stc}(R)$.

## 2.5 Concluding Remarks

This chapter provided a survey of the basic theory of binary relations by extending the works of Graham et al. (1972) and Fishburn (1978). We focused on how operators work over binary relations. The results of this chapter will be used in the analysis presented in the subsequent chapters.

We provide two remarks to conclude this chapter. First, we interpreted a binary relation as a preference of some person in this book. Although a preference interpretation is particularly important for economic theory, there are various interpretations, which is significant for other social sciences. Consider the ownership relation $O$: "$(x, y) \in O$" means "$x$ owns $y$." Here, the set $X$ of objects consists of persons and things. Persons can own things, but things cannot own persons. Thus, a type of asymmetry works for $O$. Moreover, it is important to distinguish natural persons from legal persons. Legal persons can be owned by natural persons or other legal persons, but no one can own natural persons. The working of $O$ is strongly dependent on the nature of objects. Such exercises suggest that our approach is useful in other interpretations, and further development of the theory is needed. Therefore, we have various potential applications for the theory of binary relations.

Second, we discuss only operators for binary relations, and do not consider a more general structure of mathematical relations. However, a ternary relation may be useful to understanding social structures. Such a relation is a subset of $X \times X \times X$. For example, a betweenness relation $B$ must be a ternary relation. If $(x, y, z) \in B$,

$y$ is intermediate between $x$ and $z$. This relation can represent economic structures or physiological states. We believe that there are many classes of relations, which are important in the social sciences. Extending the analysis of operators to such classes is left to future work.

## References

Berge, C. (1963). *Topological spaces: Including a treatment of multi-valued functions, vector spaces and convexity*. London: Oliver and Boyd.

Bossert, W., Sprumont, Y., & Suzumura, K. (2005). Consistent rationalizability. *Economica*, *72*(286), 185–200.

Bossert, W., & Suzumura, K. (2010). *Consistency, choice and rationality*. Cambridge MA: Harvard University Press.

Cato, S. (2012). Complements and consistent closures. *Discrete Mathematics*, *312*(6), 1218–1221.

Fishburn, P. C. (1978). Operations on binary relations. *Discrete Mathematics*, *21*(1), 7–22.

Graham, R. L., Knuth, D. E., & Motzkin, T. S. (1972). Complements and transitive closures. *Discrete Mathematics*, *2*(1), 17–29.

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