Chapter 2
Oka's First Coherence Theorem

We study the local properties of holomorphic functions. The main object is Oka's Coherence Theorem that plays the most fundamental and important role in analytic function theory in several variables. The notion of a coherent sheaf gives rise to a fundamental terminology in a broad area of modern Mathematics. Oka originally termed the notion as "idéal de domaines indéterminés". The wording "coherence" comes from H. Cartan's naming, "faisceau cohérent".

The word "cohérent" means "holding a logical compatibility", different from the Japanese counter-part "Rensetu", which is not a direct translation of "Coherent" and means "being contacted continuously". In view of the original notion of "de domaines indéterminés" or "Coherent", the Japanese wording "Rensetu" sounds appropriate for the meaning.

2.1 Weierstrass' Preparation Theorem

Let $\mathrm{P}\Delta(a; r) \subset \mathbf{C}^n$ be a polydisk and let $f \in \mathscr{O}(\mathrm{P}\Delta(a; r))$. Assume that $f \neq 0$ ($f(z) \not\equiv 0$). Then $f(z)$ is expanded to a power series as follows:

$$
\begin{aligned}
f(z) = \sum_{\lambda} c_\lambda (z-a)^\lambda &= \sum_{\nu=\nu_0}^{\infty} P_\nu(z-a), \\
P_\nu(z-a) &= \sum_{|\lambda|=\nu} c_\lambda (z-a)^\lambda \\
&\qquad \text{(a homogeneous polynomial of degree } \nu\text{)}, \\
P_{\nu_0}(z-a) &\not\equiv 0.
\end{aligned}
$$

The degree ν_0 of the first term $P_{\nu_0}(z-a)$ is called the *order of zero* of f at a, denoted by $\mathrm{ord}_a f$.

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For the sake of simplicity, we let $a = 0$ by translation. Assume that $f(0) = 0$ ($\nu_0 \geq 1$). Take a vector $v \in \mathbf{C}^n \setminus \{0\}$ such that $P_{\nu_0}(v) \neq 0$. For $\zeta \in \mathbf{C}$ we have

$$f(\zeta v) = \sum_{\nu=\nu_0}^{\infty} \zeta^{\nu} P_{\nu}(v) = \zeta^{\nu_0}(P_{\nu_0}(v) + \zeta P_{\nu_0+1}(v) + \cdots).$$

By a linear transformation of the coordinate system we choose a new coordinate system $z = (z_1, \ldots, z_n)$, so that $v = (0, \ldots, 0, 1)$. We write

$$\mathrm{P}\Delta(0; r) = \mathrm{P}\Delta_{n-1} \times \Delta(0; r_n) \subset \mathbf{C}^{n-1} \times \mathbf{C},$$

and for the coordinate system

$$\begin{aligned} z &= (z', z_n) \in \mathrm{P}\Delta_{n-1} \times \Delta(0; r_n), \\ 0 &= (0, 0). \end{aligned}$$

With respect to this coordinate system we assume the following conditions.

2.1.1 (i) f is holomorphic in a neighborhood of the closed polydisk $\overline{\mathrm{P}\Delta(0; r)}$, and the homogeneous polynomial expansion $f(z) = \sum_{\nu=\nu_0}^{\infty} P_{\nu}(z)$ satisfies that $P_{\nu_0}(0, 1) \neq 0$ and

$$f(0, z_n) = z_n^{\nu_0}(P_{\nu_0}(0, 1) + z_n P_{\nu_0+1}(0, 1) + \cdots).$$

(ii) Take $r_n > 0$ sufficiently small, so that $\{|z_n| \leq r_n;\ f(0, z_n) = 0\} = \{0\}$.

(iii) If we take small $r_1, \ldots, r_{n-1} > 0$, depending on r_n, the roots z_n of $f(z', z_n) = 0$ for every $z' \in \overline{\mathrm{P}\Delta}_{n-1}$ are contained in the disk $\Delta(0; r_n)$; in particular, $|f(z', z_n)| > 0$ for all $(z', z_n) \in \overline{\mathrm{P}\Delta}_{n-1} \times \{|z_n| = r_n\}$.

For $\underline{f}_0 \in \mathscr{O}_{\mathbf{C}^n,0}$, a polydisk $\mathrm{P}\Delta(0; r)$ satisfying 2.1.1 (i)–(iii) above is called the *standard polydisk* of $\underline{f}_0$ or f. The coordinate system $z = (z_1, \ldots, z_n)$ is called the *standard coordinate system* of $\underline{f}_0$.

Remark 2.1.2 (i) The standard polydisks of $\underline{f}_0$ form a basis of neighborhoods about 0, because $r_n > 0$ can be chosen arbitrarily small and then, depending on it, r_j, $1 \leq j \leq n-1$, are chosen arbitrarily small.

(ii) Since $\{v \in \mathbf{C}^n;\ P_{\nu_0}(v) = 0\}$ contains no interior point, the standard coordinate system and the standard polydisk can be chosen to be the same for finitely many $\underline{f_k}_0 \in \mathscr{O}_{\mathbf{C}^n,0} \setminus \{0\}$, $1 \leq k \leq l (< \infty)$, with $f_k(0) = 0$.

(iii) The direction vector $v \in \mathbf{C}^n \setminus \{0\}$ can be chosen to be the same for countably many $\underline{f_k}_0 \in \mathscr{O}_{\mathbf{C}^n,0} \setminus \{0\}$ with $f_k(0) = 0$, $k = 1, 2, \ldots$ For, with denoting $P_{k\nu_k}(z)$ the first non-zero term in the homogeneous polynomial expansion of $f_k(z)$, the set $A = \bigcup_{k=1}^{\infty} \{P_{k\nu_k}(v) = 0\}$ is a countable union of closed subsets containing no interior point. Baire's Category Theorem implies that A contains no interior point. Therefore, $\mathbf{C}^n \setminus A \neq \emptyset$, and one may take $v \in \mathbf{C}^n \setminus A$. It follows that for the countable family $\{\underline{f_{k}}_0\}$ one can take the same standard coordinate system of all $\underline{f_{k}}_0$. (It is not possible in general to take the common standard neighborhood of all $\underline{f_{k}}_0$.)

We write $\mathscr{O}(E)$ for the set of all holomorphic functions in neighborhoods of a closed subset $E \subset \mathbf{C}^n$. We define the sup-norm of a function g on a subset W of the domain of definition by

$$\|g\|_W = \sup_{z \in W} |g(z)|.$$

Theorem 2.1.3 (Weierstrass' Preparation Theorem) *Let* $\underline{f}_0 \in \mathscr{O}_{\mathbf{C}^n,0} \setminus \{0\}$, $f(0) = 0$, $p = \mathrm{ord}_0 f$, *and let* $\mathrm{P}\Delta = \mathrm{P}\Delta_{n-1} \times \Delta(0; r_n)$ $(\ni z = (z', z_n))$ *be the standard polydisk of* f.

(i) *There exist unique holomorphic functions,* $a_j \in \mathscr{O}(\overline{\mathrm{P}\Delta_{n-1}})$ *with* $a_j(0) = 0$, $1 \le j \le p$, *and zero-free* $u \in \mathscr{O}(\overline{\mathrm{P}\Delta})$ *such that*

$$(2.1.4) \qquad f(z) = f(z', z_n) = u(z)\Big(z_n^p + \sum_{j=1}^{p} a_j(z') z_n^{p-j}\Big),$$
$$(z', z_n) \in \overline{\mathrm{P}\Delta_{n-1}} \times \overline{\Delta(0; r_n)}.$$

(ii) *For every* $\varphi \in \mathscr{O}(\mathrm{P}\Delta)$ *there are unique holomorphic functions,* $a \in \mathscr{O}(\mathrm{P}\Delta)$ *and* $b_j \in \mathscr{O}(\mathrm{P}\Delta_{n-1})$, $1 \le j \le p$, *satisfying*

$$(2.1.5) \qquad \varphi(z) = af + \sum_{j=1}^{p} b_j(z') z_n^{p-j}, \quad z = (z', z_n) \in \mathrm{P}\Delta_{n-1} \times \Delta(0; r_n).$$

(iii) *In* (ii) *there is a constant* $M > 0$ *depending only on* f, *independent of* φ, *such that*

$$\|a\|_{\mathrm{P}\Delta} \le M\|\varphi\|_{\mathrm{P}\Delta}, \qquad \|b_j\|_{\mathrm{P}\Delta_{n-1}} \le M\|\varphi\|_{\mathrm{P}\Delta}.$$

Proof (i) For $k \in \mathbf{Z}^+$ we set

$$(2.1.6) \qquad \sigma_k(z') = \frac{1}{2\pi i} \int_{|z_n| = r_n} z_n^k \frac{\frac{\partial f}{\partial z_n}(z', z_n)}{f(z', z_n)} dz_n, \quad z' \in \overline{\mathrm{P}\Delta_{n-1}}.$$

It follows that $\sigma_k \in \mathscr{O}(\overline{\mathrm{P}\Delta_{n-1}})$. By the residue theorem $\sigma_0(z') \in \mathbf{Z}$ and hence the continuity implies

$$\sigma_0(z') \equiv \sigma_0(0) = \frac{1}{2\pi i} \int_{|z_n| = r_n} \frac{\frac{\partial f}{\partial z_n}(0, z_n)}{f(0, z_n)} dz_n = p.$$

Therefore, as $z' \in \mathrm{P}\Delta_{n-1}$ is fixed, the number of roots of $f(z', z_n) = 0$ with counting multiplicities is identically p. We write $\zeta_1(z'), \ldots, \zeta_p(z')$ for them with counting multiplicities. Again from the residue theorem we get

$$\sigma_k(z') = \sum_{j=1}^{p} (\zeta_j(z'))^k, \quad k = 0, 1, \ldots.$$

We set the elementary symmetric polynomial of degree ν in $\zeta_1(z'), \ldots, \zeta_p(z')$:

$$a_\nu(z') = (-1)^\nu \sum_{1 \le j_1 < \cdots < j_\nu \le p} \zeta_{j_1}(z') \cdots \zeta_{j_\nu}(z').$$

Then, $a_\nu(z') \in \mathbf{Q}[\sigma_1(z'), \ldots, \sigma_\nu(z')]$ (cf. Exercise 1 at the end of this chapter). For instance, when $p = 2$,

$$a_2(z') = \zeta_1(z')\zeta_2(z') = \frac{1}{2}\sigma_1(z')^2 - \frac{1}{2}\sigma_2(z').$$

Noting all $\zeta_j(0) = 0$, we see that $a_\nu \in \mathscr{O}(\overline{\mathrm{P}\Delta}_{n-1})$, $a_\nu(0) = 0$, $\nu \ge 1$. We put

$$W(z', z_n) = \prod_{j=1}^{p} \left(z_n - \zeta_j(z')\right) = z_n^p + \sum_{j=1}^{p} a_j(z') z_n^{p-j}. \tag{2.1.7}$$

Then, $W(z', z_n) \in \mathscr{O}(\overline{\mathrm{P}\Delta}_{n-1})[z_n] \subset \mathscr{O}(\overline{\mathrm{P}\Delta}_{n-1} \times \mathbf{C})$. With $s_n > r_n$ sufficiently close to r_n, and with every $z' \in \overline{\mathrm{P}\Delta}_{n-1}$ fixed, the roots of $W(z', z_n) = 0$ and $f(z', z_n) = 0$ in $|z_n| < s_n$ are the same with counting multiplicities,[1] so that the following integral formulae hold in $|z_n| < s_n$:

$$u(z', z_n) = \frac{f(z', z_n)}{W(z', z_n)} = \frac{1}{2\pi i} \int_{|\zeta_n| = s_n} \frac{f(z', \zeta_n)}{W(z', \zeta_n)} \cdot \frac{d\zeta_n}{\zeta_n - z_n},$$
$$v(z', z_n) = \frac{W(z', z_n)}{f(z', z_n)} = \frac{1}{2\pi i} \int_{|\zeta_n| = s_n} \frac{W(z', \zeta_n)}{f(z', \zeta_n)} \cdot \frac{d\zeta_n}{\zeta_n - z_n}.$$

It follows from these integral formulae that $u, v \in \mathscr{O}(\overline{\mathrm{P}\Delta}_{n-1} \times \{|z_n| \le r_n\})$. Furthermore, since in a neighborhood of $|z_n| = r_n$ the denominators of u, v are zero-free, $u \cdot v = 1$ holds. Therefore by the uniqueness of analytic continuation, $u \cdot v = 1$ on $\overline{\mathrm{P}\Delta}$. One sees that u is zero-free on $\overline{\mathrm{P}\Delta}$. There is a constant $C > 0$ such that

$$C^{-1} \le |u(z)| \le C, \qquad z \in \overline{\mathrm{P}\Delta}. \tag{2.1.8}$$

Thus it follows that

[1] By the arguments up to here one sees that the quotient $f(z', z_n)/W(z', z_n)$ with each fixed z' is a zero-free holomorphic function in $|z_n| < s_n$; however, as (z', z_n) runs freely, even its continuity is unclear.

$$f(z', z_n) = u(z)\Big(z_n^p + \sum_{j=1}^{p} a_j(z')z_n^{p-j}\Big) = u(z)W(z', z_n),$$

$u \in \mathscr{O}(\overline{\mathrm{P}\Delta})$, $a_\nu \in \mathscr{O}(\overline{\mathrm{P}\Delta_{n-1}})$, and $a_\nu(0) = 0$.

We confirm that $u(z)$ and $W(z', z_n)$ are uniquely determined as elements of $\mathscr{O}_{\mathbf{C}^n,0}$. Suppose that

$$\begin{aligned} \underline{f}_0 &= \underline{u}_0 \cdot \underline{\Big(z_n^p + \textstyle\sum_{j=1}^p a_j(z')z_n^{p-j}\Big)}_0 \\ &= \underline{\tilde{u}}_0 \cdot \underline{\Big(z_n^p + \textstyle\sum_{j=1}^p \tilde{a}_j(z')z_n^{p-j}\Big)}_0 . \end{aligned}$$

Let $\widetilde{\mathrm{P}\Delta}_{n-1} \times \Delta(0; \tilde{r}_n)$ be a standard polydisk for which the above expressions make sense. For each fixed $z' \in \widetilde{\mathrm{P}\Delta}_{n-1}$, the roots of two equations

$$z_n^p + \sum_{j=1}^{p} a_j(z')z_n^{p-j} = 0,$$
$$z_n^p + \sum_{j=1}^{p} \tilde{a}_j(z')z_n^{p-j} = 0$$

are identical with counting multiplicities, and then

$$a_j(z') = \tilde{a}_j(z'), \quad 1 \le j \le p.$$

Hence, $u(z) = \tilde{u}(z)$ follows.

(ii) We may assume that

$$\begin{aligned} f = W(z', z_n) &= z_n^p + \sum_{\nu=1}^{p} a_\nu(z')z_n^{p-\nu} \\ &= \sum_{\nu=0}^{p} a_\nu(z')z_n^{p-\nu} \ \in \mathscr{O}(\overline{\mathrm{P}\Delta_{n-1}})[z_n]. \end{aligned}$$

Here, we put $a_0(z') = 1$. For $\varphi \in \mathscr{O}(\mathrm{P}\Delta_{n-1} \times \Delta(0; r_n))$ we set

$$a(z', z_n) = \frac{1}{2\pi i}\int_{|\zeta_n|=t_n} \frac{\varphi(z', \zeta_n)}{W(z', \zeta_n)}\frac{d\zeta_n}{\zeta_n - z_n}, \tag{2.1.9}$$
$$(z', z_n) \in \mathrm{P}\Delta_{n-1} \times \Delta(0; r_n),$$

where $|z_n| < t_n < r_n$. Since $a(z', z_n)$ is independent of the choice of t_n close to r_n, $a(z', z_n) \in \mathscr{O}(\mathrm{P}\Delta_{n-1} \times \Delta(0; r_n))$ is determined. For $z' \in \mathrm{P}\Delta_{n-1}$, $|z_n| < t_n$ we write

$$
\begin{aligned}
(2.1.10)\quad & \varphi(z', z_n) - a(z', z_n)W(z', z_n) \\
&= \frac{1}{2\pi i}\int_{|\zeta_n|=t_n} \varphi(z', \zeta_n)\frac{d\zeta_n}{\zeta_n - z_n} - \frac{W(z', z_n)}{2\pi i}\int_{|\zeta_n|=t_n} \frac{\varphi(z', \zeta_n)}{W(z', \zeta_n)}\frac{d\zeta_n}{\zeta_n - z_n} \\
&= \frac{1}{2\pi i}\int_{|\zeta_n|=t_n} \varphi(z', \zeta_n)\left\{1 - \frac{W(z', z_n)}{W(z', \zeta_n)}\right\}\frac{d\zeta_n}{\zeta_n - z_n} \\
&= \frac{1}{2\pi i}\int_{|\zeta_n|=t_n} \varphi(z', \zeta_n)\frac{\sum_{\nu=0}^{p-1} a_\nu(z')\left(\zeta_n^{p-\nu} - z_n^{p-\nu}\right)}{W(z', \zeta_n)(\zeta_n - z_n)}d\zeta_n \\
&= \frac{1}{2\pi i}\int_{|\zeta_n|=t_n} \frac{\varphi(z', \zeta_n)}{W(z', \zeta_n)}\left\{\sum_{\nu=0}^{p-1} a_\nu(z')\left(\zeta_n^{p-\nu-1} + \zeta_n^{p-\nu-2}z_n + \cdots\right.\right. \\
&\qquad\qquad\qquad \left.\left. + z_n^{p-\nu-1}\right)\right\}d\zeta_n \\
&= b_1(z')z_n^{p-1} + b_2(z')z_n^{p-2} + \cdots + b_p(z'),
\end{aligned}
$$

where $b_\nu(z')$ are given by

$$
(2.1.11)\qquad b_\nu(z') = \frac{1}{2\pi i}\int_{|\zeta_n|=t_n} \frac{\varphi(z', \zeta_n)}{W(z', \zeta_n)}\left(\sum_{h=0}^{\nu-1} a_h(z')\zeta_n^{\nu-1-h}\right) d\zeta_n.
$$

It follows from this expression that $b_\nu(z') \in \mathscr{O}(\mathrm{P}\Delta_{n-1})$, $1 \le \nu \le p$ (independent of t_n). Therefore,

$$
(2.1.12)\qquad \varphi(z', z_n) = a(z', z_n)W(z', z_n) + \sum_{\nu=1}^{p} b_\nu(z')z_n^{p-\nu}.
$$

Next, we show the uniqueness. Suppose that

$$
(2.1.13)\qquad \varphi(z', z_n) = \tilde{a}(z', z_n)W(z', z_n) + \sum_{\nu=1}^{p} \tilde{b}_\nu(z')z_n^{p-\nu}.
$$

Subtracting the both sides of (2.1.12) and (2.1.13) and shifting terms, we assume that

$$
(a(z', z_n) - \tilde{a}(z', z_n))W(z', z_n) = \sum_{\nu=1}^{p}(\tilde{b}_\nu(z') - b_\nu(z'))z_n^{p-\nu} \not\equiv 0.
$$

Then for a fixed $z' \in \widetilde{\mathrm{P}\Delta}_{n-1}$ the left-hand side has at least p roots with counting multiplicities. The right-hand side has at most $p-1$ roots with counting multiplicities; this is absurd. Hence,

$$
\tilde{b}_\nu(z') = b_\nu(z'), \qquad \tilde{a}(z', z_n) = a(z', z_n).
$$

(iii) We show the estimates. In (2.1.9)–(2.1.11) there is a constant $\delta > 0$ such that

$$|W(z', \zeta_n)| \geq \delta > 0, \quad z' \in \mathrm{P}\Delta_{n-1}, \ |\zeta_n| = t_n \ (\nearrow r_n). \tag{2.1.14}$$

By (2.1.11) there is a constant $M > 0$, depending only on $\sup_{\mathrm{P}\Delta_{n-1}} |a_\nu|$, p and r_n such that

$$|b_\nu(z') z_n^{p-\nu}| \leq M\delta^{-1} \|\varphi\|_{\mathrm{P}\Delta}, \qquad \|b_\nu\|_{\mathrm{P}\Delta_{n-1}} \leq M\delta^{-1} \|\varphi\|_{\mathrm{P}\Delta}.$$

It follows from (2.1.10) that for each $z' \in \mathrm{P}\Delta_{n-1}$, $|z_n| = t_n (< r_n)$

$$|\varphi(z', z_n) - a(z', z_n) W(z', z_n)| \leq pM\delta^{-1} \|\varphi\|_{\mathrm{P}\Delta}.$$

From this we obtain

$$|a(z', z_n) W(z', z_n)| \leq (pM\delta^{-1} + 1) \|\varphi\|_{\mathrm{P}\Delta}.$$

By (2.1.14)

$$|a(z', z_n)| \leq (pM\delta^{-1} + 1)\delta^{-1} \|\varphi\|_{\mathrm{P}\Delta}.$$

Letting $t_n \nearrow r_n$, we see by the maximum principle (Theorem 1.2.17) that

$$\|a\|_{\mathrm{P}\Delta} \leq (pM\delta^{-1} + 1)\delta^{-1} \|\varphi\|_{\mathrm{P}\Delta}.$$

To obtain an estimate replacing W with the original f, we write

$$\begin{aligned}\varphi &= aW + \sum_{\nu=1}^{p} b_\nu(z') z_n^{p-\nu} \\ &= \left(\frac{a}{u}\right) \cdot f + \sum_{\nu=1}^{p} b_\nu(z') z_n^{p-\nu}.\end{aligned}$$

It follows from (2.1.8) that

$$C^{-1} \leq |u| \leq C.$$

Therefore, finally there is a positive constant $M' = M'(f)$ independent of φ such that

$$\begin{aligned}\|a\|_{\mathrm{P}\Delta} &\leq M' \|\varphi\|_{\mathrm{P}\Delta}, \\ \|b_\nu\|_{\mathrm{P}\Delta_{n-1}} &\leq M' \|\varphi\|_{\mathrm{P}\Delta}, \qquad 1 \leq \nu \leq p.\end{aligned}$$ □

Remark 2.1.15 Let the notation be as in Theorem 2.1.3. Moreover, assume that $p = 1$. Then, by the Implicit Function Theorem 1.2.41, equation $f(z', z_n) = 0$ has a unique solution $z_n = g(z')$ in a neighborhood of 0 with $g(0) = 0$. But this is just an existence theorem. Applying Theorem 2.1.3 for $\varphi = z_n$, we have an integral representation formula of the solution (cf. 2.1.6):

$$g(z') = \frac{1}{2\pi i} \int_{|z_n|=r_n} z_n \frac{\frac{\partial f}{\partial z_n}(z', z_n)}{f(z', z_n)} dz_n, \quad z' \in \overline{\mathrm{P}\Delta}_{n-1}.$$

Definition 2.1.16 (i) Letting $\mathrm{P}\Delta_{n-1} \subset \mathbf{C}^{n-1}$, we call the z_n-polynomial with coefficients in $\mathscr{O}(\overline{\mathrm{P}\Delta}_{n-1})$

$$W(z', z_n) = z_n^p + \sum_{\nu=1}^{p} a_\nu(z') \cdot z_n^{p-\nu},$$
$$a_\nu \in \mathscr{O}(\overline{\mathrm{P}\Delta}_{n-1}), \quad a_\nu(0) = 0$$

a *Weierstrass polynomial* (in z_n). Considering the induced germ

$$W = z_n^p + \sum_{\nu=1}^{p} \underline{a_\nu}_0 \cdot z_n^{p-\nu} \in \mathscr{O}_{\mathrm{P}\Delta_{n-1},0}[z_n],$$

we also call this a Weierstrass polynomial (in z_n).

(ii) Write (2.1.7) as $f(z', z_n) = uW(z', z_n)$ with unit u (i.e., $^\exists u^{-1}$) and Weierstrass polynomial $W(z', z_n)$. We call $f(z', z_n) = uW(z', z_n)$ the *Weierstrass decomposition* of f at 0, which is unique.

2.2 Local Rings

For the sake of simplicity we write $\mathscr{O}_{\mathbf{C}^n,a} = \mathscr{O}_{n,a}$ ($a \in \mathbf{C}^n$). This is an integral local ring (Sect. 1.3.3 (4)). In this section we investigate the algebraic properties in more detail.

2.2.1 Preparations from Algebra

Here we describe elementary properties on polynomial rings: Cf. Nagata [45], Morita [43], Lang [38] for general references.

Theorem 2.2.1 (Gauss) *The polynomial ring of a finite number of variables over a unique factorization domain is again a unique factorization domain.*

Theorem 2.2.2 (Hilbert) *The polynomial ring of a finite number of variables over a Noetherian ring is again Noetherian.*

A module M over a ring A is said to be Noetherian if every submodule of M is finitely generated over A.

Lemma 2.2.3 *If a ring A is Noetherian, then so is the p-th Cartesian product A^p for $p \in \mathbf{N}$.*

Proof We use induction on p. By the assumption, the case of $p = 1$ is trivial.

Suppose that $p > 1$ and that it folds for $p - 1$. Let $M \subset A^p$ be a submodule, and let

$$\pi : A^p \to A$$

be the projection for the first factor. The submodule $\pi(M) \subset A$ is generated by finitely many elements $u_i \in \pi(M)$, $1 \le i \le k$. Take $U_i \in \pi^{-1}u_i$ $(1 \le i \le k)$. Then it suffices to show that $M \cap \operatorname{Ker}\pi$ is finitely generated. Since $\operatorname{Ker}\pi \cong A^{p-1}$, the induction hypothesis implies the finite generation of $M \cap \operatorname{Ker}\pi$. □

We consider a polynomial ring $A[X]$ of a variable X over an integral domain A. Let $f(X)$ and $g(X)$ be two elements of $A[X]$ given by

$$\begin{aligned} &f(X) = a_0X^m + a_1X^{m-1} + \cdots + a_m, \quad m \ge 1,\\ &g(X) = b_0X^n + b_1X^{n-1} + \cdots + b_n, \quad n \ge 1,\\ &a_0b_0 \ne 0. \end{aligned}$$

We define the *resultant* $R(f, g) \in A$ of $f(X)$ and $g(X)$ to be the $(m+n)$-determinant

$$R(f,g) = \begin{vmatrix} a_0 & a_1 & \cdots\cdots & a_m & & & \\ & a_0 & a_1 & \cdots\cdots & a_m & & \\ & & \ddots & & & \ddots & \\ & & & a_0 & a_1 & \cdots\cdots & a_m \\ b_0 & b_1 & \cdots\cdots & b_n & & & \\ & b_0 & b_1 & \cdots\cdots & b_n & & \\ & & \ddots & & & \ddots & \\ & & & b_0 & b_1 & \cdots\cdots & b_n \end{vmatrix} \begin{matrix} \\ n\text{-tuple} \\ \\ \\ m\text{-tuple} \\ \\ \end{matrix} . \tag{2.2.4}$$

Here, the blank spaces are supposed to be filled with zeros.

Theorem 2.2.5 *For $f(X)$ and $g(X)$ above, there are $\varphi(X), \psi(X) \in A[X]$ such that $\deg \varphi < n$, $\deg \psi < m$, and*

$$\varphi(X)f(X) + \psi(X)g(X) = R(f, g) \ (\in A).$$

Proof In (2.2.4), multiply the jth column by X^{m+n-j} and then add it to the last column for $j = 1, 2, \ldots, m+n-1$. Then we have

(2.2.6)
$$R(f,g) = \begin{vmatrix} a_0 & a_1 & \cdots\cdots & a_m & & & X^{n-1}f(X) \\ & a_0 & a_1 & \cdots\cdots & a_m & & X^{n-2}f(X) \\ & & \ddots & & & \ddots & \vdots \\ & & & a_0 & a_1 & \cdots\cdots & f(X) \\ b_0 & b_1 & \cdots\cdots & b_n & & & X^{m-1}g(X) \\ & b_0 & b_1 & \cdots\cdots & b_n & & X^{m-2}g(X) \\ & & \ddots & & & \ddots & \vdots \\ & & & b_0 & b_1 & \cdots\cdots & g(X) \end{vmatrix}.$$

Expanding this with respect to the last column, we get the required $\varphi(x)$ and $\psi(x)$. □

Let K be the algebraic closure of the quotient field of A.

Theorem 2.2.7 (i) *Two equations* $f(X) = 0$ *and* $g(X) = 0$ *share a common root in* K *if and only if* $R(f,g) = 0$.
(ii) *Let* A *be a unique factorization domain, Then* $f(X)$ *and* $g(X)$ *share a common prime factor if and only if* $R(f,g) = 0$.

Proof (i) Let $\alpha \in K$ be a common zero of f and g. Then Theorem 2.2.5 with the substitution $X = \alpha$ implies that

$$A \ni R(f,g) = \varphi(\alpha)f(\alpha) + \psi(\alpha)g(\alpha) = 0 \text{ in } K.$$

Since $A \hookrightarrow K$, $R(f,g) = 0$ in A.

On the other hand, if $R(f,g) = 0$, then $\varphi(X)f(X) = -\psi(X)g(X)$. Since $\deg\varphi < n$, some root of $g(X) = 0$ in K must be a root of $f(X) = 0$.

(ii) By the assumption and Theorem 2.2.1 $A[X]$ is a unique factorization domain. Suppose that $f(X)$ and $g(X)$ share a common prime factor $h(X)$. Then by Theorem 2.2.5, $h(X)$ must be a prime factor of $R(f,g) \in A$; this is absurd unless $R(f,g) = 0$. If $R(f,g) = 0$, then $\varphi(X)f(X) = -\psi(X)g(X)$. By the degree comparison as in (i), it is impossible that all prime factors of g are those of φ with counting multiplicities; i.e., there is a prime factor of g which divides $f(X)$. □

We take the roots $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$ in K of $f(X) = 0$ and $g(X) = 0$, respectively with counting multiplicities. Then,

(2.2.8)
$$f(X) = a_0 \prod_{i=1}^{m} (X - \alpha_i),$$

(2.2.9)
$$g(X) = b_0 \prod_{j=1}^{n} (X - \beta_j).$$

Lemma 2.2.10 *We have*

$$R(f, g) = a_0^n b_0^m \prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i - \beta_j) = a_0^n \prod_{i=1}^{m} g(\alpha_i) = b_0^m \prod_{j=1}^{n} f(\beta_j).$$

Proof Note that a_i/a_0 (resp. b_j/b_0) are expressed by elementary symmetric polynomials of $\alpha_1, \ldots, \alpha_m$ (resp. $\beta_1, \ldots, \beta_n$). It is also noted that $R(f, g)$ is a homogeneous polynomial of degree n (resp. m) in a_i (resp. b_j). Therefore, $R(f, g)$ is a polynomial in elementary symmetric polynomials in α_i and β_j multiplied with $a_0^n b_0^m$.

Consider α_i, β_j as undetermined elements. If $\alpha_i = \beta_j$, then f and g have a common factor $(x - \alpha_i)$, and so by Theorem 2.2.7, $R(f, g) = 0$. Therefore, as a polynomial in α_i and β_j, $R(f, g)$ can be divided by $(\alpha_i - \beta_j)$. It follows that $R(f, g)$ can be divided by $\prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i - \beta_j)$. Now we check the coefficient of $(\beta_1 \cdots \beta_n)^n$ in each of $R(f, g)$ and $\prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i - \beta_j)$. Note that

$$\frac{b_n}{b_0} = (-1)^n \beta_1 \cdots \beta_n,$$

$$R(f, g) = a_0^n b_0^m \left(\left(\frac{b_n}{b_0} \right)^m + \cdots \right) = a_0^n b_0^m \left((-1)^{mn} (\beta_1 \cdots \beta_n)^m + \cdots \right),$$

$$\prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i - \beta_j) = (-1)^{mn} (\beta_1 \cdots \beta_n)^m + \cdots .$$

Thus we get the required coefficient $a_0^n b_0^m$ in the first equality. The rest follows from (2.2.8) and (2.2.9). □

The *discriminant* of $f(X)$ is defined by

$$\Delta(f) = a_0^{2m-2} \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

The formal derivative of $f(X)$ is defined by

$$\begin{aligned} f'(X) &= m a_0 X^{m-1} + \cdots + a_{m-1} \\ &= a_0 \sum_{i=1}^{m} (X - \alpha_1) \cdots (X - \alpha_{i-1})(X - \alpha_{i+1}) \cdots (X - \alpha_m), \end{aligned}$$

where $(X - \alpha_0) = (X - \alpha_{m+1}) = 1$ as convention. Then we have:

Theorem 2.2.11 $R(f, f') = (-1)^{\frac{m(m-1)}{2}} a_0 \Delta(f)$.

Proof Since $f'(\alpha_i) = a_0 \prod_{j \neq i} (\alpha_i - \alpha_j)$, it follows from Lemma 2.2.10 that

$$R(f, f') = a_0^{m-1} \prod_{i=1}^{m} f'(\alpha_i) = a_0^{2m-1} \prod_{i=1}^{m} \prod_{j \neq i} (\alpha_i - \alpha_j)$$
$$= (-1)^{\frac{m(m-1)}{2}} a_0 \Delta(f). \qquad \square$$

2.2.2 Properties of $\mathscr{O}_{n,a}$

Theorem 2.2.12 *The ring $\mathscr{O}_{n,a}$ is a unique factorization domain.*

Proof We may assume $a = 0$. We proceed by induction on n.

(1) Assume that $n = 1$. Every $\underline{f}_0 \in \mathscr{O}_{1,0}$, $\underline{f}_0 \neq 0$ is uniquely represented in a neighborhood of 0 as

$$f(z) = z^p h(z), \qquad p \in \mathbf{Z}^+, \; h(0) \neq 0, \tag{2.2.13}$$

where $\underline{h}_0$ is a unit. The germ $\underline{z}_0$ is a prime element, and p is uniquely determined by $\underline{f}_0$. Therefore, $\mathscr{O}_{1,0}$ is a unique factorization domain.

(2) Suppose $n \geq 2$ and that the statement holds for $n - 1$. By Weierstrass' Preparation Theorem 2.1.3, every $\underline{f}_0 \in \mathscr{O}_{n,0}$ is reduced to a Weierstrass polynomial up to a unit:

$$\underline{f}_0 = z_n^p + \sum_{\nu=1}^{p} \underline{a_\nu}_0 \cdot z_n^{p-\nu} \in \mathscr{O}_{n-1,0}[z_n].$$

By the induction hypothesis, $\mathscr{O}_{n-1,0}$ is a unique factorization domain, and then by Theorem 2.2.1, the polynomial ring $\mathscr{O}_{n-1,0}[z_n]$ over it is a unique factorization domain. It remains to show the equivalence between the reducibility or irreducibility in $\mathscr{O}_{n-1,0}[z_n]$ and that in $\mathscr{O}_{n,0}$. We prove:

Lemma 2.2.14 *Let $\underline{f}_0 \in \mathscr{O}_{n-1,0}[z_n]$ be a Weierstrass polynomial. Assume that there are elements $\underline{g}_0, \underline{h}_0 \in \mathscr{O}_{n,0}$ satisfying $\underline{f}_0 = \underline{h}_0 \cdot \underline{g}_0$. Then there exist Weierstrass polynomials $W_1(z', z_n)$, $W_2(z', z_n)$ such that in a neighborhood of* 0*,*

$$f(z', z_n) = W_1(z', z_n) \cdot W_2(z', z_n), \quad \frac{g}{W_1} \cdot \frac{h}{W_2} = 1.$$

$\because$) We may assume that g and h are non-units. We take a polydisk $\mathrm{P}\Delta$ so that $f(z', z_n) = g(z', z_n) \cdot h(z', z_n) \in \mathscr{O}(\overline{\mathrm{P}\Delta})$. Note that $f(0, z_n) = z_n^p \not\equiv 0$. We may take $\mathrm{P}\Delta$ to be the standard polydisk of f. Then,

$$g(0, z_n) \not\equiv 0, \qquad h(0, z_n) \not\equiv 0.$$

Therefore we may assume that PΔ is the standard polydisk for g and h. Let

$$g(z', z_n) = u_1 W_1(z', z_n), \qquad h(z', z_n) = u_2 W_2(z', z_n).$$

be the Weierstrass decompositions of g and h at 0, respectively. We see that

$$f(z', z_n) = u_1 u_2 W_1(z', z_n) W_2(z', z_n),$$

and that $W_1 W_2$ is a Weierstrass polynomial. By the uniqueness of Weierstrass decomposition

$$u_1 u_2 = 1, \qquad f = W_1 W_2. \qquad \triangle$$

Continued Proof of Theorem 2.2.12: Lemma 2.2.14 implies that if a Weierstrass polynomial $\underline{f}_0$ is reducible in $\mathscr{O}_{n,0}$, then it is reducible in $\mathscr{O}_{n-1,0}[z_n]$, and the factors are again Weierstrass polynomials. $\square$

The following lemma will be needed later.

Lemma 2.2.15 *Let $Q(z', z_n) \in \mathscr{O}_{n-1,0}[z_n]$ be a Weierstrass polynomial, and let $R \in \mathscr{O}_{n-1,0}[z_n]$. If $R = Q \cdot \underline{g}_0$ with $\underline{g}_0 \in \mathscr{O}_{n,0}$, then $\underline{g}_0 \in \mathscr{O}_{n-1,0}[z_n]$.*

Proof Assume that all the functions above are holomorphic in a neighborhood of a closed polydisk $\overline{\mathrm{P}\Delta}$. Since the leading coefficient of $Q(z', z_n)$ as a polynomial in z_n is 1, the Euclidean algorithm implies that

$$R = \varphi Q + \psi, \qquad \varphi, \psi \in \mathscr{O}_{n-1,0}[z_n], \qquad \deg_{z_n} \psi < p = \deg_{z_n} Q. \tag{2.2.16}$$

We may assume that $\mathrm{P}\Delta = \mathrm{P}\Delta_{n-1} \times \Delta_{(n)}$ ($\Delta_{(n)} = \{|z_n| < r_n\}$) is a standard polydisk for Q. With every $z' \in \mathrm{P}\Delta_{n-1}$ fixed, $Q(z', z_n) = 0$ has p zeros with counting multiplicities. Therefore, R has at least p zeros with counting multiplicities. By (2.2.16) ψ has at least p zeros with counting multiplicities, too. Since $\deg \psi < p$, $\psi \equiv 0$. Therefore we obtain

$$R = \varphi Q = gQ,$$
$$(\varphi - g)Q = 0, \qquad Q \neq 0.$$

Since $\mathscr{O}_{n,0}$ is an integral domain, $\varphi - g = 0$. Thus we see that $g = \varphi \in \mathscr{O}_{n-1,0}[z_n]$. $\square$

Lemma 2.2.17 *Let f, g be holomorphic functions in a neighborhood $0 \in \mathbf{C}^n$. If $\underline{f}_0$ and $\underline{g}_0$ are mutually prime (i.e., $(\underline{f}_0, \underline{g}_0) = 1$), then there is a neighborhood U of 0 such that $(\underline{f}_b, \underline{g}_b) = 1$ for $b \in U$.*

Proof Let $\mathrm{P}\Delta = \mathrm{P}\Delta_{n-1} \times \Delta_{(n)}$ ($\Delta_{(n)} = \Delta(0, r_n)$) be a standard polydisk for f and g. Then there are Weierstrass polynomials $P(z', z_n)$, $Q(z', z_n)$ and units $u, v \in \mathscr{O}(\overline{\mathrm{P}\Delta})$ such that

$$f = uP, \qquad g = vQ. \tag{2.2.18}$$

It follows from the assumption that $\underline{P}_0$ and $\underline{Q}_0$ are mutually prime, By Lemma 2.2.14 this is equivalent to saying that they are mutually prime as elements of the ring $\mathscr{O}_{n-1,0}[z_n]$. By Theorem 2.2.7 this is again equivalent to the resultant

$$R(P(z', z_n), Q(z', z_n)) = R(z') \neq 0 \in \mathscr{O}_{n-1,0};$$

therefore $\underline{R}_{b'} \neq 0$ at every $b' \in \mathrm{P}\Delta_{n-1}$. This implies that P and Q are mutually prime in $\mathscr{O}_{n-1,b'}[z_n]$. Thus, $\underline{f}_b$ and $\underline{g}_b$ are mutually prime at every $b \in \mathrm{P}\Delta$. □

From this lemma the next follows immediately:

Proposition 2.2.19 *Let $U \subset \mathbf{C}^n$ be an open set and let $f, g \in \mathscr{O}(U)$. Then, $\{a \in U; (\underline{f}_a, \underline{g}_a) = 1\}$ is an open subset.*

N.B. Even if $\underline{f}_a$ is irreducible, $\underline{f}_b$ is not necessarily irreducible for b in a neighborhood of a. For example, we take $f(x, y, z) = x^2 - zy^2$. This is irreducible at the origin, but at any point $w = (0, 0, c)$ with $c \neq 0$ it is factorized as $\underline{f}_w = \underline{(x + \sqrt{z}y)}_w \cdot \underline{(x - \sqrt{z}y)}_w$.

Theorem 2.2.20 *The ring $\mathscr{O}_{n,a}$ is noetherian, and so is $\mathscr{O}_{n,a}^p$ ($p \geq 2$).*

Proof If $\mathscr{O}_{n,a}$ is noetherian, Lemma 2.2.3 implies the latter half.

We show that $\mathscr{O}_{n,a}$ is noetherian. We set $a = 0$ and use induction on n.

(1) $n = 1$. We take any ideal $\mathscr{I} \subset \mathscr{O}_{1,0}$. Assume that $\mathscr{I} \neq \{0\}, \mathscr{O}_{1,0}$. We decompose $\underline{f}_0 \in \mathscr{I} \setminus \{0\}$ as (2.2.13), and denote the least p by the same p. Then,

$$\mathscr{I} = z^p \cdot \mathscr{O}_{1,0}.$$

That is, $\mathscr{I}$ is a principal ideal. Let $\mathscr{I}_1 \subset \mathscr{I}_2 \subset \mathscr{I}_3 \subset \cdots$ be a sequence of increasing ideals. Then we have that

$$\mathscr{I}_\nu = z^{p_\nu} \mathscr{O}_{1,0}, \quad p_1 \geq p_2 \geq p_3 \geq \cdots.$$

Therefore there is a natural number ν_0 such that $p_{\nu_0} = p_{\nu_0+1} = \cdots$. Therefore the sequence stabilizes at $\mathscr{I}_{\nu_0} = \mathscr{I}_\nu$, $\nu \geq \nu_0$.

(2) $n \geq 2$. Assume that it holds for $n-1$; i.e., $\mathscr{O}_{n-1,0}$ is noetherian. Theorem 2.2.2 implies that $\mathscr{O}_{n-1,0}[z_n]$ is noetherian. Let

$$\mathscr{I}_1 \subset \mathscr{I}_2 \subset \cdots \subset \mathscr{I}_\nu \subset \cdots$$

be an increasing sequence of ideals of $\mathscr{O}_{n,0}$. We show that there is a number $\nu_0 \in \mathbf{N}$ at which it stabilizes, $\mathscr{I}_{\nu_0} = \mathscr{I}_\nu$ ($^\forall \nu \geq \nu_0$).

Suppose that there exists no such ν_0. After taking a subsequence, we get

$$\mathscr{I}_1 \subsetneqq \mathscr{I}_2 \subsetneqq \cdots \subsetneqq \mathscr{I}_\nu \subsetneqq \cdots.$$

For every ν we take $\underline{f_\nu}_0 \in \mathscr{I}_\nu \setminus \mathscr{I}_{\nu-1}$ (we put $\mathscr{I}_0 = \{0\}$). By Remark 2.1.2 (iii) there is a standard coordinate system $(z', z_n) \in \mathbf{C}^{n-1} \times \mathbf{C}$ for all f_ν. Let

$$\underline{f_\nu}_0 = \underline{u_\nu}_0 \cdot W_\nu(z', z_n), \quad \nu = 1, 2, \ldots$$

be Weierstrass decompositions of $\underline{f_\nu}_0$ at 0. We may assume that $\underline{f_\nu}_0 = W_\nu(z', z_n)$, and then set

$$\mathscr{I}'_\nu = \sum_{\mu=1}^{\nu} W_\mu(z', z_n) \cdot \mathscr{O}_{n-1,0}[z_n],$$

which are ideals of $\mathscr{O}_{n-1,0}[z_n]$. By the definition, $\mathscr{I}'_\nu \subsetneqq \mathscr{I}'_{\nu+1}$, $\nu = 1, 2 \ldots$ Since $\mathscr{O}_{n-1,0}[z_n]$ is Noetherian, this is a contradiction. □

2.3 Analytic Subsets

Here we give the definition of an analytic subset, and study the elementary properties. Let $U \subset \mathbf{C}^n$ be an open set. We begin with a definition:

Definition 2.3.1 A subset $A \subset U$ is called an *analytic subset* or an *analytic set* if for every point $a \in U$ there are a neighborhood $V \subset U$ and finitely many holomorphic functions $f_j \in \mathscr{O}(V)$, $1 \leq j \leq l$, satisfying

$$A \cap V = \{z \in V;\ f_1(z) = \cdots = f_l(z) = 0\}.$$

By definition, analytic sets are closed.

Remark 2.3.2 Let $n = 1$ and let U be a domain. Then, a subset of U is analytic if and only if either it is U itself, or it is discrete without accumulation point in U.

Theorem 2.3.3 *Let U be a domain. If an analytic subset $A \subset U$ contains an interior point, then $U = A$.*

Proof Let A' be the subset of all interior points of A. Then, $A' \neq \emptyset$ and open in U. Take a point $a \in \overline{A'} \cap U$. Then there are a connected neighborhood V of a in U, and finitely many holomorphic functions $f_j \in \mathscr{O}(V)$, $1 \leq j \leq l$, such that

$$A \cap V = \{f_1 = \cdots = f_l = 0\}.$$

There exists $b \in V \cap A'$. The definition implies an existence of a neighborhood $W \subset A \cap V$ of b with $W \cap A = W$. That is, $f_j|_W(z) \equiv 0$, $1 \le j \le l$. By the Identity Theorem 1.2.14, $f_j(z) \equiv 0$, $1 \le j \le l$. Therefore $V \cap A = V$, and $a \in A'$. We saw that $A'(\subset U)$ is open and closed. Since U is connected, $A' = U$ follows. The inclusion relations, $A \supset A' = U \supset A$, imply $A = U$. □

Theorem 2.3.4 (Riemann's Extension Theorem) *Let U be a domain of $\mathbf{C}^n$ and let $A \subsetneq U$ be a proper analytic subset. Let $f \in \mathscr{O}(U \setminus A)$ be a holomorphic function bounded around every point of A; i.e., there is a neighborhood V of a such that the restriction $f|_{V \setminus A}$ is bounded. Then, f is uniquely extended holomorphically over the whole U.*

Proof The case of $n = 1$ holds by Remark 2.3.2 and Theorem 1.1.11.

Let $n \ge 2$. Take any $a \in A$. By a translation we may assume $a = 0$. By the assumption there are a neighborhood V of 0 and $\phi \in \mathscr{O}(V) \setminus \{0\}$ such that

$$A \cap V \subset \{\phi = 0\}.$$

We may assume that V is a standard polydisk $\mathrm{P}\Delta = \mathrm{P}\Delta_{n-1} \times \Delta(0; r_n)$ for ϕ. For $z' \in \mathrm{P}\Delta_{n-1}$ with $|z_n| = r_n$, $\phi(z', z_n) \ne 0$; i.e., $(\mathrm{P}\Delta_{n-1} \times \{|z_n| = r_n\}) \cap A = \emptyset$. With fixed $z' \in \mathrm{P}\Delta_{n-1}$, $\phi(z', z_n) = 0$ has at most finitely many zeros in $\Delta(0; r_n)$, around which $f(z', z_n)$ is bounded. Theorem 1.1.11 implies that it is holomorphic over $\Delta(0; r_n)$. Therefore we may write

$$f(z', z_n) = \frac{1}{2\pi i} \int_{|\zeta_n| = r_n} \frac{f(z', \zeta_n)}{\zeta_n - z_n} d\zeta_n, \quad |z_n| < r_n.$$

Since $f(z', \zeta_n)$ with $|\zeta_n| = r_n$ is holomorphic in z', The above integral expression implies that $f(z', z_n)$ is, in fact, holomorphic in $\mathrm{P}\Delta$. □

Theorem 2.3.5 *Let U be a domain of $\mathbf{C}^n$ and let $A \subsetneq U$ be a proper analytic subset. Then $U \setminus A$ is a domain.*

Proof Suppose that $U \setminus A$ is not connected. Then there are non-empty open subsets V_1, V_2 of U such that

$$U \setminus A = V_1 \cup V_2, \qquad V_1 \cap V_2 = \emptyset.$$

Define $f \in \mathscr{O}(U \setminus A)$ as follows:

$$f(z) = \begin{cases} 0, & z \in V_1, \\ 1, & z \in V_2. \end{cases}$$

By Theorem 2.3.4, f is uniquely extended to $\tilde{f} \in \mathscr{O}(U)$. Since $\tilde{f}|_{V_1} \equiv 0$, the Identity Theorem 1.2.14 implies $\tilde{f}(z) \equiv 0$, and this is a contradiction. □

Definition 2.3.6 (Geometric ideal sheaf) Let $U \subset \mathbf{C}^n$ be an open subset, and let $A \subset U$ be an analytic subset. Putting $E = A$ in Sect. 1.3.3 (5), we define the *ideal sheaf* $\mathscr{I}\langle A\rangle$ of the analytic subset A. We call $\mathscr{I}\langle A\rangle_z (\subset \mathscr{O}_{U,z})$ the *geometric ideal* of the germ $\underline{A}_z$ of the analytic subset A at $z \in U$. At $z \in U \setminus A$ the stalk $\mathscr{I}\langle A\rangle_z = \mathscr{O}_{U,z}$. In general, the ideal sheaf of an analytic subset is called a *geometric ideal sheaf* (*idéal géométrique de domaines indéterminés*[2]).

2.4 Coherent Sheaves

We begin with the definition of coherent sheaves in general.

Let X be a topological space, let $\mathscr{A} \to X$ be a sheaf of rings and let $\mathscr{S} \to X$ be a sheaf of $\mathscr{A}$-modules.

Definition 2.4.1 The sheaf $\mathscr{S}$ of $\mathscr{A}$-modules is said to be *locally finite* over $\mathscr{A}$ if for every $x \in X$ there exist a neighborhood $U \ni x$ and a finite number of sections $\sigma_j \in \Gamma(U, \mathscr{S})$, $1 \le j \le l$, satisfying

$$\mathscr{S}|_U = \sum_{j=1}^{l} \mathscr{A}|_U \cdot \sigma_j,$$

that is,

$$\mathscr{S}_y = \sum_{j=1}^{l} \mathscr{A}_y \cdot \sigma_j(y), \qquad {}^\forall y \in U.$$

In this case, $\{\sigma_j\}_{j=1}^l$ is called a *(locally) finite generator system* of the sheaf $\mathscr{S}$ of $\mathscr{A}$-modules on U, and we say that $\mathscr{S}$ is generated by $\{\sigma_j\}_{j=1}^l$ or by σ_j $(\in \Gamma(U, \mathscr{S}))$, $1 \le j \le l$, in U.

Definition 2.4.2 (*Relation sheaf*) A *relation sheaf* of $\mathscr{S}$ is a sheaf $\mathscr{R}((\tau_j)_{1\le j\le q})$ as follows:

(i) Let $U \subset X$ be an open subset.
(ii) Let $\tau_j \in \Gamma(U, \mathscr{S})$, $1 \le j \le q (< \infty)$, be finitely many sections.
(iii) Let $\mathscr{R}(\tau_1, \ldots, \tau_q) = \mathscr{R}((\tau_j)_{1\le j\le q}) \subset (\mathscr{A}|_U)^q$ be a sheaf of $\mathscr{A}|_U$-modules defined by

$$\mathscr{R}((\tau_j)_{1\le j\le q}) = \bigcup_{x\in U} \left\{ \big(a_1, \ldots, a_q\big) \in (\mathscr{A}_x)^q \,;\ \sum_{j=1}^{q} a_j \tau_j(x) = 0 \right\}. \tag{2.4.3}$$

[2] K. Oka referred to the notion in this way in Oka VII.

In the case when a finite number of linear relations (2.4.3) defined by $\tau_{(\lambda)} = (\tau_{(\lambda)j})_{1\le j\le q}$, $\lambda = 1, \ldots, l(<\infty)$, are imposed, we call

$$\mathscr{R}\left((\tau_{(\lambda)j})_{1\le j\le q};\ 1\le\lambda\le l\right) = \bigcap_{\lambda=1}^{l}\mathscr{R}\left((\tau_{(\lambda)j})_{1\le j\le q}\right)$$

a *simultaneous relation sheaf*; this is a relation sheaf of the product sheaf $\mathscr{S}^l$.

Definition 2.4.4 (*Coherence*) We say that a sheaf $\mathscr{S}$ of $\mathscr{A}$-modules is a *coherent sheaf* of ($\mathscr{A}$-) modules over $\mathscr{A}$, or *coherent over* $\mathscr{A}$ if the following two conditions are satisfied:

(i) $\mathscr{S}$ is locally finite over $\mathscr{A}$.
(ii) Every relation sheaf of $\mathscr{S}$ is locally finite over $\mathscr{A}$.

N.B. In the definition of coherence, the coherence of the base sheaf of rings, $\mathscr{A}$ itself is not required.

Definition 2.4.5 (*Coherent sheaf*) Let $\Omega \subset \mathbf{C}^n$ be an open subset. We call a coherent sheaf of modules over $\mathscr{O}_\Omega$ simply a *coherent sheaf* over Ω.

N.B. In some references a coherent sheaf defined as above is called a "coherent analytic sheaf". In this book, a "coherent sheaf" over a complex domain means a coherent sheaf of modules over a sheaf of germs of holomorphic functions, unless otherwise mentioned.

We first study the general properties.

Proposition 2.4.6 (Point–local generation) *Let $\mathscr{S}$ be a locally finite sheaf of $\mathscr{A}$-modules over X. If finitely many elements $\underline{\gamma_j}_a \in \mathscr{S}_a$, $1 \le j \le l$, generate $\mathscr{S}_a$ over $\mathscr{A}_a$ at a point $a \in X$, then there is a neighborhood $U \ni a$ such that each $\underline{\gamma_j}_a$ has a representative $\gamma_j \in \Gamma(U, \mathscr{S})$, and the equality*

$$\mathscr{S}_x = \sum_{j=1}^{l} \mathscr{A}_x \cdot \gamma_j(x), \qquad {}^\forall x \in U$$

holds; i.e., γ_j, $1 \le j \le l$, generate $\mathscr{S}$ over U. In particular, this holds for a coherent sheaf $\mathscr{S}$ of modules over $\mathscr{A}$.

Proof Because of the local finiteness of $\mathscr{S}$ there is a locally finite generator system $\{\sigma_k\}_{k=1}^m \subset \Gamma(V, \mathscr{S})$ over a neighborhood $V(\ni a)$. By assumption we may write

$$\sigma_k(a) = \sum_{j=1}^{l} \underline{f_{kj}}_a \underline{\gamma_j}_a, \quad \underline{f_{kj}}_a \in \mathscr{A}_a,\ 1 \le k \le m.$$

Take a neighborhood $U \subset V$ of a such that all $\underline{\gamma_j}_a$ and $\underline{f_{kj}}_a$ have the representatives γ_j, f_{kj} over V. After shrinking U if necessary, it follows from Proposition 1.3.4 (ii) that

$$\sigma_k(x) = \sum_{j=1}^{l} \underline{f_{kj}}_x \gamma_j(x), \quad {}^\forall x \in U, \ 1 \le k \le m.$$

Since $\{\sigma_k(x)\}$ generates $\mathscr{S}_x$ over $\mathscr{A}_x$ $(x \in U)$, $\{\gamma_j(x)\}$ generates $\mathscr{S}_x$ over $\mathscr{A}_x$. □

Proposition 2.4.7 *Let $\mathscr{S}$ be a coherent sheaf of modules over $\mathscr{A}$.*

(i) *A subsheaf of $\mathscr{S}$ (i.e., subsheaf of $\mathscr{A}$-submodules of $\mathscr{S}$) is coherent over $\mathscr{A}$ if and only if it is locally finite over $\mathscr{A}$.*
(ii) *$\mathscr{S}^N$ $(N = 2, 3, \ldots)$ are coherent over $\mathscr{A}$. That is, every simultaneous relation sheaf of $\mathscr{S}$ is coherent over $\mathscr{A}$.*

Proof (i) It remains to show the local finiteness of relation sheaves, but they are so since $\mathscr{S}$ is coherent.

(ii) We use induction on N. The case of $N = 1$ is the assumption.

Let $N \ge 2$, and suppose that it holds for $N-1$. The local finiteness of $\mathscr{S}^N$ follows from that of $\mathscr{S}$. Let $U \subset X$ be an open subset, and let $F_i \in \Gamma(U, \mathscr{S}^N)$, $1 \le i \le q$, be a finite number of sections. It suffices to show the local finiteness of the relation sheaf

$$\mathscr{R} = \left\{ (a_i) \in \mathscr{A}_x^q \subset \mathscr{A}^q; \ \sum_{i=1}^{q} a_i \underline{F_i}_x = 0, \ x \in U \right\}.$$

With the expressions $F_i = (F_{i1}, \ldots, F_{iN})$, $\mathscr{R}$ $(\subset (\mathscr{A}|_U)^q)$ is determined by

$$(a_i) \in \mathscr{A}_x^q, \quad \sum_{i=1}^{q} a_i \underline{F_{ij}}_x = 0, \quad 1 \le j \le N, \ x \in U.$$

We first consider the case for $j = 1$. Denote by $\mathscr{R}_1 \subset (\mathscr{A}|_U)^q$ the relation sheaf defined by

$$(a_i) \in \mathscr{A}_x^q, \quad \sum_{i=1}^{q} a_i \underline{F_{i1}}_x = 0, \quad x \in U.$$

Then $\mathscr{R} \subset \mathscr{R}_1$, and since $\mathscr{S}$ is coherent over $\mathscr{A}$, $\mathscr{R}_1$ is locally finite over $\mathscr{A}$. For every point $a \in U$ there are a neighborhood $V \subset U$ of a and a locally finite generator system $\{\phi^{(\lambda)}\}_{\lambda=1}^{L}$ of $\mathscr{R}_1|_V$ with $\phi^{(\lambda)} \in \Gamma(V, \mathscr{R}_1)$. Set $\phi^{(\lambda)} = (\phi_i^{(\lambda)})_{1 \le i \le q}$. At every point $x \in V$ an element of $\mathscr{R}_{1x}$,

$$(a_i) = \left(\sum_{\lambda} \underline{c_\lambda}_x \cdot \phi_i^{(\lambda)}(x) \right), \qquad \underline{c_\lambda}_x \in \mathscr{A}_x$$

belongs to $\mathscr{R}_x$ if and only if

(2.4.8) $$\sum_i \sum_\lambda \underline{c_\lambda}_x \cdot \phi_i^{(\lambda)}(x) \cdot \underline{F_{ij}}_x = 0, \quad 1 \le j \le N.$$

We consider this as a linear relation on $\left(\underline{c_\lambda}_x\right)$. For $j = 1$ it is already satisfied because of the choice of $\phi_i^{(\lambda)}$. Therefore, simultaneous relation (2.4.8), in fact, consists of $N - 1$ relations. The induction hypothesis implies that in a neighborhood $W(\subset V)$ of a such $\left(\underline{c_\lambda}_x\right)$ is written by a linear sum of finitely many sections $\gamma^{(\nu)} = \left(\gamma_\lambda^{(\nu)}\right)$ with $\gamma_\lambda^{(\nu)} \in \Gamma(W, \mathscr{A})$. Therefore, the sections $(a_i^{(\nu)}) = \sum_\lambda \gamma_\lambda^{(\nu)} \cdot \phi_i^{(\lambda)}$ generate $\mathscr{R}_x$ at every $x \in W$ over $\mathscr{A}_x$. □

Proposition 2.4.9 *Let $\mathscr{S}$ be a coherent sheaf of modules over $\mathscr{A}$. If $\mathscr{F}_i \subset \mathscr{S}$, $i = 1, 2$, are coherent subsheaves of modules over $\mathscr{A}$, so is the intersection $\mathscr{F}_1 \cap \mathscr{F}_2$.*

Proof For every point $a \in X$ there are a neighborhood $U(\subset X)$ and locally finite generator systems of $\mathscr{F}_i$, $i = 1, 2$,

$$\alpha_j \in \Gamma(U, \mathscr{F}_1), \quad 1 \le j \le l,$$
$$\beta_k \in \Gamma(U, \mathscr{F}_2), \quad 1 \le k \le m.$$

At any point $b \in U$, $\gamma \in \mathscr{S}_b$ belongs to $\mathscr{F}_{1b} \cap \mathscr{F}_{2b}$ if and only if it is written as

$$\gamma = \sum_j a_j \alpha_j(b) = \sum_k b_k \beta_k(b),$$
$$a_j, b_k \in \mathscr{A}_b.$$

This is equivalent to

(2.4.10) $$\sum_j a_j \alpha_j(b) + \sum_k b_k(-\beta_k(b)) = 0,$$
$$\gamma = \sum_j a_j \alpha_j(b).$$

The above expression defines a relation sheaf of $\mathscr{S}$ with (a_j, b_k) being unknowns, which is denoted by $\mathscr{R} := \mathscr{R}(\ldots, \alpha_j, \ldots, -\beta_k, \ldots)$. Since $\mathscr{S}$ is coherent over $\mathscr{A}$, $\mathscr{R}$ is locally finite over $\mathscr{A}$. Taking a smaller $U \ni a$ if necessary, we may assume that $\mathscr{R}|_U$ is generated by a finite number of $\eta^{(h)} \in \Gamma(U, \mathscr{R})$, $1 \le h \le L$. With $\eta^{(h)} := (a_j^{(h)}, b_k^{(h)})$, $(\mathscr{F}_1 \cap \mathscr{F}_2)|_U$ is generated by

$$\xi^{(h)} := \sum_j a_j^{(h)} \alpha_j = \sum_k b_k^{(h)} \beta_k \in \Gamma(U, \mathscr{F}_1 \cap \mathscr{F}_2), \quad 1 \le h \le L.$$ □

Example 2.4.11 We introduce a non-coherent example given in Oka VII. Consider a hypersurface $X = \{z = w\}$ in $\mathbf{C}^2$ with variables z, w. Taking two concentric balls

$B_i = \{|z|^2 + |w|^2 < r_i^2\}$ $(r_1 < r_2)$, we set $X_0 = X \cap B_2 \setminus B_1$. Let $\Gamma_1 = \partial B_1$ be the boundary hypersphere. Let $\mathscr{B}(U)$ be the set of all holomorphic functions $f(z, w)$ on an open subset $U \subset B_2$ such that $f(z, w)/(z - w)$ is holomorphic at every point of $U \cap X_0$. With the natural restrictions $\{\mathscr{B}(U)\}$ forms a presheaf, which defines a sheaf $\mathscr{B}$ of $\mathscr{O}_{B_2}$-modules; in fact, the sheaf $\mathscr{B}$ is an ideal sheaf of $\mathscr{O}_{B_2}$. By the construction we have

$$\text{(2.4.12)} \qquad \mathscr{B}_a = \begin{cases} \mathscr{O}_{n,a} \cdot \underline{(z - w)}_a, & a \in X_0, \\ \mathscr{O}_{n,a}, & a \in B_2 \setminus X_0. \end{cases}$$

Then $\mathscr{B}$ is not locally finite in any neighborhood of a point $a \in X_0 \cap \Gamma_1$. Suppose that it is locally finite in a neighborhood of a. Then there are a neighborhood U of a and finitely many $f_j \in \mathscr{B}(U)$, $1 \le j \le N$, such that

$$\mathscr{B}_b = \sum_{j=1}^{N} \mathscr{O}_{n,b} \cdot \underline{f_j}_b, \quad {}^\forall b \in U.$$

But, because of $f_j(z, z) \equiv 0$, $\mathscr{B}_b \neq \mathscr{O}_{n,b}$ for $b \in U \cap X \setminus X_0$, which contradicts (2.4.12).

Example 2.4.13 We may construct a non-coherent relation sheaf as follows. We take an open ball B of $\mathbf{C}^n$ with center at the origin. Denote by Γ its boundary. Let $\chi(a)$ denote the characteristic function of B; that is, on B, $\chi = 1$, and $\chi = 0$ on $\mathbf{C}^n \setminus B$. Set

$$\mathscr{R} = \left\{ \underline{f}_a \in \mathscr{O}_{n,a};\ \underline{f}_a \cdot \underline{\chi}_a = 0, \quad a \in \mathbf{C}^n \right\}.$$

Then,

$$\mathscr{R}_a = \begin{cases} 0, & a \in B \cup \Gamma, \\ \mathscr{O}_{n,a}, & a \notin B \cup \Gamma. \end{cases}$$

Therefore $\mathscr{R}$ is not coherent about $a \in \Gamma$.

In the above example, the length of the relation is one. To make the length two, we set $\phi(a) = 1 - \chi(a)$ and

$$\mathscr{S} = \left\{ \underline{f}_a \oplus \underline{g}_a \in \mathscr{O}_{n,a} \oplus \mathscr{O}_{n,a};\ \underline{f}_a \cdot \underline{\chi}_a + \underline{g}_a \cdot \underline{\phi}_a = 0, \quad a \in \mathbf{C}^n \right\} \subset \mathscr{O}_n^2.$$

Then,

$$\mathscr{S}_a = \begin{cases} 0 \oplus \mathscr{O}_{n,a}, & a \in B, \\ 0 \oplus 0, & a \in \Gamma, \\ \mathscr{O}_{n,a} \oplus 0, & a \notin B \cup \Gamma. \end{cases}$$

Therefore, $\mathscr{S}$ is not locally finite about any point of Γ.

If one requires the differentiability for χ and ϕ, it suffices to take a C^∞ function such that

$$\chi(a) > 0, \quad a \in B; \ \chi(a) = 0, \quad a \notin B.$$

Then, the same conclusion is obtained.

Example 2.4.14 For examples of coherent sheaves, we have $\mathscr{O}_{\mathbf{C}^n}$ which we are going to show, and a geometric ideal sheaf $\mathrm{i}\langle A\rangle$ (cf. Definition 2.3.6 and Sect. 6.5), but the proofs are not easy.

2.5 Oka's First Coherence Theorem

The following theorem is called *Oka's Coherence Theorem*, but in the present book we call this *Oka's First Coherence Theorem*.[3] It is impossible to explain the meaning of this theorem in a few lines.

Reinhold Remmert, who is a well-known German complex analysist, describes it in *Encyclopedia of Mathematics* [69] published in 1994 as follows:

> It is no exaggeration to claim that Oka's theorem became a landmark in the development of function theory of several complex variables.

Theorem 2.5.1 (Oka's First Coherence Theorem 1948, Oka [62] VII) *The sheaf $\mathscr{O}^N_{\mathbf{C}^n}$ $(N \geq 1)$ is coherent.*

Proof We proceed by induction on $n \geq 0$ with general $N \geq 1$. We write $\mathscr{O}_{\mathbf{C}^n} = \mathscr{O}_n$.

(a) $n = 0$: In this case, it is a matter of a finite-dimensional vector space over $\mathbf{C}$.
(b) $n \geq 1$: Suppose that $\mathscr{O}^N_{n-1}$ is coherent for every $N \geq 1$.

By Proposition 2.4.7 (ii) it suffices to show the case of $N = 1$.

The problem is local and it is sufficient to prove Definition 2.4.4 (ii). Taking an open subset $\Omega \subset \mathbf{C}^n$ and $\tau_j \in \mathscr{O}(\Omega) \cong \Gamma(\Omega, \mathscr{O}_n)$, $1 \leq j \leq q$, we consider the relation sheaf $\mathscr{R}(\tau_1, \ldots, \tau_q)$ defined by

(2.5.2) $$\underline{f_1}_z \underline{\tau_1}_z + \cdots + \underline{f_q}_z \underline{\tau_q}_z = 0, \quad \underline{f_j}_z \in \mathscr{O}_{n,z}, \ z \in \Omega.$$

What we want to show is:

Claim 2.5.3 *For every point $a \in \Omega$ there are a neighborhood $V \subset \Omega$ of a and finitely many sections $s_k \in \Gamma(V, \mathscr{R}(\tau_1, \ldots, \tau_q))$, $1 \leq k \leq l$, such that*

$$\mathscr{R}(\tau_1, \ldots, \tau_q)_b = \sum_{k=1}^{l} \mathscr{O}_{n,b} \cdot s_k(b), \quad {}^\forall b \in V.$$

[3] For the reason, cf. "Historical supplements" at the end of this chapter and Chap. 9.

For the proof we may assume that $a = 0$. If an element $\underline{\tau_j}_0 = 0$, then the j-th component of $\mathscr{R}(\tau_1, \ldots, \tau_q)|_V \subset (\mathscr{O}_V)^q$ is just $\mathscr{O}_V$ in a neighborhood V of 0, and so we may assume that $\underline{\tau_j}_0 \neq 0$, $1 \leq j \leq q$.

By Theorem 2.2.12, $\mathscr{O}_{n,0}$ is a unique factorization domain. We may divide $\underline{\tau_j}_0$, $1 \leq j \leq q$, by the common factors, and may assume that there is no common factor among them (this procedure is not necessary for the proof itself, but may possibly decrease the computational complexity of the algorithm to obtain the finite generator system).

Let p_j be the order of zero of τ_j at 0, and set

$$p = \max_{1 \leq j \leq q} p_j,$$
$$p' = \min_{1 \leq j \leq q} p_j \geq 0.$$

After reordering the indices, we may assume that

$$p' = p_1.$$

The following sections are clearly solutions of (2.5.2) and are sections of $\mathscr{R}$:

$$t_i = (\tau_i, 0, \ldots, 0, \overset{i\text{-th}}{-\tau_1}, 0, \ldots, 0) \in \Gamma(\mathrm{P}\Delta, \mathscr{R}), \quad 2 \leq i \leq q, \tag{2.5.4}$$

which we call the *trivial solutions*.

Take a common standard polydisk $\mathrm{P}\Delta = \mathrm{P}\Delta_{n-1} \times \Delta_{(n)}$ with $\Delta_{(n)} = \{|z_n| < r_n\}$ for all τ_j. By Weierstrass's Preparation Theorem 2.1.3 at 0 one can transfer a unit factor of τ_j to f_j in (2.5.2) so that all τ_j may be assumed to be Weierstrass polynomials:

$$\tau_j = P_j(z', z_n) = \sum_{\nu=0}^{p_j} a_{j\nu}(z') z_n^\nu = \sum_{\nu=0}^{p} a_{j\nu}(z') z_n^\nu \in \mathscr{O}(\mathrm{P}\Delta_{n-1})[z_n], \tag{2.5.5}$$
$$a_{j\nu}(0) = 0 \ (\nu < p_j), \quad a_{jp_j} = 1, \quad a_{j\nu} = 0 \ (p_j < \nu \leq p).$$

If $p_j = 0$ ($\underline{\tau_j}_0$ is a unit), $P_j = 1$. We set

$$\mathscr{R} = \mathscr{R}(P_1, \ldots, P_q). \tag{2.5.6}$$

Here, the trivial solutions are

$$T_i = (P_i, 0, \ldots, 0, \overset{i\text{-th}}{-P_1}, 0, \ldots, 0) \in \Gamma(\mathrm{P}\Delta, \mathscr{R}), \quad 2 \leq i \leq q.$$

It suffices to show the local finiteness of $\mathscr{R}$. We perform a division algorithm for an unknown vector $\alpha = (\alpha_j) \in \mathscr{R}$ with respect to the trivial solutions T_i ($2 \leq i \leq q$);

more precisely, we perform a division algorithm for α_j with respect to P_1 (cf. (2.5.11) and Remark 2.5.23 below).

Take an arbitrary point $b = (b', b_n) \in \mathrm{P}\Delta_{n-1} \times \Delta_{(n)}$. We call an element of $\mathscr{O}_{n-1,b'}[z_n]$ a *z_n-polynomial-like germ*. In the same way, we call $\alpha = (\alpha_1, \ldots, \alpha_q) \in \mathscr{O}^q_{n,b}$ consisting of z_n-polynomial-like germs α_j a *polynomial-like element*, and $f = (f_j)$ with $(f_j)_{1 \le j \le q} \in (\mathscr{O}(\mathrm{P}\Delta_{n-1})[z_n])^q$ a *z_n-polynomial-like section*. We set

$$\deg \alpha = \deg_{z_n} \alpha = \max_j \deg_{z_n} \alpha_j,$$
$$\deg f = \deg_{z_n} f = \max_j \deg_{z_n} f_j.$$

Then we have:

2.5.7 The trivial solutions T_i are z_n-polynomial-like sections of $\deg T_i \le p$.

We now show the following:

Lemma 2.5.8 (Degree structure) *Let the notation be as above. Then an element of $\mathscr{R}_b$ is written as a finite linear sum of the trivial solutions, T_i, $2 \le i \le q$, and z_n-polynomial-like elements $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_q)$ of $\mathscr{R}_b$ with coefficients in $\mathscr{O}_{n,b}$ such that*

$$\deg \alpha_1 < p,$$
$$\deg \alpha_j < p', \quad 2 \le j \le q.$$

N.B. If $p' = 0$, then there is no term of α.

$\because$) By making use of Weierstrass' Preparation Theorem at b we decompose P_1 into a unit u and a Weierstrass polynomial Q:

$$P_1(z', z_n) = u \cdot Q(z', z_n - b_n), \qquad \deg Q = d \le p_1.$$

Lemma 2.2.15 implies $u \in \mathscr{O}_{n-1,b'}[z_n]$. Therefore,

$$\deg_{z_n} u = p_1 - d. \tag{2.5.9}$$

Take an arbitrary $f = \left(f_1, \ldots, f_q\right) \in \mathscr{R}_b$. By Weierstrass' Preparation Theorem 2.1.3 (ii) we have

$$\begin{aligned} f_i &= c_i Q + \beta_i, \quad 1 \le i \le q, \\ & c_i \in \mathscr{O}_{n,b}, \quad \beta_i \in \mathscr{O}_{n-1,b'}[z_n], \\ & \deg_{z_n} \beta_i \le d - 1. \end{aligned} \tag{2.5.10}$$

Since $u \in \mathscr{O}_{n,b}$ is a unit, with $\tilde{c}_i := c_i u^{-1}$ we get

$$f_i = \tilde{c}_i P_1 + \beta_i, \qquad 1 \le i \le q. \tag{2.5.11}$$

By making use of this we perform the following calculation:

$$
\begin{aligned}
(2.5.12)\qquad & (f_1, \dots, f_q) + \tilde{c}_2 T_2 + \cdots + \tilde{c}_q T_q \\
& = (\tilde{c}_1 P_1 + \beta_1, \tilde{c}_2 P_1 + \beta_2, \dots, \tilde{c}_q P_1 + \beta_q) \\
& \quad + (\tilde{c}_2 P_2, -\tilde{c}_2 P_1, 0, \dots, 0) \\
& \quad + \ \cdots \\
& \quad + (\tilde{c}_q P_q, 0, \dots, 0, -\tilde{c}_q P_1) \\
& = \left(\sum_{i=1}^{q} \tilde{c}_i P_i + \beta_1, \beta_2, \dots, \beta_q \right) \\
& = \left(g_1, \beta_2, \dots, \beta_q \right).
\end{aligned}
$$

Here we put $g_1 = \sum_{i=1}^{q} \tilde{c}_i P_i + \beta_1 \in \mathscr{O}_{n,b}$. Note that $\beta_i \in \mathscr{O}_{n-1,b'}[z_n],\ 2 \le i \le q$. Since $\left(g_1, \beta_2, \dots, \beta_q\right) \in \mathscr{R}_b$,

$$
(2.5.13)\qquad g_1 P_1 = -\beta_2 P_2 - \cdots - \beta_q P_q \in \mathscr{O}_{n-1,b'}[z_n].
$$

Remark 2.5.14 It should be noticed that if $p_1 = 0$, then $P_1 = 1$, $\beta_i = 0$, $1 \le i \le q$, and $g_1 = 0$; the proof is finished in this case.

In general, it follows from the expression of the right-hand side of (2.5.13) that

$$
\deg_{z_n} g_1 P_1 \le \max_{2 \le i \le q} \deg_{z_n} \beta_i + \max_{2 \le i \le q} \deg_{z_n} P_i \le d + p - 1.
$$

On the other hand, $g_1 P_1 = g_1 u Q$ and Q is a Weierstrass polynomial at b. Again by Lemma 2.2.15 we see that

$$
\begin{aligned}
& \alpha_1 := g_1 u \in \mathscr{O}_{n-1,b'}[z_n], \\
(2.5.15)\qquad & \deg_{z_n} \alpha_1 = \deg_{z_n} g_1 P_1 - \deg_{z_n} Q \\
& \qquad \le d + p - 1 - d = p - 1.
\end{aligned}
$$

Setting $\alpha_i = u\beta_i$ for $2 \le i \le q$. we have by (2.5.9) and (2.5.10) that

$$
(2.5.16)\qquad \deg_{z_n} \alpha_i \le p_1 - d + d - 1 = p_1 - 1 = p' - 1, \quad 2 \le i \le q,
$$

and then by (2.5.12) that

$$
(2.5.17)\qquad f = -\sum_{i=2}^{q} \tilde{c}_i T_i + u^{-1}(\alpha_1, \alpha_2, \dots, \alpha_q). \qquad \triangle
$$

Until now we have not used the induction hypothesis. Now we are going to use it to prove the existence of a locally finite generator system of those $(\alpha_1, \dots, \alpha_q)$ appearing in (2.5.17). We write

(2.5.18)
$$\alpha_1 = \sum_{\nu=0}^{p-1} \underline{c_{1\nu}(z')}_{b'} z_n^\nu, \quad \underline{c_{1\nu}(z')}_{b'} \in \mathscr{O}_{n-1,b'},$$
$$\alpha_i = \sum_{\nu=0}^{p'} \underline{c_{i\nu}(z')}_{b'} z_n^\nu, \quad \underline{c_{i\nu}(z')}_{b'} \in \mathscr{O}_{n-1,b'}, \ 2 \le i \le q.$$

By $\mathscr{S}$ we denote the sheaf of all these $(\alpha_1, \dots, \alpha_q)$ over $\mathrm{P}\Delta = \mathrm{P}\Delta_{n-1} \times \Delta_{(n)}$ satisfying

(2.5.19)
$$\alpha_1 P_1 + \alpha_2 P_2 + \cdots + \alpha_q P_q = 0.$$

The left-hand side above is a z_n-polynomial-like element of degree at most $p + p' - 1$, and relation (2.5.19) is equivalent to the nullity of all $p + p'$ coefficients. With the expression in (2.5.5) we have

(2.5.20)
$$\sum_{i=1}^{q} \sum_{k+h=\nu}{}' \underline{a_{ik}(z')}_{b'} \cdot \underline{c_{ih}(z')}_{b'} = 0 \in \mathscr{O}_{n-1,b'}, \quad 0 \le \nu \le p + p' - 1,$$

where $\sum'$ stands for the sum over those indices h, k to which some elements $\underline{a_{ik}(z')}_{b'}$, $\underline{c_{ih}(z')}_{b'}$ correspond. Then (2.5.20) defines a $(p + p')$-simultaneous linear relation sheaf $\widetilde{\mathscr{S}}$ in $\mathscr{O}_{\mathrm{P}\Delta_{n-1}}^{p+p'}$ with $p + p'(q - 1)$ unknowns, c_{ih}'s. The induction hypothesis implies that $\widetilde{\mathscr{S}}$ is coherent, and hence there is a locally finite generator system of $\widetilde{\mathscr{S}}$ over a polydisk neighborhood $\widetilde{\mathrm{P}\Delta}_{n-1} \subset \mathrm{P}\Delta_{n-1}$ of 0. Therefore we infer from (2.5.18) that $\mathscr{S}$ has a locally finite generator system $\{\pi_\mu\}_{\mu=1}^M$ over $\widetilde{\mathrm{P}\Delta} := \widetilde{\mathrm{P}\Delta}_{n-1} \times \Delta_{(n)} (\subset \mathrm{P}\Delta_{n-1} \times \Delta_{(n)} = \mathrm{P}\Delta)$.

Thus, the finite system $\{T_i\}_{i=2}^q \cup \{\pi_\mu\}_{\mu=1}^M$ generates $\mathscr{R}$ over $\widetilde{\mathrm{P}\Delta}$. □

Definition 2.5.21 A sheaf $\mathscr{S}$ of $\mathscr{A}$-modules over X is said to be *locally free with finite rank* if for every point $x \in X$ there are a neighborhood U of x and $p \in \mathbf{N}$ such that $\mathscr{S}|_U \cong \mathscr{A}_U^p$.

The following statement is immediate from Oka's First Coherence Theorem 2.5.1.

Corollary 2.5.22 *A sheaf of $\mathscr{O}_\Omega$-modules which is locally free with finite rank is coherent.*

Remark 2.5.23 In (2.5.17) we obtained the degree estimates such that $\deg_{z_n} T_i \le p = \max_j \deg_{z_n} P_j$, and $\deg_{z_n}(\alpha_1, \alpha_2, \dots, \alpha_q) < p$, where, furthermore, only the first element α_1 is of $\deg_{z_n} < p$ and for the others,

$$\deg_{z_n} \alpha_i < p' = \min_j \deg_{z_n} P_j, \quad 2 \le i \le q$$

(see (2.5.15) and (2.5.16)). Therefore, as noticed in the proof, if $p_1 = 0$, the trivial solutions T_i, $2 \le i \le q$, form a local finite generator system of $\mathscr{R}$; if $p_1 = 1$, these α_i are constants. The argument presented here is due to [54]. This seems not to have been widely observed before and gives a slight improvement of the proof of Oka VII (1948). In the most-known references such as K. Oka [62] VII, H. Cartan [10], R. Narasimhan [48], L. Hörmander [33], T. Nishino [49], etc., division algorithm (2.5.11) is performed with respect to an element P_{j_0} with the maximum degree $p(= p_{j_0})$, so that the degree estimate for α_i is less than p; it is, however, more natural to use P_1 in (2.5.11) than P_{j_0}. For the proof with P_{j_0} does not reduce to the easiest one when $p' = 0$, where the trivial solutions already form a local generator system of the relation sheaf (cf. Remark 2.5.14). As seen in Exercise 4 at the end of this chapter, when $n = 1$, it is reduced to the case of $p' = 0$. The proof presented above reflects some merit as p' is small.

Example 2.5.24 Let $(z, w) \in \mathbf{C}^2$ and set

$$\begin{aligned} F_1(z,w) &= w + z, \\ F_2(z,w) &= w^2 + z^2 w + z^3 e^z, \\ F_3(z,w) &= w^3 + zw^2 + z^2 \tan z. \end{aligned}$$

These are Weierstrass polynomials in w about the origin 0 without common factor. Let $\mathscr{R}(F_1, F_2, F_3)$ be the relation sheaf defined by

$$f_1 F_1 + f_2 F_2 + f_3 F_3 = 0. \tag{2.5.25}$$

We shall obtain a locally finite generator system of $\mathscr{R}(F_1, F_2, F_3)$ about 0. If one uses the division algorithm of the maximum degree, that is, by F_3, the computation is rather involved. But, by the division algorithm of the minimum degree, $\deg_w F_1 = \min\{\deg_w F_i\} = 1$, it is carried out easily as follows.

By Weierstrass' Preparation Theorem 2.1.3 we set

$$f_i = c_i F_1 + \beta_i, \quad \beta_i \in \mathscr{O}_z, \ 1 \le i \le 3,$$

where $\mathscr{O}_z$ stands for the set of holomorphic functions only in variable z. Let

$$T_2 = \begin{pmatrix} -F_2 \\ F_1 \\ 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} -F_3 \\ 0 \\ F_1 \end{pmatrix}$$

be the trivial solutions. Then it follows that

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} - c_2 T_2 - c_3 T_3 = \begin{pmatrix} \sum_{i=1}^3 c_i F_i + \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} g_1 \\ \beta_2 \\ \beta_3 \end{pmatrix},$$

where $g_1 = \sum_{i=1}^{3} c_i F_i + \beta_1$. We have

$$g_1 F_1 + \beta_2 F_2 + \beta_3 F_3 = 0. \tag{2.5.26}$$

We infer from the degree comparison that $\deg g_1 \leq 2$, and then set

$$g_1(z, w) = g_{12}(z)w^2 + g_{11}(z)w + g_{10}(z).$$

Substituting this in (2.5.26), we get

$$\begin{aligned}(g_{12}w^2 + g_{11}w + g_{10})(w + z) + \beta_2(w^2 + z^2 w + z^3 e^z)\\ + \beta_3(w^3 + zw^2 + z^2 \tan z) = 0.\end{aligned}$$

As a polynomial of degree 3 in w, we get

$$\begin{aligned}&(g_{12} + \beta_3)w^3 + (g_{12}z + g_{11} + \beta_2 + \beta_3 z)w^2\\ &\quad + (g_{11}z + g_{10} + \beta_2 z^2)w + g_{10}z + \beta_2 z^3 e^z + \beta_3 z^2 \tan z = 0.\end{aligned}$$

All the coefficients are 0:

$$\begin{aligned} g_{12} + \beta_3 &= 0,\\ g_{12}z + g_{11} + \beta_2 + \beta_3 z &= 0,\\ g_{11}z + g_{10} + \beta_2 z^2 &= 0,\\ g_{10} + \beta_2 z^2 e^z + \beta_3 z \tan z &= 0.\end{aligned}$$

Here the last one is already divided by z. With a matrix we obtain that

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1\\ z & 1 & 0 & 1 & z\\ 0 & z & 1 & z^2 & 0\\ 0 & 0 & 1 & z^2 e^z & z \tan z \end{pmatrix} \begin{pmatrix} g_{12}\\ g_{11}\\ g_{10}\\ \beta_2\\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}. \tag{2.5.27}$$

Elementary transforms of matrices yield that

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 1 & z^2 - z & -z^2\\ 0 & 0 & 0 & 1 & \frac{z+\tan z}{1-z+ze^z} \end{pmatrix} \begin{pmatrix} g_{12}\\ g_{11}\\ g_{10}\\ \beta_2\\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}.$$

Here it is noted that $\frac{z+\tan z}{1-z+ze^z}$ is holomorphic in a neighborhood of 0. Therefore we have

$$\begin{pmatrix} g_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} -w^2 + \frac{z+\tan z}{1-z+ze^z} w + z^2 + (z^2 - z)\frac{z+\tan z}{1-z+ze^z} \\ -\frac{z+\tan z}{1-z+ze^z} \\ 1 \end{pmatrix} \beta_3.$$

Thus, $\mathscr{R}(F_1, F_2, F_3)$ is generated in a neighborhood of 0 by the finite system

$$\left\{ T_2, T_3, \begin{pmatrix} -w^2 + \frac{z+\tan z}{1-z+ze^z} w + z^2 + (z^2 - z)\frac{z+\tan z}{1-z+ze^z} \\ -\frac{z+\tan z}{1-z+ze^z} \\ 1 \end{pmatrix} \right\}.$$

Historical Supplements

L. Bers, well-known in Teichmüller moduli theory, closes the preface of his lecture notes on the theory of several complex variables at Courant Institute, New York University [6] (1964) with the following sentence:

> Every account of the theory of several complex variables is largely a report on the ideas of Oka. This one is no exception.

The pillar of Oka's ideas is "Oka's First Coherence Theorem 2.5.1". There is no way to describe the proof of Theorem 2.5.1 other than "really marvelous".

In the preface of their basic book [28], H. Grauert and R. Remmert, the German luminaries in complex analysis of 20th century, write:

> Of greatest importance in Complex Analysis is the concept of a coherent analytic sheaf.

And they list the following as the *four fundamental coherence theorems*:

(i) The structure sheaf $\mathscr{O}_X$ of a complex space X is coherent (cf. Sect. 6.9 (of the present book)).
(ii) The geometric ideal sheaf $\mathscr{I}\langle A\rangle$ of an analytic subset A is coherent (cf. Sect. 6.5).
(iii) The normalization sheaf $\hat{\mathscr{O}}_X$ of a complex space is coherent (cf. Sect. 6.10).
(iv) The direct image sheaf of a coherent sheaf through a proper holomorphic map is coherent.

In Oka VII (1950) and VIII (1951) K. Oka proved the first three Coherence Theorems (the first (i) is a direct consequence of the coherence of $\mathscr{O}_{\mathbf{C}^n}$ and (ii)). As for the second one, it is often attributed to H. Cartan in many literatures (cf., e.g., [29]), but as discussed in Chap. 9, this result had been clearly announced in Oka VII (received 1948), so that it would be proved "*without any additional assumption*" in the next paper (i.e., Oka VIII, where in fact Oka gave the proof). As one reads the papers, one naturally finds that Oka VII and VIII form one set of papers. As already mentioned in the Preface, a key part of the proof of (ii) was already discussed and proved in Oka VII: In Sect. 3 it was formulated and discussed as Problème (K) and proved in Sect. 6. When he completed the paper, Oka VII, he had the proof of the Second Coherence Theorem in hand.

Because of these historical developments of mathematical comprehension, we call here the first three in order:

- *Oka's First Coherence Theorem* (for $\mathscr{O}_{\mathbf{C}^n}$ with $X = \mathbf{C}^n$),
- *Oka's Second Coherence Theorem* (for $\mathscr{I}\langle A\rangle$),
- *Oka's Third Coherence Theorem* (for $\hat{\mathscr{O}}_X$).

It would be the closest to the actual history that between Oka VII and VIII, H. Cartan gave his own proof to Oka's Second Coherence Theorem. For more details, see Chap. 9.

Exercises

1. For $t_j \in \mathbf{C}$, $1 \le j \le n$, set

$$s_\nu = (-1)^\nu \sum_{1 \le j_1 < \cdots < j_\nu \le n} t_{j_1} \cdots t_{j_\nu}, \quad 1 \le \nu \le n,$$

$$\sigma_\mu = \sum_{j=1}^{n} t_i^\mu, \quad \mu \ge 1.$$

Show that

$$\sigma_m + \sigma_{m-1}s_1 + \cdots + \sigma_1 s_{m-1} + m s_m = 0, \quad m \le n,$$
$$\sigma_m + \sigma_{m-1}s_1 + \cdots + \sigma_{m-n}s_n = 0, \quad m > n.$$

(Hint: For the first, set $f(X) = \prod_{j=1}^{n}(1 - a_j X)$. Then, $\frac{f'(X)}{f(X)} = -\sum_{j=1}^{n} \frac{t_j}{1-t_j X} = -\sum_{\nu=0}^{\infty} \sigma_{\nu+1} X^\nu$. Then, use $f'(X) = -f(X)\sum_{\nu=0}^{\infty} \sigma_{\nu+1} X^\nu$. For the second, use $X^m + s_1 X^{m-1} + \cdots + s_m X^{m-n} = X^{m-n} \prod_{j=1}^{n}(X - a_j)$. Then, substitute $X = t_j$, $1 \le j \le n$.)

2. (1) Let $f(z, w) = \sin w + z^2$ with $(z, w) \in \mathbf{C}^2$.
   a. Show that $\operatorname{ord}_0 f = 1$.
   b. Obtain the Weierstrass decomposition of $f(z, w)$ at 0.

   (2) Consider the same for $g(z, w) = w + \sin w + z^2$.

3. Consider the implicit function $w = w(z)$ defined by $g(z, w) = 0$ given in (2) just above. By making use of (2.1.6), obtain an integral formula of $w(w(z))$ in a neighborhood of 0 in terms of $g(z, w)$.
4. Show directly Claim 2.5.3, when $n = 1$.
5. Write down the simultaneous relation (2.5.20) in the form of a matrix equation as in (2.5.27).

6. Set

$$P_1(z, w) = w + z,$$
$$P_2(z, w) = w^2 + (z^2 + z)w + z^3,$$
$$P_3(z, w) = w^3 + zw^2 + z^3.$$

Obtain a locally finite generator system of the relation sheaf $\mathscr{R}(P_1, P_2, P_3)$ about the origin.

7. Set

$$F_1(z, w) = w + ze^z,$$
$$F_2(z, w) = w^2 + z^2 w,$$
$$F_3(z, w) = w^3 + w^2 \sin z,$$
$$F_4(z, w) = w^3 + z^3 e^z.$$

Obtain a locally finite generator system of the relation sheaf $\mathscr{R}(F_1, F_2, F_3, F_4)$ about the origin.

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