Preface

McKean and I determined all possible boundary conditions at 0 for the Brownian motion in (0, ∞) and discussed the construction of the sample functions of the Markov processes corresponding to the boundary conditions [1]. The jumping-in measure k appearing in the boundary condition has to satisfy

$$\int_0^\infty (b \wedge 1)k(db) < \infty. \tag{1}$$

This conditions turns out to be

$$\int_0^\infty (s(b) \wedge 1)k(db) < \infty \tag{2}$$

for the diffusion in (0, ∞) with the generator

$$\mathscr{G} = \frac{d}{dm}\frac{d}{ds} \tag{3}$$

if we have

$$s(0) > -\infty,\ m(0,1) < \infty \text{ and } s(\infty) = \infty, \tag{4}$$

as we discussed in that paper. A few years ago, J. Lamperti raised the following question in connection with his work on branching processes.

What condition should the jumping-in measure k satisfy in case $m(0,1) = \infty$ *in (4)?*

By intuitive argument, I conjectured that the condition would be

$$\int_0^\infty E_b(1 - e^{-\sigma_0})k(db) < \infty, \quad \sigma_0 = \text{hitting time for } 0$$

or equivalently

$$\int_0^\infty \left(\int_0^b m(\xi, 1)\, ds(\xi) \wedge 1 \right) k(db) < \infty. \tag{5}$$

The purpose of this lecture is to solve this problem for the general Markov process with reasonable conditions by introducing the notion of the Poisson point process attached to the Markov process and to derive (2) and (5) as its special cases.

Let $Y_t(\omega)$ be a homogeneous Lévy process with paths increasing only with jumps.

Then,

$$E(e^{-\alpha Y_t}) = e^{-t \int_0^\infty (1 - e^{-\alpha u}) n(du)} \tag{6}$$

where n is the Lévy measure of the process and

$$\int_0^\infty (u \wedge 1) n(du) < \infty. \tag{7}$$

Let D_ω be the discontinuity points of Y_t and consider the random set

$$G(\omega) = \{(t, Y_{t+}(\omega) - Y_{t-}(\omega)), t \in D_\omega\}.$$

This is a countable set in $T \times U$, $T = U = [0, \infty)$. It is well known that

(a) The number $\#(E \cap G)$ of points in $E \cap G$ is Poisson distributed with the mean:

$$\int_E dt\, n(du)$$

for every Borel set E in $T \times U$ (a random variable $\equiv \infty$ is regarded as Poisson distributed with mean $= \infty$) and

(b) $\#(E_i \cap G)$, $i = 1, 2, \ldots, n$ are independent for disjoint Borel sets E_i in $T \times U$.

These two conditions characterize the probability law of the random set G_ω.

Instead of considering the random set G_ω, we can consider the *point process* $X_\bullet(\omega)$ where $X_t(\omega)$ is defined only on D_ω and

$$X_t(\omega) = Y_{t+}(\omega) - Y_{t-}(\omega) \quad \text{for} \quad t \in D_\omega$$

for each ω. Then, G_ω is the graph of the path of X.. A point process in general is a random process whose sample function is defined only on a countable subset of the time interval depending on the sample.

The values of a point process need not be real. We can consider a point process whose values are taken from a general measurable space U. Let n be an arbitrary σ-finite measure on U. Then, a point process whose values are in U is called a *Poisson point process* with *characteristic measure* n, if its graph $G = G_X$ satisfies the conditions (a) and (b) mentioned above. We can define Poisson point processes in a qualitative way and derive (a) and (b) from the definition, as we shall do in this note.

In case the total measure $n(U)$ is finite, the domain of the definition of the sample function of the Poisson point process with characteristic measure n is a discrete set a.s. and its structure is simple. This was discussed by K. Matthes, J. Kersten, and P. Pranken [2–4]. It is a generalization of the point process arising from a compound Poisson process.

If $f : U \to U_1$ is measurable and if X is a Poisson point process: $T \to U$ with characteristic measure n, then the composition $f \cdot X$ is also a Poisson point process with characteristic measure nf^{-1}.

Let X_t be a Markov process on a locally compact metric space S and a($\in S$) be a fixed state. Let $A(t)$ be a local time process of X_t at a. Then, $A^{-1}(t)$ is a homogeneous Lévy process with increasing paths such that $P_a(A^{-1}(0) = 0) = 1$.

Let X_t^0 be a Markov process obtained by stopping X_t at the hitting time σ_a of X_t for a. σ_a is the same as the hitting time σ_a^0 of X_t^0 for a.

Let U be the space of all right continuous functions with left limits. We will define a point process $\boldsymbol{X}$: $\mathrm{T} \equiv [0, \infty) \to U$ by

$$\begin{aligned} D_{\boldsymbol{X}_\omega} &(= \text{the domain of } \boldsymbol{X}_\omega) \\ &= \text{the set of all discontinuity points of } A^{-1}(t) \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{X}_{\omega,t}(s) &= X(s + A^{-1}(t-)) \quad &&\text{if } s \le A^{-1}(t+) - A^{-1}(t-) \\ &= a &&\text{if } s \ge A^{-1}(t+) - A^{-1}(t-) \end{aligned}$$

for $t \in D_{\boldsymbol{X}_\omega}$ (see the pictures in Sect. 2.2). We can use the strong Markov property of X_t to prove that $\boldsymbol{X}_\omega$ is a Poisson point process: $T \to U$.

Let us introduce a function $e : U \to S$ by

$$e(u) = u(0).$$

Then, $e \cdot \boldsymbol{X}$ is also a Poisson point process, and its characteristic measure is denoted by k and is called the *jumping-in measure* of X_t. Then, the characteristic measure $n_{\boldsymbol{X}}$ of $\boldsymbol{X}$ proves to be

$$n_{\boldsymbol{X}}(V) = \int_S k(db) P_b(X_\bullet^0 \in V), \quad V \subset U$$

when $X_\bullet^0$ denotes the sample path of the stopped process X_t^0.

Let $h(u) = \inf\{t;\ u(t) = a\}$. Then, $h \cdot \boldsymbol{X}$ is also a Poisson point process with characteristic measure $n_{\boldsymbol{X}} \cdot h^{-1}$ and the jump part of $A^{-1}(t)$ is equal to

$$\sum_{\substack{s \in D_{\boldsymbol{X}} \\ s \le t}} (h \cdot \boldsymbol{X})_s.$$

Using (7), we have

$$\int_0^\infty (t \wedge 1) n_{\boldsymbol{X}} \cdot h^{-1}(dt) < \infty$$

i.e.,

$$\int_S k(db) E_b(\sigma_a^0 \wedge 1) < \infty.$$

Since the construction of a Poisson point process with a given characteristic measure is easy, we can discuss the construction of the Markov process X_t from its *stopped process*, its *jumping-in measure*, and its *stagnancy rate* (=the coefficient of t in the continuous part of $A^{-1}(t)$) if X_t has no continuous exit from a.

To discuss the case that a continuous exit from a is allowed, we will be faced with a more difficult problem. Roughly speaking, if we can determine all possible processes X_t with continuous exit only for their stopped process X_t^0 given (e.g., one-dimensional diffusion case), then we can determine all possible processes with both continuous exit and discontinuous exit. However, we will not discuss this problem in this note.

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