Chapter 2
Application to Markov Processes

2.1 Problem

Let X_t be a standard Markov process with the state space S. The time interval $[0, \infty)$ is denoted by T. Let a be a fixed state and σ_a the hitting time for a. We impose the following four assumptions.

(A-1) $P_b(\sigma_a < \infty) = 1$;
(A-2) $E_b(\sigma_a \wedge 1) \to 0$ as $b \to a$;
(A-3) $\inf_{b \in U^c} E_b(\sigma_a \wedge 1) > 0$ for every neighborhood U of a;
(A-4) a is a *discontinuous exit state*.

We will explain the meaning of this condition.

s is called an *exit time* from a for the path $(X_t(\omega))$ if, for every $\varepsilon > 0$,

$$\{t : X_t(\omega) = a\} \cap (s - \varepsilon, s) \neq \emptyset$$

and if, for some $\varepsilon > 0$,

$$\{t : X_t(\omega) = a\} \cap (s, s + \varepsilon) = \emptyset.$$

All exit times from a for the path $(X_t(\omega))$ form a countable set depending on ω.

An exit time s from a for the path $(X_t(\omega))$ is called a *continuous* or *discontinuous* exit time according as

$$X_s(\omega) = a \quad \text{or} \quad X_s(\omega) \neq a.$$

a is called a *discontinuous exit state* if all exit times from a for the path $(X_t(\omega))$ are discontinuous a.s. with respect to P_a.

K. Itô, *Poisson Point Processes and Their Application to Markov Processes*,
SpringerBriefs in Probability and Mathematical Statistics,
DOI 10.1007/978-981-10-0272-4_2

Let $X^0_t = X_{t\wedge\sigma_a}$. Since the hitting time σ^0_a of the path $(X^0_t(\omega))$ is the same as σ_a, the conditions (A-1), (A-2) and (A-3) are equivalent to

(A^0-1) $P_b(\sigma^0_a < \infty) = 1$;
(A^0-2) $E_b(\sigma^0_a \wedge 1) \to 0$, as $b \to a$;
(A^0-3) $\inf_{b\in U^c} E_b(\sigma^0_a \wedge 1) > 0$ for every neighborhood U of a.

By the strong Markov property of (X_t), the probability laws of the path (X_t) is determined by the probability laws of the path (X^0_t) and the probability law of the path (X_t) starting at a. Symbolically we have

$$\text{p.l. of } (X_t) = \text{ p.l. of } (X^0_t) + \text{ p.l. of } (X_t) \text{ starting at } a. \tag{2.1}$$

Since the path (X_t) starting at a behaves outside of a in the same way as the path (X^0_t), the union relation in (2.1) is no disjoint union. We want to extract some information I from the probability law of the path (X_t) starting at a to obtain a symbolic information relation

$$\text{p.l. of}(X_t) = \text{p.l. of}(X^0_t) + I \quad \text{(disjoint union)}. \tag{2.1$'$}$$

In the subsequent sections we will prove that I consists of two elements: *jumping-in measure* $k(db)$ and *stagnancy rate* m.

2.2 The Poisson Point Process Attached to a Markov Process at a State *a*

We use the same notations as in Sect. 2.1 and impose the conditions (A-1), (A-2), (A-3) and (A-4).

Let $A(t)$ be a local time of (X_t) at a. By our assumptions (A-1) and (A-2), $A(t)$ is determined up to a multiplicative constant and we have

$$P_b(A(t) < \infty \text{ for every } t) = 1$$

and

$$P_b(A(t) \to \infty \text{ as } t \to \infty) = 1.$$

We refer the reader to Blumenthal and Getoor [1] for the definition and the properties of local times.

Let U be the space of all right continuous functions: $T \to S$ with left limits. The sample path of (X_t) belongs to U a.s. for every starting point.

From now on we will refer to P_a for the probability law of $X_t(\omega)$ unless the contrary is explicitly stated. Let us define a point process $\boldsymbol{X} : T \to U$ by

$$D_{\boldsymbol{X}} = \{A(s) : s \text{ moves over all exit times from } a \text{ for the path}\},$$
$$\boldsymbol{X}_\omega(t) = (X \circ \theta_{A^{-1}(t-)})^0 \quad \text{for} \quad t \in D_{\boldsymbol{X}},$$

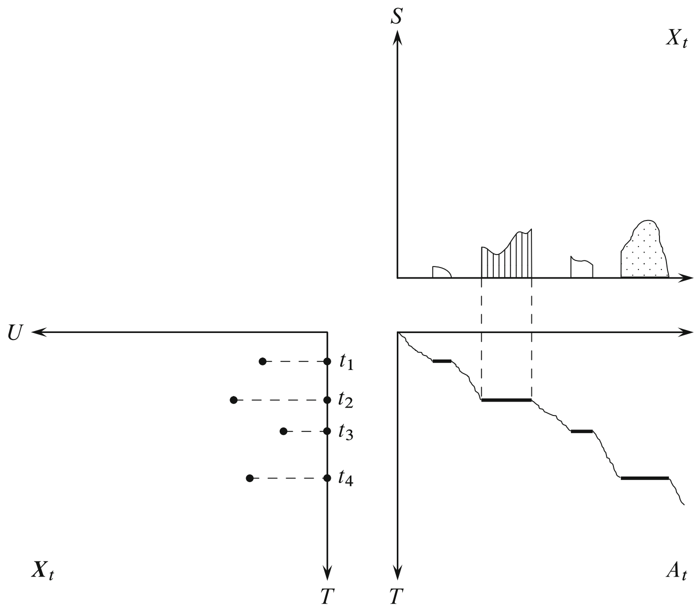


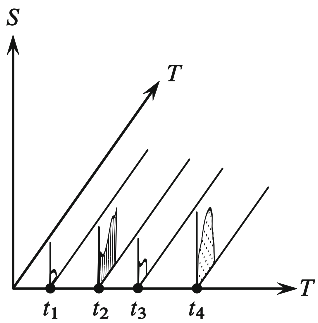
Fig. 2.1 X_t, A_t and $\boldsymbol{X}_t$

where θ_t is a shift operator and 0 is a stopping operator. Note that $D_{\boldsymbol{X}}$ consists of all values of $A(t)$ corresponding to the flat t-intervals of $A(t)$ and that $\boldsymbol{X}_\omega(t)$ is a function : $T \to S$ belonging to U for ω and t fixed and (Fig. 2.1)

$$\boldsymbol{X}_\omega(t)(s) = X_{s \wedge \sigma_a(\theta_\tau \omega)}(\theta_\tau \omega) \quad \text{for} \quad s \in T, \quad \tau = A^{-1}(t-).$$

Figure 2.2 is an intuitive picture of $\boldsymbol{X}_t$.

Fig. 2.2 An intuitive picture of $\boldsymbol{X}_t$



Theorem and Definition 2.2.1 *The point process $\boldsymbol{X}$ defined above is a Poisson point process: $T \to U$; it is called* the Poisson point process attached to the Markov process (X_t).

Proof Let $\{\mathscr{B}_t\}$ be a family of sub-σ-algebras of $\mathscr{B}$ in the definition of the Markov process (X_t). Since $A^{-1}(t+)$ and $A^{-1}(t-) = \sup_n A^{-1}(t - \frac{1}{n}+)$ are both stopping times, $\mathscr{B}_{A^{-1}(t-)}$ and $\mathscr{B}_{A^{-1}(t+)}$ are well-defined.

Let $\mathscr{B}_t(\boldsymbol{X})$ be the σ-algebra generated by $\boldsymbol{X}|_d[0, t)$. Then

$$\mathscr{B}_t(\boldsymbol{X}) \subset \mathscr{B}_{A^{-1}(t-)} \subset \mathscr{B}_{A^{-1}(t+)}.$$

For t fixed, we have

$$P(X(A^{-1}(t+)) = a) = 1.$$

Thus, for $B \in \mathscr{B}_t(\boldsymbol{X})$ and $M \in \mathscr{P}$, we have

$$P_a(B \cap ((\theta_t \boldsymbol{X}) \in M)) = P_a(B) P_a(\boldsymbol{X} \in M)$$

because of the additivity of $A(t)$, the strong Markov property of (X_t) and the definition of $\boldsymbol{X}$. Thus $\boldsymbol{X}$ has renewal property. This implies that $\boldsymbol{X}$ is differential and stationary.

To prove the σ-discreteness of $\boldsymbol{X}$ we will introduce a map $h : U \to T$ by

$$h(u) = \inf\{t : u(t) = a\}.$$

If u is the path of $(X_t(\omega))$, then $h(u)$ will be $\sigma_a(\omega)$.

Let U_n be the set of all u such that

$$h(u) > \frac{1}{n}.$$

By (A-4) we have

$$\boldsymbol{X} = \bigvee_n \boldsymbol{X}|_r U_n \quad \text{a.s.}$$

Since

$$N(\boldsymbol{X}, [0, t) \times U_n) \leq n \cdot A^{-1}(t-) < \infty \quad \text{a.s.},$$

$\boldsymbol{X}_\omega|_r U_n$ is discrete a.s. for each n. $\boldsymbol{X}$ is therefore σ-discrete. □

2.3 The Jumping-In Measure and the Stagnancy Rate

Let us consider a map $e : U \to S$ by

$$e(u) = u(0), \quad u \in U.$$

Since the path of X_t has no discontinuities of the second kind and since $A^{-1}(t) < \infty$ for $t < \infty$, the distance $\rho(X_t, a)$ between X_t and a can be larger than $\varepsilon(>0)$ a finite number of times during $[0, A^{-1}(t))$ a.s. for $t < \infty$. This implies that $e \cdot \boldsymbol{X}$ is a σ-discrete point process. By Theorem 1.6.1, we see that $e \cdot \boldsymbol{X}$ is a Poisson point process.

Definition 2.3.1 The characteristic measure k of $e \cdot \boldsymbol{X}$ is called the *jumping-in measure* of the Markov process X_t from a.

It is obvious that $k = n_{\boldsymbol{X}} e^{-1}$. Since $n_{\boldsymbol{X}}$ is concentrated on the paths starting at points in $S - \{a\}$ by (A-4), k is concentrated on $S - \{a\}$. It is obvious that the total measure of k is the same as that of $n_{\boldsymbol{X}}$. Since $\boldsymbol{X}$ is σ-discrete, the total measure of k is σ-finite.

Since $A^{-1}(t)$ is known to be an increasing homogeneous Lévy process (=a subordinator), it can be written as

$$A^{-1}(t) = m \cdot t + J(t), \quad m > 0,$$

when $J(t)$ is a pure jump process.

Definition 2.3.2 The coefficient m is called the *stagnancy rate* of the Markov process (X_t).

The following theorem shows that the characteristic measure $n_{\boldsymbol{X}}$ is determined by the measure k and the probability law of the path of (X_t^0).

Theorem 2.3.3

$$n_{\boldsymbol{X}}(V) = \int_S k(db) P_b(X^0 \in V)$$

where X^0 denotes the path $(X_t^0(\omega), t \in T)$.

Proof Let S_i denote the set $\{b \in S : \rho(a, b) > 1/i\}$ for $i = 1, 2, \ldots$. Then $\cup S_i = S - \{a\}$. Let $U_i = \{u \in U : u(0) \in S_i\} = e^{-1}(S_i)$. Then U_i increases with i and the limit U_∞ is the space of all paths in U starting from points in $S - \{a\}$. We have

$$\boldsymbol{X} = \boldsymbol{X}|_r U_\infty = \bigvee_i \boldsymbol{X}_i, \quad \boldsymbol{X}_i = \boldsymbol{X}|_r U_i$$

by (A-4). The set $A^{-1}(D_{\boldsymbol{X}_i}) \cap [0, A^{-1}(t+)]$ is included in the set of the time points $s \in [0, A^{-1}(t+)]$ for which $\rho(X_s, X_{s-}) > 1/i$. Since the sample path of (X_t) has no discontinuity points of the second kind, the latter set is finite and so is $A^{-1}(D_{\boldsymbol{X}_i}) \cap [0, A^{-1}(t+)]$. This implies $D_{\boldsymbol{X}_i} \cap [0, t]$ is finite. $\boldsymbol{X}_i$ is therefore a discrete Poisson point process.

By Theorem 1.4.1, we have

$$n_{\boldsymbol{X}_i}(V_i) = \lambda_i P_a(\boldsymbol{X}_i(\tau_i) \in V_i), \quad V_i \in \mathscr{U}_i \equiv U_i \cap \mathscr{U},$$

where $\lambda_i = n_{\boldsymbol{X}_i}(U_i)$ and τ_i is the smallest element in $D_{\boldsymbol{X}_i}$. By the definition we have

$$\boldsymbol{X}_i(\tau_i) = \boldsymbol{X}(\tau_i) = (X \circ \theta_{\sigma_i})^0, \quad \sigma_i = A^{-1}(\tau_i -).$$

Since $n_{\boldsymbol{X}_i} = n_{\boldsymbol{X}}|U_i$ and since σ_i is a stopping time with respect to $\{\mathscr{B}_t\}$, we have, for $V \in \mathscr{U}$,

$$\begin{aligned} n_{\boldsymbol{X}}(U_i \cap V) &= \lambda_i P_a((X \circ \theta_{\sigma_i})^0 \in V \cap U_i) \\ &= \lambda_i \int_{S_i} P_a(X_{\sigma_i} \in db) P_b(X^0 \in V \cap U_i) \end{aligned}$$

Set $V = e^{-1}(B_i)$, $B_i \in \mathscr{S}_i \equiv S_i \cap \mathscr{S}$. Then $V \subset e^{-1}(S_i) = U_i$ and so

$$k(B_i) = \lambda_i P_a(X_{\sigma_i} \in B_i).$$

Thus we have

$$n_{\boldsymbol{X}}(U_i \cap V) = \int_{S_i} k(db) P_b(X^0 \in V \cap U_i).$$

Letting $i \uparrow \infty$, we have

$$n_{\boldsymbol{X}}(V) = \int_{S-\{a\}} k(db) P_b(X^0 \in V) = \int_S k(db) P_b(X^0 \in V),$$

which completes the proof. □

The jumping-in measure k is not arbitrary. We have:

Theorem 2.3.4 *k is concentrated on $S - \{a\}$ and*

$$\int_S E_b(\sigma_a^0 \wedge 1) k(db) < \infty.$$

Proof $h \cdot \boldsymbol{X}$ is also a Poisson point process whose integrated process is the discontinuous part of the increasing homogeneous Lévy process $A^{-1}(t)$. Therefore $h \cdot \boldsymbol{X}$ is summable and so

$$\int_0^\infty (1 \wedge t) n_{h \cdot \boldsymbol{X}}(dt) < \infty$$

by virtue of Theorem 1.7.3. Since $n_{h\cdot X} = n_X h^{-1}$, this can be written

$$\int_0^{\infty} (1 \wedge t) \int_S k(db) P_b(\sigma_a^0 \in dt) < \infty$$

by the previous theorem, namely

$$\int_S k(db) E_b(\sigma_a^0 \wedge 1) < \infty.$$

□

Remark By this theorem and the condition (A-3), we have

$$k(U^c) < \infty$$

for every neighborhood U of a.

If $k(S) < \infty$, then $\boldsymbol{X}$ is discrete. Then the set $\{t : X_t(\omega) = a\}$ is a sequence of disjoint intervals ordered linearly and $A(t, \omega)$ is the sojourn time at a singleton $\{a\}$ up to a multiplicative constant. Thus we have $m > 0$ in the decomposition:

$$A^{-1}(t) = mt + J(t),$$

$J(t)$ being the discontinuous part of $A^{-1}(t)$. Therefore we obtain:

Theorem 2.3.5 *$m \geq 0$ in general, and $m > 0$ in case $k(S) < \infty$.*

$A(t)$ is determined up to a multiplicative constant and m and k depend on which version of $A(t)$ we take. Let $A_i(t)$, $i = 1, 2$ be two versions of $A(t)$ and write the corresponding m and k as m_i and k_i, $i = 1, 2$. Then we have a constant $c > 0$ such that

$$A_2(t) = cA_1(t).$$

Consider the decompositions

$$A_i^{-1}(s) = m_i s + J_i(s), \quad i = 1, 2.$$

Then

$$A_2^{-1}(cs) = A_1^{-1}(s),$$
$$m_2 cs + J_2(cs) = m_1 s + J_1(s)$$

and so

$$m_2 = \frac{1}{c} m_1.$$

Writing $\#A$ for the number of points in A, we have

$$\begin{aligned}
\varepsilon \cdot k_2(B) &= E_a[\#\{s : 0 \le s \le \varepsilon,\ e(\boldsymbol{X}_s) \in B\}] \\
&= E_a[\#\{s : 0 \le s \le \varepsilon,\ X(A_2^{-1}(s-)) \in B\}] \\
&= E_a[\#\{t : 0 \le A_2(t) \le \varepsilon,\ X(t) \in B\}] \\
&= E_a[\#\{t : 0 \le cA_1(t) \le \varepsilon,\ X(t) \in B\}] \\
&= E_a\left[\#\left\{t : 0 \le A_1(t) \le \frac{1}{c}\varepsilon,\ X(t) \in B\right\}\right] \\
&= \frac{1}{c}\varepsilon k_1(B)
\end{aligned}$$

and so

$$k_2 = \frac{1}{c}k_1.$$

Thus we have:

Theorem 2.3.6 *If $A_2(t) = cA_1(t)$, then $m_2 = \frac{1}{c}m_1$ and $k_2 = \frac{1}{c}k_1$.*

Therefore m and k are determined up to a common multiplicative constant.

To have m and k determined uniquely, we have to take a standard version of the local time $A(t)$.

Definition 2.3.7 $A(t)$ is called *standard* if

$$E_a\Big(\int_0^\infty e^{-t}\, dA(t)\Big) = 1,$$

in which case

$$E_b\Big(\int_0^\infty e^{-t}\, dA(t)\Big) = E_b(e^{-\sigma_a^0}) \quad \text{for every } b.$$

The m and k that correspond to the standard $A(t)$ are called the *standard stagnancy rate* and the *standard jumping-in measure*.

Theorem 2.3.8 *The standard stagnancy rate m and the stagnancy jumping-in measure k satisfy the following conditions.*

(a) $m \ge 0$ *in general and* $m > 0$ *in case* $k(S) < \infty$.
(b) k *is concentrated on* $S - \{a\}$ *and*

  (i) $\int_S k(db)E_b(\sigma_a^0 \wedge 1) < \infty$;
  (ii) $m + \int_S k(db)E_b(1 - e^{-\sigma_a^0}) = 1$.

Proof By Theorems 2.3.4 and 2.3.5 it is enough to prove (b)-(ii). Since m and k are standard, the corresponding $A(t)$ satisfies

$$E_a\Big(\int_0^\infty e^{-t}\,dA(t)\Big) = 1.$$

But the left side is

$$\begin{aligned} E_a\Big(\int_0^\infty e^{-A^{-1}(t)}\,dt\Big) &= \int_0^\infty E_a(e^{-A^{-1}(t)}\,dt) \\ &= \int_0^\infty e^{-mt-t\cdot\int_0^\infty(1-e^{-s})\int_S k(db)P_b(\sigma_a^0\in ds)}\,dt \end{aligned}$$

(see the proof of Theorem 2.3.4 and use the Lévy–Khinchin formula)

$$\begin{aligned} &= \Big(m + \int_0^\infty (1-e^{-s})\int_S k(db)P_b(\sigma_a^0 \in ds)\Big)^{-1} \\ &= \Big(m + \int_S k(db)E_b(1-e^{-\sigma_a^0})\Big)^{-1}. \end{aligned}$$

This proves (ii). □

2.4 The Existence and Uniqueness Theorem

Suppose that X_t is a standard Markov process with the state space S and that a is a fixed state. We assume (A-1), (A-2), (A-3) and (A-4) in Sect. 2.1.

Let $X_t^0 = X_{t\wedge\sigma_a}$, m the standard stagnancy rate and k the jumping-in measure for X_t. Then we have proved:

(i) X_t^0 is a standard Markov process which satisfies (A^0-1), (A^0-2), (A^0-3).
(ii) m and k satisfy (a) and (b) in Theorem 2.3.8.

Now we want to construct X_t for X_t^0, m and k given.

Theorem 2.4.1 *Suppose that X_t^0, m and k satisfy (i) and (ii). Then there exists a standard Markov process X_t satisfying (A-1), (A-2) and (A-3) such that $X_{t\wedge\sigma_a}$ is equivalent to X_t^0 and that the standard stagnancy rate and the standard jumping-in measure are respectively equal to m and k. Such X_t is unique up to equivalence.*

Proof of existence First we will construct the Poisson point process $\boldsymbol{X}$ attached to the Markov process X_t that is to be constructed.

Let U be the space of all right continuous functions: $T \to S$ with left limits. Define a σ-finite measure n on U by

$$n(V) = \int_S k(db)P_b(X^0 \in V)$$

and construct a Poisson point process $\boldsymbol{X} : T \to U$ with $n_{\boldsymbol{X}} = n$ by Theorem 1.3.5, or by Theorem 1.5.4.

Set

$$\tilde{A}(s) = ms + \sum_{\substack{\alpha \le s \\ \alpha \in D_X}} h(\boldsymbol{X}(\alpha))$$

where $h(u) = \inf\{\alpha \in T : u(\alpha) = a\}$.

Define $Y(t)$ as follows.

$$Y(t) = \begin{cases} \boldsymbol{X}(s)(t - \tilde{A}(s-)) & \text{if } \tilde{A}(s-) \le t < \tilde{A}(s) \\ a & \text{if } \tilde{A}(s-) = t = \tilde{A}(s). \end{cases}$$

Now define the probability law P_a of the path of X_t starting at a by

$$P_a(X_\bullet \in V) = P(Y(\cdot) \in V)$$

and the probability law P_b of the path of X_t starting at a general state b by

$$P_b(X_{\bullet \wedge \sigma_a} \in V_1, X \circ \theta_{\sigma_a} \in V_2) = P_b(X^0_\bullet \in V_1) P_a(X_\bullet \in V_2).$$

It is needless to say that the definition of P_a is suggested by Fig. 2.1 and that the definition of P_b is suggested by the strong Markov property.

First we will prove that

$$P(\tilde{A}(s) < \infty \text{ for every } s \text{ and } \tilde{A}(\infty) = \infty) = 1, \tag{2.2}$$

so that $Y(t)$ is well-defined for every t. If $k(S) = 0$, then $m > 0$ and

$$\tilde{A}(s) = ms < \infty \quad \text{and} \quad A(\infty) = \infty.$$

If $k(S) > 0$, then $h \cdot \boldsymbol{X}$ is a Poisson point process with

$$n_{h \cdot X} = nh^{-1} = \int_S k(db) P_b(\sigma^0_a \in \cdot).$$

Since

$$\int_0^\infty (t \wedge 1) n_{h \cdot X}(dt) = \int_S k(db) E_b(\sigma^0_a \wedge 1) < \infty,$$

$h \cdot \boldsymbol{X}$ is summable and so

$$J(s) = \sum_{\substack{\alpha \le s \\ \alpha \in D_X}} h(\boldsymbol{X}(\alpha)) < \infty$$

for every $s < \infty$ and $J(s)$ is a homogeneous Lévy process with increasing paths. Since $n_{h \cdot X}([0, \infty)) = k(S) > 0$, we have

$$P(J(\infty) = \infty) = 1.$$

This proves (2.2).

Now we will prove that the process X_t defined above is a standard Markov process with (A-1), (A-2), (A-3) and (A-4).

Case 1. $k(S) < \infty$. In this case we have $m > 0$.

Since

$$n_X(U) = \int_S k(db) P_b(X^0_\bullet \in U) = k(S),$$

X is discrete.

Set

$$D_X = \{\tau_1 < \tau_1 + \tau_2 < \tau_1 + \tau_2 + \tau_3 < \cdots\}$$

and set

$$\xi_i = \boldsymbol{X}(\tau_1 + \tau_2 + \cdots + \tau_i), \quad i = 1, 2, \ldots.$$

Then $\tau_1, \tau_2, \ldots, \xi_1, \xi_2, \ldots$ are independent and

$$P(\tau_i > t) = e^{-tk(S)},$$
$$P(\xi_i \in V) = \frac{1}{k(S)} \int_S k(db) P_b(X^0(\cdot) \in V), \quad V \in \mathscr{U}.$$

In other words the probability law of ξ_i is the probability law of the path of X^0 with the initial distribution $k(db)/k(S)$.

By the definition of $Y(t)$ we have $Y(t) = a$ for

$$\begin{aligned} &m\tau_1 + h(\xi_1) + \cdots + m\tau_{i-1} + h(\xi_{j-1}) \\ \le\, &t < m\tau_1 + h(\xi_1) + \cdots + m\tau_{i-1} + h(\xi_{i-1}) + m\tau_i \end{aligned}$$

and

$$Y(t) = \xi_i(t - m\tau_1 - h(\xi_1) - \cdots - m\tau_{i-1} - h(\xi_{i-1}) - m\tau_i)$$

for

$$\begin{aligned} &m\tau_1 + h(\xi_1) + \cdots + m\tau_{i-1} + h(\xi_{i-1}) + m\tau_i \\ \le\, &t < m\tau_1 + h(\xi_1) + \cdots + m\tau_i + h(\xi_i). \end{aligned}$$

Since

$$P(m\tau_i > t) = P(\tau_i > t/m) = e^{-tk(S)/m},$$

$X(t)$ can be described as follows. If it starts at a, it stays at a for an exponential holding time with the parameter $= k(S)/m$, then jumps into db with probability

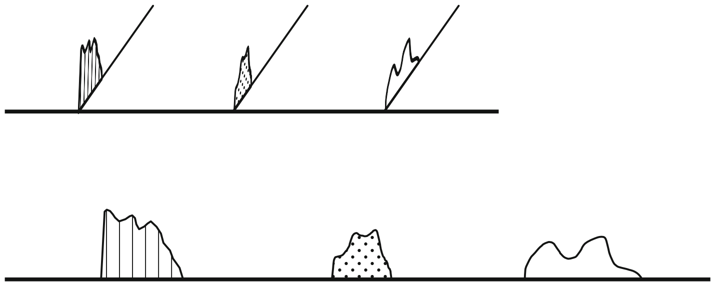


Fig. 2.3 Markov process from a point process

$k(db)/k(S)$ and moves in the same way as X_t^0 does until it hits a, it will repeat the same motion afterwards independently of its past history. If it starts at $b \neq a$, it performs the same motion as X_t^0 until it hits a and then it will act as above. We can verify the strong Markov property of this motion by routine. It is easy to check the other properties of X_t stated above (see Fig. 2.3).

Case 2. $k(S) = \infty$. Everything can be verified by routine except the fact that the sample path of $Y(t)$ belongs to U a.s. Since it is obvious that $Y(t)$ is right continuous and has left limits as far as it is in $S - \{a\}$, the only fact that needs proof is that the set of s such that

$$\sigma_\varepsilon(\boldsymbol{X}(s)) < \infty, \quad \sigma_\varepsilon(u) = \inf\{t : \rho(a, u(t)) \geq \varepsilon\}$$

forms a discrete set a.s. for every $\varepsilon > 0$. Since $\boldsymbol{X}(s)(t) = a$ for $t \geq h(\boldsymbol{X}(s))$ a.s., $\sigma_\varepsilon(\boldsymbol{X}(s)) < \infty$ is equivalent to

$$\sigma_\varepsilon(\boldsymbol{X}(s)) < h(\boldsymbol{X}(s))$$

a.s. It is therefore enough to prove that

$$\boldsymbol{X}|_r V_\varepsilon, \quad V_\varepsilon = \{u : \sigma_\varepsilon(u) < h(u)\}$$

is discrete a.s., namely that

$$n_{\boldsymbol{X}}(V_\varepsilon) < \infty.$$

Set

$$\delta = \inf\{E_b(\sigma_a^0 \wedge 1) : \rho(b, a) \geq \varepsilon\}.$$

Then $\delta > 0$ by (A^0-3).

Observe that

$$\begin{aligned}
\int_U h(u) \wedge 1\, n_X(du) &\geq \int_{V_\varepsilon} h(u) \wedge 1\, n_X(du) \\
&\geq \int_{V_\varepsilon} (h(u) - \sigma_\varepsilon(u)) \wedge 1\, n_X(du) \\
&= \int_V (h(u) - \sigma_\varepsilon(u)) \wedge 1 \int_S k(db) P_b(X^0 \in du) \\
&= \int_S k(db) E_b[(\sigma_a^0 - \sigma_\varepsilon(X^0)) \wedge 1,\ \sigma_a^0 > \sigma_\varepsilon(X^0)] \\
&= \int_S k(db) E_b[E_{X(\sigma_\varepsilon(X^0))}(\sigma_a^0 \wedge 1),\ \sigma_a^0 > \sigma_\varepsilon(X^0)] \\
&\geq \delta \int_S k(db) P_b(\sigma_a^0 > \sigma_\varepsilon(X^0)) \\
&= \delta \int_S k(db) P_b(X^0 \in V_\varepsilon) \\
&= \delta n_X(V_\varepsilon)
\end{aligned}$$

and that

$$\begin{aligned}
\int_U h(u) \wedge 1\, n_X(du) &= \int_U h(u) \wedge 1 \int_S k(db) P_b(X^0 \in du) \\
&= \int_S k(db) E_b(h(X^0) \wedge 1) \\
&= \int_S k(db) E_b(\sigma_a^0 \wedge 1).
\end{aligned}$$

Thus we have $n_X(V_\varepsilon) < \infty$.

The *proof of uniqueness* is easy, because the probability law of the path of (X_t^0) and k determine n_X and so the probability law of X, which, combined with m determines the probability law of the path of X_t. □

2.5 The Resolvent Operator and the Generator of the Markov Process Constructed in Sect. 2.4

The generator of a Markov process is defined in many ways which are not always equivalent to each other. We will adopt the following definition due to E.B. Dynkin.

Let X_t be a Markov process with right continuous paths. The *transition probability* $p(t, b, E)$ is defined by

$$p(t, b, E) = P_b(X_t \in E),$$

and the *transition operator* p_t is defined by

$$p_t f(b) = \int_S p(t, b, dc) f(c) = E_b(f(X_t)).$$

p_t carries the space $\mathbb{B}(S)$ of all bounded real Borel measurable functions into itself. It has the *semi-group property*:

$$p_{t+s} = p_t p_s, \quad p_0 = I \text{ (=identity operator)}.$$

The *resolvent operator* (*potential operator of order* α) R_α $(\alpha > 0)$ is defined by

$$R_\alpha = \int_0^\infty e^{-\alpha t} p_t \, dt$$

i.e.,

$$R_\alpha f(b) = \int_0^\infty e^{-\alpha t} p_t f(b) \, dt = E_b\Big(\int_0^\infty e^{-\alpha t} f(X_t) \, dt\Big).$$

It satisfies the *resolvent equations*:

$$R_\alpha - R_\beta + (\alpha - \beta) R_\alpha R_\beta = 0.$$

The *Dynkin subspace* $\mathbb{L}$ of $\mathbb{B}(S)$ is defined by

$$\mathbb{L} = \{f \in \mathbb{B}(S) : \lim_{t \downarrow 0} p_t f(b) = f(b) \text{ for every } b\}.$$

$\mathbb{L}$ is a linear subspace of $\mathbb{B}(S)$.

Because of the right continuity of the path of X_t we have

$$\mathbb{C}(S) \subset \mathbb{L} \subset \mathbb{B}(S),$$

$\mathbb{C}(S)$ being the space of all bounded continuous real functions on S.

It is easy to see that

$$p_t \mathbb{L} \subset \mathbb{L}, \quad R_\alpha \mathbb{L} \subset \mathbb{L}.$$

In view of this fact we will regard p_t and R_α as operators : $\mathbb{L} \to \mathbb{L}$, unless the contrary is stated explicitly.

By virtue of the resolvent equation $\mathscr{R} = R_\alpha \mathbb{L}$ is independent of α. $R_\alpha : \mathbb{L} \to \mathscr{R}$ is 1–1 and so R_α^{-1} is well-defined.

Definition 2.5.1 The generator $\mathscr{G}$ of (X_t) is defined by

$$\mathscr{D}(\mathscr{G}) = \Big\{ f \in \mathbb{L} : \frac{1}{t}(p_t f - f) \text{ converges boundedly as } t \to 0 \\ \text{to a function } \in \mathbb{L} \Big\}$$

and

$$\mathscr{G} f(b) = \lim_{t \downarrow 0} \frac{1}{t}(p_t f(b) - f(b)), \quad f \in \mathscr{D}(\mathscr{G}).$$

Theorem 2.5.2 $\mathscr{D}(\mathscr{G}) = \mathscr{R} = R_\alpha \mathbb{L}$, $\mathscr{G} f = \alpha f - R_\alpha^{-1} f$, $f \in \mathscr{D}(\mathscr{G})$.

Let X_t be a standard Markov process and a be a fixed state. We assume that (A-1), (A-2) and (A-3) are satisfied. Let $X_t^0 = X_{t \wedge \sigma_a}$. Then X_t^0 is also a standard Markov process with (A^0-1), (A^0-2) and (A^0-3). We will denote the transition operator, the resolvent operator and the generator of X_t respectively by p_t, R_α and $\mathscr{G}$ and the corresponding operators for X_t^0 are denoted by p_t^0, R_α^0 and $\mathscr{G}^0$.

Theorem 2.5.3 $\mathscr{D}(\mathscr{G}) \subset \mathscr{D}(\mathscr{G}^0)$ *and*

$$\mathscr{G} f(b) = \mathscr{G}^0 f(b), \quad b \neq a,$$
$$\mathscr{G}^0 f(a) = 0.$$

Proof If $f \in \mathscr{D}(\mathscr{G})$, then

$$f = R_\alpha g, \quad g \in \mathbb{L}.$$

By Dynkin's formula we have

$$\begin{aligned} f(b) &= E_b\Big(\int_0^\infty e^{-\alpha t} g(X_t)\, dt\Big) \\ &= E_b\Big(\int_0^{\sigma_a} e^{-\alpha t} g(X_t)\, dt\Big) + E_b(e^{-\alpha \sigma_a} f(X_{\sigma_a})) \\ &= E_b\Big(\int_0^{\sigma_a} e^{-\alpha t} g(X_t)\, dt\Big) + E_b(e^{-\alpha \sigma_a}) f(a). \end{aligned}$$

Set

$$g^0(b) = \begin{cases} g(b) & b \neq a, \\ \alpha R_\alpha g(a) = \alpha f(a) & b = a. \end{cases}$$

Then

$$\begin{aligned} R_\alpha^0 g^0(b) &= E_b\Big(\int_0^\infty e^{-\alpha t} g^0(X_t^0)\, dt\Big) \\ &= E_b\Big(\int_0^{\sigma_a} e^{-\alpha t} g(X_t)\, dt\Big) + E_b(e^{-\alpha \sigma_a}) R_\alpha^0 g^0(a). \end{aligned}$$

Since

$$R^0_\alpha g^0(a) = \int_0^\infty e^{-\alpha t} \alpha f(a)\, dt = f(a),$$

we have

$$f(b) = R^0_\alpha g^0(b).$$

To complete the proof of $\mathscr{D}(\mathscr{G}) \subset \mathscr{D}(\mathscr{G}^0)$, we need only prove that g^0 belongs to the Dynkin space $\mathbb{L}^0$ of X^0_t. Since $X^0_t = a$ for $t \geq \sigma_a$, we have

$$p^0_t g^0(a) = g^0(a) \longrightarrow g^0(a) \quad \text{as} \quad t \downarrow 0.$$

Suppose $b \neq a$. Then $P_b(\sigma_a > 0) = 1$ and so

$$\lim_{t \downarrow 0} P_b(\sigma_a \leq t) = 0.$$

Therefore

$$\begin{aligned}
&|p^0_t g^0(b) - g^0(b)| \\
=&|E_b(g^0(X^0_t) - g^0(b))| \\
=&|E_b(g(X_t), t < \sigma_a) + E_b(g^0(a), t \geq \sigma_a) - g(b)| \\
=&|E_b(g(X_t)) - E_b(g(X_t), t \geq \sigma_a) + E_b(g^0(a), t \geq \sigma_a) - g(b)| \\
=&|E_b(g(X_t)) - g(b)| + (\|g\| + |g^0(a)|) P_b(t \geq \sigma_a) \longrightarrow 0,
\end{aligned}$$

where $\|g\| = \sup_{c \in S} |g(c)|$. Since

$$f = R_\alpha g = R^0_\alpha g^0,$$

we have

$$\mathscr{G} f = \alpha f - g, \quad \mathscr{G}^0 f = \alpha f - g^0$$

and so

$$\mathscr{G}^0 f(b) = \mathscr{G} f(b) \quad \text{for} \quad b \neq a$$

and

$$\mathscr{G}^0 f(a) = \alpha f(a) - g^0(a) = 0. \qquad \square$$

Let X_t be the Markov process constructed from X^0_t, m and k in Sect. 2.4. The resolvent and the generator for X_t are denoted respectively by R_α and $\mathscr{G}$ and the corresponding operators for X^0_t are denoted respectively by R^0_α and $\mathscr{G}^0$.

We will discuss the relation between $(R_\alpha, \mathscr{G})$ and $(R^0_\alpha, \mathscr{G}^0)$. Let us make three cases.

Case 1. $k(S) = 0$. In this trivial case a is a trap for X_t and (X_t) is equivalent to (X^0_t), so that

$$R_\alpha = R^0_\alpha \quad \text{and} \quad \mathscr{G} = \mathscr{G}^0.$$

Case 2. $0 < k(S) < \infty$. ($m > 0$ in this case.) a is an exponential holding state with the rate $k(S)/m$.

Theorem 2.5.4 *If* $0 < k(S) < \infty$, *then*

$$R_\alpha g(b) = R^0_\alpha g(b) + E_b(e^{-\alpha\sigma^0_a})R_\alpha g(a) \quad \textit{for} \quad b \neq a; \tag{2.3}$$

$$R_\alpha g(a) = \frac{mg(a) + \displaystyle\int_S k(db)R^0_\alpha g(b)}{\alpha m + \displaystyle\int_S k(db)E_b(1 - e^{-\alpha\sigma^0_a})}; \tag{2.4}$$

$$\mathscr{G} f(b) = \mathscr{G}^0 f(b) \quad \textit{for} \quad b \neq a; \tag{2.5}$$

$$m\mathscr{G} f(a) = \int_S k(db)(f(b) - f(a)). \tag{2.6}$$

Proof Equation (2.3) is obvious by Dynkin's formula.

To prove (2.4), set

$$f(a) = R_\alpha g(a) \quad \text{and} \quad f^0(a) = R^0_\alpha g(a).$$

The Poisson point process $\boldsymbol{X}$ attached to (X_t) is discrete. Let σ be the first point in $D_{\boldsymbol{X}}$ and τ be the first exit time from a for (X_t). Let Y_t be the process derived from $\boldsymbol{X}$ in Sect. 2.4. By Dynkin's formula, we have

$$\begin{aligned}
f(a) &= E_a\Big(\int_0^\infty e^{-\alpha t} g(X_t)\,dt\Big) \\
&= E_a\Big(\int_0^\tau e^{-\alpha t} g(X_t)\,dt\Big) + E_a(e^{-\alpha\tau} f(X_\tau)) \\
&= E\Big(\int_0^{m\sigma} e^{-\alpha t} g(a)\,dt\Big) + E[e^{-\alpha m\sigma} f(\boldsymbol{X}_\sigma(0))] \\
&= g(a)E\Big[\frac{1 - e^{-\alpha m\sigma}}{\alpha}\Big] + E[e^{-\alpha m\sigma}]E[f(\boldsymbol{X}_\sigma(0))].
\end{aligned}$$

Observe

$$E(e^{-\alpha m\sigma}) = \int_0^\infty e^{-\alpha mt} e^{-tk(S)} k(S)\,dt$$
$$= \frac{k(S)}{\alpha m + k(S)}$$

and

$$E[f(\boldsymbol{X}_\sigma(0))] = \frac{1}{k(S)} \int_S k(db) f(b).$$

Therefore we have

$$f(a) = \frac{mg(a) + \int_S k(db) f(b)}{\alpha m + k(S)}, \tag{2.7}$$

which, combined with (2.3), implies

$$f(a) = \frac{mg(a) + \int_S k(db) f^0(b) + \int_S k(db) E_b(e^{-\alpha\sigma_a^0}) f(a)}{\alpha m + k(S)}.$$

Solving this for $f(a)$, we have

$$f(a) = \frac{mg(a) + \int_S k(db) f^0(b)}{\alpha m + \int_S k(db) E_b(1 - e^{-\alpha\sigma_a^0})},$$

which proves (2.4). Equation (2.5) is obvious by Theorem 2.5.3.

It follows from (2.7) that

$$m(\alpha f(a) - g(a)) = \int_S k(db)(f(b) - f(a)),$$

which proves (2.6). □

Case 3. $k(S) = \infty$. a is an instantaneous state for (X_t).

Theorem 2.5.5 *Theorem 2.5.4 holds also in case* $k(S) = \infty$:

$$\left(\int_S k(db)(f(b) - f(a))\right) = \lim_{\varepsilon \downarrow 0} \int_{\rho(b,a) > \varepsilon} k(db)(f(b) - f(a))$$

with the following proviso. If $m > 0$, *(2.4) holds for* g *with*

$$\lim_{b \to a} g(b) = g(a) \tag{2.8}$$

and (2.6) holds for $f = R_\alpha g$ with g satisfying the same condition.

Proof (2.3) and (2.5) are obvious. Let $\varepsilon > 0$ and set

$$\begin{aligned}
S^{1,\varepsilon} &= \{b \in S : \rho(b, a) \geq \varepsilon\},\\
S^{2,\varepsilon} &= S - S^{1,\varepsilon},\\
U^{i,\varepsilon} &= \{u \in U : u(0) \in S^{i,\varepsilon}\}, \quad i = 1, 2,\\
\boldsymbol{X}^{i,\varepsilon} &= \boldsymbol{X}|_r U^{i,\varepsilon}, \quad i = 1, 2.
\end{aligned}$$

Let $Y^{2,\varepsilon}(t)$ be the process derived from $\boldsymbol{X}^{2,\varepsilon}$ in the same way as Y_t was derived from $\boldsymbol{X}$ in Sect. 2.4. Since we fix ε for the moment, we omit ε in $S^{i,\varepsilon}$, $U^{i,\varepsilon}$ etc.

Let

$$J(t, \boldsymbol{X}) = \sum_{\substack{s \leq t \\ s \in D_{\boldsymbol{X}}}} h(\boldsymbol{X}_s).$$

Similarly for $J(t, \boldsymbol{X}^i)$. $\boldsymbol{X}^1$ is discrete. Let σ be the first element in $D_{\boldsymbol{X}^1}$.

Noticing that

$$s \in D_{\boldsymbol{X}}, \quad s < \sigma \Longrightarrow s \in D_{\boldsymbol{X}^2},$$

we have

$$J(\sigma-, \boldsymbol{X}) = J(\sigma-, \boldsymbol{X}^2), \quad \boldsymbol{X}_\sigma = \boldsymbol{X}^1_\sigma$$

and

$$Y_t = Y^2_t \quad \text{for} \quad t < m\sigma + J(\sigma-, \boldsymbol{X}) = m\sigma + J(\sigma-, \boldsymbol{X}^2).$$

$$\begin{aligned}
f(a) &\equiv R_\alpha g(a)\\
&= E\Big(\int_0^\infty e^{-\alpha t} g(Y_t)\, dt\Big)\\
&= E\Big(\int_0^{m\sigma + J(\sigma-, \boldsymbol{X})} e^{-\alpha t} g(Y_t)\, dt\Big)\\
&\quad + E\Big[e^{-\alpha m\sigma - \alpha J(\sigma-, \boldsymbol{X})} \int_0^{h(\boldsymbol{X}_\sigma)} e^{-\alpha t} g(\boldsymbol{X}_\sigma(t))\, dt\Big]\\
&\quad + E\Big[e^{-\alpha m\sigma - \alpha J(\sigma-, \boldsymbol{X}) - \alpha h(\boldsymbol{X}_\sigma)} \int_0^\infty e^{-\alpha t} g(Y(t, \theta_\sigma \boldsymbol{X}|_d(0, \infty)))\, dt\Big]\\
&= E\Big(\int_0^{m\sigma + J(\sigma-, \boldsymbol{X}^2)} e^{-\alpha t} g(Y^2_t)\, dt\Big)\\
&\quad + E\Big[e^{-\alpha m\sigma - \alpha J(\sigma-, \boldsymbol{X}^2)} \int_0^{h(\boldsymbol{X}^1_\sigma)} e^{-\alpha t} g(\boldsymbol{X}^1_\sigma(t))\, dt\Big]\\
&\quad + E\Big[e^{-\alpha m\sigma - \alpha J(\sigma-, \boldsymbol{X}^2) - \alpha h(\boldsymbol{X}^1_\sigma)} \int_0^\infty e^{-\alpha t} g(Y(t, \theta_\sigma \boldsymbol{X}|_d(0, \infty)))\, dt\Big]\\
&= I_1 + I_2 + I_3.
\end{aligned}$$

$\boldsymbol{X}^1$ and $\boldsymbol{X}^2$ are independent. σ and $\boldsymbol{X}^1_\sigma$ are $\mathscr{B}(\boldsymbol{X}^1)$ measurable and independent of each other. $\boldsymbol{X}^2$, σ and $\boldsymbol{X}^1_\sigma$ are therefore independent of each other. Thus we have

$$\begin{aligned} I_2 &= E[e^{-\alpha m\sigma - \alpha J(\sigma-,\boldsymbol{X}^2)}]E\Big[\int_0^{h(\boldsymbol{X}^1_\sigma)} e^{-\alpha t} g(\boldsymbol{X}^1_\sigma(t))\,dt\Big] \\ &= \int_0^\infty P(\sigma\in dt)e^{-\alpha mt}E[^{-\alpha J(t-,\boldsymbol{X}^2)}]\int_{S^1}\frac{k(db)}{k(S^1)}E_b\Big(\int_0^{\sigma_a^0} e^{-\alpha t} g(X_t^0)\,dt\Big). \end{aligned}$$

Since $J(t, \boldsymbol{X}^2)$ is a Lévy process increasing with jumps whose Lévy measure is equal to

$$n_{h\cdot \boldsymbol{X}^2}(dt) = \int_{S^2} k(db)P_b(\sigma_a^0 \in dt),$$

we have

$$\begin{aligned} E[e^{-\alpha J(t-,\boldsymbol{X}^2)}] &= e^{-t\int_0^\infty (1-e^{-\alpha s})n_{h\cdot \boldsymbol{X}^2}(ds)} \\ &= e^{-t\int_{S^2} k(db)E_b(1-e^{-\alpha\sigma_a^0})}. \end{aligned}$$

It is obvious that

$$P(\sigma\in dt) = e^{-k(S^1)t}k(S^1)\,dt.$$

Therefore

$$I_2 = \frac{\displaystyle\int_{S^1} k(db)E_b\Big(\int_0^{\sigma_a^0} e^{-\alpha t} g(X_t^0)\,dt\Big)}{\displaystyle\alpha m + k(S^1) + \int_{S^2} k(db)E_b(1-e^{-\alpha\sigma_a^0})}.$$

By the strong renewal property of $\boldsymbol{X}$ we have

$$\begin{aligned} I_3 &= E[e^{-\alpha m\sigma - \alpha J(\sigma-,\boldsymbol{X}^2) - \alpha h(\boldsymbol{X}^1_\sigma)}]E\Big[\int_0^\infty e^{-\alpha t} g(Y_t)\,dt\Big] \\ &= E[e^{-\alpha m\sigma - \alpha J(\sigma-,\boldsymbol{X}^2)}]E[e^{-\alpha h(\boldsymbol{X}^1_\sigma)}]f(a) \\ &= \frac{\displaystyle f(a)\int_{S^1} k(db)E_b(e^{-\alpha\sigma_a^0})}{\displaystyle\alpha m + k(S^1) + \int_{S^2} k(db)E_b(1-e^{-\alpha\sigma_a^0})}. \end{aligned}$$

Thus we have

$$f(a) = I_1 + \frac{\displaystyle\int_{S^1} k(db)\Big[E_b\Big(\int_0^{\sigma_a^0} e^{-\alpha t} g(X_t^0)\,dt\Big) + E_b(e^{-\alpha\sigma_a^0})f(a)\Big]}{\displaystyle\alpha m + k(S^1) + \int_{S^2} k(db)E_b(1-e^{-\alpha\sigma_a^0})}. \tag{2.9}$$

To evaluate I_1, consider

$$\begin{aligned}
f^2(a) &\equiv E\Big(\int_0^\infty e^{-\alpha t} g(Y_t^2)\,dt\Big)\\
&= I_1 + E\Big(e^{-\alpha m\sigma - \alpha J(\sigma-,\boldsymbol{X}^2)} \int_0^\infty e^{-\alpha t} g(Y(t, \theta_\sigma \boldsymbol{X}^2|_d(0,\infty)))\,dt\Big)\\
&= I_1 + \int_0^\infty P(\sigma \in ds)\\
&\quad \times E\Big(e^{-\alpha m s - \alpha J(s-,\boldsymbol{X}^2)} \int_0^\infty e^{-\alpha t} g(Y(t, \theta_s \boldsymbol{X}^2|_d(0,\infty)))\,dt\Big)\\
&= I_1 + \int_0^\infty P(\sigma \in ds)\\
&\quad \times E(e^{-\alpha m s - \alpha J(s-,\boldsymbol{X}^2)}) E\Big(\int_0^\infty e^{-\alpha t} g(Y(t, \boldsymbol{X}^2))\,dt\Big)\\
&\qquad \text{(by the renewal property of} \boldsymbol{X}^2\text{)}\\
&= I_1 + \int_0^\infty P(\sigma \in ds) E(e^{-\alpha m s - \alpha J(s-,\boldsymbol{X}^2)}) f^2(a)\\
&= I_1 + E(e^{-\alpha m\sigma - \alpha J(\sigma-,\boldsymbol{X}^2)}) f^2(a).
\end{aligned}$$

This implies

$$\begin{aligned}
I_1 &= f^2(a)[1 - E(e^{-\alpha m\sigma - \alpha J(\sigma-,\boldsymbol{X}^2)})]\\
&= f^2(a)\Big[1 - \frac{k(S^1)}{\alpha m + k(S^1) + \int_{S^2} k(db) E_b(1 - e^{-\alpha\sigma_a^0})}\Big]\\
&= f^2(a)\frac{\alpha m + \int_{S^2} k(db) E_b(1 - e^{-\alpha\sigma_a^0})}{\alpha m + k(S^1) + \int_{S^2} k(db) E_b(1 - e^{-\alpha\sigma_a^0})}.
\end{aligned}$$

From (2.9) we have

$$\begin{aligned}
f(a) =& f^2(a)\frac{\alpha m + \int_{S^2} k(db) E_b(1 - e^{-\alpha\sigma_a^0})}{\alpha m + k(S^1) + \int_{S^2} k(db) E_b(1 - e^{-\alpha\sigma_a^0})}\\
&+ \frac{\int_{S^1} k(db)\Big[E_b\Big(\int_0^{\sigma_a^0} e^{-\alpha t} g(X_t^0)\,dt\Big) + E_b(e^{-\alpha\sigma_a^0}) f(a)\Big]}{\alpha m + k(S^1) + \int_{S^2} k(db) E_b(1 - e^{-\alpha\sigma_a^0})}.
\end{aligned} \tag{2.10}$$

Solving this for f we have

$$f(a)=\frac{f^2(a)\Big(\alpha m+\int_{S^2}k(db)E_b(1-e^{-\alpha\sigma_a^0})\Big)+\int_{S^1}k(db)E_b\Big(\int_0^{\sigma_a^0}e^{-\alpha t}g(X_t^0)\,dt\Big)}{\alpha m+\int_S k(db)E_b(1-e^{-\alpha\sigma_a^0})}. \tag{2.11}$$

Let $\varepsilon\downarrow 0$, then

$$\int_{S^1}k(db)E_b\Big(\int_0^{\sigma_a^0}e^{-\alpha t}g(X_t^0)\,dt\Big)\longrightarrow\int_S k(db)E_b\Big(\int_0^{\sigma_a^0}e^{-\alpha t}g(X_t^0)\,dt\Big);$$

notice that

$$\begin{aligned}&\int_S k(db)\Big|E_b\Big(\int_0^{\sigma_a^0}e^{-\alpha t}g(X_t^0)\,dt\Big)\Big|\\ \le&\|g\|\int_S k(db)E_b(1-e^{-\alpha\sigma_a^0})\\ <&\infty,\quad \|\ \ \|=\text{sup. norm}\end{aligned}$$

by virtue of $\int_S k(db)E_b(\sigma_a^0\wedge 1)<\infty$. It is obvious that

$$\int_{S^2}k(db)E_b(1-e^{-\alpha\sigma_a^0})\longrightarrow 0\quad\text{as}\quad\varepsilon\downarrow 0.$$

It follows from (2.10) and (2.3) that

$$\begin{aligned}f(a)=f^2(a)&\frac{\alpha m+\int_{S^2}k(db)E_b(1-e^{-\alpha\sigma_a^0})}{\alpha m+k(S^1)+\int_{S^2}k(db)E_b(1-e^{-\alpha\sigma_a^0})}\\ &+\frac{\int_{S^1}k(db)f(b)}{\alpha m+k(S^1)+\int_{S^2}k(db)E_b(1-e^{-\alpha\sigma_a^0})}\end{aligned}$$

so that

$$\begin{aligned}&m(\alpha f(a)-\alpha f^2(a))+f(a)\int_{S^2}k(db)E_b(1-e^{-\alpha\sigma_a^0})\\ =&f^2(a)\int_{S^2}k(db)E_b(1-e^{-\alpha\sigma_a^0})+\int_{S^1}k(db)(f(b)-f(a)).\end{aligned} \tag{2.12}$$

If $m = 0$, we can derive (2.4) and (2.6) from (2.11) and (2.12), letting $\varepsilon \downarrow 0$ and noticing that

$$|f^2(a)| \equiv |f^{2,\varepsilon}(a)| \le \|g\|/\alpha.$$

If $m > 0$, we need only prove that

$$\lim_{\varepsilon \downarrow 0} f^{2,\varepsilon}(a) = \frac{g(a)}{\alpha}, \tag{2.13}$$

in order to derive (2.4) and (2.6) from (2.11) and (2.12).

Let $\eta > 0$ and set

$$\begin{aligned}
V^1 &= V^{1,\eta} = \{u \in U : \sup_t \rho(u(t), a) \ge \eta\},\\
V^2 &= V^{2,\eta} = U - V^1,\\
\boldsymbol{Y}^i &= \boldsymbol{Y}^{i,\varepsilon,\eta} = \boldsymbol{X}^{2,\varepsilon}|_r V^{i,\eta}, \quad i = 1, 2.
\end{aligned}$$

By the argument in the last step of the existence proof of Theorem 2.4.1, we have

$$\begin{aligned}
\lambda \equiv \lambda_{\varepsilon,\eta} &\equiv n_{\boldsymbol{Y}^1}(V^1) = n_{\boldsymbol{X}^{2,\varepsilon}}(V^{1,\eta})\\
&\le \Big(\inf_{\rho(b,a)>\eta} E_b(\sigma_a^0 \wedge 1)\Big)^{-1} \int_{S^{2,\varepsilon}} k(db) E_b(\sigma_a^0 \wedge 1)\\
&\longrightarrow 0, \quad \varepsilon \downarrow 0 \quad \text{for } \eta \text{ fixed.}
\end{aligned}$$

$\boldsymbol{Y}^1$ is a discrete Poisson point process. Let $\tau = \tau_{\varepsilon,\eta}$ be the first element in $D_{\boldsymbol{Y}^1}$. Then τ is exponentially distributed with rate $= \lambda_{\varepsilon,\eta}$. Using the same argument as in deriving (2.9), we obtain

$$\begin{aligned}
&\Big| f^{2,\varepsilon}(a) - \frac{g(a)}{\alpha} \Big|\\
&\le E\Big(\int_0^\infty e^{-\alpha t} g_0(Y_t^{2,\varepsilon})\, dt\Big), \quad g_0(b) = |g(b) - g(a)|\\
&= E\Big(\int_0^{m\tau + J(\tau-, \boldsymbol{Y}^2)} e^{-\alpha t} g_0(Y(t, \boldsymbol{Y}^2))\, dt\Big)\\
&\quad + E(e^{-\alpha m\tau - \alpha J(\tau-, \boldsymbol{Y}^2)}) E\Big(\int_0^{h(\boldsymbol{Y}_\tau^1)} e^{-\alpha t} g_0(\boldsymbol{Y}_\tau^1(t))\, dt\Big)\\
&\quad + E(e^{-\alpha m\tau - \alpha J(\tau-, \boldsymbol{Y}^2) - \alpha h(\boldsymbol{Y}_\tau^1)}) E\Big(\int_0^\infty e^{-\alpha t} g_0(Y_t^{2,\varepsilon})\, dt\Big).
\end{aligned}$$

Since $\rho(Y(t, \boldsymbol{Y}^2), a) < \eta$ for $0 < t < m\tau + J(\tau-, \boldsymbol{Y}^2)$, we have

$$\Big| f^{2,\varepsilon}(a) - \frac{g(a)}{\alpha} \Big| \le \delta(\eta)\frac{1}{\alpha} + E(e^{-\alpha m\tau})\frac{\|g_0\|}{\alpha} + E(e^{-\alpha m\tau})\frac{\|g_0\|}{\alpha}$$

where $\delta(\eta) = \sup\{g_0(b), \rho(b,a) < \eta\} \to 0$ $(\eta \downarrow 0)$ by (2.8). Since τ is exponentially distributed with rate $\lambda_{\varepsilon,\eta}$, we have

$$E(e^{-\alpha m\tau}) = \int_0^\infty e^{-\alpha m t} e^{-\lambda_{\varepsilon,\eta} t} \lambda_{\varepsilon,\eta}\, dt = \frac{\lambda_{\varepsilon,\eta}}{\alpha m + \lambda_{\varepsilon,\eta}} \longrightarrow 0, \quad \varepsilon \downarrow 0$$

by $m > 0$. Thus we have

$$\limsup_{\varepsilon\downarrow 0} \left| f^{2,\varepsilon}(a) - \frac{g(a)}{\alpha} \right| \le \delta(\eta) \cdot \frac{1}{\alpha} \longrightarrow 0, \quad \eta \downarrow 0.$$

This completes the proof. □

2.6 Examples

Example 1 Let $S = [0,\infty)$ and X^0 be a diffusion in S stopped at 0 such that the generator of X^0 is

$$\mathscr{G}^0 = \frac{d}{dm}\frac{d}{dx}.$$

Let 0 be an *exit or regular* (i.e., *exit* in Feller's new terminology) *boundary*, i.e.,

$$\int_0^1 m(\xi,1)\, d\xi < \infty.$$

Then X^0 satisfies (A^0-1), (A^0-2) and (A^0-3) in Sect. 2.1; notice that

$$\inf_{\rho(b,0)>\varepsilon} E_b(\sigma_0^0 \wedge 1) = E_\varepsilon(\sigma_0^0 \wedge 1) > 0.$$

We will investigate the condition (i) in Theorem 2.3.8:

$$\int_S k(db) E_b(\sigma_0^0 \wedge 1) < \infty. \tag{2.14}$$

This is equivalent to

$$\int_S k(db) E_b(1 - e^{-\sigma_0^0}) < \infty.$$

Since $u(b) = E_b(e^{-\sigma_0^0})$ is a decreasing positive solution of

$$\frac{d}{dm}\frac{d}{dx}u = u, \quad u(0) = 1,$$
$$u'(1) - u'(\xi) = \int_\xi^1 u(\xi)m(d\xi) \approx m(\xi, 1) \quad (\xi \downarrow 0),$$
$$u(0) - u(b) \approx \int_0^b (m(\xi, 1) - u'(1))\, d\xi$$

($\alpha(\xi) \approx \beta(\xi)$ $(\xi \downarrow 0)$ means that we have $c_1, c_2 > 0$ independent of ξ such that $c_1\beta(\xi) < \alpha(\xi) < c_2\beta(\xi)$ near $\xi = 0$).

Case 1. (regular case) If 0 is a *regular* (i.e., *exit and entrance* in Feller's new terminology) *boundary*, i.e., $m(0, 1) < \infty$, then

$$E_b(1 - e^{-\sigma_0^0}) = u(0) - u(b) \approx b \quad (b \downarrow 0).$$

Since $E_b(1 - e^{-\sigma_0^0}) \to 1$ as $b \to \infty$, $E_b(1 - e^{-\sigma_0^0}) \approx b \wedge 1$ in $0 < b < \infty$. Therefore our condition (2.14) turns out to be

$$\int_0^\infty k(db)(b \wedge 1) < \infty.$$

Case 2. (exit case) If 0 is an *exit* (i.e., *exit and non-entrance* in Feller's new terminology) *boundary*, i.e., $m(0, 1) = \infty$, then (2.14) turns out to be

$$\int_0^\infty k(db)\Big[\int_0^b m(\xi, 1)\, d\xi \wedge 1\Big] < \infty.$$

Example 2 Let $S = [0, \infty)$ and X^0 be a deterministic motion with constant speed "-1". Then $P_b(\sigma_0^0 = b) = 1$ and so (2.14) is written as

$$\int_0^\infty k(db)(b \wedge 1) < \infty.$$

Reference

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