Chapter 2
Order Statistics

2.1 Introduction

Order Statistics naturally appear in real life whenever we need to arrange observations in ascending order; say for example prices arranged from smallest to largest, scores scored by a player in last ten innings from smallest to largest and so on. The study of order statistics needs special considerations due to their natural dependence. The study of order statistics has attracted many statistician in the past. Formerly, order statistics are defined in the following.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution $F(x)$ and so all X_i are i.i.d. random variables having same distribution $F(x)$. The arranged sample $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ is called the *Ordered Sample* and the rth observation in the ordered sample; denoted as $X_{r:n}$ or $X_{(r)}$; is called the rth *Order Statistics.* The realized ordered sample is written as $x_{1:n} \leq x_{2:n} \leq \ldots \leq x_{n:n}$. The distribution of rth order statistics and joint distribution of rth and sth order statistics are given below.

2.2 Joint Distribution of Order Statistics

The joint distribution of all order statistics plays an important role in deriving several special distributions of individual and group of order statistics. The joint distribution of all order statistics is easily derived from the marginal distributions of available random variables. We know that if we have a random sample of size n from a distribution function $F(x)$ as $X_1, X_2, \ldots, X_n$ then the joint distribution of all sample observations is

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i);$$

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where $f(x_i)$ is the density function of X_i. Now since all possible ordered permutations of $X_1, X_2, \ldots, X_n$ can be done in $n!$ ways, therefore the joint density function of all order statistics is readily written as

$$f(x_{1:n}, x_{2:n}, \ldots, x_{n:n}) = n! \prod_{i=1}^{n} f(x_i). \tag{2.1}$$

The joint density function of all order statistics given in (2.1) is very useful in deriving the marginal density function of a single and group of order statistics.

The joint density function of all order statistics is useful in deriving the distribution of a set of order statistics. Specifically, the joint distribution of r order statistics; $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{r:n}$; is derived as below

$$\begin{aligned} f_{1,\ldots r:n}(x_1, \ldots, x_r) &= \int_{x_r}^{\infty} \cdots \int_{x_r}^{x_{r+3}} \int_{x_r}^{x_{r+2}} f(x_{1:n}, x_{2:n}, \ldots, x_{n:n}) \\ &\quad \times dx_{r+1} \cdots dx_n \\ &= \int_{x_r}^{\infty} \cdots \int_{x_r}^{x_{r+3}} \int_{x_r}^{x_{r+2}} n! \prod_{i=1}^{n} f(x_i) dx_{r+1} \cdots dx_n \\ &= n! \prod_{i=1}^{r} f(x_i) \int_{x_r}^{\infty} \cdots \int_{x_r}^{x_{r+3}} \int_{x_r}^{x_{r+2}} \prod_{i=r+1}^{n} f(x_i) \\ &\quad \times dx_{r+1} \cdots dx_n \\ &= \frac{n!}{(n-r)!} \left[\prod_{i=1}^{r} f(x_i) \right] [1 - F(x_r)]^{n-r}. \end{aligned} \tag{2.2}$$

Expression (2.2) can be used to obtain the joint marginal distribution of any specific number of order statistics.

The distribution of a single order statistics and joint distribution of two order statistics has found many applications in diverse areas of life. In the following we present the marginal distribution of a single order statistics.

2.3 Marginal Distribution of a Single Order Statistics

The marginal distribution of rth order statistics $X_{r:n}$ can be obtained in different ways. The distribution can be obtained by first obtaining the distribution function of $X_{r:n}$ and then that distribution function can be used to obtain the density function of $X_{r:n}$ as given in Arnold, Balakrishnan and Nagaraja (2008) and David and Nagaraja (2003). We obtain the distribution function of $X_{r:n}$ by first obtaining distribution function of $X_{n:n}$; the largest observation; and $X_{1:n}$; the smallest observation.

The distribution function of $X_{n:n}$ is denoted as $F_{n:n}(x)$ and is given as

$$\begin{aligned} F_{n:n}(x) &= P\{X_{n:n} \le x\} \\ &= P\{all\ X_i \le x\} = F^n(x). \end{aligned} \tag{2.3}$$

Again the distribution function of $X_{1:n}$; denoted as $F_{1:n}(x)$; is

$$\begin{aligned} F_{n:n}(x) &= P\{X_{1:n} \le x\} = 1 - P\{X_{1:n} > x\} \\ &= 1 - P\{all\ X_i > x\} = 1 - [1 - F(x)]^n. \end{aligned} \tag{2.4}$$

Now the distribution function of $X_{r:n}$; the rth order statistics; is denoted as $F_{r:n}(x)$ and is given as

$$\begin{aligned} F_{r:n}(x) &= P\{X_{r:n} \le x\} \\ &= P\{atleast\ r\ of\ X_i\ are\ less\ than\ or\ equal\ to\ x\} \\ &= \sum_{i=r}^{n} \binom{n}{i} F^i(x)[1 - F(x)]^{n-i}. \end{aligned} \tag{2.5}$$

Now using the relation

$$\sum_{i=r}^{n} \binom{n}{i} p^i (1-p)^{n-i} = \int_0^p \frac{n!}{(r-1)!(n-r)!} t^{r-1}(1-t)^{n-r} dt;$$

the distribution function of $X_{r:n}$ is given as

$$\begin{aligned} F_{r:n}(x) &= \sum_{i=r}^{n} \binom{n}{i} F^i(x)[1 - F(x)]^{n-i} \\ &= \int_0^{F(x)} \frac{n!}{(r-1)!(n-r)!} t^{r-1}(1-t)^{n-r} dt \\ &= \int_0^{F(x)} \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)} t^{r-1}(1-t)^{n-r} dt \\ &= \frac{1}{B(r, n-r+1)} \int_0^{F(x)} t^{r-1}(1-t)^{n-r} dt \\ &= I_{F(x)}(r, n-r+1); \end{aligned} \tag{2.6}$$

where $I_x(a, b)$ is incomplete Beta Function ratio. From (2.6) we see that the distribution function of $X_{r:n}$ resembles with the distributions proposed by Eugene, Lee and Famoye (2002). Expression (2.6) is valid either if sample has been drawn from a discrete distribution. An alternative form for the distribution function of $X_{r:n}$ is given as

$$\begin{aligned}F_{r:n}(x) &= \sum_{i=r}^{n}\binom{n}{i}F^{i}(x)[1-F(x)]^{n-i}\\ &= \sum_{i=r}^{n}\binom{n}{i}F^{i}(x)\sum_{k=0}^{n-i}(-1)^{k}\binom{n-i}{k}F^{k}(x)\\ &= \sum_{i=r}^{n}\sum_{k=0}^{n-i}(-1)^{k}\binom{n}{i}\binom{n-i}{k}F^{i+k}(x). \qquad (2.7)\end{aligned}$$

Assuming that $X_i's$ are absolutely continuous, the density function of $X_{r:n}$; denoted by $f_{r:n}(x)$; is easily obtained from (2.6) as below

$$\begin{aligned}f_{r:n}(x) &= \frac{d}{dx}F_{r:n}(x)\\ &= \frac{d}{dx}\left[\frac{1}{B(r,n-r+1)}\int_0^{F(x)} t^{r-1}(1-t)^{n-r}dt\right]\\ &= \frac{1}{B(r,n-r+1)}\left[\frac{d}{dx}\left\{\int_0^{F(x)} t^{r-1}(1-t)^{n-r}dt\right\}\right]\\ &= \frac{1}{B(r,n-r+1)}f(x)F^{r-1}(x)[1-F(x)]^{n-r}\\ &= \frac{n!}{(r-1)!(n-r)!}f(x)F^{r-1}(x)[1-F(x)]^{n-r}. \qquad (2.8)\end{aligned}$$

The density function of $X_{1:n}$ and $X_{n:n}$ can be immediately written from (2.8) as

$$f_{1:n}(x) = nf(x)[1-F(x)]^{n-1}$$

and

$$f_{n:n}(x) = nf(x)F^{n-1}(x).$$

The distribution of $X_{r:n}$ can also be derived by using the multinominal distribution as under:

Recall that probability mass function of multinomial distribution is

$$P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k) = \frac{n!}{y_1!y_2!\ldots y_k!}p_1^{y_1}p_2^{y_2}\ldots p_k^{y_k};$$

and can be used to compute the probabilities of joint occurrence of events. Now the place of $x_{r;n}$ in ordered sample can be given as

$$\underbrace{x_{1:n} \le x_{2:n} \le \ldots \le x_{r-1:n}}_{\substack{r-1\ observations\\ Event\ 1}} \le \underbrace{x_{r:n}}_{Event\ 2} \le \underbrace{x_{r+1:n} \le \ldots \le x_{n:n}}_{\substack{n-r\ observations\\ Event\ 3}}$$

In the above probability of occurrence of *Event 1* is $F(x)$, that of *Event 2* is $f(x)$ and probability of *Event 3* is $[1 - F(x)]$. Hence the joint occurrence of above three events; which is equal to density of $X_{r:n}$ is

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x) F^{r-1}(x)[1 - F(x)]^{n-r};$$

which is (2.8). When the density $f(x)$ is symmetrical about μ then the distributions of rth and $(n-r+1)th$ order statistics are related by relation

$$f_{r:n}(\mu + x) = f_{n-r+1:n}(\mu - x).$$

Above relation is very useful in moment relations of order statistics.

When the sample has been drawn from a discrete distribution with distribution function $F(x)$ then the density of $X_{r:n}$ can be obtained as below

$$\begin{aligned} f_{r:n}(x) &= F_{r:n}(x) - F_{r:n}(x-1) \\ &= I_{F(x)}(r, n-r+1) - I_{F(x-1)}(r, n-r+1) \\ &= P\big\{F(x-1) < T_{r,n-r+1} < F(x)\big\} \\ &= \frac{1}{B(r, n-r+1)} \int_{F(x-1)}^{F(x)} u^{r-1}(1-u)^{n-r} du. \end{aligned} \tag{2.9}$$

Expression (2.9) is the probability mass function of $X_{r:n}$ when sample is available from a discrete distribution. The probability mass function of $X_{r:n}$ can also be written in binomial sum as under

$$\begin{aligned} f_{r:n}(x) &= F_{r:n}(x) - F_{r:n}(x-1) \\ &= \sum_{i=r}^{n} \binom{n}{i} F^i(x)[1 - F(x)]^{n-i} \\ &\quad - \sum_{i=r}^{n} \binom{n}{i} F^i(x-1)[1 - F(x-1)]^{n-i} \\ &= \sum_{i=r}^{n} \binom{n}{i} \big\{F^i(x)[1 - F(x)]^{n-i} \\ &\quad - F^i(x-1)[1 - F(x-1)]^{n-i}\big\}. \end{aligned} \tag{2.10}$$

We now obtain the joint distribution of two ordered observations, namely $X_{r:n}$ and $X_{s:n}$ for $r \leq s$; in the following.

2.4 Joint Distribution of Two Order Statistics

Suppose we have random sample of size n from $F(x)$ and observations are arranged as $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$. The joint distribution function of $X_{r:n}$ and $X_{s:n}$ for $r \leq s$ is given by Arnold et al. (2008) as

$$\begin{aligned} F_{r,s:n}(x_r, x_s) &= P(X_{r:n} \leq x_r, X_{s:n} \leq x_s) \\ &= P(atleast\ r\ of\ X_i\ are\ less\ than\ or\ equal\ to\ x_r \\ &\quad and\ atleast\ s\ of\ X_i\ are\ less\ than\ or\ equal\ to\ x_s) \\ &= \sum_{j=s}^{n}\sum_{i=r}^{s} P(Exactly\ r\ of\ X_i\ are\ less\ than\ or\ equal\ to\ x_r \\ &\quad and\ exactly\ s\ of\ X_i\ are\ less\ than\ or\ equal\ to\ x_s) \\ &= \sum_{j=s}^{n}\sum_{i=r}^{s} \frac{n!}{i!(j-i)!(n-j)!} F^i(x_r) \\ &\quad \times [F(x_s) - F(x_r)]^{j-i}[1 - F(x_s)]^{n-j}. \end{aligned}$$

Now using the relation

$$\begin{aligned} &\sum_{j=s}^{n}\sum_{i=r}^{s} \frac{n!}{i!(j-i)!(n-j)!} p_1^i (p_2 - p_1)^{j-i}(1 - p_2)^{n-j} \\ &= \int_0^{p_1}\int_{t_1}^{p_2} \frac{n!}{(r-1)!(s-r-1)!(n-s)!} t_1^{r-1}(t_2 - t_1)^{s-r-1}(1 - t_2)^{n-s} dt_2 dt_1; \end{aligned}$$

we can write the joint distribution function of two order statistics as

$$\begin{aligned} F_{r,s:n}(x_r, x_s) = &\int_0^{F(x_r)}\int_{t_1}^{F(x_s)} \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \\ &\times t_1^{r-1}(t_2 - t_1)^{s-r-1}(1 - t_2)^{n-s} dt_2 dt_1 \\ &; -\infty < x_r < x_s < \infty; \end{aligned} \quad (2.11)$$

which is incomplete bivariate beta function ratio. Expression (2.11) holds for both discrete and continuous random variables. When $F(x)$ is absolutely continuous then density function of $X_{r:n}$ and $X_{s:n}$ can be obtained from (2.11) and is given as

$$\begin{aligned} f_{r,s:n}(x_r, x_s) &= \frac{d^2}{dx_r dx_s} F_{r,s:n}(x_r, x_s) \\ &= \frac{d^2}{dx_r dx_s}\Big[\int_0^{F(x_r)}\int_{t_1}^{F(x_s)} \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \\ &\quad t_1^{r-1}(t_2 - t_1)^{s-r-1}(1 - t_2)^{n-s} dt_2 dt_1\Big] \end{aligned}$$

or

$$f_{r,s:n}(x_r, x_s) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \times$$
$$\frac{d^2}{dx_r dx_s}\left[\int_0^{F(x_r)}\int_{t_1}^{F(x_s)} t_1^{r-1}(t_2-t_1)^{s-r-1}(1-t_2)^{n-s}dt_2dt_1\right]$$
$$= \frac{1}{B(r, s-r, n-s+1)} f(x_r)f(x_s)F^{r-1}(x_r)$$
$$\times\left[F(x_s)-F(x_r)\right]^{s-r-1}[1-F(x_s)]^{n-s}$$

or

$$f_{r,s:n}(x_r, x_s) = C_{r,s,n} f(x_r)f(x_s)F^{r-1}(x_r)[F(x_s)-F(x_r)]^{s-r-1}$$
$$\times[1-F(x_s)]^{n-s}, \; -\infty < x_r < x_s < \infty, \tag{2.12}$$

where $C_{r,s,n} = [B((r, s-r, n-s+1))]^{-1} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

The joint probability mass function $P\{X_{r:n} = x; X_{s:n} = y\}$ of $X_{r:n}$ and $X_{s:n}$ can be obtained by using the fact that

$$f_{r,s:n}(x, y) = F_{r,s:n}(x, y) - F_{r,s:n}(x-1, y)$$
$$-F_{r,s:n}(x, y-1) + F_{r,s:n}(x-1, y-1)$$
$$= P\{F(x-1) < T_r \le F(x), F(y-1) < T_s \le F(y)\}$$
$$= C_{r,s,n}\int\int_B v^{r-1}(w-v)^{s-r-1}(1-w)^{n-s}dvdw \; ; \tag{2.13}$$

where integration is over the region

$$\{(v, w) : v \le w, F(x-1) \le v \le F(x), F(y-1) \le w \le F(y)\}\cdot$$

Consider the joint distribution of two order statistics as

$$f_{r,s:n}(x_r, x_s) = C_{r,s,n} f(x_r)f(x_s)F^{r-1}(x_r)\left[F(x_s)-F(x_r)\right]^{s-r-1}$$
$$\times[1-F(x_s)]^{n-s};$$

where $C_{r,s,n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$. Using $r = 1$ and $s = n$ the joint density of smallest and largest observation is readily written as

$$f_{1,n:n}(x_1, x_n) = n(n-1)f(x_1)f(x_n)[F(x_n)-F(x_1)]^{n-2}. \tag{2.14}$$

Further, for $s = r + 1$ the joint distribution of two contiguous order statistics is

$$f_{r,r+1:n}(x_r, x_{r+1}) = \frac{n!}{(r-1)!(n-r-1)!} f(x_r) f(x_{r+1}) F^{r-1}(x_r) \times \Big[1 - F(x_{r+1})\Big]^{n-r-1}. \quad (2.15)$$

Analogously, the joint distribution of any k order statistics $X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_k:n}$; for $x_1 \le x_2 \le \cdots \le x_k$; is

$$f_{r_1,r_2,\ldots,r_k:n}(x_1, x_2, \cdots, x_k) = n! \prod_{j=0}^{k} \left\{ \frac{\big[F(x_{r+1}) - F(x_r)\big]^{r_{j+1}-r_j-1}}{\big(r_{j+1} - r_j - 1\big)!} \right\} \times \left\{ \prod_{j=1}^{k} f\big(x_j\big) \right\}. \quad (2.16)$$

where $x_0 = -\infty, x_{n+1} = +\infty, r_0 = 0$ and $r_{n+1} = n + 1$. Expression (2.16) can be used to obtain joint distribution of any number of ordered observations.

Example 2.1 A random sample is drawn from Uniform distribution over the interval [0, 1]. Obtain distribution of rth order statistics and joint distribution of two order statistics.

Solution: The density and distribution function of $U(0, 1)$ are

$$f(u) = 1; \;\; F(u) = u.$$

The distribution of rth order statistics is

$$f_{r:n}(u) = \frac{1}{B(r, n-r+1)} f(u) F^{r-1}(u)[1 - F(u)]^{n-r} = \frac{1}{B(r, n-r+1)} u^{r-1}(1-u)^{n-r};$$

which is a Beta random variable with parameters r and $n - r + 1$. Again the joint distribution of two order statistics is

$$f_{r,s:n}(u_r, u_s) = \frac{1}{B(r, s-r, n-s+1)} f(u_r) f(u_s) F^{r-1}(u_r) \times \Big[F(u_s) - F(u_r)\Big]^{s-r-1} [1 - F(u_s)]^{n-s} = \frac{1}{B(r, s-r, n-s+1)} u_r^{r-1}(u_s - u_r)^{s-r-1}(1 - u_s)^{n-s}.$$

The joint distribution of largest and smallest observation is immediately written as

$$f_{r,s:n}(u_r, u_s) = n(n-1)(u_n - u_1)^{n-2}.$$

Example 2.2 A random sample of size n is drawn from the standard power function distribution with density

$$f(x) = vx^{v-1}; 0 < x < 1, v > 0.$$

Obtain the distribution of rth order statistics and joint distribution of rth and sth statistics.

Solution: For given distribution we have

$$F(x) = \int_0^x f(t)dt = \int_0^x vt^{v-1}dt = x^v; 0 < x < 1.$$

Now distribution of rth order statistics is

$$\begin{aligned} f_{r:n}(x) &= \frac{1}{B(r, n-r+1)} f(x)F^{r-1}(x)[1 - F(x)]^{n-r} \\ &= \frac{1}{B(r, n-r+1)} vx^{rv-1}(1 - x^v)^{n-r}. \end{aligned}$$

The distribution function of rth order statistics is readily written as

$$\begin{aligned} F_{r:n}(x) &= \frac{1}{B(r, n-r+1)} \int_0^{F(x)} t^{r-1}(1-t)^{n-r}dt \\ &= \frac{1}{B(r, n-r+1)} \int_0^{x^v} t^{r-1}(1-t)^{n-r}dt \\ &= \sum_{i=r}^{n} \binom{n}{i} x^{iv}(1 - x^v)^{n-i}. \end{aligned}$$

The joint distribution of $X_{r:n}$ and $X_{s:n}$ is

$$\begin{aligned} f_{r,s:n}(x_r, x_s) &= \frac{1}{B(r, s-r, n-s+1)} f(x_r)f(x_s)F^{r-1}(x_r) \\ &\quad \times \Big[F(x_s) - F(x_r)\Big]^{s-r-1} [1 - F(x_s)]^{n-s} \\ &= C_{r,s,n} v^2 x_r^{rv-1} x_s^{v-1} \left(x_s^v - x_r^v\right)^{s-r-1} \left(1 - x_s^v\right)^{n-s}. \end{aligned}$$

where $C_{r,s,n} = [B(r, s-r, n-s+1)]^{-1} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

2.5 Distribution of Range and Other Measures

Suppose a random sample of size n is available from $F(x)$ and let $X_{r:n}$ be the rth order statistics. Further let $X_{s:n}$ be sth order statistics with $r < s$. The joint density function of $X_{r:n}$ and $X_{s:n}$ is

$$f_{r,s:n}(x_r, x_s) = C_{r,s,n} f(x_r) f(x_s) F^{r-1}(x_r)\Big[F(x_s) - F(x_r)\Big]^{s-r-1} \times [1 - F(x_s)]^{n-s}.$$

Using above we can obtain the density of $W_{rs} = X_{s:n} - X_{r:n}$ by making the transformation $w_{rs} = x_s - x_r$. The joint density of w_{rs} and x_r in this case is

$$f_{W_{rs}}(w_{rs}) = C_{r,s,n} f(x_r) f(x_r + w_{rs}) F^{r-1}(x_r) \times [F(x_r + w_{rs}) - F(x_r)]^{s-r-1}[1 - F(x_r + w_{rs})]^{n-s}.$$

The marginal density of w_{rs} is

$$f_{W_{rs}}(w_{rs}) = C_{r,s,n} \int_{-\infty}^{\infty} f(x_r) f(x_r + w_{rs}) F^{r-1}(x_r) \times [F(x_r + w_{rs}) - F(x_r)]^{s-r-1}[1 - F(x_r + w_{rs})]^{n-s} dx_r.$$

When $r = 1$ and $s = n$ then above result provide the density function of *Range* (w) in a sample of size n and is given as

$$f_W(w) = n(n-1) \int_{-\infty}^{\infty} f(x_r) f(x_r + w)[F(x_r + w) - F(x_r)]^{n-2} dx_r. \quad (2.17)$$

The distribution function of sample range can be easily obtained from (2.17) and is

$$\begin{aligned} F_W(w) &= n \int_{-\infty}^{\infty} f(x_r) \int_0^w (n-1) f\big(x_r + w^/\big) \times \big[F\big(x_r + w^/\big) - F(x)\big]^{n-2} dw^/ dx_r \\ &= n \int_{-\infty}^{\infty} f(x_r) \big[F\big(x_r + w^/\big) - F(x_r)\big]^{n-1}\Big|_{w^/=0}^{w^/=w} dx_r \\ &= n \int_{-\infty}^{\infty} f(x_r)[F(x_r + w) - F(x_r)]^{n-1} dx_r. \end{aligned} \quad (2.18)$$

Again suppose that number of observations in sample are even; say $n = 2m$; then we know that the sample median is

$$\widetilde{X} = \frac{1}{2}\big[X_{m:n} + X_{m+1:n}\big].$$

The distribution of median can be obtained by using joint distribution of two contiguous order statistics and is given as

$$f_{m,m+1:n}(x_m, x_{m+1}) = C^{/}_{m,n} f(x_m) f(x_{m+1}) F^{m-1}(x_m)\big[1 - F(x_{m+1})\big]^{m-1};$$

where $C^{/}_{m,n} = \frac{n!}{[(m-1)!]^2}\cdot$ Now making the transformation $\widetilde{x} = \frac{1}{2}\big[x_m + x_{m+1}\big]$ and $y = x_m$ the jaccobian of transformation is 2 and hence the joint density of $\widetilde{x}$ and y is

$$f_{\widetilde{X}Y}(\widetilde{x}, y) = 2C^{/}_{m,n} f(y) f(2\widetilde{x} - y) F^{m-1}(y)[1 - F(2\widetilde{x} - y)]^{m-1}.$$

The marginal density of sample median is, therefore

$$f_{\widetilde{X}}(\widetilde{x}) = 2C^{/}_{m,n} \int_{-\infty}^{\widetilde{x}} f(y) f(2\widetilde{x} - y) F^{m-1}(y)[1 - F(2\widetilde{x} - y)]^{m-1} dy \qquad (2.19)$$

The density of sample median for an odd sample size; say $n = 2m + 1$; is simply the density of mth order statistics for a sample of size $2m + 1$.

Example 2.3 Obtain the density function of sample range for a sample of size n from uniform distribution with density

$$f(x) = 1; 0 < x < 1.$$

Solution: The distribution of sample range for a sample of size n from distribution $F(x)$ is

$$f_W(w) = n(n-1) \int_{-\infty}^{\infty} f(x_r) f(x_r + w)[F(x_r + w) - F(x_r)]^{n-2} dx_r.$$

Now for uniform distribution we have

$$f(x) = 1;\ F(x) = x.$$

So

$$f(x_r + w) = 1;\ F(x_r + w) = (x_r + w),$$

hence the density function of range is

$$\begin{aligned} f_W(w) &= n(n-1) \int_0^{1-w} [(x_r + w) - x_r]^{n-2} dx_r \\ &= n(n-1) \int_0^{1-w} w^{n-2} dx_r \\ &= n(n-1) w^{n-2}(1 - w); 0 < w < 1. \end{aligned}$$

Example 2.4 Obtain the density function of range in a sample of size 3 from exponential distribution with density

$$f(x) = e^{-x}; x > 0.$$

Solution: The distribution of sample range for a sample of size n from distribution $F(x)$ is

$$f_W(w) = n(n-1)\int_{-\infty}^{\infty} f(x_r)f(x_r+w)[F(x_r+w) - F(x_r)]^{n-2}dx_r.$$

which for $n = 3$ becomes

$$f_W(w) = 6\int_{-\infty}^{\infty} f(x_r)f(x_r+w)[F(x_r+w) - F(x_r)]dx_r.$$

Now for exponential distribution we have

$$f(x) = e^{-x} \text{ and } F(x) = 1 - e^{-x},$$

hence

$$f(x_r) = e^{-x_r} \text{ and } F(x_r) = 1 - e^{-x_r},$$
$$f(w+x_r) = e^{-(w+x_r)} \text{ and } F(w+x_r) = 1 - e^{-(w+x_r)}.$$

Using these in above expression we have

$$\begin{aligned} f_W(w) &= 6\int_0^{\infty} e^{-x_r}e^{-(w+x_r)}\left[e^{-x_r} - e^{-(w+x_r)}\right]dx_r \\ &= 6e^{-w}\left(1 - e^{-w}\right)\int_0^{\infty} e^{-3x_r}dx_r \\ &= 2e^{-w}\left(1 - e^{-w}\right); w > 0, \end{aligned}$$

as required density function of range.

Example 2.5 Obtain the density function of median in a sample of size $n = 2m$ from exponential distribution with density function

$$f(x) = e^{-x}; x > 0.$$

Solution: The distribution of sample median for a sample of size $n = 2m$ from distribution $F(x)$ is

$$f_{\widetilde{X}}(\widetilde{x}) = 2C^{/}_{m,n}\int_{-\infty}^{\widetilde{x}} f(y)f(2\widetilde{x} - y)F^{m-1}(y)[1 - F(2\widetilde{x} - y)]^{m-1}dy;$$

where $C^{/}_{m,n} = \frac{n!}{[(m-1)!]^2}$. For the given distribution we have

$$f(y) = e^{-y};\ F(y) = 1 - e^{-y}$$
$$f(2\widetilde{x} - y) = e^{-(2\widetilde{x}-y)};\ F(2\widetilde{x} - y) = 1 - e^{-(2\widetilde{x}-y)}.$$

Substituting these values in above equation we have

$$\begin{aligned} f_{\widetilde{X}}(\widetilde{x}) &= 2C^{/}_{m,n} \int_0^{\widetilde{x}} e^{-y} e^{-(2\widetilde{x}-y)} \left(1 - e^{-y}\right)^m \\ &\quad \times \left[e^{-(2\widetilde{x}-y)}\right]^{m-1} dy \\ &= 2C^{/}_{m,n} e^{-2\widetilde{x}} e^{-2(m-1)\widetilde{x}} \int_0^{\widetilde{x}} e^{-2y} \left(1 - e^{-y}\right)^m \\ &\quad \times e^{(m-1)y} dy \\ &= 2C^{/}_{m,n} e^{-2m\widetilde{x}} \int_0^{\widetilde{x}} e^{-(3-m)y} \left(1 - e^{-y}\right)^m dy. \end{aligned}$$

Now expanding $\left(1 - e^{-y}\right)^m$ we have

$$\begin{aligned} f_{\widetilde{X}}(\widetilde{x}) &= 2C^{/}_{m,n} e^{-2m\widetilde{x}} \int_0^{\widetilde{x}} e^{-(3-m)y} \sum_{h=0}^{m} (-1)^h \binom{m}{h} e^{-hy} dy \\ &= 2C^{/}_{m,n} e^{-2m\widetilde{x}} \sum_{h=0}^{m} (-1)^h \binom{m}{h} \int_0^{\widetilde{x}} e^{-[(3-m)+h]y} dy \\ &= \frac{2C^{/}_{m,n}}{(3-m)+h} \sum_{h=0}^{m} (-1)^h \binom{m}{h} e^{-2m\widetilde{x}} \left(1 - e^{-[(3-m)+h]\widetilde{x}}\right), \end{aligned}$$

as required density of median.

2.6 Conditional Distributions of Order Statistics

The conditional distribution plays very important role in studying behavior of a random variable when information about some other variable(s) is available. The study of conditional distributions is easily extended in the case of order statistics. The conditional distributions of order statistics provide certain additional information about them and we present these conditional distributions in the following theorems as discussed in Arnold et al. (2008).

We know that when we have a bivariate distribution; say $f(x, y)$; of two random variables X and Y, then the conditional distribution of random variable Y given X is given as

$$f(y|x) = \frac{f(x, y)}{f_2(x)};$$

where $f_2(x)$ is the marginal distribution of X. Analogously, the conditional distributions in case of order statistics can be easily defined; say for example the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ is defined as

$$f(x_s|x_r) = \frac{f_{r,s:n}(x_r, x_s)}{f_{r:n}(x_r)},$$

where $f_{r,s:n}(x_r, x_s)$ is joint distribution of $X_{r:n}$ and $X_{s:n}$ and $f_{r:n}(x)$ is marginal distribution of $X_{r:n}$. The conditional distributions of order statistics are discussed in the following.

Theorem 2.1 *Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be order statistics for a sample of size n from an absolutely continuous distribution $F(x)$ then the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$; for $r < s$; is same as the distribution of $(s-r)th$ order statistics from a sample of size $(n-r)$ from a distribution $F(x)$ which is truncated on the left at x_r.*

Proof The marginal distribution of $X_{r:n}$ and the joint distribution of $X_{r:n}$ and $X_{s:n}$ are given in (2.8) and (2.12) as

$$f_{r:n}(x_r) = \frac{n!}{(r-1)!(n-r)!} f(x_r) \Big[F(x)\Big]^{r-1} \Big[1 - F(x_r)\Big]^{n-r};$$

and

$$f_{r,s:n}(x_r, x_s) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} f(x_r) f(x_s) \Big[F(x_r)\Big]^{r-1} \times \Big[F(x_s) - F(x_r)\Big]^{s-r-1} [1 - F(x_s)]^{n-s}.$$

Now the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ is

$$\begin{aligned} f(x_s|x_r) &= \frac{f_{r,s:n}(x_r, x_s)}{f_{r:n}(x_r)} \\ &= \Bigg\{ \frac{n!}{(r-1)!(s-r-1)!(n-s)!} f(x_r) f(x_s) \Big[F(x_r)\Big]^{r-1} \\ &\quad \times [F(x_s) - F(x_r)]^{s-r-1} [1 - F(x_s)]^{n-s} \Bigg\} / \\ &\quad \frac{n!}{(r-1)!(n-r)!} f(x_r) \Big[F(x_r)\Big]^{r-1} [1 - F(x_r)]^{n-r} \end{aligned}$$

or

$$\begin{aligned} f(x_s|x_r) &= \frac{(n-r)!}{(s-r-1)!(n-s)!} f(x_s)[1-F(x_s)]^{n-s} \\ &\quad \times \frac{[F(x_s)-F(x_r)]^{s-r-1}}{[1-F(x_r)]^{n-r}} \\ &= \frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{f(x_s)}{1-F(x_r)} \\ &\quad \times \left[\frac{F(x_s)-F(x_r)}{1-F(x_r)}\right]^{s-r-1} \left[\frac{1-F(x_s)}{1-F(x_r)}\right]^{n-s}. \end{aligned} \tag{2.20}$$

Noting that $\frac{f(x_s)}{\{1-F(x_r)\}}$ and $\frac{\{F(x_s)-F(x_r)\}}{\{1-F(x_r)\}}$ are respectively the density and distribution function of a random variable whose distribution is truncated at left of x_r completes the proof.

Theorem 2.2 *Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be order statistics for a sample of size n from an absolutely continuous distribution $F(x)$ then the conditional distribution of $X_{r:n}$ given $X_{s:n} = x_s$; for $r < s$; is same as the distribution of rth order statistics from a sample of size $(s-1)$ from a distribution $F(x)$ which is truncated on the right at x_s.*

Proof The marginal distribution of $X_{r:n}$ and the joint distribution of $X_{r:n}$ and $X_{s:n}$ are given in (2.8) and (2.12). Now the conditional distribution of $X_{r:n}$ given $X_{s:n} = x_s$ is

$$\begin{aligned} f(x_r|x_s) &= \frac{f_{r,s:n}(x_r, x_s)}{f_{s:n}(x_s)} \\ &= \left\{ \frac{n!}{(r-1)!(s-r-1)!(n-s)!} f(x_r) f(x_s) \Big[F(x_r)\Big]^{r-1} \right. \\ &\quad \left. \times [F(x_s)-F(x_r)]^{s-r-1}[1-F(x_s)]^{n-s} \right\} / \\ &\quad \frac{n!}{(s-1)!(n-s)!} f(x_s)[F(x_s)]^{s-1}[1-F(x_s)]^{n-s} \end{aligned}$$

or

$$\begin{aligned} f(x_r|x_s) &= \frac{(s-1)!}{(r-1)!(s-r-1)!} f(x_r) \Big[F(x_r)\Big]^{r-1} \\ &\quad \times \Big[F(x_s)-F(x_r)\Big]^{s-r-1} \frac{1}{[F(x_s)]^{s-1}} \\ &= \frac{(s-1)!}{(r-1)!(s-r-1)!} \frac{f(x_r)}{F(x_s)} \left[\frac{F(x_r)}{F(x_s)}\right]^{r-1} \\ &\quad \times \left[1-\frac{F(x_r)}{F(x_s)}\right]^{s-r-1}. \end{aligned} \tag{2.21}$$

Proof immediately follows by noting that $f(x_r)/F(x_s)$ and $F(x_r)/F(x_s)$ are respectively the density and distribution function of a random variable whose distribution is truncated at right of x_s.

Theorem 2.3 *Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be order statistics for a sample of size n from an absolutely continuous distribution $F(x)$ then the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ and $X_{t:n} = x_t$; for $r < s < t$; is same as the distribution of $(s-r)th$ order statistics for a sample of size $(t-r-1)$ from a distribution $F(x)$ which is doubly truncated on the left at x_r and on the right at x_t.*

Proof The joint distribution of $X_{r:n}$, $X_{s:n}$ and $X_{t:n}$ is obtained from (2.16) as

$$
\begin{aligned}
f_{r,s,t:n}(x_r, x_s, x_t) = {} & \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!} f(x_r) f(x_s) f(x_t) \\
& \times \Big[F(x_r)\Big]^{r-1} \Big[F(x_s) - F(x_r)\Big]^{s-r-1} \Big[F(x_t) - F(x_s)\Big]^{t-s-1} \\
& \times \Big[1 - F(x_t)\Big]^{n-t}.
\end{aligned}
$$

Also the joint distribution of $X_{r:n}$ and $X_{t:n}$ is

$$
\begin{aligned}
f_{r,t:n}(x_r, x_t) = {} & \frac{n!}{(r-1)!(t-r-1)!(n-t)!} f(x_r) f(x_t) \Big[F(x_r)\Big]^{r-1} \\
& \times [F(x_t) - F(x_r)]^{t-r-1} [1 - F(x_t)]^{n-t}.
\end{aligned}
$$

Now the conditional distribution of $X_{s:n}$ given $X_{r:n}$ and $X_{t:n}$ is

$$
\begin{aligned}
f(x_s | x_r, x_t) = {} & \frac{f_{r,s,t:n}(x_r, x_s, x_t)}{f_{r,t:n}(x_r, x_t)} \\
= {} & \Bigg\{ \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!} \\
& \times f(x_r) f(x_s) f(x_t) \Big[F(x_r)\Big]^{r-1} \Big[F(x_s) - F(x_r)\Big]^{s-r-1} \\
& \times [F(x_t) - F(x_s)]^{t-s-1} [1 - F(x_t)]^{n-t} \Bigg\} / \\
& \Bigg\{ \frac{n!}{(r-1)!(t-r-1)!(n-t)!} f(x_r) f(x_t) [F(x_r)]^{r-1} \\
& \times [F(x_t) - F(x_r)]^{t-r-1} [1 - F(x_t)]^{n-t} \Bigg\}
\end{aligned}
$$

or

$$
\begin{aligned}
f(x_s|x_r, x_t) &= \frac{(t-r-1)!}{(s-r-1)!(t-s-1)!} f(x_s)\Big[F(x_s) - F(x_r)\Big]^{s-r-1} \\
&\quad \times \Big[F(x_t) - F(x_s)\Big]^{t-s-1} \frac{1}{\Big[F(x_t) - F(x_r)\Big]^{t-r-1}} \\
&= \frac{(t-r-1)!}{(s-r-1)!(t-s-1)!} \frac{f(x_s)}{F(x_t) - F(x_r)} \\
&\quad \times \left[\frac{F(x_s) - F(x_r)}{F(x_t) - F(x_r)}\right]^{s-r-1} \times \left[\frac{F(x_t) - F(x_s)}{F(x_t) - F(x_r)}\right]^{t-s-1}. \qquad (2.22)
\end{aligned}
$$

Proof immediately follows by noting that $\frac{f(x_s)}{F(x_t)-F(x_r)}$ and $\frac{F(x_s)-F(x_r)}{F(x_t)-F(x_r)}$ are respectively the density and distribution function of a random variable whose distribution is doubly truncated from left at x_r and at the right of x_s.

Theorem 2.4 *Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be order statistics for a sample of size n from an absolutely continuous distribution $F(x)$ then the conditional distribution of $X_{r:n}$ and $X_{s:n}$ given $X_{t:n} = x_t$; for $r < s < t$; is same as the joint distribution of rth and sth order statistics for a sample of size $(t-1)$ from a distribution $F(x)$ which is truncated on the right at x_t.*

Proof The joint distribution of $X_{r:n}$, $X_{s:n}$ and $X_{t:n}$ is obtained from (2.16) as

$$
\begin{aligned}
f_{r,s,t:n}(x_r, x_s, x_t) &= \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!} f(x_r)f(x_s)f(x_t) \\
&\quad \times \Big[F(x_r)\Big]^{r-1}\Big[F(x_s) - F(x_r)\Big]^{s-r-1}\Big[F(x_t) - F(x_s)\Big]^{t-s-1} \\
&\quad \times [1 - F(x_t)]^{n-t}.
\end{aligned}
$$

Also the marginal distribution of $X_{t:n}$ is

$$
f_{t:n}(x_t) = \frac{n!}{(t-1)!(n-t)!} f(x_t)[F(x_t)]^{t-1}[1 - F(x_t)]^{n-t}.
$$

Now the conditional distribution of $X_{r:n}$ and $X_{s:n}$ given $X_{t:n}$ is

$$
\begin{aligned}
f(x_r, x_s|x_t) &= \frac{f_{r,s,t:n}(x_r, x_s, x_t)}{f_{t:n}(x_t)} \\
&= \left\{\frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!}\right. \\
&\quad \times f(x_r)f(x_s)f(x_t)[F(x_r)]^{r-1}\Big[F(x_s) - F(x_r)\Big]^{s-r-1} \\
&\quad \left.\times [F(x_t) - F(x_s)]^{t-s-1}[1 - F(x_t)]^{n-t}\right\}/ \\
&\quad \left\{\frac{n!}{(t-1)!(n-t)!} f(x_t)[F(x_t)]^{t-1}\right. \\
&\quad \left.\times [1 - F(x_t)]^{n-t}\right\}
\end{aligned}
$$

or

$$
\begin{aligned}
f(x_r, x_s|x_t) &= \frac{(t-1)!}{(r-1)!(s-r-1)!(t-s-1)!} f(x_r) f(x_s)[F(x_r)]^{r-1} \\
&\quad \times \left[F(x_s) - F(x_r)\right]^{s-r-1} \frac{\left[F(x_t) - F(x_s)\right]^{t-s-1}}{[F(x_t)]^{t-1}} \\
&= \frac{(t-1)!}{(r-1)!(s-r-1)!(t-s-1)!} \frac{f(x_r)}{F(x_t)} \frac{f(x_s)}{F(x_t)} \\
&\quad \times \left[\frac{F(x_r)}{F(x_t)}\right]^{r-1} \left[\frac{F(x_s)}{F(x_t)} - \frac{F(x_r)}{F(x_t)}\right]^{s-r-1} \\
&\quad \times \left[1 - \frac{F(x_s)}{F(x_t)}\right]^{t-s-1}. \qquad (2.23)
\end{aligned}
$$

The proof is complete by noting that $f(x_r)/F(x_t)$ and $F(x_r)/F(x_t)$ are respectively the density and distribution function of a random variable whose distribution is truncated at the right at x_t.

Theorem 2.5 *Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be order statistics for a sample of size n from an absolutely continuous distribution $F(x)$ then the conditional distribution of $X_{s:n}$ and $X_{t:n}$ given $X_{r:n} = x_r$; for $r < s < t$; is same as the joint distribution of $(s-r)th$ and $(t-r)th$ order statistics for a sample of size $(n-r)$ from a distribution $F(x)$ which is truncated on the left at x_r.*

Proof The joint distribution of $X_{r:n}$, $X_{s:n}$ and $X_{t:n}$ is obtained from (2.16) as

$$
\begin{aligned}
f_{r,s,t:n}(x_r, x_s, x_t) &= \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!} f(x_r) f(x_s) f(x_t) \\
&\quad \times [F(x_r)]^{r-1} \left[F(x_s) - F(x_r)\right]^{s-r-1} \left[F(x_t) - F(x_s)\right]^{t-s-1} \\
&\quad \times [1 - F(x_t)]^{n-t}.
\end{aligned}
$$

Also the marginal distribution of $X_{r:n}$ is

$$
f_{r:n}(x_r) = \frac{n!}{(r-1)!(n-r)!} f(x_r)[F(x_r)]^{r-1}[1 - F(x_r)]^{n-r}.
$$

Now the conditional distribution of $X_{s:n}$ and $X_{t:n}$ given $X_{r:n}$ is

$$
\begin{aligned}
f(x_s, x_t|x_r) &= \frac{f_{r,s,t:n}(x_r, x_s, x_t)}{f_{r:n}(x_t)} \\
&= \left\{ \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!} \right. \\
&\quad \times f(x_r) f(x_s) f(x_t)[F(x_r)]^{r-1} \left[F(x_s) - F(x_r)\right]^{s-r-1}
\end{aligned}
$$

$$\times [F(x_t) - F(x_s)]^{t-s-1}[1 - F(x_t)]^{n-t}\Big\}/$$
$$\left\{\frac{n!}{(r-1)!(n-r)!} f(x_r)[F(x_r)]^{r-1} \times [1 - F(x_r)]^{n-r}\right\}$$

or

$$\begin{aligned} f(x_s, x_t|x_r) &= \frac{(n-r)!}{(s-r-1)!(t-s-1)!(n-t)!} f(x_s)f(x_t) \\ &\quad \times \Big[F(x_s) - F(x_r)\Big]^{s-r-1}[F(x_t) - F(x_s)]^{t-s-1} \\ &\quad \times [1 - F(x_t)]^{n-t}\frac{1}{[1 - F(x_r)]^{n-r}} \\ &= \frac{(n-r)!}{(s-r-1)!(t-s-1)!(n-t)!}\frac{f(x_s)}{1 - F(x_r)} \\ &\quad \times \frac{f(x_t)}{1 - F(x_r)}\left[\frac{F(x_s) - F(x_r)}{1 - F(x_r)}\right]^{s-r-1} \\ &\quad \times \left[\frac{F(x_t) - F(x_s)}{1 - F(x_r)}\right]^{t-s-1}\left[\frac{1 - F(x_t)}{1 - F(x_r)}\right]^{n-t}. \end{aligned} \tag{2.24}$$

The proof is complete by noting that $\frac{f(x_s)}{1-F(x_r)}$ and $\frac{F(x_s)-F(x_r)}{1-F(x_r)}$ are respectively the density and distribution function of a random variable whose distribution is truncated at the left of x_r.

Theorems 2.1 to 2.5 provide some interesting results about the conditional distributions of order statistics from a distribution $F(x)$. From all these theorems we can see that the conditional distributions in order statistics are simply the marginal and joint distributions of corresponding order statistics obtained from the truncated parent distribution and appropriately modified sample size.

Example 2.6 A random sample of size n is drawn from the Weibull distribution with density function

$$f(x) = \alpha x^{\alpha-1}\exp(-x^{\alpha});\ x, \alpha > 0.$$

Obtain the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ and conditional distribution of $X_{r:n}$ given $X_{s:n} = x_s$.

Solution: The conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ is given as

$$\begin{aligned} f(x_s|x_r) &= \frac{(n-r)!}{(s-r-1)!(n-s)!}\frac{f(x_s)}{1 - F(x_r)} \\ &\quad \times \left[\frac{F(x_s) - F(x_r)}{1 - F(x_r)}\right]^{s-r-1}\left[\frac{1 - F(x_s)}{1 - F(x_r)}\right]^{n-s}. \end{aligned}$$

Also the conditional distribution of $X_{r:n}$ given $X_{s:n} = x_s$ is

$$f(x_r|x_s) = \frac{(s-1)!}{(r-1)!(s-r-1)!} \frac{f(x_r)}{F(x_s)} \left[\frac{F(x_r)}{F(x_s)}\right]^{r-1} \times \left[1 - \frac{F(x_r)}{F(x_s)}\right]^{s-r-1}.$$

Now for the given distribution we have

$$F(x) = \int_0^x f(t)dt = \int_0^x \alpha t^{\alpha-1} \exp(-t^\alpha)dt = 1 - \exp(-x^\alpha);\ x, \alpha > 0.$$

So, the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ is

$$f(x_s|x_r) = \frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{\alpha x_s^{\alpha-1} e^{-x_s^\alpha}}{e^{-x_r^\alpha}} \times \left(\frac{e^{-x_r^\alpha} - e^{-x_s^\alpha}}{e^{-x_r^\alpha}}\right)^{s-r-1} \left(\frac{e^{-x_s^\alpha}}{e^{-x_r^\alpha}}\right)^{n-s}. = \frac{(n-r)!\alpha x_s^{\alpha-1} e^{-(n-s+1)x_s^\alpha}}{(s-r-1)!(n-s)!} \frac{\left(e^{-x_r^\alpha} - e^{-x_s^\alpha}\right)^{s-r-1}}{e^{-(n-r)x_r^\alpha}}.$$

Again, the conditional distribution of $X_{r:n}$ given $X_{s:n}$ is

$$f(x_r|x_s) = \frac{(s-1)!}{(r-1)!(s-r-1)!} \frac{\alpha x_r^{\alpha-1} e^{-x_r^\alpha}}{1 - e^{-x_s^\alpha}} \left[\frac{1 - e^{-x_r^\alpha}}{1 - e^{-x_s^\alpha}}\right]^{r-1} \times \left[1 - \frac{1 - e^{-x_r^\alpha}}{1 - e^{-x_s^\alpha}}\right]^{s-r-1},$$

or

$$f(x_r|x_s) = \frac{(s-1)!}{(r-1)!(s-r-1)!} \frac{\alpha x_r^{\alpha-1} e^{-x_r^\alpha}}{\left(1 - e^{-x_s^\alpha}\right)^{s-1}} \times \left(1 - e^{-x_r^\alpha}\right)^{r-1} \left(e^{-x_r^\alpha} - e^{-x_s^\alpha}\right)^{s-r-1},$$

Above conditional distributions can also be derived from the parent truncated distribution.

2.7 Order Statistics as Markov Chain

In the previous section we have presented the conditional distributions of order statistics. The conditional distributions of order statistics enable us to study their additional behavior. One of the popular property which is based upon the conditional distributions of order statistics is that the order statistics follows the Markov chain. We prove this property of order statistics in the following.

We know that a sequence of random variables $X_1, X_2, \ldots, X_r, X_s$ has Markov chain property if the conditional distribution of X_s given $X_1 = x_1, X_2 = x_2, \ldots, X_r = x_r$ is same as the conditional distribution of X_s given $X_r = x_r$, that is if

$$f(x_s|X_1 = x_1, \ldots, X_r = x_r) = f(x_s|X_r = x_r).$$

Now to show that the order statistics follow the Markov chain we need to show that the conditional distribution of sth order statistics given the information of rth order statistics is same as the conditional distribution of sth order statistics given the joint information of first r order statistics. From Theorem 2.1 we know that the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ is

$$\begin{aligned} f(x_s|x_r) &= \frac{(n-r)!}{(s-r-1)!(n-s)!}\frac{f(x_s)}{1-F(x_r)} \\ &\times \left[\frac{F(x_s)-F(x_r)}{1-F(x_r)}\right]^{s-r-1}\left[\frac{1-F(x_s)}{1-F(x_r)}\right]^{n-s}. \end{aligned} \tag{2.25}$$

Also, the conditional distribution of of $X_{s:n}$ given $X_{1:n} = x_1, X_{2:n} = x_2, \ldots, X_{r:n} = x_r$ is

$$f(x_s|x_1, \ldots x_r) = \frac{f_{1,2,\ldots r,s:n}(x_1, \ldots, x_r, x_s)}{f_{1,2,\ldots r:n}(x_1, \ldots, x_r)}.$$

The joint distribution of first r order statistics is given in (2.26) as

$$f_{1,\ldots r:n}(x_1, \ldots, x_r) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r} f(x_i)\right][1-F(x_r)]^{n-r}. \tag{2.26}$$

Now the joint distribution of $X_{1:n}, X_{2:n}, \ldots, X_{r:n}$ and $X_{s:n}$ is obtained as

$$\begin{aligned} f_{1,\ldots r,s:n}(x_1, \ldots, x_r, x_s) &= \int_{x_s}^{\infty}\cdots\int_{x_s}^{x_{s+3}}\int_{x_s}^{x_{s+2}}\int_{x_r}^{x_s}\cdots\int_{x_r}^{x_{r+3}}\int_{x_r}^{x_{r+2}} \\ &\times f_{1,\ldots n:n}(x_1, \ldots, x_n) \\ &\times dx_{r+1}\cdots dx_{s-1}d_{s+1}\ldots dx_n \\ &= \int_{x_s}^{\infty}\cdots\int_{x_s}^{x_{s+3}}\int_{x_s}^{x_{s+2}}\int_{x_r}^{x_s}\cdots\int_{x_r}^{x_{r+3}}\int_{x_r}^{x_{r+2}} \\ &\times n!\prod_{i=1}^{n} f(x_i)dx_{r+1}\cdots dx_n \end{aligned}$$

or

$$f_{1,\ldots r,s:n}(x_1,\ldots,x_r,x_s) = n!\left[\prod_{i=1}^{r} f(x_i)\right] f(x_s)$$
$$\times\left\{\int_{x_r}^{x_s}\cdots\int_{x_r}^{x_{r+3}}\int_{x_r}^{x_{r+2}}\prod_{i=r+1}^{s-1} f(x_i)dx_{r+1}\ldots dx_{s-1}\right\}$$
$$\times\left\{\int_{x_s}^{\infty}\cdots\int_{x_s}^{x_{s+3}}\int_{x_s}^{x_{s+2}}\prod_{i=s+1}^{n} f(x_i)dx_{s+1}\ldots dx_n\right\}$$

or

$$f_{1,\ldots r,s:n}(x_1,\ldots,x_r,x_s) = \frac{n!}{(s-r-1)!(n-s)!}\left[\prod_{i=1}^{r} f(x_i)\right] f(x_s)$$
$$\times\left[F(x_s)-F(x_r)\right]^{s-r-1}[1-F(x_s)]^{n-s}.$$

Hence the conditional distribution of $X_{s:n}$ given $X_{1:n}=x_1, X_{2:n}=x_2,\ldots,X_{r:n}=x_r$ is

$$f(x_s|x_1,\ldots x_r) = \left\{\frac{n!}{(s-r-1)!(n-s)!}\left[\prod_{i=1}^{r} f(x_i)\right] f(x_s)\right.$$
$$\left.\left[F(x_s)-F(x_r)\right]^{s-r-1}[1-F(x_s)]^{n-s}\right\}/$$
$$\left\{\frac{n!}{(n-r)!}\left[\prod_{i=1}^{r} f(x_i)\right][1-F(x_r)]^{n-r}\right\}$$

or

$$f(x_s|x_1,\ldots x_r) = \frac{(n-r)!}{(s-r-1)!(n-s)!} f(x_s)$$
$$\times\left[F(x_s)-F(x_r)\right]^{s-r-1}\frac{[1-F(x_s)]^{n-s}}{[1-F(x_r)]^{n-r}}$$
$$= \frac{(n-r)!}{(s-r-1)!(n-s)!}\frac{f(x_s)}{1-F(x_r)}$$
$$\times\left[\frac{F(x_s)-F(x_r)}{1-F(x_r)}\right]^{s-r-1}\left[\frac{1-F(x_s)}{1-F(x_r)}\right]^{n-s};$$

which is (2.25). Hence order statistics from a distribution $F(x)$ form a Markov chain. The transition probabilities of order statistics are easily computed from conditional distributions. We know that the transition probability is computed as

$$P(X_{r+1:n} \geq y|X_{r:n} = x) = \int_y^\infty f(x_{r+1}|x_r = x)dx_{r+1}.$$

Now the conditional distribution of $X_{r+1:n}$ given $X_{r:n} = x_r$ is obtained from (2.25) by using $s = r + 1$ as

$$\begin{aligned} f(x_{r+1}|x_r) &= \frac{(n-r)!}{(n-r-1)!}\frac{f(x_{r+1})}{1-F(x_r)}\left[\frac{1-F(x_s)}{1-F(x_r)}\right]^{n-r-1} \\ &= (n-r)\frac{f(x_{r+1})}{1-F(x_r)}\left[\frac{1-F(x_s)}{1-F(x_r)}\right]^{n-r-1}. \end{aligned}$$

Hence the transition probability is

$$\begin{aligned} P(X_{r+1:n} \geq y|X_{r:n} = x) &= \int_y^\infty f(x_{r+1}|x_r = x)dx_{r+1} \\ &= \int_y^\infty (n-r)\left[\frac{1-F(x_s)}{1-F(x)}\right]^{n-r-1} \\ &\times \frac{f(x_{r+1})}{1-F(x)}dx_{r+1} \\ &= \frac{n-r}{\{1-F(x)\}^{n-r}}\int_y^\infty \{1-F(x_{r+1})\}^{n-r-1} \\ &\times f(x_{r+1})dx_{r+1} \end{aligned}$$

or

$$\begin{aligned} P(X_{r+1} \geq y|X_r = x) &= \frac{n-r}{\{1-F(x)\}^{n-r}} \times \left.\frac{-\{1-F(x_{r+1})\}^{n-r}}{n-r}\right|_y^\infty \\ &= \left[\frac{1-F(y)}{1-F(x)}\right]^{n-r}. \end{aligned}$$

We can readily see that the transition probabilities depends upon value of n and r.

2.8 Moments of Order Statistics

The probability distribution of order statistics is like conventional probability distribution and hence the moments from these distributions can be computed in usual way. Specifically, the pth raw moment of rth order statistics is computed as

$$\begin{aligned} \mu_{r:n}^p &= \int_{-\infty}^\infty x_r f_{r:n}(x)dx \\ &= \frac{1}{B(r, n-r+1)}\int_{-\infty}^\infty x_r^p f(x_r)F^{r-1}(x_r)[1-F(x_r)]^{n-r}dx. \quad (2.27) \end{aligned}$$

Using the probability integral transform property of order statistics, the moments can also be written as

$$\mu_{r:n}^{p} = \frac{1}{B(r, n-r+1)} \int_0^1 \{F^{-1}(u_r)\}^p u_r^{r-1}(1-u_r)^{n-r} du; \tag{2.28}$$

where u_r is rth Uniform Order Statistics. The mean and variance of $X_{r:n}$ can be computed from (2.27) or (2.28). Further, the joint pth and qth order moments of two order statistics; $X_{r:n} = x_1$ and $X_{s:n} = x_2$; are computed as

$$\begin{aligned} \mu_{r,s:n}^{p,q} &= E\left(X_{r:n}^p X_{s:n}^q\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} x_1^p x_2^q f_{r,s:n}(x_1, x_2) dx_1 dx_2 \\ &= C_{r,s:n}\int_{-\infty}^{\infty}\int_{-\infty}^{x_2} x_1^p x_2^q f(x_1) f(x_2) F^{r-1}(x_1) \\ &\quad \times \left[F(x_2) - F(x_1)\right]^{s-r-1} [1 - F(x_2)]^{n-s} dx_1 dx_2. \end{aligned} \tag{2.29}$$

Using probability integral transform, we have

$$\begin{aligned} \mu_{r,s:n}^{p,q} &= C_{r,s:n}\int_0^1\int_0^v \{F^{-1}(u)\}^p\{F^{-1}(v)\}^q u^{r-1}(v-u)^{s-r-1} \\ &\quad \times (1-v)^{n-s} du dv. \end{aligned} \tag{2.30}$$

The pth and qth joint central moments are given as

$$\begin{aligned} \sigma_{r,s:n}^{p,q} &= E\left[(X_{r:n} - \mu_{r:n})^p (X_{s:n} - \mu_{s:n})^q\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} (x_1 - \mu_{r:n})^p (x_2 - \mu_{s:n})^q f_{r,s:n}(x_1, x_2) dx_1 dx_2 \\ &= C_{r,s:n}\int_0^1\int_0^v \left[F^{-1}(u) - \mu_{r:n}\right]^p \left[F^{-1}(v) - \mu_{s:n}\right]^q \\ &\quad \times u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s} du dv. \end{aligned} \tag{2.31}$$

Specifically, for $p = q = 1$; the quantity $\sigma_{r,s:n}$ is called covariance between $X_{r:n}$ and $X_{s:n}$. Also if $F(x)$ is symmetrical; say about 0; then following relations holds

$$\mu_{r:n}^{p} = (-1)^p \mu_{n-r+1:n}^{p}$$

and

$$\mu_{r,s:n}^{p,q} = (-1)^{p+q} \mu_{n-s+1,n-r+1:n}^{p,q}.$$

Further, the moments of linear combinations of order statistics can be easily obtained.

The conditional distributions of order statistics provide basis for computation of conditional moments of order statistics. Specifically, the pth conditional moment of sth order statistics; $X_{s:n}$; given $X_{r:n} = x_r$ is obtained by using the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ as

$$\begin{aligned}\mu^p_{s|r:n} &= E\left(X^p_{s:n}|x_r\right) = \int_{x_r}^{\infty} x_s^p f(x_s|x_r)dx_s \\ &= \frac{(n-r)!}{(s-r-1)!(n-s)![1-F(x_r)]^{n-r}} \\ &\quad \times \int_{x_r}^{\infty} x_s^p f(x_s)\Big[F(x_s)-F(x_r)\Big]^{s-r-1} \\ &\quad \times [1-F(x_s)]^{n-s}dx_s. \end{aligned} \tag{2.32}$$

The conditional mean and variance can be obtained from (2.32). Further, the pth conditional moment of $X_{s:n}$ given $X_{r:n} = x_r$ and $X_{t:n} = x_t$ is computed from the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ and $X_{t:n} = x_t$ as

$$\begin{aligned}\mu^p_{s|r,t:n} &= E\left(X^p_{s:n}|x_r, x_t\right) = \int_{x_r}^{x_t} x_s^p f(x_s|x_r, x_t)dx_s \\ &= \frac{(t-r-1)!}{(s-r-1)!(t-s-1)![F(x_t)-F(x_r)]^{t-r-1}} \\ &\quad \times \int_{x_r}^{\infty} x_s^p f(x_s)\Big[F(x_s)-F(x_r)\Big]^{s-r-1} \\ &\quad \times [F(x_t)-F(x_s)]^{t-s-1}dx_s. \end{aligned} \tag{2.33}$$

In similar way the joint conditional moments of order statistics can be defined by using corresponding conditional distribution.

Example 2.7 A random sample has been obtained from the density

$$f(x) = vx^{v-1}; 0 < x < 1; v > 0.$$

Obtain expression for pth moment of rth order statistics and that for joint pth and qth moment of rth and sth order statistics. Hence or otherwise obtain mean and variance of $X_{r:n}$ and covariance between $X_{r:n}$ and $X_{s:n}$.

Solution: The distribution of rth order statistics is given as

$$f_{r:n}(x_r) = \frac{1}{B(r, n-r+1)} f(x_r)F^{r-1}(x_r)[1-F(x_r)]^{n-r}.$$

For given distribution we have $F(x) = x^v$ and hence the density of $X_{r:n}$ is

$$f_{r:n}(x_r) = \frac{1}{B(r, n-r+1)} vx_r^{rv-1}\left(1-x_r^v\right)^{n-r}.$$

Now we have

$$
\begin{aligned}
\mu_{r:n}^{p} &= E\left(X_{r:n}^{p}\right) = \int_{-\infty}^{\infty} x_r^p f_{r:n}(x_r)dx_r \\
&= \frac{1}{B(r,n-r+1)}\int_0^1 x_r^p v x_r^{rv-1}\left(1-x_r^v\right)^{n-r}dx_r \\
&= \frac{v}{B(r,n-r+1)}\int_0^1 x_r^{p+rv-1}\left(1-x_r^v\right)^{n-r}dx_r \\
&= \frac{v}{B(r,n-r+1)}\int_0^1 x_r^{p+rv-1}\left(1-x_r^v\right)^{n-r}dx_r \\
&= \frac{v}{B(r,n-r+1)}\int_0^1 x_r^{p+v(r-1)}x_r^{v-1}\left(1-x_r^v\right)^{n-r}dx_r
\end{aligned}
$$

Making the transformation $x_r^v = y$ we have $v x_r^{v-1}dx_r = dy$, hence

$$
\begin{aligned}
\mu_{r:n}^{p} &= \frac{1}{B(r,n-r+1)}\int_0^1 y^{\frac{p}{v}+r-1}(1-y)^{n-r}dx_r \\
&= \frac{B\left(r+\frac{p}{v},n-r+1\right)}{B(r,n-r+1)} = \frac{\Gamma(n+1)\Gamma\left(\frac{p}{v}+r\right)}{\Gamma(r)\Gamma\left(n+\frac{p}{v}+1\right)}.
\end{aligned}
$$

The mean is readily written as

$$
\mu_{r:n} = \frac{\Gamma(n+1)\Gamma\left(\frac{1}{v}+r\right)}{\Gamma(r)\Gamma\left(n+\frac{1}{v}+1\right)}.
$$

Again the joint distribution of $X_{r:n}$ and $X_{s:n}$ is

$$
f_{r,s:n}(x_r,x_s) = C_{r,s,n}v^2 x_r^{rv-1}x_s^{v-1}\left(x_s^v - x_r^v\right)^{s-r-1}\left(1-x_s^v\right)^{n-s}.
$$

The product moments are therefore

$$
\begin{aligned}
\mu_{r,s:n}^{p,q} &= E\left(X_{r:n}^{p}X_{s:n}^{q}\right) = \int_0^1\int_0^{x_s} x_r^p x_s^q f_{r,s:n}(x_r,x_s)dx_r dx_s \\
&= C_{r,s:n}v^2\int_0^1\int_0^{x_s} x_r^p x_s^q x_r^{rv-1}x_s^{v-1}\left(x_s^v - x_r^v\right)^{s-r-1} \\
&\quad \times \left(1-x_s^v\right)^{n-s}dx_r dx_s \\
&= C_{r,s:n}v\int_0^1 x_s^q x_s^{v-1}\left(1-x_s^v\right)^{n-s} \\
&\quad \times \left\{v\int_0^{x_s} x_r^{p+v(r-1)}x_r^{v-1}\left(x_s^v - x_r^v\right)^{s-r-1}dx_r\right\}dx_s \\
&= C_{r,s:n}v\int_0^1 x_s^q x_s^{v-1}\left(1-x_s^v\right)^{n-s}\{vI(x_r)\}dx_s
\end{aligned}
$$

Now consider

$$\begin{aligned}
vI(x_r) &= v\int_0^{x_s} x_r^{p+v(r-1)} x_r^{v-1}\left(x_s^v - x_r^v\right)^{s-r-1} dx_r \\
&= v x_s^{v(s-r-1)} \int_0^{x_s} x_r^{p+v(r-1)} x_r^{v-1}\left(1 - \frac{x_r^v}{x_s^v}\right)^{s-r-1} dx_r.
\end{aligned}$$

Making the transformation $y = x_r^v / x_s^v$ we have

$$\begin{aligned}
vI(x_r) &= x_s^{(s-r-1)} \int_0^1 y^{p/v+(r-1)}(1-y)^{s-r-1} dy \\
&= x_s^{p+v(s-2)} B\left(\frac{p}{v} + r, s - r\right) \\
&= x_s^{p+v(s-2)} \frac{\Gamma\left(r + \frac{p}{v}\right)\Gamma(s-r)}{\Gamma\left(s + \frac{p}{v}\right)}.
\end{aligned}$$

Using above value of $vI(x_r)$ in above equation we have

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= C_{r,s:n} \frac{\Gamma(s-r)\Gamma\left(r + \frac{p}{v}\right)}{\Gamma\left(s + \frac{p}{v}\right)} v \int_0^1 x_s^{p+q+v(s-1)} x_s^{v-1}\left(1 - x_s^v\right)^{n-s} \\
&= C_{r,s:n} \frac{\Gamma(s-r)\Gamma\left(r + \frac{p}{v}\right)}{\Gamma\left(s + \frac{p}{v}\right)} \times \frac{(n-s)\Gamma(n-s)\Gamma\left(\frac{p+q}{v} + s\right)}{\Gamma\left(\frac{p+q}{v} + n + 1\right)}
\end{aligned}$$

Now using the value of $C_{r,s:n}$ we have

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= \frac{1}{B(r, s-r, n-s+1)} \times \frac{\Gamma(s-r)\Gamma\left(r + \frac{p}{v}\right)}{\Gamma\left(s + \frac{p}{v}\right)} \\
&\quad \times \frac{(n-s)\Gamma(n-s)\Gamma\left(\frac{p+q}{v} + s\right)}{\Gamma\left(\frac{p+q}{v} + n + 1\right)} \\
&= \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(s-r)\Gamma(n-s+1)} \times \frac{\Gamma(s-r)\Gamma\left(r + \frac{p}{v}\right)}{\Gamma\left(s + \frac{p}{v}\right)} \\
&\quad \times \frac{(n-s)\Gamma(n-s)\Gamma\left(\frac{p+q}{v} + s\right)}{\Gamma\left(\frac{p+q}{v} + n + 1\right)} \\
&= \frac{\Gamma(n+1)\Gamma\left(r + \frac{p}{v}\right)\Gamma\left(\frac{p+q}{v} + s\right)}{\Gamma(r)\Gamma\left(s + \frac{p}{v}\right)\Gamma\left(\frac{p+q}{v} + n + 1\right)}; r < s.
\end{aligned}$$

The covariance can be easily obtained from above.

Example 2.8 Find pth moment of rth order statistics for standard exponential distribution.

Solution: The density and distribution function of standard exponential distribution are

$$f(x) = e^{-x} \text{ and } F(x) = 1 - e^{-x}.$$

The pth moment of rth order statistics is

$$\begin{aligned}
\mu_{r:n}^{p} &= E\left(X_{r:n}^{p}\right) = \int_{-\infty}^{\infty} x^{p} f_{r:n}(x)dx \\
&= C_{r:n} \int_{0}^{\infty} x^{p} e^{-x}\left(1-e^{-x}\right)^{r-1}\left(e^{-x}\right)^{n-r} dx \\
&= \sum_{j=0}^{r-1} (-1)^{j} C_{r:n} \binom{r-1}{j} \int_{0}^{\infty} x^{p} e^{-x(n-r+j+1)} dx \\
&= \sum_{j=0}^{r-1} (-1)^{j} \frac{n!}{j!(r-1-j)!(n-r)!} \frac{\Gamma(p+1)}{(n-r+j+1)^{p+1}}.
\end{aligned}$$

The mean and variance can be obtained from above.

Example 2.9 A random sample of size n is drawn from the Weibull distribution with density

$$f(x) = \alpha x^{\alpha-1} \exp(-x^{\alpha});\ x, \alpha > 0.$$

Obtain the expression for pth conditional moment of $X_{s:n}$ given $X_{r:n} = x_r$.

Solution: The pth conditional moment of $X_{s:n}$ given $X_{r:n} = x_r$ is given in (2.34) as

$$\begin{aligned}
\mu_{s|r:n}^{p} &= E\left(X_{s:n}^{p} | x_r\right) = \int_{x_r}^{\infty} x_s^{p} f(x_s | x_r) dx_s \\
&= \frac{(n-r)!}{(s-r-1)!(n-s)![1-F(x_r)]^{n-r}} \int_{x_r}^{\infty} x_s^{p} f(x_s) \\
&\quad \times \left[F(x_s) - F(x_r)\right]^{s-r-1} [1-F(x_s)]^{n-s} dx_s.
\end{aligned} \tag{2.34}$$

Now for given distribution we have

$$f(x) = \alpha x^{\alpha-1} \exp(-x^{\alpha})$$

and

$$F(x) = \int_{0}^{x} f(t)dt = \int_{0}^{x} \alpha t^{\alpha-1} \exp(-t^{\alpha}) dt = 1 - e^{-x^{\alpha}}.$$

Hence the pth conditional moment of $X_{r:s}$ given $X_{r:n} = x_r$ is

$$\mu^p_{s|r:n} = \frac{K_{r,s:n}}{(e^{-x_r^\alpha})^{n-r}} \int_{x_r}^{\infty} x_s^p \alpha x_s^{\alpha-1} e^{-x_s^\alpha} \left(e^{-x_r^\alpha} - e^{-x_s^\alpha}\right)^{s-r-1} \times \left(e^{-x_s^\alpha}\right)^{n-s} dx_s$$

$$= \frac{\alpha K_{r,s:n}}{e^{-(n-r)x_r^\alpha}} \int_{x_r}^{\infty} x_s^{p+\alpha-1} e^{-(n-s+1)x_s^\alpha} \sum_{j=0}^{s-r-1} (-1)^j \times \binom{s-r-1}{j} e^{-(s-r-j-1)x_r^\alpha} e^{-jx_s^\alpha} dx_s;$$

or

$$\mu^p_{s|r:n} = \alpha K_{r,s:n} \sum_{j=0}^{s-r-1} \frac{1}{e^{-(n-s+j+1)x_r^\alpha}} (-1)^j \binom{s-r-1}{j} \times \int_{x_r}^{\infty} x_s^{p+\alpha-1} e^{-(n-s+j+1)x_s^\alpha} dx_s$$

Making the transformation $(n-s+j+1)x_s^\alpha = y$ we have

$$\mu^p_{s|r:n} = \alpha K_{r,s:n} \sum_{j=0}^{s-r-1} \frac{1}{e^{-(n-s+j+1)x_r^\alpha}} (-1)^j \binom{s-r-1}{j} \times \int_{(n-s+j+1)x_r^\alpha}^{\infty} \frac{y^{p/\alpha}}{(n-s+j+1)^{p/\alpha+1}} e^{-y} dy$$

$$= \alpha K_{r,s:n} \sum_{j=0}^{s-r-1} \frac{1}{e^{-(n-s+j+1)x_r^\alpha}} (-1)^j \binom{s-r-1}{j} \times \Gamma_{(n-s+j+1)x_r^\alpha}\left(\frac{p}{\alpha} + 1\right);$$

where $K_{r,s:n} = \frac{(n-r)!}{(s-r-1)!(n-s)!}$. The conditional mean and variance can be obtained from above expression. Using $\alpha = 1$ in above expression we can obtain the expression for pth conditional moment of $X_{s:n}$ given $X_{r:n} = x_r$ for Exponential distribution.

2.9 Recurrence Relations and Identities for Moments of Order Statistics

In previous section we have discussed about single, product and conditional moments of order statistics in detail. The moments of order statistics posses certain additional characteristics in that they are related with each other in certain way. In this section we will give some relationships which exist among moments of order statistics. The relationships among moments of order statistics enable us to compute certain moments of higher order statistics on the basis of lower order statistics and/or on the basis of lower order moments. Some of the relationships which exist among moments of order statistics are distribution specific and some of the relationships are free from the underlying parent distribution of the sample. In the following we discuss some of the popular relationships which exist among moments of order statistics. We present both type of relationships; that is distribution free relationships and distribution specific relationships.

2.9.1 *Distribution Free Relations Among Moments of Order Statistics*

The moments of order statistics have several interesting relationships and identities which hold irrespective of the parent distribution. Some relations hold among moments of order statistics and several relations connect moments of order statistics with moments of the distribution from where sample has been drawn. In the following we first present some relationships that hold among single and product moments of order statistics and latter we will give some identities which hold between moments of order statistics and population moments.

We first give an interesting relationship which hold between distribution of rth order statistics and other single ordered observations and a relationship that connect joint distribution of two rth and sth order statistics with joint distribution of other ordered observations in the following.

We know that the distribution of rth order statistics is

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}$$

or

$$\begin{aligned} f_{r:n}(x) &= \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r-1}[1-F(x)] \\ &= \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{(n-1)-r} \\ &\quad - \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{(r+1)-1}[1-F(x)]^{n-(r+1)} \end{aligned}$$

or

$$f_{r:n}(x) = \frac{n}{n-r}\frac{(n-1)!}{(r-1)!(n-r-1)!}f(x)[F(x)]^{r-1}[1-F(x)]^{(n-1)-r}$$
$$-\frac{r}{n-r}\frac{n!}{r!(n-r-1)!}f(x)[F(x)]^{(r+1)-1}[1-F(x)]^{n-(r+1)}$$
$$= \frac{n}{n-r}f_{r:n-1}(x) - \frac{r}{n-r}f_{r+1:n}(x).$$

So we have following relationship for distribution of rth order statistics and other ordered observations

$$(n-r)f_{r:n}(x) = nf_{r:n-1}(x) - rf_{r+1:n}(x). \tag{2.35}$$

In similar way we can show that

$$(n-s)f_{r,s:n}(x_1,x_2) = nf_{r,s:n-1}(x_1,x_2) - rf_{r+1,s+1:n}(x_1,x_2)$$
$$-(s-r)f_{r,s+1:n}(x_1,x_2). \tag{2.36}$$

The relationships given in (2.35) and (2.36) enable us to derive two important relationships which hold between single and product moments of order statistics. The relationships are given below.

We know that the pth moment of rth order statistics is

$$\mu^p_{r:n} = E\left(X^p_{r:n}\right) = \int_{-\infty}^{\infty} x^p f_{r:n}(x)dx.$$

Using the relationship given in (2.35) we have

$$(n-r)\mu^p_{r:n} = \int_{-\infty}^{\infty} x^p\{nf_{r:n-1}(x) - rf_{r+1:n}\}dx$$
$$= n\int_{-\infty}^{\infty} x^p f_{r:n-1}(x)dx - r\int_{-\infty}^{\infty} x^p f_{r+1:n}(x)dx$$
$$= n\mu^p_{r:n-1} - r\mu^p_{r+1:n}$$

or

$$r\mu^p_{r+1:n} = n\mu^p_{r:n-1} - (n-r)\mu^p_{r:n}. \tag{2.37}$$

The relation (2.37) was derived by Cole (1951) and is equally valid for expectation of function of order statistics; that is

$$rE\left[\{g(X_{r+1:n})\}^p\right] = nE\left[\{g(X_{r:n-1})\}^p\right] - (n-r)E\left[\{g(X_{r:n})\}^p\right];$$

also holds. Again consider the product moments of order statistics as

$$\mu_{r,s:n}^{p,q} = E\left(X_{r:n}^{p} X_{s:n}^{q}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^q f_{r,s:n}(x_1, x_2)dx_1 dx_2.$$

Using the relationship (2.36) we have

$$\mu_{r,s:n}^{p,q} = \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^q \left\{\frac{n}{n-s} f_{r,s:n-1}(x_1, x_2) - \frac{r}{n-s} \times f_{r+1,s+1:n}(x_1, x_2) - \frac{(s-r)}{n-s} f_{r,s+1:n}(x_1, x_2)\right\} dx_1 dx_2$$

or

$$\begin{aligned}(n-s)\mu_{r,s:n}^{p,q} = n &\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^q f_{r,s:n-1}(x_1, x_2)dx_1 dx_2 \\ -r &\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^q f_{r+1,s+1:n}(x_1, x_2)dx_1 dx_2 \\ -(s-r) &\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^q f_{r,s+1:n}(x_1, x_2)dx_1 dx_2\end{aligned}$$

or

$$(n-s)\mu_{r,s:n}^{p,q} = n\mu_{r,s:n-1}^{p,q} - r\mu_{r+1,s+1:n}^{p,q} - (s-r)\mu_{r,s+1:n}^{p,q}$$

or

$$r\mu_{r+1,s+1:n}^{p,q} = n\mu_{r,s:n-1}^{p,q} - (n-s)\mu_{r,s:n}^{p,q} - (s-r)\mu_{r,s+1:n}^{p,q}. \tag{2.38}$$

which is due to Govindarajulu (1963). Again the relationship (2.38) is equally valid for function of product moments of order statistics. The relationship (2.37) further provide us following interesting relationship for n even

$$\frac{1}{2}\left(\mu_{\frac{n}{2}+1:n}^{p} + \mu_{\frac{n}{2}:n}^{p}\right) = n\mu_{\frac{n}{2}:n-1}^{p}; \tag{2.39}$$

which can be easily proved by using $r = \frac{n}{2}$ in (2.37). The relationship (2.39) also provide following interesting result for symmetric parent distribution

$$\begin{aligned}\mu_{\frac{n}{2}:n-1}^{p} &= \mu_{\frac{n}{2}:n}^{p} \ for\ p\ even \\ &= 0\ for\ p\ odd.\end{aligned}$$

The relationships given in (2.37) and (2.38) enable us to compute moments of order statistics recursively. We have certain additional relationships available for moments of order statistics which are based upon the sum of moments of lower order statistics. We present those relationships, which are due to Srikantan (1962); in the following theorems.

Theorem 2.6 *For any parent distribution following relation holds between moments of order statistics*

$$\mu_{r:n}^{p} = \sum_{i=r}^{n}(-1)^{i-r}\binom{i-1}{r-1}\binom{n}{i}\mu_{i:i}^{p};\ r = 1, 2, \ldots, n-1. \tag{2.40}$$

Proof We have

$$\begin{aligned}
\mu_{r:n}^{p} &= E\left(X_{r:n}^{p}\right) = \int_{-\infty}^{\infty} x^{p} f_{r:n}(x)dx \\
&= \int_{0}^{1} \left\{F^{-1}(u)\right\}^{p} f_{r:n}(u)du \\
&= \frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty}\left\{F^{-1}(u)\right\}^{p} u^{r-1}(1-u)^{n-r}du
\end{aligned}$$

Now expanding $(1-u)^{n-r}$ using binomial expansion we have

$$\begin{aligned}
\mu_{r:n}^{p} &= \frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty}\left\{F^{-1}(u)\right\}^{p} u^{r-1}\sum_{j=0}^{n-r}(-1)^{i}\binom{n-r}{j}u^{j}du \\
&= \sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty}\left\{F^{-1}(u)\right\}^{p} u^{r+j-1}du \\
&= \sum_{j=0}^{n-r}(-1)^{j}\frac{(n-r)!}{j!(n-r-j)!}\frac{n!(r+j)}{(r-1)!(n-r)!(r+j)}\mu_{r+j:r+j}^{p}.
\end{aligned}$$

Now writing $r+j=i$ we have

$$\begin{aligned}
\mu_{r:n}^{p} &= \sum_{i=r}^{n}(-1)^{i-r}\frac{(n-r)!}{i(i-r)!(n-r-i+r)!}\frac{n!}{(r-1)!(n-r)!}\mu_{i:i}^{p} \\
&= \sum_{i=r}^{n}(-1)^{i-r}\frac{(i-1)!n!}{i(i-1)!(i-r)!(n-i)!(r-1)!}\mu_{i:i}^{p} \\
&= \sum_{i=r}^{n}(-1)^{i-r}\binom{i-1}{r-1}\binom{n}{i}\mu_{i:i}^{p}
\end{aligned}$$

as required.

Theorem 2.7 *Following relationship holds between moments of order statistics*

$$\mu_{r:n}^p = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \mu_{1:i}^p; \; r = 2, 3, \ldots, n. \tag{2.41}$$

Proof Consider the expression for pth moment of rth order statistics as

$$\begin{aligned} \mu_{r:n}^p &= E\left(X_{r:n}^p\right) = \int_{-\infty}^{\infty} x^p f_{r:n}(x)dx \\ &= \int_0^1 \left\{F^{-1}(u)\right\}^p f_{r:n}(u)du \\ &= \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} \left\{F^{-1}(u)\right\}^p u^{r-1}(1-u)^{n-r}du. \end{aligned}$$

Writing u^{r-1} as $\{1-(1-u)\}^{r-1}$ and expanding binomially in power series of $(1-u)$ we have

$$\begin{aligned} \mu_{r:n}^p &= \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} \left\{F^{-1}(u)\right\}^p \sum_{j=0}^{r-1} (-1)^j \\ &\quad \times \binom{r-1}{j} (1-u)^j (1-u)^{n-r} du \\ &= \sum_{j=0}^{r-1} (-1)^j \frac{(r-1)!}{j!(r-1-j)!} \frac{n!}{(r-1)!(n-r)!} \\ &\quad \times \int_{-\infty}^{\infty} \left\{F^{-1}(u)\right\}^p (1-u)^{n-r+j} du \end{aligned}$$

Now using $j = r+i-n-1$ or $i = j-r+n+1$ we have

$$\begin{aligned} \mu_{r:n}^p &= \sum_{i=n-r+1}^{n} (-1)^{r+i-n-1} \frac{(r-1)!}{(r+i-n-1)!(n-i)!} \frac{n!}{r!(n-r)!} \\ &\quad \times r \int_{-\infty}^{\infty} \left\{F^{-1}(u)\right\}^p (1-u)^{i-1} du \\ &= \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \mu_{1:i}^p; \end{aligned}$$

as required.

Theorem 2.8 *Following relationship holds between moments of order statistics for* $r, s = 1, 2, \ldots, n$ *and* $r < s$

$$\mu_{r,s:n}^{p,q} = \sum_{i=r}^{s-1} \sum_{j=n-s+i+1}^{n} (-1)^{j+n-r-s+1} \binom{i-1}{r-1} \binom{j-i-1}{n-s} \binom{n}{j} \mu_{i,i+1:j}^{p,q} \quad (2.42)$$

Proof Consider the expression for product moments of order statistics as

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= E\left(X_{r:n}^{p} X_{s:n}^{q}\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} x_1^p x_2^q f_{r,s:n}(x_1, x_2) dx_1 dx_2 \\
&= C_{r,s:n} \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} x_1^p x_2^q f(x_1) f(x_2) F^{r-1}(x_1) \\
&\quad \times \left[F(x_2) - F(x_1) \right]^{s-r-1} [1 - F(x_2)]^{n-s} dx_1 dx_2.
\end{aligned}$$

or

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= C_{r,s:n} \int_0^1 \int_0^v \left\{F^{-1}(u)\right\}^p \left\{F^{-1}(v)\right\}^q u^{r-1} (v-u)^{s-r-1} \\
&\quad \times (1-v)^{n-s} du dv.
\end{aligned}$$

Now expanding $(v-u)^{s-r-1}$ in power series we have

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= C_{r,s:n} \int_0^1 \int_0^v \left\{F^{-1}(u)\right\}^p \left\{F^{-1}(v)\right\}^q u^{r-1} \\
&\quad \times \sum_{h=0}^{s-r-1} (-1)^h \binom{s-r-1}{h} v^{s-r-1-h} u^h (1-v)^{n-s} du dv \\
&= \sum_{h=0}^{s-r-1} (-1)^h \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \frac{(s-r-1)!}{h!(s-r-h-1)!} \\
&\quad \times \int_0^1 \int_0^v \left\{F^{-1}(u)\right\}^p \left\{F^{-1}(v)\right\}^q u^{r+h-1} v^{s-r-h-1} (1-v)^{n-s} du dv \\
&= \sum_{h=0}^{s-r-1} (-1)^h \frac{n!}{h!(r-1)!(s-r-h-1)!(n-s)!} \\
&\quad \times \int_0^1 \int_0^v \left\{F^{-1}(u)\right\}^p \left\{F^{-1}(v)\right\}^q u^{r+h-1} v^{s-r-h-1} (1-v)^{n-s} du dv.
\end{aligned}$$

Now using $i = h + r$ we have:

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= \sum_{i=r}^{s-1} (-1)^{i-r} \frac{n!}{(i-r)!(r-1)!(s-i-1)!(n-s)!} \\
&\quad \times \int_0^1 \int_0^v \left\{F^{-1}(u)\right\}^p \left\{F^{-1}(v)\right\}^q u^{i-1} v^{s-i-1} (1-v)^{n-s} du dv.
\end{aligned}$$

Now writing v^{s-i-1} as $\{1-(1-v)\}^{s-i-1}$ and expanding in power series we have

$$\mu_{r,s:n}^{p,q} = \sum_{i=r}^{s-1}(-1)^{i-r}\frac{n!}{(i-r)!(r-1)!(s-i-1)!(n-s)!}$$
$$\times \int_0^1\int_0^v \left\{F^{-1}(u)\right\}^p\left\{F^{-1}(v)\right\}^q u^{i-1}$$
$$\times \sum_{m=0}^{s-i-1}(-1)^m\binom{s-i-1}{m}(1-v)^m(1-v)^{n-s}dudv$$

or

$$\mu_{r,s:n}^{p,q} = \sum_{i=r}^{s-1}\sum_{m=0}^{s-i-1}(-1)^{i-r+m}\frac{n!}{(i-r)!(r-1)!(s-i-1)!(n-s)!}$$
$$\times \frac{(s-i-1)!}{m!(s-i-1-m)!}\int_0^1\int_0^v \left\{F^{-1}(u)\right\}^p\left\{F^{-1}(v)\right\}^q$$
$$\times u^{i-1}(1-v)^{n-s+m}dudv.$$
$$= \sum_{i=r}^{s-1}\sum_{m=0}^{s-i-1}\frac{(-1)^{i-r+m}n!}{(i-r)!(r-1)!m!(s-i-1-m)!(n-s)!}$$
$$\times \int_0^1\int_0^v \left\{F^{-1}(u)\right\}^p\left\{F^{-1}(v)\right\}^q u^{i-1}(1-v)^{n-s+m}dudv$$

or

$$\mu_{r,s:n}^{p,q} = \sum_{i=r}^{s-1}\sum_{m=0}^{s-i-1}\frac{(-1)^{i-r+m}n!(i-1)!(n-s+m)!}{(i-r)!(r-1)!m!(s-i-1-m)!(n-s)!}$$
$$\times (n-s+i+m+1)! \times \ \mu_{i,i+1:n-s+m+i+1}.$$

Now using $j = n-s+i+1+m$ and rearranging the terms we have

$$\mu_{r,s:n}^{p,q} = \sum_{i=r}^{s-1}\sum_{j=n-s+i+1}^{n}(-1)^{j+n-r-s+1}\binom{i-1}{r-1}\binom{j-i-1}{n-s}\binom{n}{j}\mu_{i,i+1:j}^{p,q};$$

as required.

Proceeding in the way of Theorem 2.8, we also have following relationships between product moments of order statistics.

$$\mu_{r,s:n}^{p,q} = \sum_{i=s-r}^{s-1}\sum_{j=n-s+i+1}^{n}(-1)^{n-j-r+1}\binom{i-1}{s-r-1}\binom{j-i-1}{n-s}\binom{n}{j}\mu_{1,i+1:j}^{p,q}; \tag{2.43}$$

and

$$\mu_{r,s:n}^{p,q} = \sum_{i=s-r}^{n-r} \sum_{j=r+i}^{n} (-1)^{s+j} \binom{i-1}{s-r-1} \binom{j-i-1}{r-1} \binom{n}{j} \mu_{j-i,j:j}^{p,q}. \tag{2.44}$$

The relationships given in (2.43) and (2.44) link product moments of order statistics with those based upon the product moments of order statistics based upon smaller sample sizes and on lower order product moments.

The relationships given in equations (2.40) to (2.44) are useful in computing single and product moments of a specific order statistics as a sum of moments of lower order statistics. We have some additional interesting identities which relates sum of moments of a specific order statistics with sum of moments of lower order statistics. We also have some interesting identities which relate moments of order statistics with population moments and are free from any sort of distributional assumptions. We present these identities in the following section.

2.9.2 Some Identities for Moments of Order Statistics

The moments of order statistics posses certain simple identities. These identities are based upon following very basic formulae

$$\sum_{r=1}^{n} X_{r:n}^{p} = \sum_{r=1}^{n} X_{r}^{p};\ p \geq 1 \tag{2.45}$$

and

$$\sum_{r=1}^{n} \sum_{s=1}^{n} X_{r:n}^{p} X_{s:n}^{q} = \sum_{r=1}^{n} \sum_{s=1}^{n} X_{r}^{p} X_{s}^{q};\ p,q \geq 1. \tag{2.46}$$

Now if all X_r have same distribution $F(x)$ with $E\left(X_r^p\right) = \mu_p$, variance σ^2 and $E\left(X_r^p X_s^q\right) = \mu_{p,q}$ then we have following interesting identities.

Taking expectation on (2.45) we

$$E\left(\sum_{r=1}^{n} X_{r:n}^{p}\right) = E\left(\sum_{r=1}^{n} X_{r}^{p}\right)$$

or

$$\sum_{r=1}^{n} E\left(X_{r:n}^{p}\right) = \sum_{r=1}^{n} E\left(X_{r}^{p}\right)$$

or

$$\sum_{r=1}^{n} \mu_{r:n}^{p} = \sum_{r=1}^{n} \mu_{p} = n\mu_{p}. \tag{2.47}$$

In particular

$$\sum_{r=1}^{n} \mu_{r:n} = n\mu. \tag{2.48}$$

Again taking the expectation of (2.46) we have

$$E\left(\sum_{r=1}^{n}\sum_{s=1}^{n} X_{r:n}^{p} X_{s:n}^{q}\right) = E\left(\sum_{r=1}^{n}\sum_{s=1}^{n} X_{r}^{p} X_{s}^{q}\right)$$

or

$$\sum_{r=1}^{n}\sum_{s=1}^{n} E\left(X_{r:n}^{p} X_{s:n}^{q}\right) = \sum_{r=1}^{n}\sum_{s=1}^{n} E\left(X_{r}^{p} X_{s}^{q}\right)$$

or

$$\sum_{r=1}^{n}\sum_{s=1}^{n} \mu_{r,s:n}^{p,q} = \sum_{r=1}^{n}\sum_{s=1}^{n} \mu_{p,q}.$$

Since X_r and X_s have same distribution therefore above relation can be written as

$$\sum_{r=1}^{n}\sum_{s=1}^{n} \mu_{r,s:n}^{p,q} = \sum_{r=1}^{n} \mu_{p+q} + \sum_{r=1}^{n}\sum_{s\neq r=1}^{n} \mu_{p}\mu_{q}$$

or

$$\sum_{r=1}^{n}\sum_{s=1}^{n} \mu_{r,s:n}^{p,q} = n\mu_{p+q} + n(n-1)\mu_{p}\mu_{q}. \tag{2.49}$$

In particular

$$\sum_{r=1}^{n}\sum_{s=1}^{n} \mu_{r,s:n} = n\mu_2 + n(n-1)\mu^2 = n\sigma^2 + n^2\mu^2; \tag{2.50}$$

and as a result we have

$$\begin{aligned}\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} \mu_{r,s:n} &= \frac{1}{2}\left\{\sum_{r=1}^{n}\sum_{s=1}^{n} \mu_{r,s:n} - \sum_{r=1}^{n} \mu_{r:n}^{2}\right\} \\ &= \frac{1}{2}\left\{n\mu_2 + n(n-1)\mu^2 - n\mu_2\right\} \\ &= \frac{n(n-1)}{2}\mu^2 = \binom{n}{2}\mu^2. \end{aligned} \tag{2.51}$$

We also have following identity for $n = 2$

$$\mu_{1,2:2}^{p,q} + \mu_{1,2:2}^{q,p} = 2\mu_{p}\mu_{q}; \tag{2.52}$$

which for $p = q = 1$ reduces to

$$\mu_{1,2:2} = \mu^2.$$

Also from (2.48) and (2.50) we have

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\sigma_{r,s:n} = \sum_{r=1}^{n}\sum_{s=1}^{n}\mu_{r,s:n} - \left(\sum_{r=1}^{n}\mu_{r:n}\right)\left(\sum_{s=1}^{n}\mu_{s:n}\right)$$
$$= n\sigma^2 + n^2\mu^2 - n^2\mu^2 = n\sigma^2. \quad (2.53)$$

Above identities are very useful in checking the accuracy of single and product moments of order statistics by their comparison with the population moments.

The single moments of order statistics have an additional identity which relates the sum of single moments in terms of sum of moments of lower order statistics. These identities are given in the following theorem.

Theorem 2.9 *The single moments of order statistics satisfies following identities*

$$\sum_{r=1}^{n}\frac{1}{r}\mu_{r:n}^{p} = \sum_{r=1}^{n}\frac{1}{r}\mu_{1:r} \quad (2.54)$$

and

$$\sum_{r=1}^{n}\frac{1}{n-r+1}\mu_{r:n}^{p} = \sum_{r=1}^{n}\frac{1}{r}\mu_{r:r}. \quad (2.55)$$

Proof Consider the expression for single moments of order statistics as

$$\mu_{r:n}^{p} = \int_{-\infty}^{\infty} x_r^p f_{r:n}(x_r)dx_r$$
$$= \int_0^1 \left\{F^{-1}(u)\right\}^p f_{r:n}(u)du$$
$$= \frac{n!}{(r-1)!(n-r)!}\int_0^1 \left\{F^{-1}(u)\right\}^p u^{r-1}(1-u)^{n-r}du.$$

Applying summation over both sides we have

$$\sum_{r=1}^{n}\frac{1}{r}\mu_{r:n}^{p} = \sum_{r=1}^{n}\frac{n!}{r(r-1)!(n-r)!}\int_0^1 \left\{F^{-1}(u)\right\}^p$$
$$\times u^{r-1}(1-u)^{n-r}du$$
$$= \sum_{r=1}^{n}\binom{n}{r}\int_0^1 \left\{F^{-1}(u)\right\}^p u^{r-1}(1-u)^{n-r}du$$

$$= \int_0^1 \left\{F^{-1}(u)\right\}^p \frac{1}{u}\left\{\sum_{r=1}^{n}\binom{n}{r}u^r(1-u)^{n-r}\right\}du$$
$$= \int_0^1 \left\{F^{-1}(u)\right\}^p \frac{1}{u}\left\{1-(1-u)^n\right\}du.$$

Now using the identity

$$1-(1-u)^n = u\sum_{r=1}^{n}(1-u)^{r-1};$$

we have

$$\sum_{r=1}^{n}\frac{1}{r}\mu_{r:n}^{p} = \int_0^1 \left\{F^{-1}(u)\right\}^p \sum_{r=1}^{n}(1-u)^{r-1}du$$
$$= \sum_{r=1}^{n}\frac{1}{r}r\int_0^1 \left\{F^{-1}(u)\right\}^p (1-u)^{r-1}du$$
$$= \sum_{r=1}^{n}\frac{1}{r}\mu_{1:r}^{p};$$

which is (2.54).

For second identity again consider the expression for single moments as

$$\mu_{r:n}^{p} = \frac{n!}{(r-1)!(n-r)!}\int_0^1 \left\{F^{-1}(u)\right\}^p u^{r-1}(1-u)^{n-r}du$$

or

$$\sum_{r=1}^{n}\frac{1}{n-r+1}\mu_{r:n}^{p} = \sum_{r=1}^{n}\frac{n!}{(r-1)!(n-r+1)!}$$
$$\times\int_0^1 \left\{F^{-1}(u)\right\}^p u^{r-1}(1-u)^{n-r}du$$
$$= \sum_{r=1}^{n}\binom{n}{r-1}\int_0^1 \left\{F^{-1}(u)\right\}^p u^{r-1}(1-u)^{n-r}du$$

or

$$\sum_{r=1}^{n}\frac{\mu_{r:n}^{p}}{n-r+1} = \int_0^1 \left\{F^{-1}(u)\right\}^p \frac{1}{1-u}\left\{\sum_{r=1}^{n-1}\binom{n}{r}u^r(1-u)^{n-r}\right\}du$$
$$= \int_0^1 \left\{F^{-1}(u)\right\}^p \frac{1}{1-u}\left(1-u^n\right)du.$$

Now using the identity

$$1 - u^n = (1-u)\sum_{r=1}^{n} u^{r-1};$$

we have

$$\begin{aligned}\sum_{r=1}^{n}\frac{1}{n-r+1}\mu_{r:n}^{p} &= \int_0^1 \left\{F^{-1}(u)\right\}^p \sum_{r=1}^{n} u^{r-1}du\\ &= \sum_{r=1}^{n}\frac{1}{r}r\int_0^1 \left\{F^{-1}(u)\right\}^p u^{r-1}du\\ &= \sum_{r=1}^{n}\frac{1}{r}\mu_{r:r}^{p};\end{aligned}$$

which is (2.55). Hence the theorem.

Example 2.10 Prove that in a random sample from a continuous distribution with cdf $F(x)$ following relation holds:

$$E[X_{s:n}F(X_{r:n})] = \frac{r}{n+1}\mu_{s+1:n+1}.$$

Solution: We have:

$$\begin{aligned}f_{r,s:n}(x_1,x_2) &= C_{r,s:n}f(x_1)f(x_2)[F(x_1)]^{r-1}\Big[F(x_2)-F(x_1)\Big]^{s-r-1}\\ &\quad\times[1-F(x_2)]^{n-s}.\end{aligned}$$

Now

$$\begin{aligned}E[X_{s:n}F(X_{r:n})] &= C_{r,s:n}\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}\{x_2F(x_1)\}f(x_1)f(x_2)[F(x_1)]^{r-1}\\ &\quad\times\Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s}dx_1dx_2\\ &= C_{r,s:n}\int_0^1\int_0^v F^{-1}(v)u^r(v-u)^{s-r-1}(1-v)^{n-s}dudv\\ &= C_{r,s:n}\int_0^1 F^{-1}(v)(1-v)^{n-s}\Big[\int_0^v u^r(v-u)^{s-r-1}du\Big]dv\\ &= C_{r,s:n}\int_0^1 F^{-1}(v)(1-v)^{n-s}(I)dv; \qquad (2.56)\end{aligned}$$

where $I = \int_0^v u^r(v-u)^{s-r-1}du$. Now consider

$$I = \int_0^v u^r (v-u)^{s-r-1} du$$
$$= v^{s-r-1} \int_0^v u^r \left(1 - \frac{u}{v}\right)^{s-r-1} du.$$

Making the transformation $\frac{u}{v} = w$ we have

$$I = v^s \int_0^1 w^r (1-w)^{s-r-1} dw$$
$$= v^s B(r+1, s-r) = v^s \frac{\Gamma(r+1)\Gamma(s-r)}{\Gamma(s+1)}$$
$$= v^s \frac{r!(s-r-1)!}{s!}.$$

Using above value in (2.56) we have

$$E[X_{s:n} F(X_{r:n})] = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$
$$\times \int_0^1 F^{-1}(v)(1-v)^{n-s} v^s \frac{r!(s-r-1)!}{s!} dv$$
$$= \frac{r(r-1)!n!}{(r-1)!s!(n-s)!} \int_0^1 F^{-1}(v) v^s (1-v)^{n-s} v^s$$
$$= \frac{r}{n+1} \frac{(n+1)!}{s!(n-s)!} \int_0^1 F^{-1}(v) v^{s+1-1} (1-v)^{n-s} v^s$$
$$= \frac{r}{n+1} \mu_{s+1:n+1};$$

as required.

2.9.3 *Distribution Specific Relationships for Moments of Order Statistics*

In previous section we have given some useful relations which exist among moments of order statistics irrespective of the parent distribution. There exist certain other relationships among moments of order statistics which are based upon the parent distribution from where sample has been drawn. In this section we will give some recurrence relations for single and product moments of order statistics which are limited to parent probability distribution. We first give two useful results in following theorem which can be used to derive the recurrence relations for single and product moments of order statistics for certain special distributions.

Theorem 2.10 *Following relations hold for single and product moments of order statistics from a distribution $F(x)$*

$$\mu_{r:n}^p - \mu_{r-1:n}^p = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \times [F(x)]^{r-1}[1-F(x)]^{n-r+1}dx. \tag{2.57}$$

and

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1) \times [F(x_1)]^{r-1}\Big[F(x_2)-F(x_1)\Big]^{s-r-1} \times [1-F(x_2)]^{n-s+1}dx_2dx_1. \tag{2.58}$$

where $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

Proof We know that the pth moment of rth order statistics is

$$\begin{aligned}\mu_{r:n}^p &= E\left(X_{r:n}^p\right) = \int_{-\infty}^{\infty} x^p f_{r:n}(x)dx \\ &= \int_{-\infty}^{\infty} x^p \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx \\ &= \frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^p f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx.\end{aligned}$$

Integrating above equation by parts taking $f(x)[1-F(x)]^{n-r}$ as function for integration we have

$$\begin{aligned}\mu_{r:n}^p = \frac{n!}{(r-1)!(n-r)!}\Bigg[&-x^p[F(x)]^{r-1}\frac{\{1-F(x)\}^{n-r+1}}{n-r+1}\Bigg|_{-\infty}^{\infty} \\ &-\int_{-\infty}^{\infty}\left\{px^{p-1}[F(x)]^{r-1}+(r-1)x^p[F(x)]^{r-2}f(x)\right\} \\ &\times\frac{-[1-F(x)]^{n-r+1}}{n-r+1}dx\Bigg]\end{aligned}$$

or

$$\begin{aligned}\mu_{r:n}^p = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}&\int_{-\infty}^{\infty} x^{p-1}[1-F(x)]^{n-r+1} \\ \times[F(x)]dx + \frac{(r-1)}{n-r+1}&\frac{n!}{(r-1)!(n-r)!}\end{aligned}$$

$$\times \int_{-\infty}^{\infty} x^p f(x)[1-F(x)]^{n-r+1}[F(x)]^{r-2}dx$$
$$= \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1}[F(x)]^{r-1}$$
$$\times [1-F(x)]^{n-r+1}dx + \frac{n!}{(r-2)!(n-r+1)!}\int_{-\infty}^{\infty} x^p f(x)$$
$$\times [F(x)]^{r-2}[1-F(x)]^{n-(r-1)}dx.$$

Since

$$\mu_{r-1:n}^p = \frac{n!}{(r-2)!(n-r+1)!}\int_{-\infty}^{\infty} x^p f(x)[F(x)]^{r-2}$$
$$\times [1-F(x)]^{n-(r-1)}dx,$$

hence above equation can be written as

$$\mu_{r:n}^p = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1}[F(x)]^{r-1}$$
$$\times [1-F(x)]^{n-r+1}dx + \mu_{r-1:n}^p$$

or

$$\mu_{r:n}^p - \mu_{r-1:n}^p = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1}$$
$$\times [F(x)]^{r-1}[1-F(x)]^{n-r+1}dx,$$

which is (2.57).

To prove the second result we consider the expression for product moments of order statistics as

$$\mu_{r,s:n}^{p,q} = E\left(X_{r:n}^p X_{s:n}^q\right) = \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f_{r,s:n}(x_1,x_2)dx_2dx_1$$
$$= C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times \Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s}dx_2dx_1$$
$$= C_{r,s:n}\int_{-\infty}^{\infty} x_1^p f(x_1)[F(x_1)]^{r-1}I(x_2)dx_1, \tag{2.59}$$

where

$$I(x_2) = \int_{x_1}^{\infty} x_2^q f(x_2)\Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s}dx_2.$$

Integrating above integral by parts using $f(x_2)\{1-F(x_2)\}^{n-s}$ for integration we have

$$\begin{aligned}
I(x_2) &= -x_2^q\Big[F(x_2)-F(x_1)\Big]^{s-r-1}\frac{\{1-F(x_2)\}^{n-s+1}}{n-s+1}\Big|_{x_1}^{\infty}\\
&\quad+\frac{1}{n-s+1}\int_{x_1}^{\infty}\Big[qx_2^{q-1}\Big[F(x_2)-F(x_1)\Big]^{s-r-1}\\
&\quad+(s-r-1)x_2^q\{F(x_2)-F(x_1)\}^{s-r-2}f(x_2)\Big]\\
&\quad\times\{1-F(x_2)\}^{n-s+1}dx_2\\
&= \frac{q}{n-s+1}\int_{x_1}^{\infty}x_2^{q-1}[F(x_2)-F(x_1)]^{s-r-1}\\
&\quad\times\{1-F(x_2)\}^{n-s+1}dx_2+\frac{s-r-1}{n-s+1}\int_{x_1}^{\infty}x_2^q f(x_2)\\
&\quad\times[F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1}dx_2.
\end{aligned}\tag{2.60}$$

Now using the value of $I(x_2)$ from (2.60) in (2.59) we have

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= C_{r,s:n}\int_{-\infty}^{\infty}x_1^p f(x_1)[F(x_1)]^{r-1}\Big[\frac{q}{n-s+1}\\
&\quad\times\int_{x_1}^{\infty}x_2^{q-1}\Big[F(x_2)-F(x_1)\Big]^{s-r-1}\{1-F(x_2)\}^{n-s+1}dx_2\\
&\quad+\frac{s-r-1}{n-s+1}\int_{x_1}^{\infty}x_2^q f(x_2)[F(x_2)-F(x_1)]^{s-r-2}\\
&\quad\times[1-F(x_2)]^{n-s+1}dx_2\Big]dx_1
\end{aligned}$$

or

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1^p x_2^{q-1}f(x_1)[F(x_1)]^{r-1}\\
&\quad\times\Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s+1}dx_2dx_1\\
&\quad+\frac{s-r-1}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1^p x_2^q f(x_1)f(x_2)[F(x_1)]^{r-1}\\
&\quad\times[F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1}dx_2dx_1
\end{aligned}$$

or

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1^p x_2^{q-1}f(x_1)[F(x_1)]^{r-1}\\
&\quad\times\Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s+1}dx_2dx_1
\end{aligned}$$

$$+C_{r,s-1:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times [F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1}dx_2dx_1$$

or

$$\mu_{r,s:n}^{p,q} = \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1)[F(x_1)]^{r-1}$$
$$\times \left[F(x_2)-F(x_1)\right]^{s-r-1}[1-F(x_2)]^{n-s+1}dx_2dx_1$$
$$+\mu_{r,s-1:n}^{p,q}$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1)$$
$$\times [F(x_1)]^{r-1}\left[F(x_2)-F(x_1)\right]^{s-r-1}$$
$$[1-F(x_2)]^{n-s+1}dx_2dx_1,$$

which is (2.58) and hence the theorem.

The results given in Theorem 2.10 are very useful in deriving distribution specific recurrence relations for single and product moments of order statistics. In the following subsections we have discussed recurrence relations between single and product moments of order statistics for certain distributions.

2.9.4 Exponential Distribution

The Exponential distribution has been the area of study in order statistics by many researchers. The density and distribution function of this distribution are

$$f(x) = \alpha\exp(-\alpha x);\ x,\alpha > 0$$

and

$$F(x) = 1-\exp(-\alpha x).$$

We can readily see that the density and distribution function for exponential distribution are related through the equation

$$f(x) = \alpha[1-F(x)]. \tag{2.61}$$

The recurrence relation for single and product moments of order statistics for exponential distribution are derived Joshi (1978, 1982) by using (2.57), (2.58) and (2.61) as below.

For recurrence relation between single moments of order statistics consider relation (2.57) as

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \times [F(x)]^{r-1}[1-F(x)]^{n-r+1}dx.$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \times [F(x)]^{r-1}[1-F(x)]^{n-r}[1-F(x)]dx.$$

Now using (2.61) in above equation we have

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\alpha(n-r+1)}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\alpha(n-r+1)}\mu_{r:n}^{p-1}$$

or

$$\mu_{r:n}^{p} = \mu_{r-1:n}^{p} + \frac{p}{\alpha(n-r+1)}\mu_{r:n}^{p-1}; \tag{2.62}$$

as given in Balakrishnan and Rao (1998). The recurrence relation (2.62) provide following relation as a special case for $r = 1$

$$\mu_{1:n}^{p} = \frac{p}{\alpha n}\mu_{1:n}^{p-1}.$$

The recurrence relation for product moments of order statistics for exponential distribution is readily obtained by using (2.58) as

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^{p}x_2^{q-1}f(x_1) \times [F(x_1)]^{r-1}\Big[F(x_2)-F(x_1)\Big]^{s-r-1} \times [1-F(x_2)]^{n-s+1}dx_2dx_1.$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1} C_{r,s:n} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1) \times [F(x_1)]^{r-1} \Big[F(x_2) - F(x_1) \Big]^{s-r-1} \times [1-F(x_2)]^{n-s} [1-F(x_2)] dx_2 dx_1.$$

Now using (2.61) in above equation we have

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{\alpha(n-s+1)} C_{r,s:n} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1) f(x_2) \times [F(x_1)]^{r-1} \Big[F(x_2) - F(x_1) \Big]^{s-r-1} \times [1-F(x_2)]^{n-s} dx_2 dx_1.$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{\alpha(n-s+1)} \mu_{r,s:n}^{p,q-1}$$

or

$$\mu_{r,s:n}^{p,q} = \mu_{r,s-1:n}^{p,q} + \frac{q}{\alpha(n-s+1)} \mu_{r,s:n}^{p,q-1}. \tag{2.63}$$

Using $s = r+1$ in (2.63) we have following recurrence relation for product moments of two contiguous order statistics from exponential distribution

$$\mu_{r,r+1:n}^{p,q} = \mu_{r:n}^{p+q} + \frac{q}{\alpha(n-r)} \mu_{r,r+1:n}^{p,q-1}. \tag{2.64}$$

Certain other relations can be derived from (2.63) and (2.64). Some more recurrence relations for single and product moments of order statistics from exponential distribution can be found in Joshi (1982).

2.9.5 The Weibull Distribution

The Weibull distribution has wide spread applications in almost all the areas of life. The density and distribution function for a Weibull random variable are

$$f(x) = \beta \alpha^{\beta} x^{\beta-1} \exp\big[-(\alpha x)^{\beta}\big];\ x, \alpha, \beta > 0$$

and

$$F(x) = 1 - \exp\big[-(\alpha x)^{\beta}\big].$$

We can see that the density and distribution function are related as

$$f(x) = \beta\alpha^{\beta}x^{\beta-1}[1 - F(x)]. \tag{2.65}$$

The recurrence relations for single and product moments of order statistics for Weibull distribution are derived below.

The recurrence relation for single moments is given in (2.57) as

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \times [F(x)]^{r-1}[1-F(x)]^{n-r+1}dx.$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \times [F(x)]^{r-1}[1-F(x)]^{n-r}[1-F(x)]dx.$$

Using (2.65) in above equation we have

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\alpha^{\beta}\beta(n-r+1)}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1}x^{1-\beta} \times f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\alpha^{\beta}\beta(n-r+1)}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-\beta} \times f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\alpha^{\beta}\beta(n-r+1)}\mu_{r:n}^{p-\beta}$$

or

$$\mu_{r:n}^{p} = \mu_{r-1:n}^{p} + \frac{p}{\alpha^{\beta}\beta(n-r+1)}\mu_{r:n}^{p-\beta}. \tag{2.66}$$

We can see that (2.66) reduces to (2.62) for $\beta = 1$ as it should be. Using $\beta = 2$ in (2.66) we obtain the recurrence relation for single moments of order statistics from Rayleigh distribution as

$$\mu_{r:n}^{p} = \mu_{r-1:n}^{p} + \frac{p}{2\alpha^{2}(n-r+1)}\mu_{r:n}^{p-2}. \tag{2.67}$$

Also for $r = 1$ we have following recurrence relations for single moments of first order statistics from Weibull distribution

$$\mu_{1:n}^{p} = \frac{p}{n\alpha^{\beta}\beta}\mu_{1:n}^{p-\beta}. \tag{2.68}$$

We now present the recurrence relations for product moments of order statistics for Weibull distribution. We have relation (2.58) as

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^{p}x_2^{q-1}f(x_1) \times [F(x_1)]^{r-1}\Big[F(x_2) - F(x_1)\Big]^{s-r-1} \times [1 - F(x_2)]^{n-s+1}dx_2dx_1.$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^{p}x_2^{q-1}f(x_1) \times [F(x_1)]^{r-1}\Big[F(x_2) - F(x_1)\Big]^{s-r-1} \times [1 - F(x_2)]^{n-s}[1 - F(x_2)]dx_2dx_1.$$

Using (2.65) in above equation we have

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{\alpha^{\beta}\beta(n-s+1)}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^{p}x_2^{q-\beta}f(x_1) \times f(x_2)[F(x_1)]^{r-1}\Big[F(x_2) - F(x_1)\Big]^{s-r-1} \times [1 - F(x_2)]^{n-s}dx_2dx_1.$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{\alpha^{\beta}\beta(n-s+1)}\mu_{r,s:n}^{p,q-\beta}$$

or

$$\mu_{r,s:n}^{p,q} = \mu_{r,s-1:n}^{p,q} + \frac{q}{\alpha^{\beta}\beta(n-s+1)}\mu_{r,s:n}^{p,q-\beta}. \tag{2.69}$$

We can immediately see that (2.69) reduces to (2.63) for $\beta = 1$. Using $\beta = 2$ in (2.69) we have following recurrence relation for product moments of order statistics for Rayleigh distribution

$$\mu_{r,s:n}^{p,q} = \mu_{r,s-1:n}^{p,q} + \frac{q}{2\alpha^2(n-s+1)}\mu_{r,s:n}^{p,q-2}. \tag{2.70}$$

Also for $s = r + 1$ the recurrence relation for product moments of two contiguous order for Weibull distribution is

$$\mu_{r,r+1:n}^{p,q} = \mu_{r:n}^{p+q} + \frac{q}{\alpha^\beta \beta(n-r)}\mu_{r,r+1:n}^{p,q-\beta}. \tag{2.71}$$

The relationships (2.67) and (2.69) can be used to derive recurrence relations for mean, variances and covariances of order statistics from Weibull distribution.

2.9.6 The Logistic Distribution

The Logistic distribution has density and distribution function as

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}; -\infty < x < \infty$$

and

$$F(x) = \frac{1}{1+e^{-x}}.$$

We can readily see that the density and distribution function are related as

$$f(x) = F(x)[1 - F(x)]. \tag{2.72}$$

Shah (1966, 1970) used the representation (2.72) to derive the recurrence relations for single and product moments of order statistics from the Logistic distribution. These relations are given below.

For single moments consider (2.57) as

$$\mu_{r:n}^p - \mu_{r-1:n}^p = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \times [F(x)]^{r-1}[1-F(x)]^{n-r+1}dx.$$

or

$$\begin{aligned}\mu_{r:n}^p - \mu_{r-1:n}^p &= \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1}F(x) \\ &\quad\times [F(x)]^{(r-1)-1}[1-F(x)]^{n-r}[1-F(x)]dx. \\ &= \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \\ &\quad\times F(x)[1-F(x)][F(x)]^{(r-1)-1}[1-F(x)]^{n-r}dx.\end{aligned}$$

Now using (2.72) in above equation we have

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} f(x) \times [F(x)]^{(r-1)-1}[1-F(x)]^{n-r}dx$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n(n-1)!}{(r-1)(r-2)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \times f(x)[F(x)]^{(r-1)-1}[1-F(x)]^{n-r}dx$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{np}{(r-1)(n-r+1)}\mu_{r-1:n-1}^{p-1}$$

or

$$\mu_{r:n}^{p} = \mu_{r-1:n}^{p} + \frac{np}{(r-1)(n-r+1)}\mu_{r-1:n-1}^{p-1}; \tag{2.73}$$

as the recurrence relation for single moments of order statistics from Logistic distribution.

We now present the recurrence relation for first order product moments of order statistics from Logistic distribution. For this consider

$$\begin{aligned}
\mu_{r:n} = E\left(X_{r:n}X_{s:n}^{0}\right) &= \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1 x_2^0 f_{r,s:n}(x_1,x_2)dx_2dx_1 \\
&= C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1 f(x_1)f(x_2)[F(x_1)]^{r-1} \\
&\quad \times \left[F(x_2)-F(x_1)\right]^{s-r-1}[1-F(x_2)]^{n-s}dx_2dx_1 \\
&= C_{r,s:n}\int_{-\infty}^{\infty} x_1 f(x_1)[F(x_1)]^{r-1} I(x_2)dx_1;
\end{aligned} \tag{2.74}$$

where

$$I(x_2) = \int_{x_1}^{\infty} f(x_2)\left[F(x_2)-F(x_1)\right]^{s-r-1}[1-F(x_2)]^{n-s}dx_2.$$

Using (2.72) in above equation we have

$$I(x_2) = \int_{x_1}^{\infty} F(x_2)\Big[F(x_2) - F(x_1)\Big]^{s-r-1}[1 - F(x_2)]^{n-s+1} dx_2$$
$$= \int_{x_1}^{\infty} [1 - \{1 - F(x_2)\}]\Big[F(x_2) - F(x_1)\Big]^{s-r-1}$$
$$\times [1 - F(x_2)]^{n-s+1} dx_2$$

or

$$I(x_2) = \int_{x_1}^{\infty} \Big[F(x_2) - F(x_1)\Big]^{s-r-1}[1 - F(x_2)]^{n-s+1} dx_2$$
$$- \int_{x_1}^{\infty} [F(x_2) - F(x_1)]^{s-r-1}[1 - F(x_2)]^{n-s+2} dx_2$$
$$= I_1(x_2) - I_2(x_2);$$

Using above equation in (2.74) we have

$$\mu_{r:n} = C_{r,s:n} \int_{-\infty}^{\infty} x_1 f(x_1)[F(x_1)]^{r-1} I_1(x_2) dx_1$$
$$-C_{r,s:n} \int_{-\infty}^{\infty} x_1 f(x_1)[F(x_1)]^{r-1} I_2(x_2) dx_1. \quad (2.75)$$

Now consider

$$I_1(x_2) = \int_{x_1}^{\infty} \Big[F(x_2) - F(x_1)\Big]^{s-r-1}[1 - F(x_2)]^{n-s+1} dx_2.$$

Integrating by parts, taking dx_2 as integration and rest of the function for differentiation we have

$$I_1(x_2) = \Big[F(x_2) - F(x_1)\Big]^{s-r-1}[1 - F(x_2)]^{n-s+1} x_2 \Big|_{x_1}^{\infty}$$
$$- \int_{x_1}^{\infty} x_2\Big[\Big\{(s - r - 1)[F(x_2) - F(x_1)]^{s-r-2}$$
$$\times [1 - F(x_2)]^{n-s+1} f(x_2)\Big\} - \{(n - s + 1) f(x_2)$$
$$[F(x_2) - F(x_1)]^{s-r-1}[1 - F(x_2)]^{n-s}\Big\}\Big] dx_2$$
$$= (n - s + 1) \int_{x_1}^{\infty} x_2 f(x_2)\Big[F(x_2) - F(x_1)\Big]^{s-r-1}$$
$$\times [1 - F(x_2)]^{n-s} dx_2 - (s - r - 1) \int_{x_1}^{\infty} x_2 f(x_2)$$
$$\times [F(x_2) - F(x_1)]^{s-r-2}[1 - F(x_2)]^{n-s+1} dx_2. \quad (2.76)$$

Similarly

$$\begin{aligned} I_2(x_2) = (n-s+2)\int_{x_1}^{\infty} x_2 f(x_2)\Big[F(x_2)-F(x_1)\Big]^{s-r-1} \\ \times [1-F(x_2)]^{n-s+1}dx_2 - (s-r-1)\int_{x_1}^{\infty} x_2 f(x_2) \\ \times [F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+2}dx_2. \end{aligned} \tag{2.77}$$

Now using (2.76) and (2.77) in (2.75) we have

$$\begin{aligned} \mu_{r:n} = (n-s+1)C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1x_2 f(x_1)f(x_2)[F(x_1)]^{r-1} \\ \times \Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s}dx_2dx_1 \\ -(s-r-1)C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1x_2 f(x_2)f(x_1)[F(x_1)]^{r-1} \\ \times [F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1}dx_2 \\ -(n-s+2)C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1x_2 f(x_1)f(x_2)[F(x_1)]^{r-1} \\ \times \Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s+1}dx_2dx_1 \\ +(s-r-1)C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1x_2 f(x_2)f(x_1)[F(x_1)]^{r-1} \\ \times [F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1}dx_2. \end{aligned}$$

Now rearranging the terms we have

$$\begin{aligned} \mu_{r,s:n+1} = \frac{n+1}{n-s+2}\Big[\mu_{r,s:n} - \mu_{r,s-1:n} - \frac{n-s+2}{n+1}\mu_{r,s-1:n+1} \\ -\frac{1}{n-s+2}\mu_{r:n}\Big]. \end{aligned} \tag{2.78}$$

Using $s = r+1$ in above equation the relation for product moments of two contiguous order statistics from Logistic distribution turned out to be

$$\mu_{r,r+1:n+1} = \frac{n+1}{n-r+1}\Big[\mu_{r,r+1:n} - \frac{r}{n+1}\mu^2_{r+1:n+1} - \frac{1}{n-r}\mu_{r:n}\Big]. \tag{2.79}$$

Certain other relations can be derived from (2.78) and (2.79).

2.9.7 The Inverse Weibull Distribution

The Inverse Weibull distribution is another useful distribution which has several applications in almost all the areas of life. The density and distribution function for an Inverse Weibull random variable are

$$f(x) = \frac{\beta}{x^{\beta+1}} \exp\left(-\frac{1}{x^{\beta}}\right); x, \beta > 0$$

and

$$F(x) = \exp\left(-\frac{1}{x^{\beta}}\right).$$

The density and distribution function of Inverse Weibull distribution are related as

$$f(x) = \frac{x^{\beta+1}}{\beta} F(x). \tag{2.80}$$

The recurrence relations for single and product moments of order statistics for Inverse Weibull distribution are derived below.

The recurrence relation for single moments is given in (2.57) as

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1} \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} x^{p-1} \times [F(x)]^{r-1}[1-F(x)]^{n-r+1} dx.$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1} \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} x^{p-1} F(x) \times [F(x)]^{r-2}[1-F(x)]^{n-r+1} dx.$$

Using (2.80) in above equation we have

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\beta(n-r+1)} \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} x^{p-1} x^{\beta+1} \times f(x)[F(x)]^{(r-1)-1}[1-F(x)]^{n-(r-1)} dx$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\beta(r-1)} \frac{n!}{(r-2)!(n-r+1)!} \int_{-\infty}^{\infty} x^{p+\beta} \times f(x)[F(x)]^{(r-1)-1}[1-F(x)]^{n-(r-1)} dx$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\beta(r-1)} \mu_{r-1:n}^{p+\beta}$$

or

$$\mu_{r:n}^{p} = \mu_{r-1:n}^{p} + \frac{p}{\beta(r-1)} \mu_{r-1:n}^{p+\beta}. \tag{2.81}$$

as a recurrence relation for single moments of order statistics from Inverse Weibull distribution. Using $\beta = 2$ in (2.81) we have following recurrence relation for single moments of order statistics from Inverse Rayleigh distribution

$$\mu_{r:n}^{p} = \mu_{r-1:n}^{p} + \frac{p}{2(r-1)} \mu_{r-1:n}^{p+2}. \tag{2.82}$$

We now present the recurrence relations for product moments of order statistics for Inverse Weibull distribution. We have relation (2.58) as

$$\begin{aligned}
\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1} C_{r,s:n} & \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1) \\
& \times [F(x_1)]^{r-1} \Big[F(x_2) - F(x_1) \Big]^{s-r-1} \\
& \times [1 - F(x_2)]^{n-s+1} dx_2 dx_1.
\end{aligned}$$

or

$$\begin{aligned}
\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1} C_{r,s:n} & \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1) \\
& \times [F(x_1)]^{r-1} \Big[F(x_2) - F(x_1) \Big]^{s-r-1} \\
& \times [1 - F(x_2)]^{n-s} [1 - F(x_2)] dx_2 dx_1.
\end{aligned}$$

or

$$\begin{aligned}
\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = & \frac{q}{n-s+1} C_{r,s:n} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1) [F(x_1)]^{r-1} \\
& \times \Big[F(x_2) - F(x_1) \Big]^{s-r-1} [1 - F(x_2)]^{n-s} dx_2 dx_1 \\
& - \frac{q C_{r,s:n}}{n-s+1} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1) [F(x_1)]^{r-1} F(x_2) \\
& \times \Big[F(x_2) - F(x_1) \Big]^{s-r-1} [1 - F(x_2)]^{n-s} dx_2 dx_1.
\end{aligned}$$

Now using (2.80) in above equation we have

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1} C_{r,s:n} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1)[F(x_1)]^{r-1} \times \Big[F(x_2) - F(x_1)\Big]^{s-r-1} [1-F(x_2)]^{n-s} dx_2 dx_1 - \frac{qC_{r,s:n}}{\beta(n-s+1)} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q+\beta} f(x_1) f(x_2)[F(x_1)]^{r-1} \times \Big[F(x_2) - F(x_1)\Big]^{s-r-1} [1-F(x_2)]^{n-s} dx_2 dx_1.$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1} C_{r,s:n} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1)[F(x_1)]^{r-1} \times \Big[F(x_2) - F(x_1)\Big]^{s-r-1} [1-F(x_2)]^{n-s} dx_2 dx_1 - \frac{q}{\beta(n-s+1)} \mu_{r,s:n}^{p,q+\beta}$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{qC_{r,s:n}}{n-s+1} \int_{-\infty}^{\infty} x_1^p f(x_1)[F(x_1)]^{r-1} \times I(x_2) dx_1 - \frac{q}{\beta(n-s+1)} \mu_{r,s:n}^{p,q+\beta}; \tag{2.83}$$

where

$$I(x_2) = \int_{x_1}^{\infty} x_2^{q-1} \Big[F(x_2) - F(x_1)\Big]^{s-r-1} [1-F(x_2)]^{n-s} dx_2.$$

Now integrating above equation by parts using x_2^{q-1} as function for integration we have

$$I(x_2) = \Big[F(x_2) - F(x_1)\Big]^{s-r-1} [1-F(x_2)]^{n-s} \frac{x_2^q}{q}\Big|_{x_1}^{\infty} - (s-r-1) \int_{x_1}^{\infty} \Big\{[F(x_2) - F(x_1)]^{s-r-2} [1-F(x_2)]^{n-s} f(x_2) - (n-s)[1-F(x_2)]^{n-s-1} \Big[F(x_2) - F(x_1)\Big]^{s-r-1} f(x_2)\Big\} \frac{x_2^q}{q} dx_2$$

or

$$I(x_2) = \frac{n-s}{q}\int_{x_1}^{\infty} x_2^q f(x_2)[F(x_2)-F(x_1)]^{s-r-1}$$
$$\times [1-F(x_2)]^{n-s-1}dx_2 - \frac{s-r-1}{q}\int_{x_1}^{\infty} x_2^q$$
$$\times f(x_2)[F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s}dx_2.$$

Using above in (2.83) we have

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{qC_{r,s:n}}{n-s+1}\int_{-\infty}^{\infty} x_1^p f(x_1)[F(x_1)]^{r-1}$$
$$\times \left\{ \frac{n-s}{q}\int_{x_1}^{\infty} x_2^q f(x_2)[F(x_2)-F(x_1)]^{s-r-1} \right.$$
$$\times [1-F(x_2)]^{n-s-1}dx_2 - \frac{s-r-1}{q}\int_{x_1}^{\infty} x_2^q f(x_2)$$
$$\left. \times [F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s}dx_2 \right\} dx_1$$
$$-\frac{q}{\beta(n-s+1)}\mu_{r,s:n}^{p,q+\beta}$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{(n-s)C_{r,s:n}}{n-s+1}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times \Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{(n-1)-s}dx_2dx_1$$
$$-\frac{(s-r-1)C_{r,s:n}}{n-s+1}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times [F(x_2)-F(x_1)]^{(s-1)-r-1}[1-F(x_2)]^{n-s}dx_2dx_1$$
$$-\frac{q}{\beta(n-s+1)}\mu_{r,s:n}^{p,q+\beta}$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{nC_{r,s:n-1}}{n-s+1}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times \Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{(n-1)-s}dx_2dx_1$$
$$-\frac{nC_{r,s-1:n-1}}{n-s+1}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times [F(x_2)-F(x_1)]^{(s-1)-r-1}[1-F(x_2)]^{n-s}dx_2dx_1$$
$$-\frac{q}{\beta(n-s+1)}\mu_{r,s:n}^{p,q+\beta}$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{n}{n-s+1}\mu_{r,s:n-1}^{p,q} - \frac{n}{n-s+1}\mu_{r,s-1:n-1}^{p,q} - \frac{q}{\beta(n-s+1)}\mu_{r,s:n}^{p,q+\beta}$$

or

$$\mu_{r,s:n}^{p,q} = \mu_{r,s-1:n}^{p,q} + \frac{1}{n-s+1}\left[n\mu_{r,s:n-1}^{p,q} - n\mu_{r,s-1:n-1}^{p,q} - \frac{q}{\beta}\mu_{r,s:n}^{p,q+\beta}\right]. \tag{2.84}$$

The recurrence relations for product moments of order statistics for Inverse Exponential and Inverse Rayleigh distribution can be easily obtained from (2.84) by using $\beta = 1$ and $\beta = 2$ respectively. Some other references on recurrence relations for moments of order statistics include Al-Zahrani and Ali (2014), Al-Zahrani et al. (2015), Balakrishnan et al. (2015a, b).

Example 2.11 Show that for standard exponential distribution having density $f(x) = e^{-x}$; following relation holds for moments of order statistics:

$$\mu_{r:n}^{p} = \mu_{r-1:n-1}^{p} + \frac{p}{n}\mu_{r:n}^{p-1};\quad 2 \le r \le n.$$

Solution: We have $(p-1)th$ moment of rth order statistics as:

$$\begin{aligned}\mu_{r:n}^{p-1} = E\left(X_{r:n}^{p-1}\right) &= \int_{-\infty}^{\infty} x^{p-1} f_{r:n}(x)dx \\ &= \int_{-\infty}^{\infty} x^{p-1}\frac{n!}{(r-1)!(n-r)!}f(x)[F(x)]^{r-1} \\ &\quad \times [1-F(x)]^{n-r}dx \\ &= \frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1}f(x)[F(x)]^{r-1} \\ &\quad \times [1-F(x)]^{n-r}dx.\end{aligned}$$

Now for standard exponential distribution we have $f(x) = 1 - F(x)$ so we have:

$$\mu_{r:n}^{p-1} = \frac{n!}{(r-1)!(n-r)!}\int_{0}^{\infty} x^{p-1}[F(x)]^{r-1}[1-F(x)]^{n-r+1}dx.$$

Now integrating by parts taking x^{p-1} for integration and rest of the function for differentiation we have:

$$\mu_{r:n}^{p-1} = \frac{n!}{(r-1)!(n-r)!}\left[[F(x)]^{r-1}[1-F(x)]^{n-r+1}\frac{x^p}{p}\bigg|_0^\infty \right.$$
$$- \int_0^\infty \left\{ (r-1)[F(x)]^{r-2}[1-F(x)]^{n-r+1} f(x) \right.$$
$$\left.\left. -(n-r+1)[F(x)]^{r-1}[1-F(x)]^{n-r} f(x) \right\} \frac{x^p}{p} dx \right]$$
$$= \frac{n!}{(r-1)!(n-r)!p}\left[(n-r+1)\int_0^\infty x^p f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} dx \right.$$
$$\left. -(r-1)\int_0^\infty x^p f(x)[F(x)]^{r-2}[1-F(x)]^{n-r+1} dx \right].$$

Now splitting the first integral we have:

$$\mu_{r:n}^{p-1} = \frac{n!}{(r-1)!(n-r)!p}\left[n\int_0^\infty x^p f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} dx \right.$$
$$-(r-1)\int_0^\infty x^p f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} dx$$
$$\left. -(r-1)\int_0^\infty x^p f(x)[F(x)]^{r-2}[1-F(x)]^{n-r+1} dx \right]$$
$$= \frac{n!}{(r-1)!(n-r)!p}\left[n\int_0^\infty x^p f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} dx \right.$$
$$\left. -(r-1)\int_0^\infty x^p f(x)[F(x)]^{r-2}[1-F(x)]^{n-r} dx \right]$$
$$= \frac{n}{p}\left[\frac{n!}{(r-1)!(n-r)!}\int_0^\infty x^p f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} dx \right]$$
$$- \frac{(r-1)}{p}\left[\frac{n!}{(r-1)!(n-r)!}\int_0^\infty x^p f(x)[F(x)]^{r-2}[1-F(x)]^{n-r} dx \right]$$

or

$$\mu_{r:n}^{p-1} = \frac{n}{p}\mu_{r:n}^{p} - \frac{n}{p}\mu_{r-1:n-1}^{p}$$
$$\text{or } \mu_{r:n}^{p} = \mu_{r-1:n-1}^{p} + \frac{p}{n}\mu_{r:n}^{p-1};$$

as required. Further, for $r = 1$ we have a special relation as:

$$\mu_{1:n}^{p} = \frac{p}{n}\mu_{1:n}^{p-1}.$$

Example 2.12 Show that for standard exponential distribution having density $f(x) = e^{-x}$; following relations holds for joint moments of order statistics:

$$\mu_{r:r+1:n} = \mu_{r:n}^2 + \frac{1}{n-r}\mu_{r:n};\quad 1 \le r \le n-1$$

$$\text{and } \mu_{r,s:n} = \mu_{r,s-1:n} + \frac{1}{n-s-1}\mu_{r:n};\ 1 \le r < s \le n, s-r \ge 2.$$

Solution: We first obtain the first relation. Consider:

$$\begin{aligned}
\mu_{r:n} = E\left(X_{r:n}X_{s:n}^0\right) &= \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1 x_2^0 f_{r,s:n}(x_1,x_2)dx_2dx_1 \\
&= C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_2}^{\infty} x_1 f(x_1)f(x_2)[F(x_1)]^{r-1} \\
&\quad \times \Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s}dx_2dx_1;
\end{aligned}$$

where $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$. Now for standard exponential distribution we have $f(x) = 1-F(x)$ and hence writing $f(x_2) = 1-F(x_2)$ above equation can be written as:

$$\begin{aligned}
\mu_{r:n} &= C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1 f(x_1)[F(x_1)]^{r-1}\Big[F(x_2)-F(x_1)\Big]^{s-r-1} \\
&\quad \times \Big[1-F(x_2)\Big]^{n-s+1}dx_2dx_1 \\
&= C_{r,s:n}\int_{-\infty}^{\infty} x_1 f(x_1)[F(x_1)]^{r-1}I(x_2)dx_1; \qquad (2.85)
\end{aligned}$$

where

$$I(x_2) = \int_{x_1}^{\infty}\Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s+1}dx_2.$$

Now integrating above by parts, using dx_2 for integration and rest as differentiation we have

$$\begin{aligned}
I(x_2) = {} & \Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s+1}x_2\Big|_{x_1}^{\infty} \\
& - \int_{x_1}^{\infty}\Big[\Big\{(s-r-1)[F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1}f(x_2)\Big\} \\
& \Big\{-(n-s+1)\Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s}f(x_2)\Big\}\Big]x_2dx_2
\end{aligned}$$

or

$$I(x_2) = \Big[F(x_2) - F(x_1)\Big]^{s-r-1}[1 - F(x_2)]^{n-s+1}x_2\Big|_{x_1}^{\infty}$$
$$+(n - s + 1)\int_{x_1}^{\infty} x_2\Big[F(x_2) - F(x_1)\Big]^{s-r-1}[1 - F(x_2)]^{n-s}$$
$$\times f(x_2)dx_2 - (s - r - 1)\int_{x_1}^{\infty} x_2[F(x_2) - F(x_1)]^{s-r-2}$$
$$\times [1 - F(x_2)]^{n-s+1}f(x_2)dx_2. \tag{2.86}$$

Now for $s = r + 1$ in (2.86) we have

$$I(x_2) = (n - r)\int_{x_1}^{\infty} x_2[1 - F(x_2)]^{n-r-1}f(x_2)dx_2$$
$$-x_1[1 - F(x_1)]^{n-r}. \tag{2.87}$$

Now using (2.87) in (2.85); with $C_{r,s:n} = \frac{n!}{(r-1)!(n-r-1)!}$; we have:

$$\mu_{r:n} = C_{r,s:n}\int_{-\infty}^{\infty} x_1 f(x_1)[F(x_1)]^{r-1}\Big[(n - r)\int_{x_1}^{\infty} x_2[1 - F(x_2)]^{n-r-1}$$
$$\times f(x_2)dx_2 - x_1[1 - F(x_1)]^{n-r}\Big]dx_1$$
$$= \frac{n!}{(r-1)!(n-r-1)!}(n - r)\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1x_2 f(x_1)f(x_2)$$
$$\times [F(x_1)]^{r-1}[1 - F(x_2)]^{n-r-1}dx_2dx_1$$
$$-\frac{n!}{(r-1)!(n-r-1)!}\int_{-\infty}^{\infty} x_1^2 f(x_1)[F(x_1)]^{r-1}$$
$$\times [1 - F(x_1)]^{n-r}dx_1$$

or

$$\mu_{r:n} = (n - r)\mu_{r,r+1:n} - (n - r)\mu_{r:n}^2$$

or

$$\mu_{r,r+1:n} = \mu_{r:n}^2 + \frac{1}{n - r}\mu_{r:n};$$

as required. Again for $s - r \geq 2$ we have, from (2.86):

$$I(x_2) = (n - s + 1)\int_{x_1}^{\infty} x_2\Big[F(x_2) - F(x_1)\Big]^{s-r-1}[1 - F(x_2)]^{n-s}$$
$$\times f(x_2)dx_2 - (s - r - 1)\int_{x_1}^{\infty} x_2[F(x_2) - F(x_1)]^{s-r-2}$$
$$\times [1 - F(x_2)]^{n-s+1}f(x_2)dx_2. \tag{2.88}$$

Now using (2.88) in (2.85) we have:

$$\mu_{r:n} = C_{r,s:n} \int_{-\infty}^{\infty} x_1 f(x_1)[F(x_1)]^{r-1}$$
$$\left\{(n-s+1)\int_{x_1}^{\infty} x_2\Big[F(x_2)-F(x_1)\Big]^{s-r-1}\right.$$
$$\times [1-F(x_2)]^{n-s} f(x_2)dx_2 - (s-r-1)\int_{x_1}^{\infty} x_2$$
$$\left.[F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1} f(x_2)dx_2\right\}dx_1$$

or

$$\mu_{r:n} = (n-s+1)C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1x_2 f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times \Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s}dx_2dx_1$$
$$-(s-r-1)C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1x_2 f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times [F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1} f(x_2)dx_2dx_1$$

or

$$\mu_{r:n} = (n-s+1)\mu_{r,s:n} - (n-s+1)\mu_{r,s-1:n}$$

or

$$\mu_{r,s:n} = \mu_{r,s-1:n} + \frac{1}{n-s+1}\mu_{r:n};$$

as required.

2.10 Relations for Moments of Order Statistics for Special Class of Distributions

In previous section we have discussed recurrence relations for single and product moments of order statistics for certain distributions. We have seen that the recurrence relations for single moments of order statistics can be derived from (2.89) as

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1}$$
$$\times [F(x)]^{r-1}[1-F(x)]^{n-r+1}dx. \tag{2.89}$$

Using probability integral transform above relation can be written as

$$\mu_{r:n}^{p}-\mu_{r-1:n}^{p}=\frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty}\left\{F^{-1}(t)\right\}^{p-1}\times\left\{F^{-1}(t)\right\}^{/}t^{r-1}(1-t)^{n-r+1}dt. \quad (2.90)$$

We now present a recurrence relation for single moments of order statistics for class of distributions having special structure of $\left\{F^{-1}(t)\right\}^{/}$. The relation is given in the following.

Theorem: For the class of distributions defined as

$$\left\{F^{-1}(t)\right\}^{/}=\frac{1}{d}t^{p_1}(1-t)^{q-p_1-1} \text{ on } (0,1);$$

the following relation holds for single moments of order statistics

$$\mu_{r:n}^{p}-\mu_{r-1:n}^{p}=pC(r,n,p_1,q)\mu_{r+p_1:n+q}^{p-1}; \quad (2.91)$$

where

$$C(r,n,p_1,q)=\mu_{r:n}-\mu_{r-1:n}=\frac{1}{d}\frac{\binom{n}{r-1}}{(r+p_1)\binom{n+q}{r+p_1}}.$$

Proof We have from (2.90)

$$\mu_{r:n}^{p}-\mu_{r-1:n}^{p}=\frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty}\left\{F^{-1}(t)\right\}^{p-1}\times\left\{F^{-1}(t)\right\}^{/}t^{r-1}(1-t)^{n-r+1}dt. \quad (2.92)$$

Now using the representation for $\left\{F^{-1}(t)\right\}^{/}$ we have

$$\mu_{r:n}^{p}-\mu_{r-1:n}^{p}=\frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty}\left\{F^{-1}(t)\right\}^{p-1}\times\frac{1}{d}t^{p_1}(1-t)^{q-p_1-1}t^{r-1}(1-t)^{n-r+1}dt.$$

or

$$\mu_{r:n}^{p}-\mu_{r-1:n}^{p}=\frac{1}{d}\frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty}\left\{F^{-1}(t)\right\}^{p-1}\times t^{r+p_1-1}(1-t)^{(n+q)-(r+p_1)+1}dt.$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = p\frac{1}{d}\frac{n!}{(r-1)!(n-r+1)!}\frac{(r+p_1)\binom{n+q}{r+p_1}}{(r+p_1)\binom{n+q}{r+p_1}} \times \int_{-\infty}^{\infty}\left\{F^{-1}(t)\right\}^{p-1}t^{r+p_1-1}(1-t)^{(n+q)-(r+p_1)+1}dt$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = p\frac{1}{d}\frac{\binom{n}{r-1}}{(r+p_1)\binom{n+q}{r+p_1}}(r+p_1)\binom{n+q}{r+p_1} \times \int_{-\infty}^{\infty}\left\{F^{-1}(t)\right\}^{p-1}t^{r+p_1-1}(1-t)^{(n+q)-(r+p_1)+1}dt$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = pC(r,n,p_1,q)\mu_{r+p_1:n-q}^{p-1};$$

as required. Using $p = 1$ we can readily see that

$$C(r,n,p_1,q) = \mu_{r:n} - \mu_{r-1:n} = \frac{1}{d}\frac{\binom{n}{r-1}}{(r+p_1)\binom{n+q}{r+p_1}}.$$

The class of distribution defined as

$$\left\{F^{-1}(t)\right\}^{/} = \frac{1}{d}t^{p_1}(1-t)^{q-p_1-1};$$

give rise to several special distributions for suitable choices of p_1 and q. Some of these special cases are given below.

1. For $p_1 = 0$ and $q = 0$ we have

$$\left\{F^{-1}(t)\right\}^{/} = \frac{1}{d}(1-t)^{-1}$$

or

$$F^{-1}(t) = \frac{1}{d}\log\left(\frac{1}{1-t}\right)$$

or

$$t = F(x) = 1 - e^{-dx};$$

that is the Exponential Distribution.

2. For $p_1 = 0$ and $q \neq 0$ we have

$$\left\{F^{-1}(t)\right\}^{/} = \frac{1}{d}(1-t)^{q-1}$$

or

$$F^{-1}(t) = -\frac{1}{dq}(1-t)^q$$

or

$$t = F(x) = 1 - (dqx)^{1/q};$$

which for $q < 0$ provides Pareto distribution and for $q > 0$ provides Pearson type I distribution.

3. For $p_1 \neq -1$ and $q = p + 1$ we have

$$\left\{F^{-1}(t)\right\}^{/} = \frac{1}{d}t^{q-1}$$

or

$$F^{-1}(t) = \frac{1}{dq}t^q$$

or

$$t = F(x) = (dqx)^{1/q};$$

which for $q > 0$ provides Power function distribution.

4. For $p = -1$ and $q = 0$ we have

$$\left\{F^{-1}(t)\right\}^{/} = \frac{1}{d}t^{-1}$$

or

$$F^{-1}(t) = \frac{1}{d}\log(t)$$

or

$$t = F(x) = e^{dx};$$

which is reflected Exponential distribution.

5. For $p = -1$ and $q = -1$

$$\left\{F^{-1}(t)\right\}^{/} = \frac{1}{d}t^{-1}(1-t)^{-1}$$

or

$$F^{-1}(t) = \frac{1}{d}\log\left(\frac{t}{1-t}\right)$$

or

$$t = F(x) = \frac{1}{1+e^{-dx}};$$

which is the Logistic distribution.

We can obtain other distributions for various choices of p_1 and q. The recurrence relations for these special cases can be directly obtained from (2.91) by using the corresponding values.

2.11 Reversed Order Statistics

Reversed Order Statistics appear frequently when data is arranged in descending order of magnitude, say for example marks of students arranged from highest to lowest or population of cities; in million; arranged in decreasing order. The distribution theory of such variables can be studied in the context of *Reversed Order Statistics* which appear as a special case of Dual Generalized Order Statistics; discussed in Chap. 5. The reversed order statistics and their distribution are defined in the following.

Let $x_1,\ x_2,\ \ldots, x_n$ be a random sample from a distribution $F(x)$ and suppose that the sample is arranged in descending order as $x_1 \geq x_2 \geq \cdots \geq x_n$ then this descendingly ordered sample constitute the reversed order statistics. The joint distribution of n reversed order statistics is same as the joint distribution of ordinary order statistics. The joint marginal distribution of r reversed order statistics is given as

$$f_{1(re),\ldots,r(re):n}(x_1,\ldots,x_r) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r} f(x_i)\right]\{F(x_r)\}^{n-r}. \tag{2.93}$$

Further, the marginal distribution of rth reversed order statistics and joint marginal distribution of rth and sth reversed order statistics; for $r < s$; are easily written from (2.93) as

$$f_{r(re):n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)\{F(x)\}^{n-r}\{1-F(x)\}^{r-1}. \tag{2.94}$$

and

$$\begin{aligned} f_{r(re),s(re):n}(x_1,x_2) = {} & \frac{n!}{(r-1)!(s-r-1)!(n-s)!} f(x_1)f(x_2)\{1-F(x_1)\}^{r-1} \\ & \times\ [F(x_1)-F(x_2)]^{s-r-1}\{F(x_2)\}^{n-s}. \end{aligned} \tag{2.95}$$

We can readily see that the distribution of rth reversed order statistics from distribution $F(x)$ is same as the the distribution of $(n-r+1)$th ordinary order statistics from the distribution $F(x)$.

Example 2.13 A random sample is available from the density

$$f(x) = \alpha x^{\alpha-1}\exp(-x^{\alpha});\ x, \alpha > 0$$

Obtain the marginal density function of rth reversed order statistics and joint density function of rth and sth reversed order statistics for this distribution.

Solution: The density function of rth reversed order statistics and joint density function of rth and sth reversed order statistics are given in (2.94) and (2.95) as

$$f_{r(re):n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)\{F(x)\}^{n-r}\{1-F(x)\}^{r-1}.$$

and

$$\begin{aligned} f_{r(re),s(re):n}(x_1,x_2) = & \frac{n!}{(r-1)!(s-r-1)!(n-s)!} f(x_1)f(x_2)\{1-F(x_1)\}^{r-1} \\ & \times [F(x_1)-F(x_2)]^{s-r-1}\{F(x_2)\}^{n-s}. \end{aligned}$$

Now we have

$$f(x) = \alpha x^{\alpha-1}\exp(-x^{\alpha})$$

So

$$F(x) = \int_0^x f(t)dt = \int_0^x \alpha t^{\alpha-1}\exp(-t^{\alpha})dt = 1-\exp(-x^{\alpha}).$$

The density function of rth reversed order statistics is therefore

$$\begin{aligned} f_{r(re):n}(x) &= \frac{n!}{(r-1)!(n-r)!} f(x)\{F(x)\}^{n-r}\{1-F(x)\}^{r-1} \\ &= \frac{n!}{(r-1)!(n-r)!}\alpha x^{\alpha-1}\exp(-x^{\alpha}) \\ &\quad \times \{1-\exp(-x^{\alpha})\}^{n-r}\{\exp(-x^{\alpha})\}^{r-1} \\ &= \frac{n!}{(r-1)!(n-r)!}\alpha x^{\alpha-1}\exp(-rx^{\alpha}) \\ &\quad \times \sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}\exp(-jx^{\alpha}) \end{aligned}$$

or

$$\begin{aligned} f_{r(re):n}(x) = & \frac{n!}{(r-1)!(n-r)!}\alpha x^{\alpha-1}\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j} \\ & \times \exp\left\{-x^{a}(r+j)\right\}. \end{aligned}$$

Again the joint density of rth and sth reversed order statistics is

$$\begin{aligned} f_{r(re),s(re):n}(x_1, x_2) = {} & C_{r,s:n} \alpha x_1^{\alpha-1} \exp\left(-x_1^{\alpha}\right) \alpha x_2^{\alpha-1} \exp\left(-x_2^{\alpha}\right) \\ & \times \left\{\exp\left(-x_1^{\alpha}\right)\right\}^{r-1} \left[\exp\left(-x_1^{\alpha}\right) - \exp\left(-x_2^{\alpha}\right)\right]^{s-r-1} \\ & \times \left\{1 - \exp\left(-x_2^{\alpha}\right)\right\}^{n-s}, \end{aligned}$$

or

$$\begin{aligned} f_{r(re),s(re):n}(x_1, x_2) = {} & C_{r,s:n} \alpha^2 x_1^{\alpha-1} x_2^{\alpha-1} \exp\left(-r x_1^{\alpha}\right) \exp\left(-x_2^{\alpha}\right) \\ & \times \sum_{j=0}^{s-r-1} (-1)^j \binom{s-r-1}{j} \exp\left\{-x_1^{\alpha}(s-r-j-1)\right\} \\ & \times \exp\left(-j x_2^{\alpha}\right) \sum_{k=0}^{n-s} (-1)^k \binom{n-s}{k} \exp\left(-k x_2^{\alpha}\right). \end{aligned}$$

or

$$\begin{aligned} f_{r(re),s(re):n}(x_1, x_2) = {} & C_{r,s:n} \alpha^2 x_1^{\alpha-1} x_2^{\alpha-1} \sum_{k=0}^{n-s} \sum_{j=0}^{s-r-1} (-1)^{j+k} \binom{n-s}{k} \\ & \times \binom{s-r-1}{j} \exp\left\{-x_1^{\alpha}(s-j-1)\right\} \\ & \times \exp\left\{-x_2^{\alpha}(j+k+1)\right\}, \end{aligned}$$

where $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

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