Chapter 2
Record Statistics

In this chapter some of the basic concepts and properties of the record values are presented. For simplicity the descriptions are confined here to the sequence of independent and identically distributed continuous random variables.

2.1 Introduction and Examples of Record Values

Suppose we consider the weighing of objects on a scale missing its spring. An object is placed on the scale and its weight is measured. The 'needle' indicated the correct weight but does not return to zero when the object is removed. If various objects are placed on the scale, only the weights greater than the previous ones can be recorded. These recorded weights are the upper record value sequence. If X_{ij} be the height water level of a river on the jth day of the i-th location. If one is interested to study at each location the local maximum values of X_{ij}, then the local maxima are the upper record values.

Let us consider a sequence of products that may fail under stress. We are interested to determine the minimum failure stress of the products sequentially. We test the first product until it fails with stress less than X_1 then we record its failure stress, otherwise we consider the next product. In general we will record stress X_m of the mth product if $X_m < \min(X_1, \ldots, X_{m-1})$, $m > 2$. The recorded failure stresses are the lower record values. One can go from lower records to upper records by replacing the original sequence of random variables $\{X_j\}$ by $\{-X_j, j \geq 1\}$ or if $P(X_j > 0) = 1$ by $\{1/X_i, i \geq 1\}$.

Chandler (1952) introduced the record values, record times and inter record times. He proved the interesting result that for any given distribution of the random variables the expected value of the inter record time is infinite. Feller (1952) gave some examples of record values with respect to gambling problems.

M. Ahsanullah, *Extreme Value Distributions*, Atlantis Studies in Probability and Statistics 8, DOI 10.2991/978-94-6239-222-9_2

2.1.1 Definition of Record Values and Record Times

Suppose that X_1, X_2, is a sequence of independent and identically distributed random variables with cumulative distribution function F(x). Let Y_n = max (min) $\{X_1, X_2, \ldots, X_n\}$ for $n \geq 2$. We say Yj is an upper (lower) record value of $\{X_n, n \geq 1\}$, if $Y_j > (<) Y_{j-1}, j > 2$. By definition X_1 is an upper as well as a lower record value. One can transform the upper records to lower records by replacing the original sequence of $\{X_j\}$ by $\{-X_j, j \geq 1\}$ or (if $P(X_i > 0) = 1$ for all i) by $\{1/X_i, i \geq 1\}$; the lower record values of this sequence will correspond to the upper record values of the original sequence.

The indices at which the upper record values occur are given by the record times $\{U(n)\}, n > 0$, where $U(n) = \min\{j|j > U(n-1), X_j > X_{U(n-1)}, n > 1\}$ and U(1) = 2. The record times of the sequence $\{X_n\ n \geq 1\}$ are the same as those for the sequence $\{F(X_n), n \geq 1\}$. Since F(X) has a uniform distribution, it follows that the distribution of U(n), $n \geq 1$ does not depend on F. We will denote L(n) as the indices where the lower record values occur. By our assumption U(1) = L(1) = 2. The distribution of L(n) also does not depend on F.

2.2 The Exact Distribution of Record Values

Many properties of the record value sequence can be expressed in terms of the function R(x), where $R(x) = -\ln \bar{F}(x)$,

$0 < \bar{F}(x) < 1$ and $\bar{F}(x) = 1 - F(x)$. Here 'ln' is used for the natural logarithm. If we define $F_n(x)$ as the distribution function of $X_{U(n)}$ for

$n \geq 1$, then we have

$$F_1(x) = P\left[X_{U(1)} \leq x\right] = F(x) \tag{2.2.1}$$

$$\begin{aligned} F_2(x) &= P\left[X_{U(2)} \leq x\right] \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} \sum_{i=1}^{\infty} (F(u))^{i-1} dF(u) dF(y) \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{dF(u)}{1 - F(u)} dF(y) \\ &= \int_{-\infty}^{x} R(y) dF(y) \end{aligned} \tag{2.2.2}$$

If F(x) has a density f(x), then the probability density function (pdf) of $X_{U(2)}$ is

$$f_2(x) = R(x)f(x) \tag{2.2.3}$$

The distribution function

$$\begin{aligned} F_3(x) &= P(X_{U(3)} < x) \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} \sum_{i=0}^{\infty} (F(u))^i R(u)\,dF(u)\,dF(y) \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{R(u)}{1-F(u)}\,dF(u)\,dF(y) \\ &= \int_{-\infty}^{x} \frac{(R(u))^2}{2!}\,dF(u) \end{aligned} \tag{2.2.4}$$

The pdf $f_3(x)$ of $X_{U(3)}$ is

$$f_3(x) = \frac{(R(x))^2}{2!} f(x), \; -\infty < x < \infty \tag{2.2.5}$$

It can similarly be shown that the pdf $F_n(x)$ of $X_{U(n)}$ is

$$\begin{aligned} F_n(x) &= P(X_{U(n)} < x) \\ &= \int_{-\infty}^{x} f(u_n)du_n \int_{-\infty}^{u_n} \frac{f(u_{n-1})}{1-F(u_{n-1})} du_{n-1} \int_{-\infty}^{u_2} \frac{f(u_1)}{1-F(u_1)} du_1. \\ &= \int_{-\infty}^{x} \frac{R^{n-1}(u)}{(n-1)!} dF(u), \quad -\infty < x < \infty \end{aligned}$$

This can be expressed as

$$\begin{aligned} F_n(x) &= \int_{-\infty}^{R(x)} \frac{u^{n-1}}{(n-1)!} e^{-u} du, \quad -\infty < x < \infty \\ \bar{F}_n(x) &= 1 - F_n(x) = \bar{F}(x) \sum_{j=0}^{n-1} \frac{(R(x))^j}{j!} \\ &= e^{-R(x)} \sum_{j=0}^{n-1} \frac{(R(x))^j}{j!} \end{aligned} \tag{2.2.6}$$

The pdf $f_n(x)$ of $X_{U(n)}$ is

$$f_n(x) = \frac{R^{n-1}(x)}{(n-1)!} f(x), \quad -\infty < x < \infty. \tag{2.2.7}$$

The joint pdf $f(x_1, x_2, \ldots, x_n)$ of the n record values $(X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)})$ is given by

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= r(x_1) r(x_2) \ldots r(x_{n-1}) f(x_n) \\ &\text{for} \quad -\infty < x_1 < x_2 < \ldots < x_{n-1} < x_n < \infty, \\ \text{where } r(x) &= \frac{d}{dx} R(x) = \frac{f(x)}{1 - F(x)}, \quad 0 < F(x) < 1. \end{aligned} \tag{2.2.8}$$

The function r(x) is known as hazard rate.

The joint pdf of $X_{U(i)}$ and $X_{U(j)}$ is

$$\begin{aligned} f(x_i, x_j) &= \frac{(R(x_i))^{i-1}}{(i-1)!} r(x_i) \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} f(x_j) \\ &\text{for } -\infty < x_i < x_j < \infty. \end{aligned} \tag{2.2.9}$$

In particular for i = 1 and j = n we have

$$\begin{aligned} f(x_1, x_n) &= r(x_1) \frac{(R(x_n) - R(x_1))^{n-2}}{(n-2)!} f(x_n) \\ &\text{for } -\infty < x_1 < x_n < \infty. \end{aligned}$$

Suppose we use the transformation $Y_1 = R(X_{U(i)})$ and $Y_2 = R(X_{U(i)})/R(X_{U(j)})$, $i < j$, then using (2.2.9), it can be shown that the pdf $f_2^*(y)$ of Y_2 is as follows:

$$f_2^*(y) = \frac{\Gamma(j)}{\Gamma(i)} \cdot \frac{1}{\Gamma(j-i)} \cdot y^{i-1} (1-y)^{j-i-1}, 0 < y < \infty. \tag{2.2.10}$$

Thus Y_2 is distributed as Beta distribution with parameters i and j (i.e. B(i, j−i)). The mean and variance of Y_2 are $E(Y_2) = \frac{i}{j}$ and $Var(Y_2) = \frac{ij}{(j+1)j^2}$.

If we use the transformation $V_i = R(X_{U(i)})$, then the joint pdf of V_i, i = 1,2, …, n, is

$$f(v_1, v_2, \ldots, v_n) = e^{-v_n}, \quad 0 < v_1 < v_2 < \ldots < v_n < \infty. \tag{2.2.11}$$

The joint distribution of V_m and V_r, $r > m$, is

$$f(v_m, v_r) = \frac{1}{\Gamma(m)} \cdot \frac{(v_r - v_m)^{r-m-1}}{\Gamma(r-m)} \cdot e^{-v_r}\; 0 < v_m < v_r < \infty$$
$$= 0, \textit{otherwise}.$$
$$E(V_k^l) = \int_0^\infty t^l \frac{1}{\Gamma(k)} t^{k-1} e^{-t} dt = \frac{\Gamma(k+l)}{\Gamma(k)}.$$

Thus $E(V_k) = k$ and Var $(V_k) = k$. The conditional pdf of

$$X_{U(j)} | X_{U(i)} = x_i \text{ if} \left(x_j | X_{U(i)} = x_i \right) = \frac{f_{ij}(x_i, x_j)}{f_i(x_i)}$$
$$= \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} \frac{f(x_j)}{1 - F(x_i)} \tag{2.2.12}$$
$$\text{for } -\infty < x_i < x_j < \infty.$$

For j = i + 1

$$f\left(x_{i+1} | X_{U(i)} = x_i \right) = \frac{f(x_{i+1})}{1 - F(x_i)} \quad \text{for } -\infty < x_i < x_{i+1} < \infty. \tag{2.2.13}$$

For i > 0, $1 \le k < m$, the joint conditional pdf of $X_{U(i+k)}$ and $X_{U(i+m)} \mid X_{U(i)}$ is

$$f_{i+k,i+m}\left(x, y | X_{U(i)} = z\right) = \frac{1}{\Gamma(m-k)} \cdot \frac{1}{\Gamma(k)} \cdot [R(y) - R(x)]^{m-k-1} [R(x) - R(z)]^{k-1} \frac{f(y) r(x)}{\bar{F}(z)}$$
$$\text{for } -\infty < z < x < y < \infty.$$

The marginal pdf of the nth lower record value can be derived by using the same procedure as that of the nth upper record value. Let

H(u) = −ln F(u), 0 < F(u) < 1 and $h(u) = -\frac{d}{du} H(u)$, then

$$P(X_{L(n)} \le x) = \int_{-\infty}^{x} \frac{\{H(u)\}^{n-1}}{(n-1)!} dF(u) \tag{2.2.14}$$

and the corresponding the pdf $f_{(n)}$ can be written as

$$f_{(n)}(x) = \frac{(H(x))^{n-1}}{(n-1)!} f(x). \tag{2.2.15}$$

The joint pdf of $X_{L(1)}, X_{L(2)}, \ldots, X_{L(m)}$ can be written as

$$\begin{aligned} f_{(1),(2),\ldots,(m)}(x_1, x_2, \ldots, x_m) &= h(x_1)h(x_2)\ldots h(x_{m-1})f(x_m) \\ &\qquad -\infty < x_m < x_{m-1} < \ldots < x_1 < \infty \\ &= 0, \text{ otherwise.} \end{aligned} \tag{2.2.16}$$

The joint pdf of $X_{L(r)}$ and $X_{L(s)}$ is

$$f_{(r),(s)}(x, y) = \frac{(H(x))^{r-1}}{(r-1)!} \frac{[H(y) - H(x)]^{s-r-1}}{(s-r-1)!} h(x)f(y) \tag{2.2.17}$$
$$\text{for } s > r \text{ and } -\infty < y < x < \infty.$$

Using the transformations U = H(y) and W = H(x)/H(y) in (2.2.17), it can be shown easily that W is distributed as B(r, s−r).

Proceeding as in the case of upper record values, we can obtain the conditional pdfs of the lower record values.

Example 2.2.1 Let us consider the exponential distribution with pdf f(x) as

$$f(x) = e^{-x}, 0 \le x < \infty$$

and the cumulative distribution function (cdf) F(x) as

$$F(x) = 1 - e^{-x}, 0 \le x < \infty.$$

Then R(x) = x and

$$\begin{aligned} f_n(x) &= \frac{x^{n-1}}{\Gamma(n)} e^{-x}, x \ge 0 \\ &= 0, \text{otherwise} \end{aligned}$$

The joint pdf of $X_{U(m)}$ and $X_{U(n)}$, n > m is

$$\begin{aligned} f_{m,n}(x, y) &= \frac{x^{m-1}}{\Gamma(m)\Gamma(n-m)} (y - x)^{n-m-1} e^{-y} \\ &\qquad 0 \le x < y < \infty \\ &= 0, \text{otherwise.} \end{aligned}$$

The conditional pdf of $X_{U(n)} \mid X_{U(m)} = x)$ is

$$\begin{aligned} f\left(y | X_{U(m)} = x\right) &= \frac{(y - x)^{n-m-1}}{\Gamma(n-m)} e^{-(y-x)} \\ &\qquad 0 \le x < y < \infty \\ &= 0, \text{otherwise.} \end{aligned}$$

Thus the conditional distribution of $X_{U(n)} - X_{U(m)}$ given $X_{U(m)}$ is the same as the unconditional distribution of $X_{U(n-m)}$ for n > m.

Example 2.2.2 Suppose that the random variable X has the Gumbel distribution with pdf $f(x) = e^{-x}e^{-e^{-x}}, -\infty < x < \infty$. Let $F_{(n)}$ and $f_{(n)}$ be the cdf and pdf of $X_{L(n)}$. It is easy to see that

$$F_{(n)}(x) = \int_{-\infty}^{x} \frac{e^{-nu}}{\Gamma(n)} e^{-e^{-u}} du$$

and

$$f_{(n)}(x) = \frac{e^{-nx}}{\Gamma(n)} e^{-e^{-x}}, \quad -\infty < x < \infty.$$

Let $f_{(m,\ n)}(x, y)$ be the joint pdf of $X_{L(m)}$ and $X_{L(n)}$, m < n. Using (2.2.16), we get for the Gumbel distribution

$$f_{(m,n)}(x,y) = \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} \frac{e^{-mx}}{\Gamma(m)} e^{-y} e^{-e^{-y}},$$
$$-\infty < y < x < \infty$$

Thus the conditional pdf $f_{(n|m)}(y|x)$ of $X_{L(n)} \mid X_{L(m)} = x$ is given by

$$f(n|m)(y|x) = \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} e^{-y} e^{-(e^{-y} - e^{-x})},$$
$$-\infty < y < x < \infty$$

2.3 Moments of Record Values

Let μ_n^r *and* $\mu_{(n)}^r$ be the rth moment of $X_{U(n)}$ and $X_{L(n)}$ respectively, then

$$\mu_n^r = \int_{-\infty}^{\infty} x^r \frac{(R(x))^{n-1}}{\Gamma(n)} f(x) dx \text{ and}$$
$$\mu_{(n)}^r = \int_{-\infty}^{\infty} x^r \frac{(H(x))^{n-1}}{\Gamma(n)} f(x) dx$$

$Var(X_{U(n)}) = \mu_n^2 - (\mu_n^1)^2$ and $Var(X_{L(n)}) = \mu_{(n)}^2 - (\mu_{(n)}^1)^2$. We will denote

$$\mu_{m,n}^{r,s} = E\left(X_{U(m)}^r X_{U(n)}^s\right)$$

and

$$\mu^{r,s}_{(m),(n)} = E\left(X^r_{L(m)} X^s_{L(n)}\right)$$

$$\mathrm{Cov}\left(\mathrm{X}_{\mathrm{U(m)}}, \mathrm{X}_{\mathrm{U(n)}}\right) = \mu^{1,1}_{m,n} - \mu^1_m\, \mu^1_n$$

and

$$\mathrm{Cov}\left(\mathrm{X}_{\mathrm{L(m)}}, \mathrm{X}_{\mathrm{L(n)}}\right) = \mu^{1,1}_{(m),(n)} - \mu^1_{(m)}\, \mu^1_{(n)}$$

If we take $V_k = R(X_{U(k)})$, k = 1, 2, .., then

$$E(V_m V_r) = \int_0^\infty \int_0^y xy \frac{x^{m-1}}{\Gamma(m)} \frac{(y-x)^{r-m-1}}{\Gamma(r-m)} \mathrm{e}^{-y} dx dy$$

Using the transformation t = yx and w = y, we get on simplification

$$E(V_m V_r) = \frac{\Gamma(m+1)}{\Gamma(m)} \frac{\Gamma(r-m)}{\Gamma(r+1)} \frac{\Gamma(r+2)}{\Gamma(r-m)} = \mathrm{m(r+1), m<r.}$$

$$\mathrm{Cov(V_m V_r) = m(r+1) - m\,r = m = Var(V_m), m<r.}$$

If $\rho_{m,n}$ = the correlation coefficient between V_m and V_n, m < n, is

$$\rho_{m,n} = \sqrt{m/n}$$

Example 2.3.1 For the exponential distribution with $\mathrm{f(x) = e^{-x}, 0 \le x < \infty}$.

$$\mu^r_n = \int_0^\infty x^r \frac{x^{n-1}}{(n-1)!} \mathrm{e}^{-x} dx = \frac{(n+r-1)!}{(n-1)!}$$
$$= n^{(r)},\ where\, x^{(k)} = x(x+1)(x+2)\ldots(x+k-1), k > 0,$$
$$= x^{(k)} = 1\ if\ k = 0$$

Thus $E(X_{U(n)}) = n$, $Var(X_{U(n)}) = n(n + 1) - n^2 = n$.
For m < n,

$$\mu^{r,s}_{m,n} = \int_0^\infty \int_x^\infty x^r y^s \frac{x^{m-1}}{\Gamma(m)\Gamma(n-m)} (y-x)^{n-m-1} \mathrm{e}^{-y} dx\, dy$$
$$= \sum_{k=0}^{s} \frac{\Gamma(m+r+s-k)}{\Gamma(m)} \frac{\Gamma(n-m+k)}{\Gamma(n-m)}$$

and $\mathrm{Cov}(X_{U(m)}, X_{U(n)}) = \mu_{m,n}^{1,1} - \mu_m^1 \mu_n^1 = nm + m - nm = m = \mathrm{Var}(X_{U(m)})$. Let $\rho_{m,n}$ be the correlation between $X_{U(n)}$ and $X_{U(m)}$, then

$$\rho_{m,n} = \sqrt{\frac{m}{n}}.$$

It can easily be shown that $E[X_{U(n)} - X_{U(m)}]^r = (n - m)^{(r)}$.

Example 2.3.2 For the Gumbel distribution with $f(x) = e^{-x}e^{-e^{-x}}, -\infty < x < \infty$, $E(X_{L(r)}) = \int_{-\infty}^{\infty} x \frac{e^{-rx}}{\Gamma(r)} e^{-e^{-x}} dx = -\frac{d}{dr} \ln \Gamma(r) = -\psi(r)$, where $\psi(r)$ is the Psi (Digamma) function. Thus

$$\begin{aligned} E(X_{L(r)}) &= \upsilon_r^* \\ \upsilon_1^* &= \upsilon \\ \upsilon_j^* &= \upsilon_{j-1}^* - (j-1)^{-1}, j \geq 2. \end{aligned}$$

Here υ is the Euler's constant. Let $f_{(m),(n)}(x, y)$ be the joint pdf of $X_{L(m)}$ and $X_{L(n)}$, m < n. Using (2.2.17), we get for the Gumbel distribution

$$f_{(m),(n)}(x, y) = \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} \frac{e^{-x}}{\Gamma(m)} e^{-my} e^{-e^{-y}}$$
$$-\infty < y < x < \infty.$$

Thus the conditional pdf $f_{(n|m)}(y|x)$ of $X_{L(n)} | X_{L(m)} = x$ is given by

$$f_{(n|m)}(y|x) = \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} e^{-y} e^{(-e^{-y} - e^{-x})}, -\infty < y < x < \infty$$

$$\begin{aligned} E(X_{L(m)} X_{L(n)}) &= \int_{-\infty}^{\infty} \int_{y}^{\infty} xy f_{(m),(n)}(x, y) dx\, dy \\ &= \int_{-\infty}^{\infty} \int_{y}^{\infty} xy \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} \frac{e^{-mx}}{\Gamma(m)} e^{-y} e^{-e^{-y}} dx\, dy. \end{aligned}$$

Substituting y−x = t, we get on simplification

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{y}^{\infty} xy \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} \frac{e^{-mx}}{\Gamma(m)} e^{-y} e^{-e^{-y}} dx\, dy \\ &= E\left(X_{L(n)}^2\right) + E(T) E(X_{L(n)}) \end{aligned}$$

where $E(T) = \int_0^\infty \Gamma(n)(\Gamma(m)\Gamma(n-m-1))^{-1}(1-e^{-t})^{n-m-1}e^{-mt}\,dt$

Similarly it can be shown that

$$E(X_{L(m)}) = E(X_{L(n)}) + E(T)$$

Thus $\mathrm{Cov}\left(X_{L(m)}X_{L(n)}\right) = \mathrm{Var}(X_{L(n)})$ and

$$\begin{aligned}\mathrm{Var}(X_{L(r)}) &= \int_{-\infty}^{\infty} x^2 f_{(r)}(x)dx\_\left(\int_0^\infty x f_{(r)}(x)dx\right)^2 \\ &= \frac{d}{dr}\psi(r) \\ &= \frac{\pi^2}{6} - \sum_{k=1}^{r-1}\frac{1}{k^2}, \mathrm{k} > 1\end{aligned}$$

and

$$= \frac{\pi^2}{6}\,\mathrm{for\,k} = 2.$$

Let $\mathrm{Var}(X_{L(r)}) = V^*_{r,r}$, r = 1, 2, …, then

$$\begin{aligned}V^*_{1,1} &= \frac{\pi^2}{6} \\ V^*_{j,j} &= V^*_{j-1,j-1} - (j-1)^{-2}, \quad j \geq 2\end{aligned}$$

Further

$$\begin{aligned}E(X_{L(m)}) &= E(X_{L(n)}) + \sum_{p=m}^{n-1}\frac{1}{p} \\ \mathrm{Var}(X_{L(n-1)}) - \mathrm{Var}(X_{L(n)}) &= (n-1)^{-2}\end{aligned}$$

Let ρ(m,n) be the correlation coefficient between $X_{L(m)}$ and $X_{L(n)}$, then

$$\rho(m,n) = \sqrt{\frac{\mathrm{Var}(X_{(n)})}{\mathrm{Var}(X_{(m)})}}.$$

Example 2.3.3 A random variable is said to have generalized Pareto distribution if its probability density function is of the following form:

$$f_0(x,\mu,\sigma,\beta) = \frac{1}{\sigma}\left(1+\beta\left(\frac{x-\mu}{\sigma}\right)\right)^{-(1+\beta^{-1})}$$
$$x \geq \mu,\ for\ \beta > 0,$$
$$\mu < x \leq \mu - \sigma\beta^{-1},\ for\ \beta < 0$$
$$= \frac{1}{\sigma}e^{-(x-\mu)\sigma^{-1}}, x \geq \mu,\ for\ \beta = 0$$
$$= 0, \quad \text{otherwise.}$$

It can be shown that for $\beta \neq 0$

$$X_{U(n)} \stackrel{d}{=} \mu - \frac{\sigma}{\beta} + \frac{\sigma}{\beta}\prod_{i=1}^{n} U_i$$

where $U_1, U_2, \ldots, U_n$ are independently and identically distributed with

$$P(U_i \leq x) = 1 - (x)^{-\beta^{-1}}, \quad x \geq 1, \beta > 0,$$
$$= (x)^{-\beta^{-1}}, \quad \beta < 0, 0 < x < 1.$$

For $\beta = 0$, we have

$$X_{U(n)} \stackrel{d}{=} \mu + \sigma\sum_{i=1}^{n} Z_i$$

where Z_1, Z_2, ….., Z_n are independently and identically distributed with P ($Z_i \leq z$) = 1 − e^{-z}, z > 0, here $\stackrel{d}{=}$ denotes the equality in distribution.

For $\beta \neq 0$, we have

$$E(X_{U(n)}) = \mu + \frac{\sigma}{\beta}\{(1-\beta)^{-n} - 1\}, \beta < 1$$
$$Var(X_{U(n)}) = \sigma^2\beta^{-2}\left\{(1-2\beta)^{-n} - (1-\beta)^{-2n}\right\}, \beta < \frac{1}{2}$$

$Cov(X_{U(m)}, X_{U(n)}) = \sigma^2\beta^{-2}(1-\beta)^{m-n}\left\{(1-2\beta)^{-m} - (1-\beta)^{-2m}\right\}$ Let $\rho_{m,n}$ be the correlation coefficient between $X_{U(m)}$ and $X_{U(n)}$, then

$$\rho_{m,n} = (1-\beta)^{m-n}\left[\frac{(1-2\beta)^{-m} - (1-\beta)^{-2m}}{(1-2\beta)^{-n} - (1-\beta)^{-2n}}\right]^{\frac{1}{2}}, \beta < 1/2.$$
$$= \{(t^m - 1)/(t^n - 1)\}^{1/2}, \quad \text{where } t = \frac{(1-\beta)^2}{1-2\beta} \text{ and } \beta < 1/2.$$

As $\beta \rightarrow 0$, $\rho_{m,n} \rightarrow \surd(m/n)$ which is the correlation coefficient between $X_{U(m)}$ and $X_{U(n)}$ when $\beta = 0$ i.e. for the exponential distribution.

Example 2.3.4 A random variable is said to have Type II extreme value distribution if its cumulative distribution function is of the following form:

$$F(x) = \mathrm{e}^{-\left(\frac{x-\mu}{\sigma}\right)^{-\delta}}, x > \mu, \sigma > 0, \delta > 0.$$

Suppose $X_{L(1)}, X_{L(2)}$ …. be the sequence of lower record values and $f_{(n)}(x)$ is the pdf of $X_{L(n)}$, n = 1, 2, …We can write

$$\begin{aligned} \mathrm{f}_{(n)}(x) &= \frac{(H(x))^{n-1}}{\Gamma(n)}\mathrm{f}(x) \\ &= \frac{\delta^n\left(\frac{x-\mu}{\sigma}\right)^{-(n\,\delta+1)}}{\sigma\,\Gamma(n)}\mathrm{e}^{-\left(\frac{x-\mu}{\sigma}\right)^{-\delta}} \end{aligned}$$

Here $\mathrm{H(x)} = -\ln \mathrm{F(x)} = \mathrm{e}^{-x}$. We can write $\frac{X_{L(n)}-\mu}{\sigma} \overset{d}{=} (W_1 + W_2 + \ldots\ldots + W_n)^{-\frac{1}{\delta}}$, where $W_1, W_2, \ldots, W_n$ are independent and identically distributed as exponential with unit mean.

Let $Y_{L(n)} = \frac{X_{L(n)}-\mu}{\sigma}$, and $U_n = W_1 + W_2 + \cdots + W_n$, then

$$\begin{aligned} \mathrm{E}(Y_{L(n)}) &= \mathrm{E}((U_n) - 1/\delta) \\ &= \int_0^\infty \frac{u^{-\frac{1}{\delta}}u^{n-1}\mathrm{e}^{-u}}{\Gamma(n)}du = \frac{\Gamma\left(n-\frac{1}{\delta}\right)}{\Gamma(n)} \\ \mathrm{E}(Y_{L(n)})^2 &= \mathrm{E}((U_n) - 2/\delta). \\ &= \int_0^\infty \frac{u^{-\frac{2}{\delta}}u^{n-1}\mathrm{e}^{-x}}{\Gamma(n)}du = \frac{\Gamma\left(n-\frac{2}{\delta}\right)}{\Gamma(n)} \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{E}(X_{L(n)}) &= \mu + \sigma\frac{\Gamma\left(n-\frac{1}{\delta}\right)}{\Gamma(n)}, \\ \mathrm{Var}(X_{L(n)}) &= \sigma^2\left[\frac{\Gamma\left(n-\frac{2}{\delta}\right)}{\Gamma(n)} - \left(\frac{\Gamma\left(n-\frac{1}{\delta}\right)}{\Gamma(n)}\right)^2\right] \end{aligned}$$

For m < n,

$$\mathrm{E}(\mathrm{Y}_{\mathrm{L(m)}}.\mathrm{Y}_{\mathrm{L(n)}}) = \int_0^\infty \int_0^\infty \frac{u^{-\frac{1}{\delta}}(u+v)^{-\frac{1}{\delta}}}{\Gamma(m)\Gamma(n-m)} \mathrm{e}^{-u} u^{m-1} \mathrm{e}^{-v} v^{n-m-1} du dv$$

Substituting

$$y_1 = u$$
$$y_2 = \frac{u}{u+v}$$

we get on simplification,

$$\begin{aligned}\mathrm{E}(\mathrm{Y}_{\mathrm{L(m)}}.\mathrm{Y}_{\mathrm{L(n)}}) &= \int_0^\infty \int_0^1 \frac{(y_1)^{n-1-\frac{2}{\delta}} \mathrm{e}^{-y_1} (1-y_2)^{n-m-1}}{\Gamma(m)\Gamma(n-m)} (y_2)^{m-1-\frac{1}{\delta}} dy_1 dy_2 \\ &= \frac{\Gamma\left(n-\frac{2}{\delta}\right)\Gamma\left(m-\frac{1}{\delta}\right)}{\Gamma(m)\Gamma\left(n-\frac{1}{\delta}\right)}\end{aligned}$$

Thus

$$\mathrm{Cov}\left(\mathrm{X}_{\mathrm{L(m)}}\mathrm{X}_{\mathrm{L(n)}}\right) = \sigma^2 \left\{ \frac{\Gamma\left(m-\frac{1}{\delta}\right)}{\Gamma(m)} \left[\frac{\Gamma\left(n-\frac{2}{\delta}\right)}{\Gamma\left(n-\frac{1}{\delta}\right)} - \frac{\Gamma\left(n-\frac{1}{\delta}\right)}{\Gamma(n)} \right] \right\}$$

We rewrite the covariance expression as $\mathrm{Cov}\left(\mathrm{X}_{\mathrm{L(m)}}\mathrm{X}_{\mathrm{L(n)}}\right) = \sigma^2 \mathrm{a_m b_n}$, where

$$a_m = \frac{\Gamma\left(m-\frac{1}{\delta}\right)}{\Gamma(m)} \text{ and } b_n = \frac{\Gamma\left(n-\frac{2}{\delta}\right)}{\Gamma\left(n-\frac{1}{\delta}\right)} - \frac{\Gamma\left(n-\frac{1}{\delta}\right)}{\Gamma(n)}, 1 \le m \le n.$$

$$\mathrm{Corr}(\mathrm{X}_{\mathrm{L(m)}}\mathrm{X}_{\mathrm{L(n)}}) = \sqrt{\frac{a_m}{a_n} \cdot \frac{b_n}{b_n}}$$

The following theorem gives the condition for the existence of the moments of the nth record value.

Theorem 2.3.1 *If* $\int_{-\infty}^{\infty} |x|^{r+\delta} dF(x) < \infty$, *for some* $\delta > 0$, *then* $E(X_{U(n)})^r$ *is finite for all* $n \geq 2$.

Proof We define the inverse function $\mathrm{R}^{-1}(\mathrm{y}) = \inf\{\mathrm{x} : \mathrm{R(x)} \ge \mathrm{y}\}$

$$\begin{aligned}\mathrm{E}\left(\left|\mathrm{X}_{\mathrm{U(n)}}\right|^{\mathrm{r}}\right) &= \int_{-\infty}^{\infty} \frac{1}{\Gamma(\mathrm{n})}|x|^{r+\delta}(R(x))^{n-1}dF(x) < \infty, \text{for } \delta > 0,\\ &= \frac{1}{\Gamma(\mathrm{n})}\int_0^{\infty} \left|R^{-1}y\right|^{r} y^{n-1}\mathrm{e}^{-y}dy\\ &= \frac{1}{\Gamma(\mathrm{n})}\left(\int_0^{\infty} \left|R^{-1}y\right|^{rp}\mathrm{e}^{-y}dy\right)^{1/p}\left(\int_0^{\infty} y^{nq}\mathrm{e}^{-y}dy\right)^{1/q}\end{aligned}$$

by Holder's inequality, where $\frac{1}{p}+\frac{1}{q}=1, p>1, q>1,$

$$= \frac{1}{\Gamma(n)}\left(\int_0^{\infty} \left|R^{-1}(y)\right|^{r+\delta}\mathrm{e}^{-y}dy\right)^{1/p}\left(\int_0^{\infty} y^{nq}\mathrm{e}^{-y}dy\right)^{1/q},$$

$$\text{where} \quad p = \frac{r+\delta}{r};$$

$$= \frac{1}{\Gamma(n)}\left(E(|x|)^{r+\delta}\right)^{1/p}\left(\int_0^{\infty} y^{nq}\mathrm{e}^{-y}dy\right)^{1/q} < \infty.$$

Theorem 2.3.2 *If E(X) = 0 and Var(X) = 1, then* $\left|\mathrm{E}(\mathrm{X}_{\mathrm{U(n+1)}})\right| \le \sqrt{\binom{2n}{n}-1}.$

Proof Let

$$\begin{aligned}\mathrm{F}^{-1}(u) &= \mathrm{Sup}\{x\colon \mathrm{F}(x) \le u\}, 0<u<1,\\ \mathrm{F}^{-1}(1) &= \mathrm{Sup}\left\{\mathrm{F}^{-1}(u), u<1\right\}\end{aligned}$$

$$0 = \mathrm{E(X)} = \int_0^{\infty} x\,\mathrm{f}(x)dx = \int_0^{\infty} \bar{\mathrm{F}}^{-1}tdt.$$

$$1 = \mathrm{E}(\mathrm{X}^2) = \int_0^{\infty} x^2\mathrm{f}(x)dx = \int_0^{\infty} \left\{\bar{\mathrm{F}}^{-1}(t)\right\}^2 dt.$$

$$\begin{aligned}\mathrm{E}(\mathrm{X}_{\mathrm{U(n+1)}}) &= \int_{-\infty}^{\infty} x\frac{\{-\ln \bar{\mathrm{F}}(x)\}^n}{\Gamma(n+1)}\mathrm{f}(x)dx\\ &= \int_0^1 \bar{\mathrm{F}}^{-1}(t)\frac{\{-\ln(1-t)\}^n}{\Gamma(n+1)}dt\\ &= \int_0^1 \bar{\mathrm{F}}^{-1}(t)\left[\frac{\{-\ln(1-t)\}^n}{\Gamma(n+1)} - \lambda\right]dt.\end{aligned}$$

Using Cauchy and Schwarz inequality, we get

$$\left|E(X_{U(n+1)})\right| \le \left\{\int_0^1 \left[\bar{F}^{-1}(t)\right]^2 dt\right\}^{\frac{1}{2}} \left\{\int_0^1 \left(\frac{(-\ln(1-t))^n}{\Gamma(n+1)} - \lambda\right)^2 dt\right\}^{\frac{1}{2}}.$$

Now

$$\int_0^1 \left\{\bar{F}^{-1}(t)\right\}^2 dt = 1,$$

and

$$\int_0^1 \left(\frac{(-\ln(1-t))^n}{\Gamma(n+1)} - \lambda\right)^2 dt = \binom{2n}{n} + \lambda^2 - 2\lambda$$

Since the minimum value of $\lambda^2 - 2\lambda$ is -1, we get

$$E(X_{U(n)})| \le \sqrt{\binom{2n}{n} - 1} \approx \frac{2^{2n}}{\sqrt{n\pi}}, \text{for large n.} \qquad (2.3.1)$$

For symmetric distribution the upper bound of $|E(X_{U(n)})|$ is smaller. The bound of the symmetric distribution is given in the following theorem.

Theorem 2.3.3 *Suppose the random variable X is symmetric about zero and has variance 1, then* $E(X_{U(n+1)}) < \frac{1}{\sqrt{2}}\left\{\binom{2n}{n} - \frac{1}{[\Gamma(n+1)]^2}\int_0^\infty [\ln(1-u)\ln u]^n du\right\}^{\frac{1}{2}}$.

Proof

$$E(X_{U(n+1)}) = \int_{-\infty}^{\infty} x\frac{\left\{-\ln \bar{F}(x)\right\}^n}{\Gamma(n+1)} f(x)dx$$

$$= \int_0^{\infty} x\frac{\left\{-\ln \bar{F}(x)\right\}^n}{\Gamma(n+1)} f(x)dx - \int_0^{\infty} x\frac{\{-\ln F(x)\}^n}{\Gamma(n+1)} f(x)dx$$

$$= \frac{1}{2\Gamma(n+1)}\int_0^1 F^{-1}(u)[\{-\ln(1-u)\}^n - \{-\ln u\}^n]du.$$

Now

$$\int_0^1 \left\{F^{-1}(u)\right\}^2 du = 1$$

and

$$\int_0^1 [\{-\ln(1-u)\}^n - \{-\ln u\}^n]^2 du$$

$$= 2\Gamma(2n+1) - 2\int_0^1 [\ln(1-u)\ln u]^n du$$

Hence using the Cauchy and Schwarz inequality, we get

$$\mathrm{E}\left|\ \mathrm{X}_{\mathrm{U(n+1)}}\right| \le \frac{1}{\sqrt{2}}\left\{\binom{2n}{n} - \frac{1}{[\Gamma(n+1)]^2}\int_0^\infty [\ln(1-u)\ln u]^n du\right\}^{\frac{1}{2}}. \qquad (2.3.2)$$

The following table gives the upper bounds of the inequalities given by (2.3.1) and (2.3.2). For large n, the ratio of the bounds as given by (2.3.2) and (2.3.1) is approximately $\sqrt{2}$.

Let

$$\mathrm{h(n)} = \frac{1}{\sqrt{2}}\left\{\binom{2n}{n} - \frac{1}{[\Gamma(n+1)]^2}\int_0^\infty [\ln(1-u)\ln u]^n du\right\}^{\frac{1}{2}},$$

$$\mathrm{g(n)} = \sqrt{\left(\frac{2n}{n}\right) - 1} \text{ and } \mathrm{b(n)} = \frac{g(n)}{h(n)}.$$

Thus g(n) is the upper bound of $|\mathrm{E}(\mathrm{X}_{\mathrm{U(n)}}|$ and h(n) is the upper bound of $\mathrm{E}(\mathrm{X}_{\mathrm{U(n)}})$, when the distribution of $\mathrm{X_i}$, i = 1, 2, … is symmetric (Table 2.1).

Nevzerov (1992) gave an interesting upper bounds of the correlation coefficient between any two upper record values. The result is given in the following theorem.

Theorem 2.3.4 *Let { X_i, i = 1, 2, …} be a sequence of independent and identically distributed random variables and suppose that for* $1 \le m < n$, $E(X_1^2(\ln(1-F(X_1)))^{j-1} < \infty$, *for j = n. Then*

Table 2.1 Values of h(n), g(n) and b(n)

N	h(n)	g(n)	b(n)
1	0.906896	1	2.102662
2	2.726929	2.236068	2.294824
3	3.162147	4.358899	2.378462
4	5.916078	8.306624	2.404076
5	12.224972	15.84298	2.411405
6	22.494185	30.380915	2.413448
7	42.424630	58.574739	2.414008
8	80.218452	113.441615	2.414159
9	155.916644	220.497166	2.41419
10	303.937494	429.831362	2.414210

$$\rho(X_{U(m)}, X_{U(n)}) \leq \sqrt{\frac{m}{n}},$$

where ρ(X, Y) is the correlation between X and Y. The equality holds if and only if X1 has an

Theorem 2.3.5 *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent and identically distributed random variables with distribution function F(x) and the corresponding density function f(x). If* $E(X_n)$, $n \geq 1$ *is finite and F belongs to the class C1, then* $E\{X_{U(m+1)} - X_{U(m)}\} \leq (\geq)\ E(X_n)$, *for any fixed m and n according as F is NBU (NWU).*

Proof From Eq. (2.2.4), we can write the $E\{X_{U(m+1)} - X_{U(m)}\}$ as

$$\begin{aligned} E\{X_{U(m+1)} - X_{U(m)}\} &= \int_0^\infty \int_0^\infty \frac{1}{\Gamma(n)} (R(u))^{n-1} f(u) \frac{\bar{F}(u+z)}{\bar{F}(u)} du dz \\ &\leq (\geq) \int_0^\infty \int_0^\infty \frac{1}{\Gamma(n)} (R(u))^{n-1} f(u) \bar{F}(z) du dz, \end{aligned}$$

according as $\bar{F}(x+y) \leq (\geq) \bar{F}(x)\bar{F}(y)$. Hence $E\{X_{U(m+1)} - X_{U(m)}\} \leq (\geq) E(X_n)$ according as F is NBU(NWU).

If F(x) has the density f(x), the ratio $r(x) = \frac{f(x)}{\bar{F}(x)}$, for $\bar{F}(x) > 0$ is called the failure (hazard) rate hazard rate, we will say F belongs to the class C_2 if the failure rate, r(x), is either monotone increasing (IFR) or monotone decreasing (DFR).

Theorem 2.3.6 *Let* $\{X_i, i = 1, 2, \ldots\}$ *be ac sequence of i.i.d. continuous non-negative rv's with common cdf F(x) and pdf f(x). Suppose that* $X_{U(1)}, X_{U(2)}, \ldots$ *are the upper record values of this sequence and* $Z_{n+1,\ n} = X_{U(n+1)} - X_{U(n)}$, $n = 1, 2, \ldots$

with $X_{U(0)} = 0$. If $E(D_{n+1})$ exists and F belongs to class C_2, then $E(Z_{n+1}) > (<) E(Z_n)$ according as F is IFR or DFR.

Proof For n = 1, 2, …, the joint pdf of $X_{U(n)}$ and $X_{U(n+1)}$ is given by

$$f_{n,n+1}(x,y) = \frac{(R(x))^{n-1}}{(n-1)!} r(x) f(y)$$
$$\text{for } -\infty < x < y < \infty.$$

The joint pdf of $X_{U(n)}$ and $Z_{n+1,n}$ is

$$f_{n,z}(x,z) = \frac{(R(x))^{n-1}}{(n-1)!} r(x) f(z+x)$$
$$\text{for } 0 < x, z < \infty.$$

Now

$$E(Z_{n+1,n}) = \int_0^\infty \int_0^\infty z \frac{(R(x))^{n-1)}}{(n-1)!} r(x) f(z + x) dx dz.$$

Since $\int_0^\infty z f(z+x) dz = \int_0^\infty \vec{F}(z+x) dz$, we obtain

$$E(Z_{n+1,n}) = \int_0^\infty \int_0^\infty \frac{(R(x))^{n-1)}}{(n-1)!} r(x) \vec{F}(z+x) dx dz.$$

On integrating by parts and using the relation R'(x) = r(x), we get

$$\begin{aligned} E(Z_{n+1,n}) &= \int_0^\infty \int_0^\infty \frac{(R(x))^n}{n!} f(z+x) dx dz \\ &= \int_0^\infty \int_0^\infty \frac{(R(x))^n}{n!} r(z+x) \bar{F}(z+x) dx dz \\ &\geq (\leq) \int_0^\infty \int_0^\infty \frac{(R(x))^n}{n!} r(z) \bar{F}(z+x) dx dz \\ &\quad \text{according as r(x) is IFR or DFR} \\ &= E\left(Z_{n+2,n+1}\right).. \end{aligned}$$

2.4 Entropies of Record Values

Let X be a continuous random variable with the pdf f(x), then the entropy H(x) of X is defined as

$$H(x) = -\int_{-\infty}^{\infty} f(x) \ln f(x) dx$$

where f(x) ln f(x) is integrable.

For a discrete random variable X taking values on x_1, x_2,…, with h probabilities p_1, p_2,… the entropy H(x) is defined as

$$H(X) = \sum_{i=1}^{\infty} p_i x_i$$

provided the summation is finite.

In the case of discrete distribution the transformation

$$Y = a + bX, -\infty < a < \infty, b > 0,$$

Does not change the probabilities p_1, p_2, … and we have

$$H(Y) = H(X)$$

In the case of continuous random variable the Y = a+bX will change the entropy of Y as

$$\begin{aligned} H(Y) &= -\int_{-\infty}^{\infty} \frac{1}{b} f\left(\frac{x-a}{b}\right) \ln f\left(\frac{x-a}{b}\right) dx \\ &= -\int_{-\infty}^{\infty} f(x) \ln \left(\frac{1}{b} f(x)\right) dx \\ &= \ln b + H(x) \end{aligned}$$

The concept of entropy has recently been used in statistical inference. Shannon was the first to compute the entropies of the normal, exponential and uniform distribution. We will discuss here the entropies of upper record values. The entropies of lower record values are similar.

Let $H_n(x)$ be the entropy of $X_{U(n)}$ for a continuous random variable, then

$$-H_n(x) = \int_{-\infty}^{\infty} f_n(x) \ln f_n(x) dx$$
$$= -\ln \Gamma(n) + (n-1) \int_{-\infty}^{\infty} \ln R(x) \frac{(R(x))^{n-1}}{\Gamma(n)} f(x) dx$$
$$+ \int_{-\infty}^{\infty} \ln f(x) \frac{(R(x))^{n-1}}{\Gamma(n)} f(x) dx$$
$$= -\ln \Gamma(n) + (n-1) \int_0^{\infty} \frac{t^{n-1}}{\Gamma(n)} e^{-t} \ln t \, dt + I,$$
$$= -\ln \Gamma(n) + (n-1)\psi(n) + I$$

where

$$I = \int_{-\infty}^{\infty} \ln f(x) \frac{(R(x))^{n-1}}{\Gamma(n)} f(x) dx = \int_{-\infty}^{\infty} f_n(x) \ln f(x) dx,$$

and $\psi(n)$ is the digamma function i.e. $\psi(n) = \frac{d}{dn} = \ln \Gamma(n) = \frac{\Gamma'(n)}{\Gamma(n)}$.

Example 2.4.1 Suppose the sequence of independent and identically distributed random variables X_n, $n \geq 1$, has the Rayleigh distribution with pdf f(x), where

$$f(x) = \frac{x}{b^2} e^{-x^2/(2b^2)}, 0 < x < \infty, \text{then}$$
$$I = \int_0^{\infty} f_n(x) \ln f(x) dx$$
$$= -2\ln b + \int_0^{\infty} \ln x \, f_n(x) dx - \int_0^{\infty} \frac{x^2}{2b^2} f_n(x) dx$$
$$= -2 \ln b + \frac{1}{2}\psi(n) + \frac{1}{2}\ln 2 + \ln b$$
$$= \frac{1}{2}\ln 2 - \ln b + \frac{1}{2}\ln \psi(n) - n.$$

Hence

$$H_n(x) = \ln\Gamma(n) - (n - 1/2)\psi(n) + \ln b - 1/2\ln 2 + n$$

Example 2.4.2 Suppose that the sequence of i.i.d. random variables X_n has the Weibull pdf, f(x) where

$$f(x) = \frac{c}{a}x^{c-1}x^{-x^c/a}, 0 < x, a, c < \infty.$$

In this case, we have

$$\begin{aligned} I &= \int_0^\infty f_n(x)\ln f(x)dx \\ &= \ln\frac{c}{a} + (c-1)\int_0^\infty \ln x f_n(x)dx - \int_0^\infty \frac{x^c}{a} f_n(x)dx \\ &= \ln\frac{c}{a} + (c-1)(\ln a + \psi(n)) - n \\ &= \ln c - \frac{1}{c}\ln a + \frac{c-1}{c}\psi(n) - n. \end{aligned}$$

Hence

$$H_n(x) = \ln\Gamma(n) - \left(n - \frac{1}{c}\right)\psi(n) - \ln c + \frac{1}{c}\ln a + n.$$

2.5 Estimation of Parameters and Predictions of Records

2.5.1 Exponential Distribution

We will consider here the two parameter exponential distribution with pdf f(x) as given by

$$\begin{aligned} f(x) &= \sigma^{-1}\exp\left(-\sigma^{-1}(x-\mu)\right), x \geq \mu \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{2.5.1}$$

2.5.1.1 Minimum Variance Linear Unbiased Estimates (MVLUE) of μ and σ

Suppose that $X(1), X(2), \ldots, X(m)$ are the m (upper) record values from $E(\mu, \sigma)$ with pdf as given in (2.5.1).

Let

$$Y_i = \sigma^{-1}(X(i) - \mu), \mathrm{i} = 1, 2, \ldots, \mathrm{m}, \text{then}$$
$$E(Y_i) = i = Var(Y_i), \quad i, = 1, 2, \ldots, m,$$

and $Cov(Y_i, Y_j) = \min(i, j)$.

Let

$$X = (X(1), X(2), \ldots, X(m)), \text{then}$$
$$E(X) = \mu L + \sigma \delta$$
$$Var(X) = \sigma^2 V,$$

where

$$L' = (1, 1, \ldots, 1)', \delta' = (1, 2, \ldots m)'$$
$$V = (V_{ij}), V_{ij} = \min(i, j), \quad i, j = 1, 2, \ldots, m.$$

The inverse $V^{-1}(= V^{ij})$ can be expressed as

$$V^{ij} = \begin{array}{lll} 2 & if & i = j = 12, \ldots m - 1 \\ 1 & if & i = j = m \\ -1 & if & |i - j| = 1, i, j = 1, 2 \ldots, m \\ 0 & & otherwise. \end{array}$$

The minimum variance linear unbiased estimates (MVLUE) $\hat{\mu}, \hat{\sigma}$ of μ and σ respectively are

$$\hat{\mu} = -\delta' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta$$
$$\hat{\sigma} = L' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta,$$

where

$$\Delta = (L'V^{-1}L)(\delta' V^{-1}\delta) - (L'V^{-1}\delta)^2$$

and

$$Var(\hat{\mu}) = \sigma^2 L'V^{-1}\delta/\Delta$$
$$Var(\hat{\sigma}) = \sigma^2 L'V^{-1}L/\Delta$$
$$Cov(\hat{\mu}, \hat{\sigma}) = -\sigma^2 L'V^{-1}\delta/\Delta.$$

It can be shown that

$$L'V^{-1} = (1,0,0,\ldots,0), \delta' V^{-1} = (0,0,0,\ldots,1), \quad \delta' V^{-1}\delta = m \quad \text{and}$$
$$\Delta = m - 1.$$

On simplification, we get

$$\hat{\mu} = (m(X(1)) - (X(m)))/(m-1)$$
$$\hat{\sigma} = (X(m) - X(1))/(m-1)$$

with

$$Var(\hat{\mu}) = m\,\sigma^2/(m-1), Var(\hat{\sigma}) = \sigma^2/(m-1) \text{ and}$$
$$Cov(\hat{\mu},\hat{\sigma}) = -\sigma^2/(m-1).$$

Exercise 2.5.1.1 If $\mu = 0$, then the MVLUE $\hat{\sigma}_0$ of σ_0 is

$$\hat{\sigma}_0 = \frac{X(m)}{m}$$

2.5.1.2 Best Linear Invariant Estimators (BLIE) of μ and σ Are

The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter μ) estimators (BLIE) $\tilde{\mu}$ $\tilde{\sigma}$ of μ and σ are

$$\tilde{\mu} = \hat{\mu} - \hat{\sigma}\left(\frac{E_{12}}{1+E_{22}}\right)$$

and

$$\tilde{\sigma} = \hat{\sigma}/(1+E_{22}),$$

where $\hat{\mu}$ and $\hat{\sigma}$ are MVLUE of μ and σ and

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu},\hat{\sigma}) \\ Cov(\hat{\mu},\hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}$$

The mean squared errors of these estimators are

$$MSE(\tilde{\mu}) = \sigma^2\left(E_{11} - E_{12}^2(! \;+\; E_{22})^{-1}\right) \text{ and}$$
$$MSE(\tilde{\sigma}) = \sigma^2 E_{22}(1+E_{22})^{-1}$$

We have

$$E(\tilde{\mu}-\mu)(\tilde{\sigma}-\sigma) = \sigma^2 E_{12}(1+E_{22})^{-1}.$$

Using the values of E_{11}, E_{12} and E_{22} from (2.3.2), we obtain

$$\hat{\mu} = ((m+1)X(1) - X(m))/m,$$
$$\hat{\sigma} = (X(m) - X(1))/m$$
$$Var(\tilde{\mu}) = \frac{m+1}{m}\,\sigma^2 \text{ and } \quad Var(\hat{\sigma}) = \frac{m-1}{m^2}\sigma^2$$

2.5.1.3 Maximum Likelihood Estimate. of μ and σ Are

The log likelihood equation based on the m upper records X(1), X(2), …, X(m) can be written as

$$\ln \mathrm{L} = -\mathrm{m}\ln\sigma - \frac{1}{\sigma}(X(m)-\mu), \mu < X(1) < X(2)\ldots < X(m) < \infty$$

The maximum likelihood estimate $\hat{\mu}_{ml}$ and $\hat{\sigma}_{ml}$ of μ and σ are respectively

$$\hat{\mu}_{ml} = \mathrm{X}(1)$$

and

$$\hat{\sigma}_{ml} = \frac{1}{m}(X(m) - X(1))$$
$$E(\hat{\mu}_{ml}) = \mu + \sigma, Var(\hat{\mu}_{ml}) = \sigma^2,$$
$$E(\hat{\sigma}_{ml}) = \frac{(m-1)\sigma}{m}, Var(\hat{\sigma}_{ml}) = \frac{(m-1)\sigma^2}{m^2}$$

and $Cov(\hat{\mu}_{ml}\hat{\sigma}_{mj}) = 0$

Exercise 2.5.1.3 Show that in the case of one parameter exponential with $F(x) = 1 - e^{=x/\sigma}$, x $\geq$ 0, $\sigma > 0$. The maximum likelihood estimate σ^*_{ml} of σ based on m upper records X(1), X(3), …, X(m) is

$$\sigma^*_{ml} = \frac{\mathrm{x(m)}}{\mathrm{m}} \ with \quad E\left(\sigma^*_{ml}\right) = \sigma + \frac{\mu}{m} \text{and}\, Var\left(\sigma^*_{ml}\right) = \frac{\sigma^2}{m}.$$

2.5.1.4 Prediction of Record Values

We will predict the sth upper record value based on the first m record values for s > m. Let $W' = (W_1, W_2, \ldots, W_m)$, where

$$\sigma^2 W_{ij} Cov(X(i), X(j)), i = 1, \ldots, m \text{ and } \alpha^* = \sigma^{-1} E(X(i) - \mu).$$

The best linear unbiased predictor of X(s) is $\hat{X}(s)$ where

$$\hat{X}(s) = \hat{\mu} + \hat{\sigma}\alpha^* + W' V^{-1}(X - \hat{\mu} L - \hat{\sigma}\delta), \hat{X}_{U(s)},$$

$\hat{\mu}, \hat{\sigma}$ are the MVLUE of μ, σ respectively. It can be shown that $W' V^{-1}(X - \hat{\mu} L - \hat{\sigma}\delta) = 0.$

$$\begin{aligned} \hat{X}(s) &= ((s-1)X(m) + (m-s)X(1))/(m-1) \\ E(\hat{X}(s)) &= \mu + s\sigma \\ Var(\hat{X}(s)) &= \sigma^2\left(m + s^2 - 2s\right)/(m-1). \end{aligned}$$

Let $\tilde{X}(s)$ be the best linear invariant predictor of $X(s)$. Then it can be shown that

$$\tilde{X}(s) = \hat{X}(s) - C_{12}(1 + E_{22})^{-1}\hat{\sigma},$$

where

$$C_{12}\sigma^2 = Cov\left(\hat{\sigma}, (L - W' V^{-1} L)\hat{\mu} + (\alpha^* - W' V^{-1}\delta)\hat{\sigma}\right)$$

and $\sigma^2 E_{22} = Var(\hat{\sigma})$. On simplification we get

$$\begin{aligned} \tilde{X}(s) &= \frac{m-s}{m} X(1) + \frac{s}{m} X(m) \\ E(\tilde{X}(s)) &= \mu + \left(\frac{ms + m - s}{m}\right)\sigma \\ Var(\tilde{X}(s)) &= \sigma^2\left(m^2 + ms^2 - s^2\right)/m^2. \end{aligned}$$

It is well known that the best (unrestricted) least squares predictor $\tilde{\tilde{X}}$ of X(s) is

$$\hat{\hat{X}}(s) = E(X(s)|X(1),\ldots,X(m)) \\ = X(m) + (s-m)\sigma\ldots$$

But $\hat{\hat{X}}_{U(s)}$ depends on the unknown parameter σ. If we substitute the minimum variance linear unbiased estimate $\hat{\sigma}$ for σ, then $\hat{\hat{X}}(s)$
becomes equal to $\hat{X}(s)$. Now

$$E\left(\hat{\hat{X}}(s)\right) = \mu + s\,\sigma = E(X(s))$$
$$Var\left(\hat{\hat{X}}(s)\right) = m\sigma^2$$

2.5.2 *Generalized Pareto Distribution*

We will consider the generalized Pareto distribution with the following pdf f(x)

$$\begin{aligned} f(x) &= \frac{1}{\sigma}\left(1+\beta\left(\frac{x-\mu}{\sigma}\right)\right)^{-1(1+\beta)^{-1}} \\ & x \geq \mu, \text{ for } \beta > 0, \\ & \mu < x < \mu - \sigma/\beta, \text{ for } \beta < 0, \\ &= \frac{1}{\sigma} e^{-1(x-\mu)\sigma^{-1}}, x \geq \mu \text{ for } \beta = 0, \quad \text{for } \sigma > 0. \\ &= 0, \text{ otherwise,} \end{aligned} \tag{2.5.2}$$

2.5.2.1 Minimum Variance Linear Unbiased Estimator of μ and σ When β Is Known

Theorem 2.5.2.1 *The minimum variance linear unbiased estimators* $\hat{\mu}$ *and* $\hat{\sigma}$ *of* μ *and* σ *based on the observed upper record values X(1), X(2), ..., X(m)*

$$\hat{\mu} = X(1)_1 - (1-\beta)^{-1}\hat{\sigma}.$$

$$\hat{\sigma} = (1-\beta)(\beta - D)^{-1}(1-2\beta)^3 X(1) + D^{-1}\beta(1-\beta)\sum_{i=2}^{m-1}(1-2\beta)^{i+1}X(i) \\ + D^{-1}(1-\beta)^2(! - 2\beta)^{m+1}X(m)$$

where

$$D = \sum_{I=2}^{m} (1-2\beta)^{i+1} \text{and } \beta < 1/2$$

Proof We assume GP(μ, σ, β) with β ≠ 0 and with finite variance. Let R be the m × 1 vector corresponding to X(i), i = 1,2, …, m, then we can write

$$E(R) = \mu L + \sigma\delta$$

where

$$R' = (X(1), X(2), \ldots, X(m))$$
$$L' = (1, 1, \ldots, 1), \delta' = (\alpha_1, \alpha_1, \alpha_1, \ldots, \alpha_m)$$
$$\alpha_i = \beta^{-1}(1-\beta)^{-i},$$

and

$$\alpha_i = \beta^{-1}(1-\beta)^{-i}, i = 1, 2, \ldots, m.$$

We can write $V(R) = \sigma^2 V, V = (V_{i,j}), V_{i,j} = \beta^{-2} a_i b_j, 1 < i < j < m$ and $V_{i,j} = V_{j,i}$. The inverse $V^{-1} (= V^{i,j})$ can be expressed as

$$V^{i+1,i} = V^{i,i+1} = -\frac{1}{a_{i+1}b_i - a_i b_{i+1}} = -(1-2\beta)^{i+1}(1-\beta), i = 1, 2, \ldots, m-1,$$
$$V^{i,i} = \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})}, i = 1, 2, \ldots, n, V^{i,j} = 0, \text{for } |i-j| > 1,$$

where $a_o = 0 = b_{n+1}$ *and* $b_o = 1 = a_{n+1}$.
On simplification, we obtain

$$V^{i,i} = (1-2\beta)^i (2 - 4\beta + \beta^2), i = 1, 2, \ldots, m-1$$

and

$$V^{m,m} = (1-2\beta)^m (1-\beta).$$

The minimum variance linear unbiased estimators (MVLUE) $\hat{\mu}, \hat{\sigma}$ of μ and σ are respectively based on the upper record values are

$$\hat{\mu} = -\delta' V^{-1}(L\delta' - \delta L')V^{-1}R/\Delta,$$

and

$$\hat{\sigma} = L'V^{-1}(L\delta' - \delta L\prime)V^{-1}R/\Delta,$$

where

$$\Delta = (\mathrm{L'V^{-1}L})(\mathrm{d'V^{-1}}\delta) - (\mathrm{L'V^{-1}}\delta)^2.$$

On substituting the values for δ and V^{-1} and subsequent simplification, it can be shown that

$$\hat{\mu} = X(1) - \hat{\sigma}(1-\beta)^{-1} \textit{ and}$$
$$\hat{\sigma} = (1-\beta)\left(\beta - D^{-1}(1-2\beta)^3 X(1)r_1\right) + D^{-1}\beta(1-\beta)\sum_{i=2}^{m}(1-2\beta)X(i)_i$$

where

$$\mathrm{D} = \sum_{i=2}^{m}(1-2\beta)^{i+1}.$$

The corresponding variances and the covariance of the estimates are

$$Var(\hat{\mu}) = \sigma^2 \frac{T}{D}$$
$$Var(\hat{\sigma}) = \sigma^2 \frac{\beta T - (1-2\beta)}{D}$$
$$Cov(\hat{\mu}, \hat{\sigma}) = \sigma^2 \frac{\left\{(1-2\beta)^2 + \beta^2 T\right\}}{D}$$

and

$$\mathrm{T} = \sum_{i=2}^{m}(1-2\beta)^i.$$

Exercise 2.5.2.1 Find the MVLUE $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ based on n upper record values X(1), X(2), …, X(n) of the Pareto Type II (Lomax) distribution with pdf f(x) as $f(x) = \frac{v}{\sigma}\left(1 + \frac{x-\mu}{\sigma}\right)^{-(v+1)}, x > \mu . \sigma > 0$ and $v > 0$,

2.5.2.2 Best Linear Invariant Estimators (BLIE)

Theorem 2.5.2.2 *The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter* μ*) estimators* $\tilde{\mu}, \tilde{\sigma}$ *of* μ *and* σ *are respectively*

$$\tilde{\mu} = \hat{\mu} - \frac{\beta T - (1 - 2\beta)}{T(1 - \beta)^2}\hat{\sigma} \text{ and}$$
$$\tilde{\sigma} = \frac{D}{T(1 - \beta)^2}\hat{\sigma}, \text{ where}$$
$$D = \sum_{i=2}^{m}(1-2\beta)^{i+1}, T = \sum_{i=1}^{m}(1-2\beta)^i.$$

and $\hat{\mu}$ *and* $\hat{\sigma}$ *are MVLUE of* μ *and* σ*.*

Proof The BLIE $\tilde{\mu}$ *and* $\tilde{\sigma}$ can be written as

$$\hat{\mu} = \hat{\mu} - \frac{E_{12}}{1 + E_{22}}\hat{\sigma}.$$

and

$$\tilde{\sigma} = \frac{1}{1 + E_{22}}\tilde{\sigma},$$

where

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu}, \hat{\sigma}) \\ Cov(\hat{\mu}, \hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}.$$

The mean squared errors of $\tilde{\mu}$ *and* $\tilde{\sigma}$ are

$$MSE(\tilde{\mu}) = \sigma^2\left(E_{11} - \frac{E_{12}^2}{1 + E_{22}}\right),$$
$$MSE(\tilde{\sigma}) = \sigma^2\left(\frac{E_{22}}{1 + E_{22}}\right).$$

Substituting the values of E_{11}, E_{12} and E_{22} in terms of β, T and D, we get the result.

2.5.2.3 Estimator of β for Known μ and σ

A Moment Estimator of β. We have seen that for μ = 0 and σ = 1. $E(X_{U(m)}) = \beta-1$ $\{(1-\beta)^{-m}-1\}$. Thus

$$E(\bar{X}) = E\{(X(1)+X(2)+\ldots\ldots+X(m)\}/m\} = \frac{1}{m\beta^2}\{(1-\beta)^{-m}-1\} - \frac{1}{\beta}$$
$$= \frac{X(m)-m}{m\beta}$$

Thus we can take $\tilde{\beta}$ as an estimator of β where

$$\tilde{\beta} = \frac{X(m)-m}{X(1)+X(2)+\ldots+X(m)}, for\, rx(1)+X(2)+\ldots+X(m) \neq 0$$

2.5.3 *Power Function Distribution*

We will consider the following pdf f(x) of power function distribution

$$f(x,\alpha,\beta,\gamma) = \gamma\,\beta^{-\gamma}(\alpha+\beta-x)^{\gamma-1}, \quad \text{for}\, \alpha<x<\alpha+\beta, \beta>0, \gamma>0, \\ = 0, \text{ otherwise.} \tag{2.5.3}$$

We will say a rv X ∈ PF(α, β, γ) if its pdf is given by (5.0.1). This is a Pearson's Type I distribution. If γ = 1, then f(x, α, β, γ) as given by (5.3.3) coincides with the uniform distribution in the interval (α, α + β). If we take $Y = (\alpha+\beta)^{\gamma}$, the Y has the uniform distribution in (0, 1). If γ is an integer, then the pdf of X as given in (5.0.1) can be consider as the pdf of ξ, where $\xi = \max(X_1, X_2, \ldots, X_\gamma)$.

2.5.3.1 The Minimum Variance Linear Unbiased Estimate of α and β When γ Is Known and γ ≠ 0

We will consider the following pdf f(x) for X.

$$f(x,\alpha,\beta,\gamma) = \gamma\,\beta^{-\gamma}(\alpha+\beta-x)^{\gamma-1}, \quad \text{for}\, \alpha<x<\alpha+\beta, \beta>0, \gamma>0, \\ = 0, \text{otherwise.}$$

We will say a rv X ∈ PF(α, β, γ) if its pdf is given by (5.0.1). This is a Pearson's Type I distribution. If γ = 1, then f(x, α, β, γ) as given by (5.0.1) coincides with the uniform distribution in the interval (α, α + β). If we take $Y = (\alpha+\beta)^{\gamma}$, the Y has the uniform distribution in (0, 1). If γ is an integer, then the pdf of X as given in (5.0.1) can be consider as the pdf of ξ, where ξ = max ($X_1, X_2, \ldots, X_\gamma$). Let X(1), X(2),…, X(m) be the first m upper records from this distribution. Let

$$W_k = c_k(X(k) - \frac{\gamma}{\gamma+1} X(k-1)), k = 1, 2, .., m$$

with X(0) = 0, and $c_k = (\gamma+1)\left(\frac{\gamma+2}{\gamma}\right)^{k/2}, k = 1, 2, \ldots, m.$
Now

$$E(W_1) = \left(\frac{\gamma+2}{\gamma}\right)^{1/2} \{(\gamma+1)\alpha + \beta\},$$
$$E(W_k) = \left(\frac{\gamma+2}{\gamma}\right)^{k/2} (\alpha + k), k = 1, 2, \ldots, m.$$
$$\mathrm{Var}(\mathrm{W_k}) = \beta^2, \mathrm{k} = 1, 2, \ldots, \mathrm{m}$$
$$\mathrm{Cov}(\mathrm{W_i W_j}) = 0, \mathrm{i} \neq \mathrm{j},\ 1 \leq \mathrm{i},\ \mathrm{j} \leq \mathrm{m}.$$

Let $W' = (W_1, W_2, \cdots, W_n)$, then $E(W) = X\theta$, where

$$X = \begin{bmatrix} (\gamma+2/\gamma)^{1/2}(\gamma+1) & \left((\gamma+2/\gamma)^{1/2}\right) \\ (\gamma+2)/\gamma & (\gamma+2)/\gamma \\ . & . \\ \left((\gamma+2/\gamma)^{n/2}\right) & \left((\gamma+2/\gamma)^{n/2}\right) \end{bmatrix}, \theta = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

We can write X'X as

$$XX' = \begin{pmatrix} (\gamma + 2)^2 + T & \gamma + 2 + T \\ \gamma + 2 + T & T \end{pmatrix}$$
$$\mathrm{T} = \sum_{k=1}^{m} \left(\frac{\gamma+2}{\gamma}\right)^k$$
$$(\mathrm{X'X})^{-1} = D_0^{-1} \begin{pmatrix} T & -(\gamma+2+T) \\ -(\gamma+2+T) & (\gamma+2)^2 + T \end{pmatrix}$$
$$D_0 = (\gamma+2)(\gamma T - \gamma - 2)$$
$$X'W = \begin{pmatrix} \frac{V_1}{V_2} \end{pmatrix}$$
$$V_1 = (\gamma(\gamma+2))^{1/2} W_1 + V_2$$
$$V_2 = \sum_{k=1}^{m} \left(\frac{\gamma+2}{\gamma}\right)^{k/2} W_k$$

Theorem 2.5.3.1 *The minimum variance unbiased estimates of α and β respectively based on Y_1, ..., Y_n(assuming γ as known) are*

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = (X'X)^{-1}X'W$$

On simplification, we get

$$\hat{\alpha} = \frac{1}{D_0}\left[(\gamma(\gamma+2)^{1/2})W_1 - \sum_{k=2}^{n}((\gamma+2)/\gamma)^{k/2}W_k\right]$$
$$\hat{\beta} = \frac{1}{D_0}\left[-(\mathrm{T}+\gamma+2)(\gamma(\gamma+2))^{1/2}W_1 + (\gamma+2)(\gamma+1)\sum_{k=1}^{n}((\gamma+2)/\gamma)^{k/2}W_k\right]$$

The variances and covariance of are given by

$$\mathrm{Var}(\hat{\alpha}) = \beta^2 T D_0^{-1},$$
$$\mathrm{Var}(\hat{\beta}) = \beta^2((\gamma+2)^2 + T)D_0^{-1}$$

and

$$\mathrm{Cov}(\hat{\alpha}, \hat{\beta}) = -\beta^2(\gamma+2+T)D_0^{-1}$$

2.5.3.2 Minimum Variance Invariance Estimators

Theorem 2.5.3.2 *The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter α) estimators* $\tilde{\alpha}$ *and* $\tilde{\beta}$ *of* α *and* β *are respectively*

$$\tilde{\alpha} = \hat{\alpha} - \frac{\gamma+2+T}{(\gamma+1)\{(\gamma+1)T - (\gamma+2)\}}\hat{\beta}$$
$$\text{and } \tilde{\beta} = \frac{D_0}{(\gamma+1)\{(\gamma+1)T - (\gamma+2)\}}\hat{\beta}$$

where

$$D_0 = (\gamma+2)\{\gamma T - (\gamma+2)\}, T = \sum_{i=1}^{m}\left(\frac{\gamma+2}{\gamma}\right)^i.$$

and $\hat{\alpha}$ *and* $\hat{\beta}$ *are MVLUEs of* α *and* β.

Proof The BLIE $\tilde{\alpha}$ *and* $\tilde{\beta}$ *of* α *and* β can be written as

$$\hat{\alpha} = \hat{\alpha} - \frac{E_{12}}{1 + E_{22}}\hat{\beta}.$$

and

$$\tilde{\beta} = \frac{1}{1 + E_{22}}\hat{\beta},$$

where

$$\begin{pmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha}, \hat{\beta}) \\ Cov(\hat{\alpha}, \hat{\beta}) & Var(\hat{\beta}) \end{pmatrix} = \gamma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}.$$

The mean squared errors of $\tilde{\alpha}$ *and* $\tilde{\beta}$ *of* α *and* β are

$$MSE(\tilde{\alpha}) = \gamma^2 \left(E_{11} - \frac{E_{12}^2}{1 + E_{22}} \right),$$
$$MSE(\hat{\beta}) = \gamma^2 \left(\frac{E_{22}}{1 + E_{22}} \right).$$

Substituting the values of E_{11}, E_{12} and E_{22} in terms of γ, we get the results.

2.5.3.3 Maximum Estimator of β for Known μ and σ

Without any loss of generality we will assume $\mu = 0$ and $\sigma = 1$. The log likelihood function can be written as

$$\ln L = m \ln \gamma - \sum_{i=1}^{m} \frac{1}{1 - x(i)} + \gamma \ln(1 - x(m))$$

Differentiating with respect γ and equating to zero, we get $\breve{\gamma}$ as the maximum likelihood estimator of γ as

$$\breve{\gamma} = \frac{m}{\ln(1 - x(m))}$$

A moment Estimator of γ. Taking $\alpha = 0$ and $\beta = 1$, we get $E(X(i)) = \left(\frac{\gamma}{\lambda + 1}\right)^i - 1$ and

$$\mathrm{E(X(1)+X(2)+\ldots+X(m))} = \gamma\left\{\left(\frac{\gamma}{\gamma+2}\right)^m - 1\right\} - m.$$

Thus we can have a moment estimator based on the m record values X(1), X(2), …, X(m) is

$$\ddot{\lambda} = \frac{X(1)+\ldots+X(m)+m}{x(m)}.$$

2.5.4 Rayleigh Distribution

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d random variables from standard Rayleigh distribution with pdf

$$\mathrm{f(x)} = x\mathrm{e}^{-x^2/2}, \mathrm{x} > 0 \tag{2.5.4}$$

and cdf

$$\mathrm{F(x)} = 1 - \mathrm{e}^{-x^2/2}, \mathrm{x} > 0$$

We say X€ RH(0,1) if the pdf of X is given by (2.5.6.1)

Theorem 2.5.4.1 Let

$$\mu_n = E(X_{U(n)}), V_{n,n} = \mathrm{Var}(X_{U(n)}) \text{ and } V_{m,n} = \mathrm{Cov}(X_{U(m)}X_{U(n)}),$$

then

$$\mu_n = \sqrt{2}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)}, V_{n,n} = 2\left[n - \left(\frac{\Gamma(n+1/2)}{\Gamma(n)}\right)^2\right] \text{ and}$$
$$V_{m,n} = 2\left[\frac{\Gamma(m+1/2)}{\Gamma(m)}\right]\left[\frac{\Gamma(n+1)}{\Gamma(n+1/2)} - \left[\frac{\Gamma(n+1/2)}{\Gamma(n)}\right]\right], \text{for } 1 \leq m \leq n.$$

Proof

$$\begin{aligned}
\mu_n &= \frac{1}{\Gamma(n)}\int_0^\infty x\{-\ln(1-F(x))\}^{n-1}f(x)dx \\
&= \frac{1}{\Gamma(n)}\int_0^\infty x\left(\frac{x^2}{2}\right)^{n-1} e^{-x^2/2}xdx \\
&= \frac{1}{\Gamma(n)}\sqrt{2}\int_0^\infty u^{1/2}u^{n-1}\mathrm{e}^{-u}du \\
&= \sqrt{2}\frac{\Gamma(n+1/2)}{\Gamma(n)}.
\end{aligned}$$

Similarly it can be shown that

$$\mu_n^2 = E\left(X_{U(n)}^2\right) = 2\frac{\Gamma(n+1)}{\Gamma(n)} = 2n$$

$$\mu_{\mathrm{m,n}} = \frac{1}{\Gamma(m)\Gamma(n-m)}\int_0^\infty\int_0^y xy\left(\frac{x^2}{2}\right)^{m-1} x\left(\frac{y^2}{2}+\frac{x^2}{2}\right)^{n-m-1} ye^{-y^2/2}dxdy$$

$$= \frac{1}{\Gamma(m)\Gamma(n-m)}\frac{2}{2^{m-1}}\int_0^\infty y\left(\frac{y^2}{2}\right)^{n-m-1} ye^{-y^2/2}I_y\,dy,$$

where

$$\mathrm{I_Y} = \int_0^y (x^2)^m\left(1-\frac{x^2}{y^2}\right)^{n-m-1} dx$$

$$= \frac{1}{2}y^{2m+!}B(m+1/2, n-m),$$

with

$$\mathrm{B(a,b)} = \frac{\Gamma(\mathrm{a})\Gamma(\mathrm{b})}{\Gamma(\mathrm{a+b})}.$$

On simplification we get

$$\mathrm{V_{n,n}} = 2\left[n-\left(\frac{\Gamma(n+1/2)}{\Gamma(n)}\right)^2\right] \text{and}$$

$$\mathrm{V_{m,n}} = \left[\frac{\Gamma(m+1/2)}{\Gamma(m)}\right]\left[\frac{\Gamma(n+1)}{\Gamma(n+1/2)}-\frac{\Gamma(n+1/2)}{\Gamma(n)}\right], \text{for } 1 \le \mathrm{m} \le \mathrm{n}.$$

$$= \left[\frac{\Gamma(m+1/2)}{\Gamma(m)}\right]\left[\frac{\Gamma(n)}{\Gamma(n+1/2)}\right]\mathrm{V_{n,n}}$$

We will consider the estimation of μ and σ based on the observed record values X(1), X(2), …, X(m) of the two parameter Rayleigh distribution with the pdf

$$\mathrm{f(x,\mu,\sigma)} = \frac{x-\mu}{\sigma}\mathrm{e}^{\frac{(x-\mu)^2}{2\sigma^2}}, \mu < x < \infty, \sigma > 0$$

2.5.4.1 Minimum Variance Linear Unbiased Estimators of μ and σ

Theorem 2.5.4.2 *The minimum variance linear unbiased estimators* $\hat{\mu}$ and $\hat{\sigma}$ *of* μ *and* σ *based on the X(1), X(2),..., X(m) are*

$$\hat{\mu} = \sum_{k=1}^{m} c_k X(k), \text{and } \hat{\sigma} \sum_{k=1}^{m} d_k X(k),$$

where

$$c_1 = \frac{3}{2}\frac{\alpha_m b_m}{D}, c_i = \frac{2}{2i}\frac{\alpha_m b_m}{D}, i = 2, 3, \ldots, m-1,$$
$$c_m = 1 - \frac{\alpha_m b_m}{2D}\left[3 + \sum_{i=2}^{m-1}\frac{1}{i}\right], d_1 = \frac{3}{2}\frac{b_m}{D}, d_i = \frac{2}{2i}\frac{b_m}{D}, i = 2, 3, \ldots, m-1,$$
$$d_m = \frac{1}{2}\frac{b_m}{D}\left\{3 + \sum_{i=2}^{m-1}\frac{1}{i}\right\},$$

where

$$\mathrm{D} = \alpha_m b_m T - 1, T = \left[\frac{3}{2} + \sum_{i=2}^{m-1}\frac{1}{2i} + (2m-1)\left(\frac{b_{m-1}}{b_m} - 1\right)\right]$$
$$\alpha_k = \sqrt{2}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k)} = a_k \text{ and } b_k = \sqrt{2}\left\{\frac{\Gamma(k+1)}{\Gamma\left(k+\frac{1}{2}\right)} - \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k)}\right\},$$
$$k=1, 2, \ldots, m.$$

Proof Let R be the m × 1 vector corresponding to X(k), ki = 1, 2, …, m, then we have

$$E(R) = \mu L + \sigma\delta$$

where

$$R' = (X(1), X(2), \ldots, X(m))$$
$$L' = (1, 1, \ldots, 1), \delta' = (\alpha_1, \alpha_1, \ldots, \alpha_m,)$$
$$\alpha_i = \sqrt{2}\frac{\Gamma(i+1/2)}{\Gamma(i)}, \ i = 1, 2, \ldots, m.$$

We can write

$$V(R) = \sigma^2 V, V = (V_{i,j}), V_{i,j} = a_i b_j, 1 < i < j < m \text{ and } V_{i,j} = V_{j,i}.$$

The inverse V^{-1} (= $V^{i,j}$) can be expressed as

$$V^{i+1,i} = V^{i,i+1} = -\frac{1}{a_{i+1}b_i - a_i b_{i+1}} = -(2i+1), i = 1,2,\ldots,m-1,$$
$$V^{i,i} = \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})}, i = 1,2,\ldots,n,$$
$$V^{i,j} = 0, \text{for} |i-j| > 1,$$

where $a_0 = 0 = b_{n+1}$ *and* $b_0 = 1 = a_{n+1}$.

On simplification, we obtain

$$V^{i,i} = \frac{8i^2+1}{2i}, i = 1, 2,\ldots,m-1,$$

and

$$V^{m,m} = (2m-1)\frac{b_{m-1}}{b_m}.$$

The minimum variance linear unbiased estimates (MVLUE) $\hat{\mu}, \hat{\sigma}$ of μ and σ respectively are

$$\hat{\mu} = -\delta' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta$$
$$\hat{\sigma} = L' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta,$$

where

$$\Delta = (L' V^{-1} L)(\delta' V^{-1}\delta) - (L' V^{-1}\delta)^2$$

and

$$Var(\hat{\mu}) = \sigma^2 L' V^{-1}\delta/\Delta,$$
$$Var(\hat{\sigma}) = \sigma^2 L' V^{-1} L/\Delta$$
$$Cov(\hat{\mu}, \hat{\sigma}) = -\sigma^2 L' V^{-1}\delta/\Delta.$$

On simplification, we obtain the MVLUE $\hat{\mu}, \hat{\sigma}$ of μ and σ.

The corresponding variances and the covariance of the estimates are

$$Var(\hat{\mu}) = \sigma^2 \frac{\alpha_n b_n}{D}$$
$$Var(\hat{\sigma}) = \sigma^2 \frac{b_n^2 T}{D}$$
$$Cov(\hat{\mu}, \hat{\sigma}) = -\sigma^2 \frac{b_n}{D}.$$

2.5.4.2 Best Linear Invariant Estimators (BLIEs) of μ and σ

Theorem 2.5.4.3 *The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter μ) estimators (BLIEs) $\tilde{\mu}$ $\tilde{\sigma}$ of μ and σ are*

$$\tilde{\mu} = \hat{\mu} - \hat{\sigma}\left(\frac{E_{12}}{1+E_{22}}\right)$$

and

$$\tilde{\sigma} = \hat{\sigma}/(1+E_{22}),$$

$\hat{\mu}$ *and* $\hat{\sigma}$ *are* MVLUE *of* μ *and* σ *and*

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu},\hat{\sigma}) \\ Cov(\hat{\mu},\hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}.$$

The mean squared errors of these estimators are

$$\text{MSE}(\tilde{\mu}) = \sigma^2\left(E_{11} - E_{12}^2(!1+E_{22})^{-1}\right)$$

and

$$\text{MSE}(\tilde{\sigma}) = \sigma^2 E_{22}(1+E_{22})^{-1}$$

Using the values of E_{11}, E_{12} and E_{22} from (2.3.4), we obtain

$$\tilde{\mu} = \hat{\mu} + \hat{\sigma}\left(\frac{b_m}{D+b_m^2 T}\right)$$

and

$$\tilde{\sigma} = \hat{\sigma}\frac{D}{D + b_m^2 T}.$$

Exercise 2.5.4.1 Show that if μ = 0, then MVLUE of σ based on X(1), X(2), …, X(m) is

$$\hat{\sigma} = cX(m)$$

where

$$c = \frac{\sigma}{E(X(m))} = \frac{1}{\sqrt{2}}\frac{\Gamma(m)}{\Gamma\left(m + \frac{1}{\gamma}\right)}$$

Exercise 2.5.4.2 Show that the minimum varaiance linear unbiased predictor $\hat{X}(s)$ of X(s) based on X(1), X(2), …, X(m), s > m is $\hat{X}(s) = \hat{\mu} + \alpha_{s\hat{\sigma}}$. Where $\hat{\mu}$ and σ are the MVLUEs of μ and σ, Respectively.

2.5.5 Two Parameter Uniform Distribution

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables from a uniform distribution with the following pdf

$$\mathrm{f(x)} = \frac{1}{\theta_1 - \theta_2}, \theta_1 < x < \theta_2 \qquad (2.5.5)$$

and cdf

$$\mathrm{F(x)} = \frac{x - \theta_1}{\theta_2 - \theta_1}, \theta_1 < x < \theta_2.$$

We will say $X \in U(\theta_1, \theta_2)$ if the pdf of X is as given in Eq (2.5.5).The pdf $f_n(x)$ of X(n) can be written as

$$\mathrm{f_n(x)} = \frac{1}{\Gamma(n)}\frac{1}{\theta_2 - \theta_1}\left\{\ln\frac{\theta_2 - \theta_1}{\theta_2 - x}\right\}^{n-1}, \theta_1 < x < \theta_2$$
$$\mathrm{E(X(m))} = 2^{-n}\theta_1 + (1 - 2^{-n})\theta_2$$
$$\mathrm{Var(X(m))} = (3^{-n} - 4^{-n})(\theta_2 - \theta_1)^2.$$

The joint pdf of X(m) and X(n) is

$$f_{m,n}(x,y) = \frac{1}{\Gamma(m)}\frac{1}{\Gamma(n-m)}\frac{1}{\theta_2-\theta_1}\frac{1}{\theta_2-x}\left\{\ln\frac{\theta_2-\theta_1}{\theta_2-x}\right\}^{m-1}\left\{\ln\frac{\theta_2-\theta_1}{\theta_2-y}\right\}^{n-m-1},$$
$$\text{for } \theta_1 < x < y < \theta_2$$

We have

$$E(X(n)||X(m)=y_m) = 2^{m-n}y_m + (1-2^{m-n})\theta_2.$$

and

$$\mathrm{Cov}(X(m)X(n)) = 2^{m-n}\mathrm{Var}\left(X_{U(m)}\right).$$

2.5.6 Minimum Variance Linear Unbiased Estimate of θ_1 and θ_2

We will consider here the estimation of θ_1 and θ_2 based on the observed m upper record values X(1), X(2), …, X(m).Consider the following transformation

$$\begin{aligned} W_1 &= X_{U(1)} \\ W_i &= (3)^{(i-1)/2}\left(X_{U(i)} - \frac{1}{2}X_{U(i-1)}\right), i = 2,3,\ldots\ldots,m \end{aligned} \tag{2.5.6}$$

It can easily be verified that

$$\begin{aligned} E(W_1) &= \frac{\theta_1+\theta_2}{2}, \\ E(W_k) &= \frac{3^{\frac{i-1}{2}}}{2}\theta_2, k = 2,3,\ldots,m. \\ E(W_i) &= \frac{3^{i-1}}{2}\theta_2, i = 2,3,\ldots,m \\ \mathrm{Var}(W_i) &= \frac{\sigma^2}{12}, i = 2,3,\ldots,m \end{aligned}$$

and

$$\mathrm{Cov}(W_i,W_j) = 0, i \neq j.$$

Let $W' = (W_1, W_2, \ldots, W_m)$, then $E(W) = H\,\theta$, where

$$H = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2}\left(3^{\frac{i-1}{2}}\right) \\ & \cdots \\ 0 & \frac{1}{2}(3)^{(m-1)} \end{bmatrix}, \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}.$$

We have

$$(H'H)^{-1} = \frac{32}{3(3^{m-1} - !)} \begin{bmatrix} \frac{3^m-1}{8} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Thus, expressing W's in terms of the X(1), X(2), …, X(m) we obtain

$$\hat{\theta}_1 = 2X(1) - \hat{\theta}_2$$

and

$$\hat{\theta}_1 = \frac{4}{3(3^{m-1} - 1)}\left(3^{m-1}X(m) - \frac{3^{m-2}}{2}x(m-1) - \ldots - \frac{3}{2}X(2) - \frac{3}{2}X(1)\right)$$

The variances covariance of these estimates are

$$Var(\hat{\theta}) = \frac{1}{9}\frac{3^m - 1}{3^{m-1} - 1}(\theta_2 - \theta_1)^2,$$
$$Var(\hat{\theta}_2) = \frac{2}{9}\frac{1}{3^{n-1} - 1}(\theta_2 - \theta_1)^2$$

and

$$Cov(\hat{\theta}_1, \hat{\theta}_2) = \frac{2}{9}\frac{1}{3^{m-1} - 1}(\theta_2 - \theta_1)^2.$$

The generalized variance $\hat{\sum}\left(\hat{\sum} = \text{var}\theta_1.\text{var}\theta_2 - (\text{cov}(\theta_1\theta_2))^2\right)$ is

$$\frac{2}{27}\cdot\frac{1}{3^{n-1} - 1}(\theta_1 - \theta_2)^2.$$

Exercise 2.5.6.1 Suppose X(1), X(2), …, X(m) are m upper record values from a one parameter uniform distribution with pdf f_U(u) as $f_U(u) = \frac{1}{\theta}$, $0 < x < \theta, \theta > 0$. Then the MVLUE $\hat{\theta}$ of θ is

$$\hat{\theta} = \frac{2}{3^n - 1}\left(2.3^{n-1}X(n) - 3^{n-2}X(n-1) - 3^{n-3}X(n-2) - \ldots - X(1)\right)$$

Proof Let X″ = (X(1), X(2), …, X(m)); We have E(X′) = $\delta\theta$
and Var(X) $= \theta^2 V$. V + (V_{ij})
where $\delta' = (\delta_1,\delta_2,\ldots,\delta_m)$, $\delta_i = 1 - \frac{1}{2^i}, i = 1,2,\ldots m$
Vii = $\frac{1}{3^i} - \frac{1}{4^i}, i = 1,2,\ldots,m$ and
Let V = (V_{ij}), then), then

$$V_{ii} = \frac{1}{3^i} - \frac{1}{4^i}, \mathrm{i} = 1,2,\ldots,\mathrm{m}$$
$$V_{ij} = 2^{i-j}\left(\frac{1}{3^i} - \frac{1}{4^i}\right), i<j<m.$$

Let $V^{-1} - (V^{ij})$, **t**hen $V^{ii} = 73^i$, **i = 1,2, .., m–1.**
$V^{mm} = 4.3^m$, $V^{i\,i+1} = -2, 3^{i+1}. = V^{i+1i}$ and $V^{ij} = 0$ for |i−j|.
The MVLUE $\hat{\sigma}$ of σ is

$$\hat{\sigma} = \frac{\delta^{/}V^{-1}X}{\delta^{/}V^{-1}\delta}$$
$$= \frac{2}{3^m - 1}\left(2.3^{m-1}X(m) - 3^{m-2}X(m-1).,,,, - -X(1)\right)$$
$$Var(\hat{\sigma}) = \frac{2\sigma^2}{3(3^n - 1)}.$$

2.5.7 One Parameter Uniform Distribution

Suppose γ = 1 and α = 0, i.e. when X is distributed uniformly in the interval (0, β), We have in this case the pdf $f_n(x)$ of X(n) as

$$f_n(x) = \frac{1}{\Gamma(n)}\left[\ln\frac{\beta}{x}\right]^{n-1}, 0<x<\beta. \tag{2.5.7}$$

Using (2.5.2.1), we obtain

$$\mathrm{E(X(n))} = (1 - 2^{-n})\beta.$$
$$\mathrm{Var(X(n))} = (3^{-n} - 4^{-n})\beta^2$$

The joint pdf of X(m) and X(n), n > m is

$$f_{m,n}(x,y)=\frac{1}{\Gamma(m)}\frac{1}{\Gamma(n-m)}\frac{1}{\beta}\frac{1}{\beta-x}\left[\ln\frac{\beta}{\beta-x}\right]^{m-1}\left[\ln\frac{\beta}{\beta-y}\right]^{n-m-1},$$

$$\mathrm{n}>\mathrm{m}>0, 0<\mathrm{x}<\mathrm{y}<\beta.$$

It follows from (2.2.6) that

$$\mathrm{E(X(n)|X(m)=x_m)=2^{m-n}x_m+(1-2^{m-n})\beta}.$$

and

$$\mathrm{Cov(X(n)X(m))=2^{m-n}Var(X(m)), m<n,\ 1<m<n}$$

The correlation coefficient $\rho_{m,n}$ of X(m) and X(n)–s

$$\rho_{m,n}=\left(\left(\frac{4}{3}\right)^m-1\right)^{\frac{1}{2}}\left(\left(\frac{4}{3}\right)^n-1\right)^{\frac{1}{2}}, m<n$$

2.5.7.1 Minimum Variance Unbiased Estimator of β

Using the following transformation

$$\begin{aligned}
\mathrm{W_1}&=\mathrm{X(1)}\\
W_i&=3^{\frac{i-1}{2}}\left(X(i)-\frac{1}{2}(X-i)\right), i=2,\ldots,n\\
\mathrm{E(W_i)}&=\mathrm{(1/2)(3)^{(i-1)/2}\beta}\\
\mathrm{Var(W_i)}&=\frac{\beta^2}{12},\\
\mathrm{Cov(W_i,W_j)}&=\mathrm{0,\ i\neq j,\ i,j=1,2,\ldots,n.}
\end{aligned}$$

Let

$$\mathrm{X}'=\left(\frac{1}{2},\frac{1}{2}(3)^{1/2},\frac{1}{2}(3),\ldots,\frac{1}{2}(3)^{n-1}\right)$$

and

$$W' = (W_1, W_2, \ldots, Wn),$$

then minimum variance linear unbiased estimator $\hat{\beta}$ of β based on the first n record values is

$$\begin{aligned}\hat{\beta} &= (X'X) - 1X'W \\ &= \frac{4}{3^n - 1}\left(\sum_{i=1}^{n} (3)^{(i-1)/2} W_i\right) \\ &= \frac{4}{3^n - 1}\left(3^{n-1}X(n) - \frac{3^{n-2}}{2}X(n-1) - \frac{3^{n-3}}{2}X(n-2) - \ldots - \frac{1}{2}X(1)\right)\end{aligned}$$

Since $X'X = \frac{3^n - 1}{8}$ and $\mathrm{Var}(W_i) = \frac{\beta^2}{12}$, we have

$$\begin{aligned}\mathrm{var}(\hat{\beta}) &= (X'X)^{-1}\frac{\beta^2}{12} \\ &= \frac{2\beta^2}{3(3^n - 1)}\end{aligned}$$

2.5.7.2 Minimum Mean Square Estimate of β

If we drop the condition of unbiasedness, then the estimator $\tilde{\beta}$, where

$$\tilde{\beta} = \frac{3(3^n - 1)}{3^{n+1} - 1}\hat{\beta}$$

has minimum mean squared error.

$$\text{Bias of } \tilde{\beta} = E(\tilde{\beta}) - \beta = \frac{2}{3^{n+1} - 1}\beta$$

and

$$\mathrm{MSE}(\hat{\beta}) = \frac{2\beta^2}{3^{n+1} - 1}$$

Exercise 2.5.7.1 Find the maximum likelihood estimate of β.

2.5.8 Prediction of Record Values

Writing

$$\mathrm{Y}_{\mathrm{n+s}} = Y_{n+s} \\ -\frac{1}{2}Y_{n+s-1} + \frac{1}{2}(Y_{n+s-2}) + \cdots + \frac{1}{2^{n+s-2}}\left(Y_2 - \frac{1}{2}Y_1\right) + \frac{1}{2^{n+s-1}}Y_1,$$

it can be shown that

$$\mathrm{Cov}(\mathrm{Y}_{\mathrm{n+s}}, \mathrm{W}_{\mathrm{i}}) = , \mathrm{i} = 1,2,\ldots,\mathrm{n}.$$

It can be shown that the best linear unbiased predictor (BLUP) of $\mathrm{Y}_{\mathrm{n+s}}$ is $\hat{Y}_{n+s}$, where

$$\hat{Y}_{n+s} = \left(1 - \frac{1}{2^{n+s}}\right)\beta + c'V^{-1}\left(W - X\hat{\beta}\right)$$

where

$$\mathrm{c}' = (c_1,c_2,\ldots,c_n), \mathrm{V}^{-1} = (\mathrm{X}'\mathrm{X})^{-1} \text{ and } \mathrm{c_i Var(W_i)} = \mathrm{Cov}(\mathrm{Y}_{\mathrm{n+s}}, \mathrm{W}_{\mathrm{i}}), \mathrm{s} > 1.$$

Thus

$$\hat{Y}_{n+s} = \left(1 - \frac{1}{2^{n+s}}\right)\hat{\beta} + \frac{8}{3^n - 1}\left[\sum_{i=1}^{n} - \frac{1}{2^{n+s-i}} \cdot \frac{W}{3^{(i-1)/2}} - \frac{\hat{\beta}}{2^s}\left(1 - \frac{1}{2^n}\right)\right]$$

The best linear (unrestricted) least square predictor of $\mathrm{Y}_{\mathrm{n+s}}$ is $\widetilde{Y}_{r+s}$, where

$$\overleftrightarrow{Y}_{r+s} = E(Y_{n+s}|Y_1,Y_2,\cdots Y_n) \\ = \frac{y_n}{2^s} + \left(1 - \frac{1}{2^s}\right)\beta,$$

Substituting $\hat{\beta}$ for β, we get the best linear least squares predictor as

$$\frac{y_n}{2^s} + \left(1 - \frac{1}{2^s}\right) \cdot \frac{4}{3^n - 1}\left(3^{n-1}y_n - \frac{1}{3}(3)^{n-2}y_{n-1} - \ldots - \frac{1}{2}y_1\right).$$

2.6 Weibull Distribution

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d random variables from standard Weibull distribution with pdf

$$f(x) = x^{\gamma-1}e^{-x^{\gamma}}, x > 0, \gamma > 0, \tag{2.6.1}$$

and cdf

$$F(x) = 1 - e^{-\frac{1}{\gamma}x^{\gamma}}, x > 0, g > 0,$$

Let μn = E(X(n), $V_{n,n}$ = Var(X(n)) and V_{mn} = Cov(X(m)X(n)), m < n, then

$$\mu_n = \gamma^{1/\gamma}\frac{\Gamma\left(n+\frac{1}{\gamma}\right)}{\Gamma(n)}, V_{n,n} = \gamma^{2/\gamma}\left\{\frac{\Gamma\left(n+\frac{2}{\gamma}\right)}{\Gamma\left(n+\frac{1}{\gamma}\right)} - \left(\frac{\sqrt{\left(n+\frac{1}{2}\right)}}{\sqrt{(n)}}\right)^2\right\}.$$

and

$$V_{m,n} = \frac{\Gamma\left(m+\frac{1}{\gamma}\right)}{\Gamma(m)}\gamma^{2/\gamma}\left\{\frac{\Gamma\left(n+\frac{2}{\gamma}\right)}{\Gamma\left(n+\frac{1}{\gamma}\right)} - \frac{\Gamma\left(n+\frac{1}{\gamma}\right)}{\Gamma(n)}\right\}, \text{for } 1 < m < n.$$

We will consider the following pdf $f(x, \mu, \sigma)$, for Weibull distribution,

$$f(x, \mu, \sigma) = \frac{(x-\mu)^{\gamma-1}}{\sigma^{\gamma}}e^{-\frac{1}{\gamma}\left(\frac{x-\mu}{\sigma}\right)^{\gamma}} \quad -\infty < \mu < x < \infty, \sigma > 0.$$

2.6.1 Minimum Variance Linear Unbiased Estimators of μ and σ

Theorem 2.6.1 *The minimum variance linear unbiased estimators* $\hat{\mu}$ and $\hat{\sigma}$ *of* μ *and* σ *based on the record values X(1),X(2,,..,X(n)are*

$$\hat{\mu} = \sum_{k=1}^{m} c_k X(k), \text{ and } \hat{\sigma} = \sum_{k=1}^{m} d_k X(k),$$

where

$$c_1 = \frac{\alpha_m b_m}{D}\frac{(\gamma+1)\gamma^{-2/\gamma}}{\Gamma\left(1+\frac{1}{\gamma}\right)}, c_i = \frac{\alpha_m b_m}{D}\gamma^{-2/\gamma}(\gamma-!)\frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)}, i = 2, 3, \ldots, m-1,$$

$$c_m = 1 - \frac{\alpha_m b_m}{D}\gamma^{-2/\gamma}\left[\frac{\gamma+1}{\Gamma\left(1+\frac{2}{\gamma}\right)} + (\gamma-1)\sum_{i=2}^{m-1}\frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)}\right],$$

$$d_1 = -\frac{b_m(\gamma+1)\gamma^{-2/\gamma}}{D},$$

$$d_i = -\frac{b_m}{D}(\gamma-1)\gamma^{-2/\gamma}\frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)}, i = 2, 3, \ldots, m-1,$$

$$d_m = \frac{b_m}{D}\gamma^{-\frac{2}{\gamma}}\left[\frac{\gamma+1}{\Gamma\left(1+\frac{2}{\gamma}\right)} + (\gamma-1)\sum_{i=2}^{m-1}\frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)}\right],$$

where

$$\mathrm{D} = \alpha_\mathrm{m}\mathrm{b}_\mathrm{m}\mathrm{T} - 1,$$

$$\mathrm{T} = \gamma^{-2/\gamma}\left[\frac{\gamma+1}{\Gamma\left(1+\frac{2}{\gamma}\right)} + (\gamma-1)\sum_{i=2}^{m-1}\frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)} + \frac{\Gamma(m)}{\Gamma\left(m+\frac{2}{\gamma}\right)}(m\gamma-\gamma-1)(m\gamma-\gamma+2)\left(\frac{b_{m-1}}{b_m}-1\right)\right]$$

$$\alpha_\mathrm{m} = \gamma^{1/\gamma}\frac{\left(m+\frac{1}{\gamma}\right)}{(m)} \text{ and } \mathrm{b}_\mathrm{m} = \gamma^{1/\gamma}\left\{\frac{\left(n+\frac{2}{\gamma}\right)}{\left(n+\frac{1}{\gamma}\right)} - \frac{\left(n+\frac{1}{\gamma}\right)}{\Gamma(n)}\right\}$$

We can write

$$\mathrm{V(R)} = \sigma^2\mathrm{V}, \mathrm{V} = (\mathrm{V}_{\mathrm{i,j}}), \mathrm{V}_{\mathrm{i,j}} = \mathrm{a_i b_j}, 1 < \mathrm{i} < \mathrm{j} < \mathrm{m} \text{ and } \mathrm{V}_{\mathrm{i,j}} = \mathrm{V}_{\mathrm{j,i}}.$$

The inverse V^{-1} (= $V^{i,j}$) can be expressed as

$$\mathrm{V}^{\mathrm{i+1,i}} = \mathrm{V}^{\mathrm{i,i+1}} = -\frac{1}{a_{i+1}b_i - a_i b_{i+1}} = -(2\mathrm{i}+1), \mathrm{i} = 1,2,\ldots,\mathrm{m}-1,$$

$$\mathrm{V}^{\mathrm{i,i}} = \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})}, i = 1,2,\ldots,n,$$

$$\mathrm{V}^{\mathrm{i,j}} = 0, \text{for} |\mathrm{i}-\mathrm{j}| > 1,$$

where $a_0 = 0 = b_{n+1}$ *and* $b_0 = 1 = a_{n+1}$. On simplification, we obtain

$$V^{i,i} = \gamma^{-2/\gamma} \frac{\Gamma(i)}{\Gamma\left(i+\frac{1}{\gamma}\right)} \left[\gamma^2(2i^2 - 2i + 1) + \gamma(4i+2) + 1\right], i = 1,2,\ldots,m-1,$$

$$V^{m,m} = \gamma^{-2/\gamma} \frac{\Gamma(n)}{\Gamma\left(n+\frac{2}{\gamma}\right)} \frac{b_{n-1}}{b_n} [(n\gamma - \gamma + 1)(n\gamma - \gamma + 2)].$$

The minimum variance linear unbiased estimates (MVLUE) $\hat{\mu}, \hat{\sigma}$ of μ and σ respectively are

$$\hat{\mu} = -\delta' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta$$
$$\hat{\sigma} = L' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta,$$

where

$$\Delta = (L'V^{-1}L)(\delta' V^{-1}\delta) - (L'V^{-1}\delta)^2,$$
$$X' = (\ X(1),X(2),\ldots.X(n))$$

and

$$Var(\hat{\mu}) = \sigma^2 L' V^{-1}\delta/\Delta,$$
$$Var(\hat{\sigma}) = \sigma^2 L' V^{-1} L/\Delta$$
$$Cov(\hat{\mu},\hat{\sigma}) = -\sigma^2 L' V^{-1}\delta/\Delta.$$

On simplification, we obtain the MVLUEs $\hat{\mu}, \hat{\sigma}$ of μ and σ. The corresponding variances and the covariance of the estimates are

$$Var(\hat{\mu}) = \sigma^2 \frac{\alpha_n b_n}{D}$$
$$Var(\hat{\sigma}) = \sigma^2 \frac{b_n^2 T}{D}$$
$$Cov(\hat{\mu}, \hat{\sigma}) = -\sigma^2 \frac{b_n}{D}.$$

Best Linear Invariant Estimators (BLIEs) of μ and σ.

Theorem 2.6.2 *The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter μ) estimators (BLIEs) $\tilde{\mu}, \tilde{\sigma}$ of μ and σ are*

$$\tilde{\mu} = \hat{\mu} - \hat{\sigma}\left(\frac{E_{12}}{1+E_{22}}\right)$$

and

$$\tilde{\sigma} = \hat{\sigma}/(1 + E_{22}),$$

where $\hat{\mu}$ *and* $\hat{\sigma}$ *are* MVLUE of μ *and* σ *and*

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu}, \hat{\sigma}) \\ Cov(\hat{\mu}, \hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}$$

The mean squared errors of these estimators are

$$\text{MSE}(\tilde{\mu}) = \sigma^2 \left(E_{11} - E_{12}^2 (1 + E_{22})^{-1} \right)$$

and

$$\text{MSE}(\tilde{\sigma}) = \sigma^2 E_{22} (1 + E_{22})^{-1}.$$

Using the values of E_{11}, E_{12} and E_{22} from (2.62.4), we obtain

$$\tilde{\mu} = \hat{\mu} + \hat{\sigma} \left(\frac{b_m}{D + b_m^2 T} \right)$$

and

$$\tilde{\sigma} = \hat{\sigma} \frac{D}{D + b_m^2 T}.$$

Exercise 2.6.2 Show that if μ = 0, then MVLUE estimator of σ based on the record values X(1),X(2),…,X(m) for known v is

$$\bar{\sigma} = c_0 X(m),$$

Springer

http://www.springer.com/978-94-6239-221-2

Extreme Value Distributions
Ahsanullah, M.
2016, VIII, 137 p. 11 illus., 10 illus. in color., Hardcover
ISBN: 978-94-6239-221-2
A product of Atlantis Press