# Chapter 2
# Thurston Maps

In this chapter, we introduce the dynamical systems that we are going to study, namely, expanding Thurston maps. We first recall briefly some history in Sect. 2.1, where we by no means intend to give a complete account of the development of the subject. We then introduce Thurston maps in Sect. 2.2 and certain cell decompositions of the 2-sphere $S^2$ induced by Thurston maps in Sect. 2.3. Next, we discuss various notions of expansion in our context and define expanding Thurston maps in Sect. 2.4. Two most important tools in the study of expanding Thurston maps are explored in the last two sections. The first tool is a natural class of metrics on the $S^2$, called visual metrics, discussed in Sect. 2.5. The second tool, discussed in Sect. 2.6, is the existence and properties of certain forward invariant Jordan curves on $S^2$, which induce nice partitions of the sphere. It is the geometric and combinatorial information we get from these tools that enables us to investigate the dynamical properties of expanding Thurston maps.

We prove in Lemma 2.12 that the union of all iterated preimages of an arbitrary point $p \in S^2$ of an expanding Thurston map is dense in $S^2$. We also summarize properties of visual metrics from [BM17], especially the relation between visual metrics and the cell decompositions, in Lemma 2.13 and the discussion that follows it. The fact that an expanding Thurston map is Lipschitz with respect to a visual metric is established in Lemma 2.15. M. Bonk and D. Meyer proved that for each expanding Thurston map $f$, there exists an $f^n$-invariant Jordan curve containing post $f$ for each sufficiently large $n \in \mathbb{N}$ depending on $f$ (see Theorem 2.16). We prove in Lemma 2.17 a slightly stronger version of this result, which carries additional combinatorial information of the Jordan curve. This lemma will be used in Chaps. 4 and 6. Finally, in Lemma 2.19, we prove that an expanding Thurston map locally expands the distance, with respect to a visual metric, between two points exponentially as long as they belong to one set in some particular partition of $S^2$ induced by a backward iteration of some Jordan curve on $S^2$. This observation, generalizing a result of M. Bonk and D. Meyer [BM17, Lemma 15.25], enables us to establish the distortion lemmas (Lemmas 5.3 and 5.4) in Sect. 5.2, which serve as the cornerstones for the mechanism of thermodynamical formalism that is essential in Chap. 5.



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## 2.1 Historical Background

The study of Thurston maps dates back to W.P. Thurston's celebrated combinatorial characterization theorem of postcritically-finite rational maps on the Riemann sphere among a class of more general continuous maps [DH93]. We call this class of continuous maps *Thurston maps* nowadays. Thurston's theorem asserts that a Thurston map is essentially a rational map if and only if there does not exist a so-called *Thurston obstruction*, i.e., a collection of simple closed curves on $S^2$ subject to certain conditions [DH93]. Due to the important and fruitful applications of Thurston's theorem, many authors have worked on extending it beyond postcritically-finite rational maps using similar combinatorial obstructions. See for example, J.H. Hubbard, D. Schleicher, M. Shishikura's work on some postcritically-finite exponential maps [HSS09]; G. Cui and L. Tan's and G. Zhang and Y. Jiang's works on hyperbolic rational maps [CT11, ZJ09]; G. Zhang's work on certain rational maps with Siegel disks [Zh08]; X. Wang's work on certain rational maps with Herman rings [Wan14]; and G. Cui and L. Tan's work on some geometrically finite rational maps [CT15].

It has since been a central theme in the study of conformal dynamical systems to search for Thurston-type theorems, i.e., characerizations of conformal dynamical systems in a wider class of dynamical systems satisfying suitable metric-topological conditions. See also [Th16, KPT15] for some remarkable recent works in this direction.

It is natural to ask for Thurston-type theorems from different points of view. One promising approach is from a point of view of metric space properties. In order to gain more precise metric estimates, groups of authors, notably M. Bonk and D. Meyer [BM17], P. Haïssinsky and K. Pilgrim [HP09], and J.W. Cannon, W.J. Floyd, and R. Parry [CFP07] started to impose natural notions of expansion in their respective contexts. These notions turned out to coincide in the context of expanding Thurston maps (see Sect. 2.4 for more details).

The existence of certain invariant Jordan curves as stated in Theorem 2.16 serves as foundation and starting point of the investigation of expanding Thurston maps. The special case of Theorem 2.16 for rational expanding Thurston maps was announced by M. Bonk during an Invited Address at the AMS Meeting at Athens, Ohio, in March 2004, where W.J. Floyd also mentioned a related result independently obtained by J.W. Cannon, W.J. Floyd, and R. Parry [CFP07]. Finally a Thurston-type theorem from a metric space point of view was obtained independently by M. Bonk and D. Meyer [BM17], and P. Haïssinsky and K. Pilgrim [HP09] in their respective contexts. See Theorem 1.1 in the case of expanding Thurston maps. Special cases of Theorem 1.1 go back to [Me02] and unpublished joint work of M. Bonk and B. Kleiner (see [BM17, Preface]).

## 2.2 Definition for Thurston Maps

Let $S^2$ denote an oriented topological 2-sphere. We first define branched covering maps on $S^2$. For detailed treatment of branched covering maps, we refer the reader to [BM17, Appendix 6].

A continuous map $f: S^2 \to S^2$ is called a *branched covering map* on $S^2$ if for each point $x \in S^2$, there exist a positive integer $d \in \mathbb{N}$, open neighborhoods $U$ of $x$ and $V$ of $y = f(x)$, open neighborhoods $U'$ and $V'$ of $0$ in $\widehat{\mathbb{C}}$, and orientation-preserving homeomorphisms $\varphi: U \to U'$ and $\eta: V \to V'$ such that $\varphi(x) = 0$, $\eta(y) = 0$, and

$$(\eta \circ f \circ \varphi^{-1})(z) = z^d$$

for each $z \in U'$. The above relation can be seen from the following diagram:

$$\begin{array}{ccc} x \in U & \xrightarrow{\ f\ } & y \in V \\ \varphi \big\downarrow & & \big\downarrow \eta \\ 0 \in U' & \xrightarrow[z \mapsto z^d]{} & 0 \in V'. \end{array}$$

The positive integer $d$ above is uniquely determined by $f$ and $x$, and is called the *local degree* of $f$ at $x$, denoted by $\deg_f(x)$. The *(topological) degree* $\deg f$ of $f$ can be calculated by

$$\deg f = \sum_{x \in f^{-1}(y)} \deg_f(x) \tag{2.1}$$

for $y \in S^2$ and is independent of $y$. If $f: S^2 \to S^2$ and $g: S^2 \to S^2$ are two branched covering maps on $S^2$, then so is $f \circ g$ (see [BM17, Lemma A.14]), and

$$\deg_{f \circ g}(x) = \deg_g(x) \deg_f(g(x)), \qquad \text{for each } x \in S^2, \tag{2.2}$$

and moreover,

$$\deg(f \circ g) = (\deg f)(\deg g). \tag{2.3}$$

A point $x \in S^2$ is a *critical point* of $f$ if $\deg_f(x) \geq 2$. The set of critical points of $f$ is denoted by $\operatorname{crit} f$. A point $y \in S^2$ is a *postcritical point* of $f$ if $y = f^n(x)$ for some $x \in \operatorname{crit} f$ and $n \in \mathbb{N}$. The set of postcritical points of $f$ is denoted by $\operatorname{post} f$. Note that $\operatorname{post} f = \operatorname{post} f^n$ for all $n \in \mathbb{N}$.

**Definition 2.1** (*Thurston maps*) A Thurston map is a branched covering map $f: S^2 \to S^2$ on $S^2$ with $\deg f \geq 2$ and $\operatorname{card}(\operatorname{post} f) < +\infty$.

*Example 2.2* We take two congruent Euclidean equilateral triangles $\Delta ABC$ and $\Delta A'B'C'$, and then paste them along the boundary with $A$ and $A'$, $B$ and $B'$, and $C$

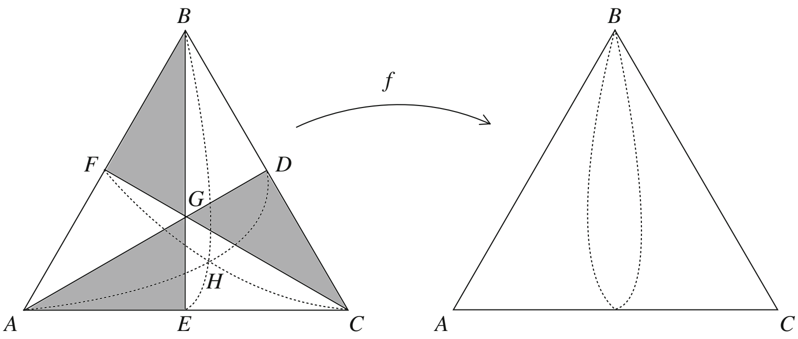


**Fig. 2.1** A Thurston map from the barycentric subdivisions

and $C'$ identified as shown in Fig. 2.1. We define a piecewise linear map $f: S^2 \to S^2$ by using the barycentric subdivision of each triangle. Firstly, the triangle $\Delta AGE$ is mapped linearly to the triangle $\Delta ABC$ on the front of the sphere $S^2$ with $f(A) = A$, $f(E) = C$, and $f(G) = B$. We can then define $f$ on $\Delta EGC$ in such a way that $f$ maps $\Delta EGC$ linearly to the triangle $\Delta C'B'A'$ on the back of the sphere with $f(E) = C' = C$, $f(G) = B' = B$, and $f(C) = A' = A$. Similarly, we can extend $f$ continuously to the whole sphere. We shade the preimages of the triangle $\Delta ABC$ on the front of the sphere that are facing us as shown in Fig. 2.1.

It is easy to see that $\operatorname{crit} f = \{A, B, C, D, E, F, G, H\}$, and $\operatorname{post} f = \{A, B, C\}$. It is then clear that $f$ is a Thurston map. We leave it as an exercise to show that $f$ is an expanding Thurston map.

## 2.3 Cell Decompositions

We now recall the notion of cell decompositions of $S^2$. A *cell of dimension n* in $S^2$, $n \in \{1, 2\}$, is a subset $c \subseteq S^2$ that is homeomorphic to the closed unit ball $\overline{\mathbb{B}^n}$ in $\mathbb{R}^n$. We define the *boundary of* $c$, denoted by $\partial c$, to be the set of points corresponding to $\partial \mathbb{B}^n$ under such a homeomorphism between $c$ and $\overline{\mathbb{B}^n}$. The *interior of* $c$ is defined to be $\operatorname{inte}(c) = c \setminus \partial c$. For each point $x \in S^2$, the set $\{x\}$ is considered a *cell of dimension* 0 in $S^2$. For a cell $c$ of dimension 0, we adopt the convention that $\partial c = \emptyset$ and $\operatorname{inte}(c) = c$.

We record the following three definitions from [BM17].

**Definition 2.3** (*Cell decompositions*) Let $\mathbf{D}$ be a collection of cells in $S^2$. We say that $\mathbf{D}$ is a *cell decomposition of* $S^2$ if the following conditions are satisfied:

(i) The union of all cells in $\mathbf{D}$ is equal to $S^2$.
(ii) If $c \in \mathbf{D}$, then $\partial c$ is a union of cells in $\mathbf{D}$.

(iii) For $c_1, c_2 \in \mathbf{D}$ with $c_1 \neq c_1$, we have $\mathrm{inte}(c_1) \cap \mathrm{inte}(c_2) = \emptyset$.
(iv) Every point in $S^2$ has a neighborhood that meets only finitely many cells in $\mathbf{D}$.

**Definition 2.4** (*Refinements*) Let $\mathbf{D}'$ and $\mathbf{D}$ be two cell decompositions of $S^2$. We say that $\mathbf{D}'$ is a *refinement* of $\mathbf{D}$ if the following conditions are satisfied:

(i) Every cell $c \in \mathbf{D}$ is the union of all cells $c' \in \mathbf{D}'$ with $c' \subseteq c$.
(ii) For every cell $c' \in \mathbf{D}'$ there exits a cell $c \in \mathbf{D}$ with $c' \subseteq c$.

**Definition 2.5** (*Cellular maps and cellular Markov partitions*) Let $\mathbf{D}'$ and $\mathbf{D}$ be two cell decompositions of $S^2$. We say that a continuous map $f : S^2 \to S^2$ is *cellular* for $(\mathbf{D}', \mathbf{D})$ if for every cell $c \in \mathbf{D}'$, the restriction $f|_c$ of $f$ to $c$ is a homeomorphism of $c$ onto a cell in $\mathbf{D}$. We say that $(\mathbf{D}', \mathbf{D})$ is a *cellular Markov partition* for $f$ if $f$ is cellular for $(\mathbf{D}', \mathbf{D})$ and $\mathbf{D}'$ is a refinement of $\mathbf{D}$.

Let $f : S^2 \to S^2$ be a Thurston map, and $\mathscr{C} \subseteq S^2$ be a Jordan curve containing $\mathrm{post}\, f$. Then the pair $f$ and $\mathscr{C}$ induces natural cell decompositions $\mathbf{D}^n(f, \mathscr{C})$ of $S^2$, for $n \in \mathbb{N}_0$, in the following way:

By the Jordan curve theorem, the set $S^2 \setminus \mathscr{C}$ has two connected components. We call the closure of one of them the *white* 0-*tile* for $(f, \mathscr{C})$, denoted by $X^0_w$, and the closure of the other one the *black* 0-*tile* for $(f, \mathscr{C})$, denoted by $X^0_b$. The set of 0-*tiles* is $\mathbf{X}^0(f, \mathscr{C}) = \{X^0_b, X^0_w\}$. The set of 0-*vertices* is $\mathbf{V}^0(f, \mathscr{C}) = \mathrm{post}\, f$. We set $\overline{\mathbf{V}}^0(f, \mathscr{C}) = \{\{x\} \,|\, x \in \mathbf{V}^0(f, \mathscr{C})\}$. The set of 0-*edges* $\mathbf{E}^0(f, \mathscr{C})$ is the set of the closures of the connected components of $\mathscr{C} \setminus \mathrm{post}\, f$. Then we get a cell decomposition

$$\mathbf{D}^0(f, \mathscr{C}) = \mathbf{X}^0(f, \mathscr{C}) \cup \mathbf{E}^0(f, \mathscr{C}) \cup \overline{\mathbf{V}}^0(f, \mathscr{C})$$

of $S^2$ consisting of *cells of level* 0, or 0-*cells*.

We can recursively define unique cell decompositions $\mathbf{D}^n(f, \mathscr{C})$, $n \in \mathbb{N}$, consisting of $n$-*cells* such that $f$ is cellular for $(\mathbf{D}^{n+1}(f, \mathscr{C}), \mathbf{D}^n(f, \mathscr{C}))$. We refer to [BM17, Lemma 5.15] for more details. We denote by $\mathbf{X}^n(f, \mathscr{C})$ the set of $n$-cells of dimension 2, called $n$-*tiles*; by $\mathbf{E}^n(f, \mathscr{C})$ the set of $n$-cells of dimension 1, called $n$-*edges*; by $\overline{\mathbf{V}}^n(f, \mathscr{C})$ the set of $n$-cells of dimension 0; and by $\mathbf{V}^n(f, \mathscr{C})$ the set $\big\{x \,\big|\, \{x\} \in \overline{\mathbf{V}}^n(f, \mathscr{C})\big\}$, called the set of $n$-*vertices*. The $k$-*skeleton*, for $k \in \{0, 1, 2\}$, of $\mathbf{D}^n(f, \mathscr{C})$ is the union of all $n$-cells of dimension $k$ in this cell decomposition.

We record Proposition 5.17 of [BM17] here in order to summarize properties of the cell decompositions $\mathbf{D}^n(f, \mathscr{C})$ defined above.

**Proposition 2.6** (M. Bonk & D. Meyer) *Let $k, n \in \mathbb{N}_0$, let $f : S^2 \to S^2$ be a Thurston map, $\mathscr{C} \subseteq S^2$ be a Jordan curve with* $\mathrm{post}\, f \subseteq \mathscr{C}$, *and* $m = \mathrm{card}(\mathrm{post}\, f)$.

(i) *The map $f^k$ is cellular for* $(\mathbf{D}^{n+k}(f, \mathscr{C}), \mathbf{D}^n(f, \mathscr{C}))$. *In particular, if $c$ is any $(n+k)$-cell, then $f^k(c)$ is an $n$-cell, and $f^k|_c$ is a homeomorphism of $c$ onto $f^k(c)$.*
(ii) *Let $c$ be an $n$-cell. Then $f^{-k}(c)$ is equal to the union of all $(n+k)$-cells $c'$ with $f^k(c') = c$.*

(iii) *The* 1*-skeleton of* $\mathbf{D}^n(f, \mathcal{C})$ *is equal to* $f^{-n}(\mathcal{C})$. *The* 0*-skeleton of* $\mathbf{D}^n(f, \mathcal{C})$ *is the set* $\mathbf{V}^n(f, \mathcal{C}) = f^{-n}(\mathrm{post}\, f)$*, and we have* $\mathbf{V}^n(f, \mathcal{C}) \subseteq \mathbf{V}^{n+k}(f, \mathcal{C})$.

(iv) $\mathrm{card}(\mathbf{X}^n(f, \mathcal{C})) = 2(\deg f)^n$, $\mathrm{card}(\mathbf{E}^n(f, \mathcal{C})) = m(\deg f)^n$, *and* $\mathrm{card}(\mathbf{V}^n(f, \mathcal{C})) \leq m(\deg f)^n$.

(v) *The* $n$*-edges are precisely the closures of the connected components of* $f^{-n}(\mathcal{C}) \setminus f^{-n}(\mathrm{post}\, f)$. *The* $n$*-tiles are precisely the closures of the connected components of* $S^2 \setminus f^{-n}(\mathcal{C})$.

(vi) *Every* $n$*-tile is an* $m$*-gon, i.e., the number of* $n$*-edges and the number of* $n$*-vertices contained in its boundary are equal to* $m$.

We also note that for each $n$-edge $e \in \mathbf{E}^n(f, \mathcal{C})$, $n \in \mathbb{N}_0$, there exist exactly two $n$-tiles $X, X' \in \mathbf{X}^n(f, \mathcal{C})$ such that $X \cap X' = e$.

For $n \in \mathbb{N}_0$, we define *the set of black* $n$*-tiles* as

$$\mathbf{X}_b^n(f, \mathcal{C}) = \{X \in \mathbf{X}^n(f, \mathcal{C}) \mid f^n(X) = X_b^0\},$$

and the *set of white* $n$*-tiles* as

$$\mathbf{X}_w^n(f, \mathcal{C}) = \{X \in \mathbf{X}^n(f, \mathcal{C}) \mid f^n(X) = X_w^0\}.$$

It follows immediately from Proposition 2.6 that

$$\mathrm{card}\left(\mathbf{X}_b^n(f, \mathcal{C})\right) = \mathrm{card}\left(\mathbf{X}_w^n(f, \mathcal{C})\right) = (\deg f)^n \tag{2.4}$$

for each $n \in \mathbb{N}_0$. Moreover, for $n \in \mathbb{N}$, we define the *set of black* $n$*-tiles contained in a white* $(n-1)$*-tile* as

$$\mathbf{X}_{bw}^n(f, \mathcal{C}) = \{X \in \mathbf{X}_b^n(f, \mathcal{C}) \mid \exists X' \in \mathbf{X}_w^{n-1}(f, \mathcal{C}),\ X \subseteq X'\},$$

the *set of black* $n$*-tiles contained in a black* $(n-1)$*-tile* as

$$\mathbf{X}_{bb}^n(f, \mathcal{C}) = \{X \in \mathbf{X}_b^n(f, \mathcal{C}) \mid \exists X' \in \mathbf{X}_b^{n-1}(f, \mathcal{C}),\ X \subseteq X'\},$$

the *set of white* $n$*-tiles contained in a black* $(n-1)$*-tile* as

$$\mathbf{X}_{wb}^n(f, \mathcal{C}) = \{X \in \mathbf{X}_w^n(f, \mathcal{C}) \mid \exists X' \in \mathbf{X}_b^{n-1}(f, \mathcal{C}),\ X \subseteq X'\},$$

and *the set of white* $n$*-tiles contained in a while* $(n-1)$*-tile* as

$$\mathbf{X}_{ww}^n(f, \mathcal{C}) = \{X \in \mathbf{X}_w^n(f, \mathcal{C}) \mid \exists X' \in \mathbf{X}_w^{n-1}(f, \mathcal{C}),\ X \subseteq X'\}.$$

In other words, for example, a black $n$-tile is an $n$-tile that is mapped by $f^n$ to the black 0-tile, and a black $n$-tile contained in a white $(n-1)$-tile is an $n$-tile that is contained in some white $(n-1)$-tile as a set, and is mapped by $f^n$ to the black 0-tile.

If we fix the cell decomposition $\mathbf{D}^n(f,\mathscr{C})$, $n \in \mathbb{N}_0$, we can define for each $v \in \mathbf{V}^n(f,\mathscr{C})$ the *n-flower of* $v$ as

$$W^n(v) = \bigcup\{\operatorname{inte}(c) \mid c \in \mathbf{D}^n(f,\mathscr{C}),\ v \in c\}. \tag{2.5}$$

Note that flowers are open (in the standard topology on $S^2$). Let $\overline{W}^n(v)$ be the closure of $W^n(v)$. We define the *set of all n-flowers* by

$$\mathbf{W}^n(f,\mathscr{C}) = \{W^n(v) \mid v \in \mathbf{V}^n(f,\mathscr{C})\}. \tag{2.6}$$

From now on, if the map $f$ and the Jordan curve $\mathscr{C}$ are clear from the context, we will sometimes omit $(f,\mathscr{C})$ in the notation above.

*Remark 2.7* For $n \in \mathbb{N}_0$ and $v \in \mathbf{V}^n$, we have

$$\overline{W}^n(v) = X_1 \cup X_2 \cup \cdots \cup X_m,$$

where $m = 2\deg_{f^n}(v)$, and $X_1, X_2, \ldots X_m$ are all the $n$-tiles that contain $v$ as a vertex (see [BM17, Lemma 5.28]). Moreover, each flower is mapped under $f$ to another flower in such a way that is similar to the map $z \mapsto z^k$ on the complex plane. More precisely, for $n \in \mathbb{N}_0$ and $v \in \mathbf{V}^{n+1}$, there exist orientation preserving homeomorphisms $\varphi\colon W^{n+1}(v) \to D$ and $\eta\colon W^n(f(v)) \to D$ such that $D$ is the unit disk on $\mathbb{C}$, $\varphi(v) = 0$, $\eta(f(v)) = 0$, and

$$(\eta \circ f \circ \varphi^{-1})(z) = z^k$$

for all $z \in D$, where $k = \deg_f(v)$. Let $\overline{W}^{n+1}(v) = X_1 \cup X_2 \cup \cdots \cup X_m$ and $\overline{W}^n(f(v)) = X'_1 \cup X'_2 \cup \cdots \cup X'_{m'}$, where $X_1, X_2, \ldots X_m$ are all the $(n+1)$-tiles that contain $v$ as a vertex, listed counterclockwise, and $X'_1, X'_2, \ldots X'_{m'}$ are all the $n$-tiles that contain $f(v)$ as a vertex, listed counterclockwise, and $f(X_1) = X'_1$. Then $m = m'k$, and $f(X_i) = X'_j$ if $i \equiv j \pmod{k}$, where $k = \deg_f(v)$. (See also Case 3 of the proof of Lemma 5.24 in [BM17] for more details.)

We denote, for each $x \in S^2$,

$$U^n(x) = \bigcup\{Y^n \in \mathbf{X}^n \mid \text{there exists } X^n \in \mathbf{X}^n \text{with } x \in X^n,\ X^n \cap Y^n \neq \emptyset\}, \tag{2.7}$$

and for each integer $m \leq -1$, set $U^m(x) = S^2$. We define the *n-partition* $O_n$ of $S^2$ induced by $(f,\mathscr{C})$ as

$$O_n = \{\operatorname{inte}(X^n) \mid X^n \in \mathbf{X}^n\} \cup \{\operatorname{inte}(e^n) \mid e^n \in \mathbf{E}^n\} \cup \overline{\mathbf{V}}^n. \tag{2.8}$$

## 2.4 Notions of Expansion for Thurston Maps

We now define two notions of expansion introduced by M. Bonk and D. Meyer [BM17].

It is proved in [BM17, Corollary 7.2] that for each expanding Thurston map $f$ (see Definition 2.10), we have $\operatorname{card}(\operatorname{post} f) \geq 3$.

**Definition 2.8** (*Joining opposite sides*) Fix a Thurston map $f$ with $\operatorname{card}(\operatorname{post} f) \geq 3$ and an $f$-invariant Jordan curve $\mathscr{C}$ containing $\operatorname{post} f$. A set $K \subseteq S^2$ *joins opposite sides* of $\mathscr{C}$ if $K$ meets two disjoint 0-edges when $\operatorname{card}(\operatorname{post} f) \geq 4$, or $K$ meets all three 0-edges when $\operatorname{card}(\operatorname{post} f) = 3$.

**Definition 2.9** (*Combinatorial expansion*) Let $f$ be a Thurston map. We say that $f$ is *combinatorially expanding* if $\operatorname{card}(\operatorname{post} f) \geq 3$, and there exists an $f$-invariant Jordan curve $\mathscr{C} \subseteq S^2$ (i.e., $f(\mathscr{C}) \subseteq \mathscr{C}$) with $\operatorname{post} f \subseteq \mathscr{C}$, and there exists a number $n_0 \in \mathbb{N}$ such that none of the $n_0$-tiles in $\mathbf{X}^{n_0}(f, \mathscr{C})$ joins opposite sides of $\mathscr{C}$.

**Definition 2.10** (*Expansion*) A Thurston map $f : S^2 \to S^2$ is called *expanding* if there exist a metric $d$ on $S^2$ that induces the standard topology on $S^2$ and a Jordan curve $\mathscr{C} \subseteq S^2$ containing $\operatorname{post} f$ such that

$$\lim_{n\to+\infty} \max\{\operatorname{diam}_d(X) \mid X \in \mathbf{X}^n(f, \mathscr{C})\} = 0.$$

We call such a Thurston map an *expanding Thurston map*.

*Remark 2.11* It is clear that if $f$ is an expanding Thurston map, so is $f^n$ for each $n \in \mathbb{N}$. We observe that being expanding is a topological property of a Thurston map and independent of the choice of the metric $d$ that generates the standard topology on $S^2$. By Lemma 6.1 in [BM17], it is also independent of the choice of the Jordan curve $\mathscr{C}$ containing $\operatorname{post} f$. More precisely, if $f$ is an expanding Thurston map, then

$$\lim_{n\to+\infty} \max\Bigl\{\operatorname{diam}_{\widetilde{d}}(X) \Bigm| X \in \mathbf{X}^n\bigl(f, \widetilde{\mathscr{C}}\bigr)\Bigr\} = 0,$$

for each metric $\widetilde{d}$ that generates the standard topology on $S^2$ and each Jordan curve $\widetilde{\mathscr{C}} \subseteq S^2$ that contains $\operatorname{post} f$.

P. Haïssinsky and K. Pilgrim developed a notion of expansion in a more general context for finite branched coverings between topological spaces (see [HP09, Sect. 2.1 and Sect. 2.2]). This applies to Thurston maps and their notion of expansion is equivalent to our notion defined above in the context of Thurston maps (see [BM17, Proposition 6.3]). Such concepts of expansion are natural analogs, in the contexts of finite branched coverings and Thurston maps, to some of the more classical versions, such as expansive homeomorphisms and forward-expansive continuous maps between compact metric spaces (see for example, [KH95, Definition 3.2.11]), and distance-expanding maps between compact metric spaces (see for example, [PU10, Chap. 4]). Our notion of expansion is not equivalent to any such classical notion in the context of Thurston maps. In fact, as mentioned in the introduction, there are subtle connections between our notion of expansion and some classical notions of weak expansion. Chapter 6 will be devoted to this topic. See Theorem 6.1 for the precise statement.

**Lemma 2.12** *Let $f: S^2 \to S^2$ be an expanding Thurston map. Then for each $p \in S^2$, the set $\bigcup_{n=1}^{+\infty} f^{-n}(p)$ is dense in $S^2$, and*

$$\lim_{n \to +\infty} \operatorname{card}(f^{-n}(p)) = +\infty. \tag{2.9}$$

*Proof* Let $\mathscr{C} \subseteq S^2$ be a Jordan curve containing $\operatorname{post} f$. Let $d$ be any metric on $S^2$ that generates the standard topology on $S^2$.

Without loss of generality, we assume that $p \in X_w^0$ where $X_w^0 \in \mathbf{X}_w^0(f, \mathscr{C})$ is the white 0-tile in the cell decompositions induced by $(f, \mathscr{C})$. The proof for the case when $p \in X_b^0$ where $X_b^0 \in \mathbf{X}_b^0(f, \mathscr{C})$ is the black 0-tile is similar.

By Proposition 2.6(ii), for each $n \in \mathbb{N}$ and each white $n$-tile $X_w^n \in \mathbf{X}_w^n(f, \mathscr{C})$, there is a point $q \in X_w^n$ with $f^n(q) = p$. Since $f$ is an expanding Thurston map,

$$\lim_{n \to +\infty} \max\{\operatorname{diam}_d(X) \mid X \in \mathbf{X}^n(f, \mathscr{C})\} = 0. \tag{2.10}$$

Then the density of the set $\bigcup_{n=1}^{+\infty} f^{-n}(p)$ follows from the observation that for each $n \in \mathbb{N}$, each black $n$-tile $X_b^n \in \mathbf{X}_b^n(f, \mathscr{C})$ intersects nontrivially with some white $n$-tile $X_w^n \in \mathbf{X}_w^n(f, \mathscr{C})$.

By the above observation, the triangular inequality, and the fact that $\operatorname{diam}_d(S^2) > 0$ and $S^2$ is connected in the standard topology, the equation (2.9) follows from (2.10). □

## 2.5 Visual Metric

For an expanding Thurston map $f$, we can fix a particular metric $d$ on $S^2$ called *visual metric for* $f$. For the existence and properties of such metrics, see [BM17, Chap. 8]. For a visual metric $d$ for $f$, there exists a unique constant $\Lambda > 1$ called the *expansion factor* of $d$ (see [BM17, Chap. 8] for more details). One major advantage of a visual metric $d$ is that in $(S^2, d)$ we have good quantitative control over the sizes of the cells in the cell decompositions discussed above. We summarize several results of this type ([BM17, Proposition 8.4, Lemmas 8.10, 8.11]) in the lemma below.

**Lemma 2.13** (M. Bonk and D. Meyer) *Let $f: S^2 \to S^2$ be an expanding Thurston map, and $\mathscr{C} \subseteq S^2$ be a Jordan curve containing* $\operatorname{post} f$. *Let $d$ be a visual metric on*

*$S^2$ for $f$ with expansion factor $\Lambda > 1$. Then there exist constants $C \geq 1$, $C' \geq 1$, $K \geq 1$, and $n_0 \in \mathbb{N}_0$ with the following properties:*

(i) *$d(\sigma, \tau) \geq C^{-1}\Lambda^{-n}$ whenever $\sigma$ and $\tau$ are disjoint $n$-cells for $n \in \mathbb{N}_0$.*
(ii) *$C^{-1}\Lambda^{-n} \leq \operatorname{diam}_d(\tau) \leq C\Lambda^{-n}$ for all $n$-edges and all $n$-tiles $\tau$ for $n \in \mathbb{N}_0$.*
(iii) *$B_d(x, K^{-1}\Lambda^{-n}) \subseteq U^n(x) \subseteq B_d(x, K\Lambda^{-n})$ for $x \in S^2$ and $n \in \mathbb{N}_0$.*
(iv) *$U^{n+n_0}(x) \subseteq B_d(x, r) \subseteq U^{n-n_0}(x)$ where $n = \lceil -\log r/\log \Lambda \rceil$ for $r > 0$ and $x \in S^2$.*
(v) *For every $n$-tile $X^n \in \mathbf{X}^n(f, \mathscr{C})$, $n \in \mathbb{N}_0$, there exists a point $p \in X^n$ such that $B_d(p, C^{-1}\Lambda^{-n}) \subseteq X^n \subseteq B_d(p, C\Lambda^{-n})$.*

*Conversely, if $\widetilde{d}$ is a metric on $S^2$ satisfying conditions* (i) *and* (ii) *for some constant $C \geq 1$, then $\widetilde{d}$ is a visual metric with expansion factor $\Lambda > 1$.*

Recall $U^n(x)$ is defined in (2.7).

In addition, we will need the fact that a visual metric $d$ induces the standard topology on $S^2$ ([BM17, Proposition 8.3]) and the fact that the metric space $(S^2, d)$ is linearly locally connected ([BM17, Proposition 18.5]). A metric space $(X, d)$ is *linearly locally connected* if there exists a constant $L \geq 1$ such that the following conditions are satisfied:

1. For all $z \in X$, $r > 0$, and $x, y \in B_d(z, r)$ with $x \neq y$, there exists a continuum $E \subseteq X$ with $x, y \subseteq E$ and $E \subseteq B_d(z, rL)$.
2. For all $z \in X$, $r > 0$, and $x, y \in X \setminus B_d(z, r)$ with $x \neq y$, there exists a continuum $E \subseteq X$ with $x, y \subseteq E$ and $E \subseteq X \setminus B_d(z, r/L)$.

We call such a constant $L \geq 1$ a *linear local connectivity constant of $d$*.

*Remark 2.14* If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational expanding Thurston map, then a visual metric is quasisymmetrically equivalent to the chordal metric on the Riemann sphere $\widehat{\mathbb{C}}$ (see [BM17, Lemma 18.10]). Here the chordal metric $\sigma$ on $\widehat{\mathbb{C}}$ is given by $\sigma(z, w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for $z, w \in \mathbb{C}$, and $\sigma(\infty, z) = \sigma(z, \infty) = \frac{2}{\sqrt{1+|z|^2}}$ for $z \in \mathbb{C}$. We also note that a quasisymmetric embedding of a bounded connected metric space is Hölder continuous (see [He01, Sect. 11.1 and Corollary 11.5]). Consequently, the classes of Hölder continuous functions on $\widehat{\mathbb{C}}$ equipped with the chordal metric and on $S^2 = \widehat{\mathbb{C}}$ equipped with any visual metric for $f$ are the same (upto a change of the Hölder exponent).

An expanding Thurston map is Lipschitz with respect to a visual metric.

**Lemma 2.15** *Let $f : S^2 \to S^2$ be an expanding Thurston map, and $d$ be a visual metric on $S^2$ for $f$ with expansion factor $\Lambda > 1$. Then $f$ is Lipschitz with respect to $d$.*

*Proof* Fix a Jordan curve $\mathscr{C} \subseteq S^2$ containing $\operatorname{post} f$. Let $x, y \in S^2$ and we assume that

$$0 < d(x, y) < K^{-1}\Lambda^{-2}, \tag{2.11}$$

where $K \geq 1$ is a constant from Lemma 2.13 depending only on $f$, $\mathscr{C}$, and $d$.

Set $m = \max\{k \in \mathbb{N}_0 \mid y \in U^k(x)\}$, where $U^k(x)$ is defined in (2.7). By Lemma 2.13(iii), the number $m$ is finite. Then $y \notin U^{m+1}(x)$. Thus by Lemma 2.13(iii),

$$\frac{1}{K}\Lambda^{-m-1} \leq d(x, y) \leq K\Lambda^{-m}.$$

By (2.11) we get $m \geq 1$. Since $f(y) \in f(U^m(x)) \subseteq U^{m-1}(f(x))$ by Proposition 2.6, we get from Lemma 2.13(iii) that

$$d(f(x), f(y)) \leq K\Lambda^{-m+1}.$$

Therefore,

$$\frac{d(f(x), f(y))}{d(x, y)} \leq \frac{K\Lambda^{-m+1}}{\frac{1}{K}\Lambda^{-m-1}} = K^2\Lambda^2,$$

and $f$ is Lipschitz with respect to $d$. □

## 2.6 Invariant Curves

A Jordan curve $\mathscr{C} \subseteq S^2$ is $f$-invariant if $f(\mathscr{C}) \subseteq \mathscr{C}$. We are interested in $f$-invariant Jordan curves that contain post $f$, since for such a curve $\mathscr{C}$, the partition $(\mathbf{D}^1(f, \mathscr{C}), \mathbf{D}^0(f, \mathscr{C}))$ is then a cellular Markov partition for $f$. According to Example 15.11 and Lemma 15.12 in [BM17], $f$-invariant Jordan curves containing post $f$ need not exist. However, M. Bonk and D. Meyer [BM17, Theorem 15.1] proved that there exists an $f^n$-invariant Jordan curve $\mathscr{C}$ containing post $f$ for each sufficiently large $n$ depending on $f$.

**Theorem 2.16** (M. Bonk & D. Meyer) *Let $f: S^2 \to S^2$ be an expanding Thurston map. Then for each $n \in \mathbb{N}$ sufficiently large, there exists a Jordan curve $\mathscr{C} \subseteq S^2$ containing* post $f$ *such that $f^n(\mathscr{C}) \subseteq \mathscr{C}$.*

We will need a slightly stronger version in Chaps. 4 and 6. Its proof is almost the same as that of [BM17, Theorem 15.1]. For the convenience of the reader, we include the proof here.

**Lemma 2.17** *Let $f: S^2 \to S^2$ be an expanding Thurston map, and $\widetilde{\mathscr{C}} \subseteq S^2$ be a Jordan curve with* post $f \subseteq \widetilde{\mathscr{C}}$. *Then there exists an integer $N(f, \widetilde{\mathscr{C}}) \in \mathbb{N}$ such that for each $n \geq N(f, \widetilde{\mathscr{C}})$ there exists an $f^n$-invariant Jordan curve $\mathscr{C}$ isotopic to $\widetilde{\mathscr{C}}$ rel.* post $f$ *such that no $n$-tile in $\mathbf{D}^n(f, \mathscr{C})$ joins opposite sides of $\mathscr{C}$.*

*Proof* By [BM17, Lemma 15.15], there exists an integer $N(f, \widetilde{\mathscr{C}}) \in \mathbb{N}$ such that for each $n \geq N(f, \widetilde{\mathscr{C}})$, there exists a Jordan curve $\mathscr{C}' \subseteq f^{-n}(\widetilde{\mathscr{C}})$ that is isotopic to $\widetilde{\mathscr{C}}$ rel. post $f$, and no $n$-tile for $(f, \widetilde{\mathscr{C}})$ joins opposite sides of $\mathscr{C}'$. Let $H: S^2 \times [0, 1] \to S^2$

be this isotopy rel. post $f$. We set $H_t(x) = H(x,t)$ for $x \in S^2$, $t \in [0,1]$. We have $H_0 = \mathrm{id}_{S^2}$ and $\mathscr{C}' = H_1(\widetilde{\mathscr{C}}) \subseteq f^{-n}(\widetilde{\mathscr{C}})$.

If $F = f^n$, then post $F$ = post $f$ and $F$ is also an expanding Thurston map ([BM17, Lemma 6.4]). Note that $F$ is cellular for $(\mathbf{D}^n(f,\widetilde{\mathscr{C}}), \mathbf{D}^0(f,\widetilde{\mathscr{C}}))$. So $\mathbf{D}^1(F,\widetilde{\mathscr{C}}) = \mathbf{D}^n(f,\widetilde{\mathscr{C}})$ (see [BM17, Lemma 5.12]). Thus no 1-cell for $(H_1 \circ F, \mathscr{C}')$ joins opposite sides of $\mathscr{C}'$, and thus $H_1 \circ F$ is combinatorially expanding for $\mathscr{C}'$. Note that $\mathscr{C}'$ contains post$(H_1 \circ F)$ = post $F$ = post $f$. By Theorem 14.2 in [BM17], there exists a homeomorphism $\phi\colon S^2 \to S^2$ that is isotopic to the identity rel. post $(H_1 \circ F)$ such that $\phi(\mathscr{C}') = \mathscr{C}'$ and $G = \phi \circ H_1 \circ F$ is an expanding Thurston map. Since $\phi \circ H_1$ is isotopic to the identity on $S^2$ rel. post $F$, the pair $F$ and $G$ are Thurston equivalent. By Theorem 11.1 in [BM17], there exists a homeomorphism $h\colon S^2 \to S^2$ that is isotopic to the identity on $S^2$ rel. $F^{-1}$(post $F$) with $F \circ h = h \circ G$. Set $\mathscr{C} = h(\mathscr{C}')$. Then $\mathscr{C}$ is a Jordan curve in $S^2$ that is isotopic to $\mathscr{C}'$ rel. $F^{-1}$(post $F$) and thus isotopic to $\widetilde{\mathscr{C}}$ rel. post $F$. Since $F(\mathscr{C}) = F(h(\mathscr{C}')) = h(G(\mathscr{C}')) = h(\phi(H_1(F(\mathscr{C}')))) \subseteq h(\phi(\mathscr{C}')) = h(\mathscr{C}') = \mathscr{C}$, we get that $\mathscr{C}$ is $F$-invariant.

Moreover, since no 1-cell for $(H_1 \circ F, \mathscr{C}')$ joins opposite sides of $\mathscr{C}'$, $H_1 \circ F(\mathscr{C}') \subseteq H_1(\widetilde{\mathscr{C}}) = \mathscr{C}'$, $\phi\colon S^2 \to S^2$ is a homeomorphism isotopic to the identity rel. post$(H_1 \circ F)$ with $\phi(\mathscr{C}') = \mathscr{C}'$, $G = \phi \circ H_1 \circ F$, we can conclude that $G(\mathscr{C}') \subseteq \mathscr{C}'$ and no 1-cell for $(G, \mathscr{C}')$ joins opposite sides of $\mathscr{C}'$. Since $h\colon S^2 \to S^2$ is a homeomorphism, $\mathscr{C} = h(\mathscr{C}')$, and $F \circ h = h \circ G$, we can finally conclude that no 1-cell for $(F, \mathscr{C})$ joins opposite sides of $\mathscr{C}$. Therefore no $n$-cell for $(f, \mathscr{C})$ joins opposite sides of $\mathscr{C}$. □

Compared with [BM17, Theorem 15.1], the above lemma carries additional combinatorial information of $\mathscr{C}$, i.e., no $n$-tile joins opposite sides of $\mathscr{C}$. In fact, we will only need the following corollary of Lemma 2.17 in Chaps. 4 and 6.

**Corollary 2.18** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map. Then there exists a constant $N(f) > 0$ such that for each $n \geq N(f)$, there exists an $f^n$-invariant Jordan curve $\mathscr{C}$ containing* post $f$ *such that no $n$-tile in $\mathbf{D}^n(f,\mathscr{C})$ joins opposite sides of $\mathscr{C}$.*

*Proof* We can choose an arbitrary Jordan curve $\widetilde{\mathscr{C}} \subseteq S^2$ containing post $f$ and set $N(f) = N(f, \widetilde{\mathscr{C}})$, and $\mathscr{C}$ an $f^n$-invariant Jordan curve containing post $f$ as in Lemma 2.17. □

We now establish a generalization of [BM17, Lemma 15.25]. It is an essential ingredient for the distortion lemmas (Lemma 5.3 and Lemma 5.4) that we will repeatedly use in Chaps. 5 and 7.

**Lemma 2.19** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\mathscr{C} \subseteq S^2$ be a Jordan curve that satisfies* post $f \subseteq \mathscr{C}$ *and $f^{n_{\mathscr{C}}}(\mathscr{C}) \subseteq \mathscr{C}$ for some $n_{\mathscr{C}} \in \mathbb{N}$. Let $d$ be a visual metric on $S^2$ for $f$ with expansion factor $\Lambda > 1$. Then there exists a constant $C_0 > 1$, depending only on $f$, $d$, $\mathscr{C}$, and $n_{\mathscr{C}}$, with the following property:*

*If* $k, n \in \mathbb{N}_0$*,* $X^{n+k} \in \mathbf{X}^{n+k}(f, \mathscr{C})$*, and* $x, y \in X^{n+k}$*, then*

$$\frac{1}{C_0} d(x, y) \leq \frac{d(f^n(x), f^n(y))}{\Lambda^n} \leq C_0 d(x, y). \tag{2.12}$$

*Proof* In this proof, we set a constant $K = 2\max\{1, l_f\}$, where $l_f$ is the Lipschitz constant of $f$ with respect to $d$. Let $N = n_{\mathscr{C}}$.

By Remark 2.11, the map $f^N$ is an expanding Thurston map. It is easy to see from Lemma 2.13 that the metric $d$ is a visual metric for the expanding Thurston map $f^N$ with expansion factor $\Lambda^N$. So by Lemma 15.25 in [BM17], there exists a constant $D \geq 1$ depending only on $f^N$, $\mathscr{C}$, and $d$ such that for each $k, l \in \mathbb{N}_0$, each $X \in \mathbf{X}^{(l+k)N}(f, \mathscr{C})$, and each pair of points $x, y \in X$, we have

$$\frac{1}{D} d(x, y) \leq \frac{d(f^{lN}(x), f^{lN}(y))}{\Lambda^{lN}} \leq D d(x, y). \tag{2.13}$$

Fix $m, l \in \mathbb{N}_0$, $s, t \in \{0, 1, \dots, N-1\}$, $X \in \mathbf{X}^{(mN+s)+(lN+t)}(f, \mathscr{C})$, and $x, y \in X$.

We prove the second inequality in (2.12) with $n = mN + s$ and $k = lN + t$ by considering the following cases depending on whether $l = 0$ or $l \geq 1$.

If $l = 0$, then by Lemma 2.15 and the fact that $K > l_f$,

$$d\left(f^{lN+t}(x), f^{lN+t}(y)\right) \leq K^t d(x, y) \leq K^{2N} d(x, y) \Lambda^{lN+t}.$$

If $l \geq 1$, then by Lemma 2.15, (2.13), and the fact that $K > l_f$,

$$\begin{aligned}
& d\left(f^{lN+t}(x), f^{lN+t}(y)\right) \\
= & d\left(f^{(l-1)N+(N-s)}\left(f^{t+s}(x)\right), f^{(l-1)N+(N-s)}\left(f^{t+s}(y)\right)\right) \\
\leq & K^{N-s} d\left(f^{(l-1)N}\left(f^{t+s}(x)\right), f^{(l-1)N}\left(f^{t+s}(y)\right)\right) \\
\leq & K^{N-s} D d\left(f^{t+s}(x), f^{t+s}(y)\right) \Lambda^{(l-1)N} \\
\leq & K^{N-s} D \left(K^{t+s} d(x, y)\right) \Lambda^{lN+t} \\
\leq & K^{2N} D d(x, y) \Lambda^{lN+t}.
\end{aligned}$$

We consider the first inequality in (2.12) with $n = mN + s$ and $k = lN + t$ now. By Proposition 2.6(i), we can choose $Y \in \mathbf{X}^{(m+l+2)N}(f, \mathscr{C})$ and two points $x', y' \in Y$ such that $f^{2N-s-t}(Y) = X$, $f^{2N-s-t}(x') = x$, and $f^{2N-s-t}(y') = y$. Note that $2N - s - t \geq 2$. Then by Lemma 2.15, (2.13), and the fact that $K > l_f$,

$$\begin{aligned}
& d\left(f^{lN+t}(x), f^{lN+t}(y)\right) \\
= & d\left(f^{lN+t}\left(f^{2N-s-t}(x')\right), f^{lN+t}\left(f^{2N-s-t}(y')\right)\right) \\
= & d\left(f^{lN+2N-s}(x'), f^{lN+2N-s}(y')\right) \\
\geq & K^{-s} d\left(f^{lN+2N}(x'), f^{lN+2N}(y')\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq K^{-s} D^{-1} d(x', y') \Lambda^{lN+2N} \\
&\geq K^{-s} D^{-1} K^{-(2N-s-t)} d(x, y) \Lambda^{lN+t} \\
&\geq K^{-2N} D^{-1} d(x, y) \Lambda^{lN+t}.
\end{aligned}
$$

Therefore,

$$
\frac{1}{C_0} d(x, y) \leq \frac{d(f^{lN+t}(x), f^{lN+t}(y))}{\Lambda^{lN+t}} \leq C_0 d(x, y),
$$

where $C_0 = K^{2N} D$ is a constant depending only on $f$, $d$, $\mathscr{C}$, and $N = n_{\mathscr{C}}$. □

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