Preface

This monograph is a contribution to the theory of second order quasilinear parabolic and hyperbolic equations with the nonlinear structure that may change from one point to another in the problem domain. In the past decade, there was an impetuous growth of interest in the study of such equations, which appear in a natural way in the mathematical modeling of various real-world phenomena and give rise to challenging mathematical problems. The aim of this work is to give an account of the known results on existence, uniqueness, and qualitative properties of solutions.

The parabolic equations studied below can be conventionally divided into several groups. Chaps. 2 and 3 are devoted to study the generalized porous medium equation

$$u_t = \operatorname{div}\left(|u|^{m(x,t)}\nabla u\right) + f(x,t) \tag{1}$$

with a given exponent $m(x,t) > -1$ and its generalizations, such as equations with lower order terms or anisotropic equations. We establish conditions of existence and uniqueness of weak solutions and show that for definite ranges of the exponent $m(x,t)$ the solutions exhibit properties typical for the solutions of equations with constant m, those of the finite speed of propagation and extinction in finite time. The former means the following: if the support of the initial data is compact, then the support of the solution remains compact for all time but may expand in space with finite speed. The latter property means that the solution corresponding to a nonzero initial datum may extinct in a finite time.

Chapters 4–6 concern the homogeneous Dirichlet problem for the nonlinear degenerate parabolic equations

$$u_t - \operatorname{div}\mathscr{A}(x,t,u,\nabla u) + \mathscr{B}(x,t,u) = 0 \tag{2}$$

with the function $\mathscr{A} \equiv (\mathscr{A}_1, \ldots, \mathscr{A}_n)$ whose components are of the form

$$\mathscr{A}_i(x,t,r,\xi) = a_i(x,t,r)|\xi_i|^{p_i(x,t)-2}\xi_i \quad \forall\, r \in \mathbb{R}, \xi \in \mathbb{R}^n, \tag{3}$$

where $p_i(x,t) \in (1,\infty)$ are given measurable functions and the coefficients a_i are Carathéodory functions, $a_i \in [a_0, a_1]$ with positive constants a_0, a_1. The function $\mathscr{B}$ is assumed to satisfy the growth condition $|\mathscr{B}(x,t,r)| \le d|r|^{\lambda} + f(x,t)$ with constants $d \ge 0$, $\lambda > 1$ and a given function f. Special attention is paid to the model case when

$$\mathscr{B}(x,t,r)r = c(x,t)|r|^{\sigma(x,t)} - f(z)r \quad \forall\, r \in \mathbb{R} \tag{4}$$

with a continuous exponent $\sigma(x,t) \in (1,\infty)$ and a given coefficient $c(x,t)$. Most of the results remain true if the operators $\mathscr{A}_i$ are substituted by the Leray-Lions operators with variable coercivity and growth conditions.

The assumptions on the functions $\mathscr{A}_i$ show that in the case of variable exponents p_i there is a gap between the coercivity and growth conditions; for this reason such equations are often termed PDEs with nonstandard growth. We extend this name to PDEs of the type (1), to PDEs with anisotropic but possibly constant nonlinearity and to equations with lower order terms of variable growth. All these equations share the same important property: they are not scaling-invariant, which makes inapplicable many of the traditional methods and requires new approaches to the study of their solvability and the analysis of the qualitative properties of solutions.

The main results of Chap. 4 are the theorems of existence and uniqueness of weak solutions. The natural analytic framework for the study is furnished by the Lebesgue and Sobolev spaces with variable exponents which are introduced in Chap. 1. This chapter collects all the information about the properties of these spaces used throughout the text. The constructed weak solutions are the so-called energy solutions which can be taken for the test-function in the corresponding integral identity. Chapter 4 contains results on the dependence of the regularity of the weak energy solutions on the regularity of the data and the nonlinearity exponents p_i and σ, and on the unique solvability of the Cauchy problem for the evolution $p(x)$-Laplace equation. In the final section we provide a review of other results and approaches to the study of parabolic equations of the type (2).

The properties of space localization are studied in Chap. 5. The study is confined to the energy solutions of Eq. (2) with the lower order terms of the form (4) with $c(x,t) \ge 0$. We show that there are two different mechanisms which can give rise to the property of finite speed of propagation. The first one is due to a suitable balance between the diffusion and absorption part of the equation. This is possible only if $c(x,t) \ge c_0 > 0$ and is expressed in terms of conditions on the variable rates of growth of the functions $\mathscr{A}_i$ and $\mathscr{B}$. The other is caused by the anisotropy of the diffusion operator and works even in the case when $c(x,t) \equiv 0$. It turns that if the anisotropic Eq. (2) combines the directions of slow and fast diffusion, $p_i > 2$ or

$p_i \in (1,2)$ respectively, the disturbances from the data run only a finite or even zero distance in the direction of the slowest diffusion.

Chapter 6 is devoted to the study of the large time behavior and the phenomenon of vanishing in a finite time for energy weak solutions of the homogeneous Dirichlet problem for Eq. (2). It is assumed that the lower order term $\mathscr{B}$ satisfies (4) with $c(x,t) \geq 0$. The study is based on the analysis of behavior of the local energy functions, which satisfy nonlinear ordinary differential equations. The effect of total vanishing in finite time is provided by a suitable relation between the exponents p_i, σ in conditions (3), (4). Besides, the same effect is possible in several situations specific for equations with the nonstandard growth conditions. It turns out that a solution may vanish in a finite time even if the equation eventually transforms into the linear heat equation, which does not admit localized solutions. Extinction in a finite time is possible also in the case when the coefficient c is allowed to vanish on a set of zero measure. Another effect is due to anisotropy: the equations of anisotropic diffusion admit solutions localized simultaneously in space and time.

In Chap. 7 we derive conditions of nonexistence of global in time bounded solutions to Eq. (2). We prove that under certain conditions on the data every energy solution becomes infinite in a finite time: there exists a finite t^* such that $\|u(\cdot,t)\|_{\infty,\Omega} \to \infty$ as $t \to t^*$. The following two versions of Eq. (2) are considered: the semilinear equation with $\mathscr{A}(x,t,\nabla u) = \nabla u$ and $\mathscr{B}$ satisfying condition (4) with superlinear growth and $c(x,t) < 0$, and the quasilinear equation with the exponents p_i and σ independent of t. The results are extended to nonlocal equations, equations that become linear as t grows to infinity, equations with nonnegative but not strictly positive coefficient $c(x,t)$.

Parabolic equations with double degeneracy are studied in Chaps. 8–10. Chapter 8 is devoted to study the homogeneous Dirichlet problem for the isotropic equations

$$u_t = \operatorname{div}\left(a|u|^{\alpha(x,t)}|\nabla u|^{p(x,t)-2}\nabla u\right) + f(x,t).$$

It is shown that under suitable conditions on the regularity of p and α this problem has a weak energy solution. Under additional restrictions on the data, the uniqueness and comparison theorems are proven for the solutions that possess better regularity and satisfy the inclusion $\partial_t u \in L^1(Q)$. It is shown that this class is nonempty and every weak solution falls into it, provided that the initial data meet additional regularity assumptions.

Anisotropic doubly degenerate equations are studied in Chaps. 9, 10. We consider the homogeneous Dirichlet problem for the equation

$$\partial_t \Psi(x,t,u) = \operatorname{div}\mathscr{A}(x,t,\nabla u) + c|u|^{\sigma(x,t)-2}u + f \tag{5}$$

with $\Psi = |u|^{m(x,t)-2}u$, given exponents m, σ and the anisotropic operator $\mathscr{A}$ that satisfies conditions (3). If the exponents p_i and m were constant and the operator $\mathscr{A}$ were isotropic, this equation would be formally equivalent to the equation studied in

Chap. 7, but in the case of Eq. (5) such a reduction is impossible and an independent analysis is required. Chapter 9 is devoted to prove the existence of strong solutions of Eq. (5) which possess the extra regularity property: $m|u|^{m-2}u_t^2 \in L^1(Q)$. Sufficient conditions for local and global in time existence of bounded strong solutions are proven and the energy relations are derived. Unlike the case of weak solutions to Eq. (2) the proof of existence of strong solutions to Eq. (5) requires certain monotonicity properties of the exponents p_i, σ, m and the coefficients a_i and $c(x,t)$.

The energy relations derived in Chap. 9 are used in Chap. 10 to study the phenomenon of extinction in a finite time, the large time behavior and the possibility of a finite time blow-up for strong solutions of Eq. (5).

In Chaps. 11, 12 we present results of the study of quasilinear and semilinear hyperbolic equations with nonstandard growth conditions. The homogeneous Dirichlet problem for the equation

$$u_t = \operatorname{div}\mathscr{A}(x,t,\nabla u) + \varepsilon\Delta u_t + c|u|^{\sigma(x,t)-2}u + f$$

with the isotropic operator $\mathscr{A}$ of the form (3) is studied in Chap. 11. Two different cases are considered: the equation with the damping term, $\varepsilon > 0$, and the equation with $\varepsilon = 0$. It is shown that for every $\varepsilon > 0$ the problem admits global or local in time weak solutions, provided that the exponents of nonlinearity satisfy certain regularity assumptions. The solutions of the damped problem may blow-up in a finite time and the blow-up moment t^* admits the two-sided bounds, which depend on the problem data but are independent of ε. If $\varepsilon = 0$, the question of existence of weak solutions is left open. Nonetheless, it is proven that in this case the problem admits the weaker solution in the sense of Young measure, which is obtained as the limit of the sequence of weak solutions to the damped problems as $\varepsilon \to 0$. Stronger results are obtained in Chap. 12 for the semilinear hyperbolic equation

$$u_{tt} = \Delta u + c(x,t)|u|^{\sigma(x,t)-2}u + f.$$

We derive sufficient conditions for local in time existence of weak and strong solutions and prove uniqueness of a strong solution. It is shown that in the case of superlinear growth, that is, if $\sigma(x,t) > 2$, every nonnegative strong solution blows-up in a finite time. The nonexistence result is extended to equations with nonlocal lower order terms, to equations which transform into the linear wave equation as $t \to \infty$ and to the case when the coefficient $c(x,t)$ is not separated away from zero in the problem domain.

The bulk of the presented material is constituted by the original results of the authors obtained in course of the past 10 years. The other pertinent results scattered in the literature are reviewed in each chapter. Although the selection and presentation of the supplementary material always reflect the interests of the authors, we expect that the reader will find them quite complete.

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