Chapter 2
A Porous Medium Equation with Variable Nonlinearity

2.1 Introduction

We devote this chapter to study the homogeneous Dirichlet problem for the semilinear parabolic equation

$$u_t = \operatorname{div}\left(a\,|u|^{\gamma(x,t)}\nabla u\right) + f(x,t,u,\nabla u) \tag{2.1}$$

with a given coefficient $a > 0$, an exponent $\gamma(x,t) > -1$ and the lower-order term f. Equation (2.1) is formally parabolic, but it degenerates or becomes singular wherever $u = 0$ and $\gamma \neq 0$. Equations of this type appear in the most natural way in various physical contexts such as mechanics of fluids and gases, the theory of heat propagation or diffusion processes. There exists an abundant literature devoted to study the questions of existence, uniqueness and qualitative properties of solutions to nonlinear parabolic equations of the type (2.1) with constant exponent of nonlinearity. The most studied prototypes are the famous porous medium equation, $u_t = \Delta u^m$, or the signed porous medium equation

$$u_t = \operatorname{div}(|u|^{m-1}\nabla u) \tag{2.2}$$

with constant exponent $m > 0$. Notwithstanding the fact that these equations have played the role of a "touchstone" for numerous methods in the theory of degenerate parabolic equations—see, e.g. [36, 64, 171, 197, 254]—not much is known about how their solutions respond to the variation of the exponent of nonlinearity. Under the assumption that the exponent m may vary from one point to another, equation (2.2) ceases to be invariant with respect to scaling, a fact which makes inapplicable many of well-developed methods in the theory of nonlinear PDEs.

We start by the analysis of the model equation

$$u_t = \operatorname{div}\left(a(x,t)\,|u|^{\gamma(x,t)}\nabla u\right) + f(x,t) \tag{2.3}$$

S. Antontsev and S. Shmarev, *Evolution PDEs with Nonstandard Growth Conditions*, Atlantis Studies in Differential Equations 4,
DOI 10.2991/978-94-6239-112-3_2

with a given variable exponent $\gamma > -1$. The existence and uniqueness theorems for the model equation (2.3) are extended then to the complete equation (2.1) with lower-order terms. Another generalization consists in extending the existence result to the class of semilinear anisotropic equations

$$u_t - \sum_{i=1}^{n} D_i \left(|u|^{\gamma_i(x,t)} D_i u\right) = f, \qquad D_i = \frac{\partial}{\partial x_i} \quad (i = 1, \ldots, n),$$

with given exponents $\gamma_i(x,t) > -1$. We study also the question of existence of bounded stationary weak solutions of quasi-linear anisotropic equations

$$-\sum_{i=1}^{n} D_i \left(a_i |u|^{\alpha_i(x)} D_i u\right) + f(x,u) = 0.$$

2.2 Model Equation: Assumptions and Results

Let us consider the following problem:

$$u_t - \operatorname{div}(|u|^{\gamma(x,t)} \nabla u) = f \quad \text{in } Q_T, \tag{2.4}$$

$$u = 0 \text{ on } \Gamma_T = \partial\Omega \times [0,T], \qquad u(x,0) = u_0(x) \text{ in } \Omega. \tag{2.5}$$

It is always assumed that γ is measurable and bounded in Q_T, and

$$-1 < \gamma^- \le \gamma(x,t) \le \gamma^+ < \infty \quad \text{a.e. in } Q_T \tag{2.6}$$

with given constants γ^- and γ^+.

Definition 2.1 A locally integrable bounded function $u(x,t)$ is called *weak solution* of problem (2.4)–(2.5) if:

(i) $u \in L^\infty(Q_T)$, $|u|^{\gamma(x,t)/2}\nabla u \in L^2(Q_T)$, $u_t \in L^2(0,T; H^{-1}(\Omega))$,
(ii) $u = 0$ on $\partial\Omega \times (0,T)$ in the sense of traces,
(iii) for every test-function $\zeta(x,t) \in C^\infty(0,T; C_0^\infty(\Omega))$, $\zeta(x,T) = 0$,

$$\int_0^T \int_\Omega (-u\zeta_t + |u|^{\gamma(x,t)} \nabla u \nabla \zeta - f\zeta)\, dx dt = \int_\Omega u_0 \zeta(x,0)\, dx. \tag{2.7}$$

The main existence result is given in the following theorem.

Theorem 2.1 *Let $\gamma(x,t) : Q_T \mapsto \mathbb{R}$ be a measurable function satisfying condition* (2.6). *Assume that $\nabla\gamma \in L^2(Q_T)$. If $f \in L^2(Q_T) \cap L^1(0,T; L^\infty(\Omega))$ and*

$$\|u_0\|_{\infty,\Omega} + \int_0^T \|f(\cdot,t)\|_{\infty,\Omega}\,dt = K(T) < \infty, \tag{2.8}$$

then problem (2.4) *and* (2.5) *has at least one weak solution in the sense of* Definition 2.1. *The solution is bounded and satisfies the estimate* $\|u\|_{\infty,Q_T} \le K(T)$ *with the constant* $K(T)$ *from* (2.8).

The uniqueness theorem is proved under stronger assumptions on the data.

Theorem 2.2 *Let the conditions of* Theorem 2.1 *be fulfilled. If the exponent* γ *satisfy the additional assumptions*

$$\gamma(x,t) \ge \gamma^- > 0 \text{ in } Q_T, \quad \sup_{x\in\overline{\Omega}} |\nabla\gamma| \in L^2(0,T), \tag{2.9}$$

then the solution of problem (2.4) *and* (2.5) *is unique.*

2.3 Regularization

Let us consider the auxiliary nonlinear parabolic problem

$$\begin{cases} Lu \equiv u_t - \operatorname{div}\left(a(\varepsilon,u,M,x,t))\nabla u\right) = f & \text{in } Q_T, \\ u(x,0) = u_0 \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T, \end{cases} \tag{2.10}$$

which depends on positive parameters $\varepsilon > 0$ and M. The coefficient a has the form

$$a = \left(\varepsilon^2 + \min\{u^2, M^2\}\right)^{\gamma(x,t)/2} \tag{2.11}$$

and (2.10) is a nondegenerate quasilinear parabolic equation because

$$0 < C'(\varepsilon, M, \gamma^\pm) \le a = \left(\varepsilon^2 + \min\{u^2, M^2\}\right)^{\gamma(x,t)/2} \le C(\varepsilon, M, \gamma^\pm).$$

A weak solution of problem (2.10) is constructed by means of the Schauder Fixed Point Theorem [187, Chap. 4, Sect. 8]. Let us consider the linear problem

$$\begin{cases} u_t - \operatorname{div}\left(\left(\varepsilon^2 + \min\{v^2, M^2\}\right)^{\gamma(x,t)/2}\nabla u\right) = \tau f & \text{in } Q_T, \\ u(x,0) = \tau\, u_0 \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T, \end{cases} \tag{2.12}$$

where $\tau \in [0,1]$ is a real parameter, and $v \in L^2(Q_T)$ is a given function. Let us denote $\mathscr{B}_R = \{v : \|v\|_{2,Q_T} < R\}$ and consider the solution u of problem (2.12) as a solution of the functional equation

$$u = \tau\,\Phi(v) \qquad \text{with} \quad \tau\,\Phi(u) : \mathscr{B}_R \times [0,1] \mapsto L^2(Q_T).$$

Solvability of problem (2.10) in a ball $\mathscr{B}_R = \{v : \|v\|_{2,Q_T} \le R\}$ will follow if we prove that

(1) the mapping $\Phi(v) : \mathscr{B}_R \mapsto \mathscr{B}_R$ is continuous and compact,
(2) for every $\tau \in [0, 1]$ the fixed points of the mapping $u = \tau \Phi(v)$ satisfy the estimate $\|v\|_{2,Q_T} \le R'$.

Let us define the Banach space $\mathbf{V}$ as the completion of $C^\infty(0, T; C_0^\infty(\Omega))$ in the norm

$$\|v\|_{\mathbf{V}} = \max_{[0,T]} \|v\|_{2,\Omega} + \|\nabla v\|_{2,Q_T}, \tag{2.13}$$

By $\mathbf{V}_0$ we denote the subset of the elements of $\mathbf{V}$ for which

$$\frac{1}{h}\int_0^{T-h} \|v(x, t+h) - v(x, t)\|_{2,\Omega}^2 dt \to 0 \quad \text{as } |h| \to 0. \tag{2.14}$$

Compactness of the mapping $\tau\Phi(v)$ follows from the classical results on solvability of linear parabolic equations with measurable coefficients [185, Chap. 3]: for every $u_0 \in L^2(\Omega)$ and $f, v \in L^2(Q_T)$, a measurable exponent $\gamma(x, t)$ satisfying condition (2.6), and any $\tau \in [0, 1]$ problem (2.12) has a unique solution $u \in \mathbf{V}_0$. Moreover, by [185, Chap. 3, Theorem 4.5] the mapping $\tau\Phi(v)$ is continuous in $\mathscr{B}_R$. For every test-function ζ satisfying the conditions of Definition 2.1

$$\int_{t_1}^{t_2}\int_\Omega (-u\zeta_t + a\,\nabla u \nabla\zeta - \tau\, f\zeta)dxdt = \int_\Omega u_0\zeta(x, 0)\,dx, \tag{2.15}$$

and $u_t \in L^2(0, T; H^{-1}(\Omega))$, $\|u\|_{L^\infty(Q_T)} \le C(R)$.

To check the fulfillment of the second condition with some $R' > 0$, it amounts to derive the a priori estimates for all possible solutions of the nonlinear problem (2.12) with $v = u$ and $\tau \in [0, 1]$. The next section is entirely devoted to deriving these estimates.

2.4 A Priori Estimates

We consider the nonlinear problems

$$\begin{cases} u_t - \operatorname{div}\,(a(\varepsilon, u, M, x, t)\nabla u) = \tau\, f \quad \text{in } Q_T, \\ u(x, 0) = \tau\, u_0 \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T, \end{cases} \tag{2.16}$$

with $\tau \in [0, 1]$ and the coefficient $a(\varepsilon, u, M, x, t)$ defined in (2.11).

Lemma 2.1 *The solution of problem* (2.16) *satisfies the estimate*

$$\|u\|_{\infty,Q_T} \leq \tau \left(\int_0^T \|f(\cdot,t)\|_{\infty,\Omega}\, dt + \|u_0\|_{\infty,\Omega} \right) \leq \tau\, K(T). \tag{2.17}$$

Proof Multiplying Eq. (2.16) by u^{2k-1} with an arbitrary $k \in \mathbb{N}$ and integrating over Ω, we arrive at the relation

$$\begin{aligned} \frac{1}{2k}\frac{d}{dt}\left(\|u(\cdot,t)\|^{2k}_{2k,\Omega}\right) + (2k-1)\int_\Omega a|\nabla u|^2\, u^{2(k-1)} dx \\ = \tau \int_\Omega u^{2k-1} f\, dx. \end{aligned} \tag{2.18}$$

By Hölder's inequality

$$\tau \left| \int_\Omega u^{2k-1} f\, dx \right| \leq \tau\ \|u(\cdot,t)\|^{2k-1}_{2k,\Omega}\ \|f(\cdot,t)\|_{2k,\Omega}\,,$$

whence

$$\begin{aligned} \|u(\cdot,t)\|^{2k-1}_{2k,\Omega}\ \frac{d}{dt}\left(\|u(\cdot,t)\|_{2k,\Omega}\right) + (2k-1)\int_\Omega a|\nabla u|^2\, u^{2(k-1)} dx \\ \leq \tau\ \|u(\cdot,t)\|^{2k-1}_{2k,\Omega}\ \|f(\cdot,t)\|_{2k,\Omega}\,. \end{aligned}$$

Simplifying and then integrating the last inequality in t, we obtain the following estimates for the solutions of problem (2.16):

$$\|u(\cdot,t)\|_{2k,\Omega} \leq \tau \left(\int_0^t \|f(\cdot,t)\|_{2k,\Omega}\, dt + \|u_0\|_{2k,\Omega} \right), \qquad k = 1, 2, \ldots .$$

Estimate (2.17) follows from this inequality as $k \to \infty$. □

Corollary 2.1 *If we choose $M > K(T)$, then*

$$\min\{u^2, M^2\} = u^2 \quad \textit{and} \quad a = (\varepsilon^2 + u^2)^{\gamma(x,t)/2}. \tag{2.19}$$

Corollary 2.2 *Let $u_0 \geq 0$ and $f \geq 0$. In this special case the solution $u(x,t)$ is nonnegative in $\overline{Q}_T$.*

Proof Set $u^- = \min\{u,\ 0\} \leq 0$. Multiplying (2.16) by u^-, integrating over Ω and taking into account the equalities $u^-(x,0) = 0,\ u^-\big|_{\Gamma_T} = 0$ we find that

$$\frac{1}{2}\frac{d}{dt}\left\|u^-(\cdot,t)\right\|_{2,\Omega} + \int_\Omega a|\nabla u^-|^2 dx \leq 0.$$

It follows that for every $t > 0$

$$0 \leq \|u^-(\cdot, t)\|_{2,\Omega} \leq \|u^-(x, 0)\|_{2,\Omega} = 0,$$

whence the assertion. □

Lemma 2.2 *The solutions of problem* (2.16) *satisfy the estimates*

$$\left\| \sqrt{a}\, \nabla u \right\|_{2,Q_T} \leq C \tag{2.20}$$

with an independent of ε *constant* C.

Proof It follows from (2.18) with $k = 1$ that for every $t \in (0, T)$ and $\tau \in [0, 1]$

$$\begin{aligned}\frac{1}{2}\|u(\cdot, t)\|^2_{2,\Omega} + \int_{Q_t} a\,|\nabla u|^2 dxdt &\leq \frac{\tau}{2}\|u_0\|^2_{2,\Omega} + \tau \int_{Q_T} |u|\,|f|\,dxdt \\ &\leq \frac{1}{2}\|u_0\|^2_{2,\Omega} + K(T)\int_0^T \|f(\cdot, t)\|_{\infty,\Omega}\,dt \\ &\leq (|\Omega| + K(T))K(T).\end{aligned}$$

□

Corollary 2.3 *The solutions of problem* (2.16) *satisfy the estimates*

$$\left\| (\varepsilon^2 + u^2)^{\gamma^+/4} \nabla u \right\|_{2,Q_T} \leq C \tag{2.21}$$

with a constant C *not depending on* ε.

Proof The inequality $|u| \leq K$ a.e. in Q_T yields

$$\left(\frac{\varepsilon^2 + u^2}{\varepsilon^2 + K^2}\right)^{\gamma(x,t)} \geq \left(\frac{\varepsilon^2 + u^2}{\varepsilon^2 + K^2}\right)^{\gamma^+}. \tag{2.22}$$

The assertion immediately follows now from Lemma 2.2. □

Lemma 2.3 *The solutions of problem* (2.16) *satisfy the estimates*

$$\|a\, \nabla u\|_{2,Q_T} \leq C$$

with a constant C *not depending on* ε. *Moreover,* $\|\nabla u\|_{2,Q_T} \leq C$ *if* $\gamma^+ \leq 0$.

Proof Let us consider the function

$$\psi(u) \equiv \int_0^u (\varepsilon^2 + s^2)^{\gamma^-/2} ds, \qquad \psi = 0 \text{ on } \Gamma_T.$$

Notice that

$$\begin{aligned} I &= \int_{Q_T} u\,\psi_t\,dxdt \\ &= \int_{Q_T} u\,u_t\,(\varepsilon^2+u^2)^{\gamma^-/2}dxdt \\ &= \int_{Q_T} \frac{\partial}{\partial t}\left(\int_0^u s\,\left(\varepsilon^2+s^2\right)^{\gamma^-/2} ds\right) dxdt \\ &= \int_\Omega \left(\int_{\tau\,u_0(x)}^{u(x,T)} s\,\left(\varepsilon^2+s^2\right)^{\gamma^-/2} ds\right) dx. \end{aligned}$$

By virtue of Lemma 2.1 $|I| \le M \equiv M(\gamma^-, K)$. Choosing in (2.15) $\zeta = \psi(u)$ for the test-function we deduce that

$$\int_{Q_T} a\,\psi'(u)\,|\nabla u|^2\,dxdt \le M + \tau\int_{Q_T} |f|\,\psi(u)\,dxdt \le M + \tau M_1 \int_{Q_T} |f|\,dxdt \tag{2.23}$$

with the constant

$$M_1 = \max\,\psi = 2\int_0^K s\,\left(\varepsilon^2+s^2\right)^{\gamma^-/2} ds.$$

It is easy to see [cf. with (2.22)] that

$$\begin{aligned} a = (\varepsilon^2+u^2)^{\frac{\gamma}{2}} &\le (\varepsilon^2+K^2)^{\frac{\gamma}{2}-\frac{\gamma^-}{2}}(\varepsilon^2+K^2)^{\frac{\gamma^-}{2}} \\ &= M_2(K,\gamma^-)\psi'(u), \qquad M_2 = const. \end{aligned}$$

Gathering this inequality with (2.23), we obtain

$$\int_{Q_T} a^2\,|\nabla u|^2\,dxdt \le M_2\int_{Q_T} a\psi'(u)|\nabla u|^2\,dxdt \le C(K,\gamma^-).$$

□

Corollary 2.4 *The solutions of problem* (2.16) *satisfy the estimates*

$$\left\|(\varepsilon^2+u^2)^{\gamma^+/2}\nabla u\right\|_{2,Q_T} \le C, \qquad \left\||u|^{\gamma^+}\nabla u\right\|_{2,Q_T} \le C \tag{2.24}$$

with an independent of ε *constant* C.

Lemma 2.4 *For the solutions of problem* (2.16) $u_t \in L^2(0,T;H^{-1}(\Omega))$ *and*

$$\|u_t\|_{L^2(0,T;H^{-1}(\Omega))} \le M$$

with an independent of ε *constant* M.

Proof It is sufficient to show that for every $\zeta \in C^{\infty}(0,T;C_0^{\infty}(\Omega))$, $\zeta(x,0) = \zeta(x,T) = 0$,

$$I = \left| \int_0^T \int_{\Omega} u\, \zeta_t \, dxdt \right| \leq M \, \|\zeta\|_{L^2(0,T;H_0^1(\Omega))}$$

with a constant M independent of ζ, ε and u. By virtue of identity (2.15) and the uniform estimates of Lemma 2.3

$$I \leq \left(\|a \nabla u\|_{2,Q_T} + \tau \, C(n,\Omega) \|f\|_{2,Q_T} \right) \, \|\nabla \zeta\|_{2,Q_T},$$

where $C(n,\Omega)$ is the best constant in the Poincaré inequality:

$$\forall \eta \in W_0^{1,2}(\Omega) \qquad \|\eta\|_{2,\Omega} \leq C(n,\Omega) \|\nabla \eta\|_{2,\Omega}.$$

□

Let us represent Eq. (2.16) in the form

$$u_t = \operatorname{div} \mathbf{G}, \tag{2.25}$$

with

$$\mathbf{G} = a\nabla u + \tau \, \mathbf{f} \in L^2(Q_T), \quad \operatorname{div} \mathbf{f} = f, \quad \mathbf{f}|_{\Gamma_T} = 0.$$

For the vector-valued function $\mathbf{f}$ we take the potential vector $\mathbf{f} = \nabla\theta$, where $\theta(x,t)$ is the solution of the Dirichlet problem for the Laplace equation

$$\Delta \theta = f \, \text{in} \, \Omega, \qquad \theta = 0 \, \text{on} \, \partial\Omega, \quad t \in (0,T). \tag{2.26}$$

The solution of problem (2.26) is considered as a function of $x \in \Omega$ depending on t as a parameter. Since $f \in L^2(Q_T)$ by assumption, $f(x,t) \in L^2(\Omega)$ for a.e. $t \in (0,T)$. For a.e. $t \in (0,T)$ problem (2.26) has a unique solution $\theta \in H_0^1(\Omega)$ and

$$\|\theta\|_{2,\Omega}^2 + \|\nabla\theta\|_{2,\Omega}^2 \leq C \, \|f\|_{2,\Omega}^2.$$

Integrating this estimate in t, we find that

$$\|\theta\|_{2,Q_T}^2 + \|\nabla\theta\|_{2,Q_T}^2 \leq C \, \|f\|_{2,Q_T}^2.$$

Equation (2.25) is fulfilled in the sense of distributions. If $\partial_{x_i} u \in L^2(Q_T)$, which corresponds to the case $\gamma^+ \leq 0$ (see Lemma 2.3), then according to [185, Chap. III, Lemma 4.1] the function $u(x,t)$ satisfies the estimates

$$\begin{cases} \|u(x+\mathbf{e}_i h, t) - u(x,t)\|_{2,Q_T^h}^2 \leq C|h|^2, \quad i = 1, \ldots, n, \\ \|u(x,t+h) - u(x,t)\|_{2,Q_T^h}^2 \leq C|h|, \end{cases} \tag{2.27}$$

where

$$Q_T^h = \{(x,t) \in Q_T : (x + \mathbf{e}_i h, t), (x, t+h) \in Q_T,\ i = 1, \dots, n\},$$

h is a scalar parameter, $\mathbf{e}_i$ are the unit vectors of the axes x_i.

Lemma 2.5 *Let $u(x,t)$ be a solution of problem (2.16) with $\gamma^+ \geq 0$. The function*

$$z(x,t) = z[u(x,t)] \equiv \int_0^{u(x,t)} |s|^{\gamma^+} ds$$

satisfies the estimates

$$\begin{cases} \|z(x + \mathbf{e}_i h, t) - z(x,t)\|^2_{2,Q_T^h} \leq C|h|^2, \quad i = 1, \dots, n, \\ \|z(x, t+h) - z(x,t)\|^2_{2,Q_T^h} \leq C|h|. \end{cases} \tag{2.28}$$

Proof Since $\nabla z = |u|^{\gamma^+} \nabla u$, estimates (2.24) yield the inclusions $\partial_{x_i} z \in L^2(Q_T)$, and the first of estimates (2.28) stems from the inequality

$$\int_{Q_T^h} \frac{|z(x + \mathbf{e}_i h, t) - z(x,t)|^2}{|h|^2} dx \leq C, \quad i = 1, \dots, n.$$

To derive the second estimate we introduce the functions

$$\delta u = u(x, t+h) - u(x,t), \qquad \delta z = z(x, t+h) - z(x,t).$$

Notice that sign $\delta u =$ sign z by the definition of z,

$$|\delta z| = \left| \int_{u(x,t)}^{u(x,t+h)} |s|^{\gamma^+} ds \right| \leq M^{\gamma^+} |\delta u| \quad \text{with } M > K(T), \tag{2.29}$$

$$|\delta z|^2 \leq (1+M)^{\gamma^+} |\delta z||\delta u| = (1+M)^{\gamma^+} \delta z \cdot \delta u,$$

$$\|\nabla(\delta z)\|_{L^2(Q_T^h)} \leq C.$$

Then

$$\begin{aligned} (1+M)^{-\gamma^+} \|\delta z\|^2_{2,\Omega} &\leq (\delta z,\ \delta u)_{2,\Omega} \\ &= \left(\delta z, \int_t^{t+h} \nabla \mathbf{G}(\cdot, \tau) d\tau \right)_{2,\Omega} \\ &= -\left(\nabla(\delta z), \int_t^{t+h} \mathbf{G}(\cdot, \tau) d\tau \right)_{2,\Omega}. \end{aligned} \tag{2.30}$$

Integrating (2.30) in $t \in (0, T-h)$, applying Hölder's inequality and then (2.29), we obtain the estimate

$$
\begin{aligned}
(1+M)^{-\gamma^+}\|\delta z\|^2_{2,Q_T^h} &\leq \int_0^{T-h}\int_\Omega |\nabla(\delta z)|^2\,dxdt \int_0^{T-h}\int_\Omega \left(\int_t^{t+h} \mathbf{G}(x,\tau)d\tau\right)^2 dxdt \\
&\leq C\int_0^{T-h}\int_\Omega \left(\int_t^{t+h} |\mathbf{G}(x,\tau)|\,d\tau\right)^2 dxdt \\
&\leq C\,|h|\left(\int_0^{T-h}\int_\Omega\int_t^{t+h} |\mathbf{G}(x,\tau)|^2\,d\tau dxdt\right) \leq C\,|h|.
\end{aligned}
$$

□

2.5 Passage to the Limit

In the proof of convergence of the sequence $\{u_\varepsilon\}$ we rely on the following result.

Lemma 2.6 ([235, Theorem 5]) *Let X, B and Y be Bahach spaces, $X \subset B \subset Y$ with compact embedding $X \subset B$. If a family of functions $\mathscr{F}$ possesses the properties*

1. *$\mathscr{F}$ is bounded in $L^p(0, T; X)$ with $1 \leq p \leq \infty$,*
2. *$\|f(x, t+h) - f(x, t)\|_{L^p(0,T-h;Y)} \to 0$ as $h \to 0$ uniformly for $f \in \mathscr{F}$,*

then $\mathscr{F}$ is relatively compact in $L^p(0, T; B)$(in $C(0, T; B)$ if $p = \infty$).

Let $\{u_\varepsilon\}$ be the sequence of solutions of problem (2.10). If $\gamma^+ \leq 0$, the sequence $\{u_\varepsilon\}$ satisfy the uniform estimates (2.27). Since the embedding $H^1_0(\Omega) \subset L^2(\Omega)$ is compact, it follows from Lemma 2.6 that $\{u_\varepsilon\}$ is precompact in $L^2(Q_T)$. Let $\gamma^+ \geq 0$. By (2.28)

$$
\|z[u_\varepsilon(x, t+h)] - z[u_\varepsilon(x, t)]\|^2_{2,Q_T^h} \to 0 \quad \text{as}\, h \to 0, \qquad \|\nabla z[u_\varepsilon]\|_{2,Q_T} \leq C,
$$

which yields precompactness of the sequence $z[u_\varepsilon(x, t)]$ in $L^2(Q_T)$. Not loosing generality we may assume that the converging subsequences coincide with the entire sequences. Convergence of $\{z_\varepsilon[u_\varepsilon]\}$ in $L^2(Q_T)$ yields convergence of $\{z[u_\varepsilon]\}$ and $\{u_\varepsilon\}$ a.e. in Q_T. Since $\|u_\varepsilon\|_{\infty,Q_T} \leq K(T)$, by the dominated convergence theorem $\{u_\varepsilon\}$ converges strongly in $L^p(Q_T)$ with any $p > 1$. Finally, by Lemma 2.4 u_t are uniformly bounded in $L^2(0, T; H^{-1}(\Omega))$, while $a|\nabla u_\varepsilon|^2$ are uniformly bounded by Lemma 2.3. Thus, there exist functions u and $\chi \in (L^2(Q_T))^n$ such that

$$
\begin{aligned}
& u_\varepsilon \to u \quad \text{a.e. in}\, Q_T, \\
& u_\varepsilon \to u \quad \text{in}\, L^p(Q_T)\, \text{with any}\, 1 < p < \infty, \\
& u_{\varepsilon t} \rightharpoonup u_t \quad \text{in}\, L^2(0, T; H^{-1}(\Omega)), \\
& a\nabla u_\varepsilon \rightharpoonup \chi \quad \text{in}\, L^2(Q_T).
\end{aligned}
\tag{2.31}
$$

Let us consider the auxiliary function $g[u] = u(\varepsilon^2 + u^2)^{\gamma/2}$. It is easy to calculate that

$$\nabla g = \left[\gamma \frac{u^2}{\varepsilon^2 + u^2} + 1\right] (\varepsilon^2 + u^2)^{\gamma/2} \nabla u + \frac{1}{2} \nabla \gamma \left[u (\varepsilon^2 + u^2)^{\gamma/2}\right] \ln (\varepsilon^2 + u^2).$$

If $\gamma(x, t) + 1 > 0$, then $\nabla g \in L^2(Q_T)$ because

$$\begin{aligned} |\nabla g|^2 &\le \left((\gamma + 1)|a \nabla u| + \frac{1}{2}(\varepsilon^2 + u^2)^{(\gamma+1)/2} |\ln (\varepsilon^2 + u^2)||\nabla\gamma|\right)^2 \\ &\le C_1|a \nabla u|^2 + C_2|\nabla\gamma|^2 \end{aligned}$$

with constants C_i depending on $\gamma^{\pm}$ and $\|u\|_{\infty, Q_T}$. It follows that there exists a function g^* such that

$$\nabla g_\varepsilon \rightharpoonup \nabla g^* \text{in} L^2(Q_T),$$

while by virtue of (2.31)

$$g_\varepsilon \equiv g[u_\varepsilon] = u_\varepsilon(\varepsilon^2 + u_\varepsilon^2)^{\gamma/2} \to u|u|^\gamma \quad \text{a.e. in } Q_T \text{ and in } L^q(Q_T) \text{ with } 1 < q < \infty.$$

Let us identify the limit ∇g^*. Notice that

$$\begin{aligned} &\frac{(1 + \gamma)u_\varepsilon^2 + \varepsilon^2}{\varepsilon^2 + u_\varepsilon^2} (\varepsilon^2 + u_\varepsilon^2)^{\gamma/2} \nabla u_\varepsilon \\ &\qquad = \nabla g_\varepsilon - \frac{1}{2} \nabla \gamma \left[u_\varepsilon (\varepsilon^2 + u_\varepsilon^2)^{\gamma/2}\right] \ln (\varepsilon^2 + u_\varepsilon^2). \end{aligned} \tag{2.32}$$

For every smooth test-function ψ

$$\begin{aligned} \int_{Q_T} \nabla g^* \cdot \psi \, dxdt &= \lim_{\varepsilon\to 0} \int_{Q_T} \nabla g_\varepsilon \cdot \psi \, dxdt \\ &= - \lim_{\varepsilon\to 0} \int_{Q_T} g_\varepsilon \cdot \nabla \psi \, dxdt = - \int_{Q_T} u \, |u|^\gamma \, \nabla \psi \, dxdt, \end{aligned}$$

whence

$$\nabla g^* = \nabla (u|u|^\gamma) = (1 + \gamma)|u|^\gamma \nabla u + \frac{1}{2} u|u|^\gamma (\ln u^2) \nabla \gamma \quad \text{a.e. in } Q_T. \tag{2.33}$$

For every test-function ζ from the conditions of Definition 2.1 we can pass to the limit as $\varepsilon \to 0$ in each term of the integral relation

$$\sum_{i=1}^{4} I_i^{(\varepsilon)} \equiv \int_{Q_T} \left(-u_\varepsilon\zeta_t + (\varepsilon^2 + u_\varepsilon^2)^{\gamma(x,t)/2} \nabla u_\varepsilon \nabla\zeta - f \zeta\right) dxdt - \int_{\Omega} u_0 \, \zeta(x, 0) \, dx = 0.$$

By virtue of (2.31)

$$I_1^{(\varepsilon)} \to -\int_{Q_T} u\,\zeta_t\,dxdt \qquad \text{as } \varepsilon \to 0.$$

According to (2.32) and (2.33)

$$\begin{aligned} I_2^{(\varepsilon)} &= \int_{Q_T} \frac{\varepsilon^2 + u_\varepsilon^2}{(1+\gamma)u_\varepsilon^2 + \varepsilon^2} \\ &\qquad \times \left[\nabla g_\varepsilon - \frac{1}{2}\left[u_\varepsilon\,(\varepsilon^2 + u_\varepsilon^2)^{\gamma/2}\right] \ln(\varepsilon^2 + u_\varepsilon^2)\,\nabla\gamma \right] \nabla\zeta\,dxdt \\ &\to \int_{Q_T} \frac{1}{\gamma+1}\left[\nabla(u|u|^\gamma) - \frac{1}{2}u|u|^\gamma\left(\ln u^2\right)\nabla\gamma\right]\nabla\zeta\,dxdt \\ &= \int_{Q_T} |u|^\gamma \nabla u\,\nabla\zeta\,dxdt \qquad \text{as } \varepsilon \to 0. \end{aligned}$$

Gathering these formulas we conclude that the limit function $u(x,t)$ is a weak solution of problem (2.4)–(2.5). The proof of Theorem 2.1 is completed.

2.6 Uniqueness of Weak Solutions

For the proof of uniqueness of weak solutions of the model problem (2.4)–(2.5) we assume that the data satisfy the conditions of Theorem 2.1 and the additional assumptions (2.9).

Let us notice that since $C^\infty(0,T;C_0^\infty(\Omega))$ is dense in the set

$$\mathscr{M} = \left\{\zeta \in L^2(0,T;H_0^1(\Omega)) : \zeta_t \in L^2(0,T;H^{-1}(\Omega))\right\},$$

the weak solutions of problem (2.4) and (2.5) satisfy identity (2.7) with the test-functions $\zeta \in \mathscr{M}$, $\zeta(x,T) = 0$ a.e. in Ω.[1]

Let us assume that there are two different solutions of problem (2.4) and (2.5) u_1 and u_2 and consider the function $u = u_1 - u_2$. This function satisfies the identity

$$\int_{Q_T} (-u\zeta_t + (|u_1|^{\gamma(x,t)}\nabla u_1 - |u_2|^{\gamma(x,t)}\nabla u_2)\nabla\zeta)\,dxdt = 0 \tag{2.34}$$

[1] Such a choice of the test-function is meaningful because every function $\zeta \in L^2(0,T;H_0^1(\Omega))$ with $\zeta_t \in L^2(0,T;H^{-1}(\Omega))$ belongs to $C([0,T];L^2(\Omega))$ after possible redefining on a set of zero measure in $(0,T)$—see [197, p. 156] or Lemma 4.5 below.

for every $\zeta \in \mathscr{M}$, $\zeta(x, T) = 0$ a.e. in Ω. Let us introduce the functions

$$W(x,t) \equiv W[u(x,t)] = \frac{1}{1+\gamma(x,t)} u\,|u|^{\gamma(x,t)},$$
$$F(u,x,t) = u\,|u|^{\gamma}\,\frac{\nabla\gamma}{1+\gamma}\left(\frac{\ln u^2}{2} - \frac{1}{1+\gamma}\right).$$

It is straightforward to check that

$$\nabla W(x,t) = |u|^{\gamma}\nabla u + u\,|u|^{\gamma}\,\frac{\nabla\gamma}{1+\gamma}\left(\frac{\ln u^2}{2} - \frac{1}{1+\gamma}\right)$$
$$\equiv |u|^{\gamma}\nabla u + F(u,x,t).$$

Since

$$|u_1|^{\gamma}\nabla u_1 - |u_2|^{\gamma}\nabla u_2 = \nabla(W[u_1] - W[u_2]) - (F(u_1,x,t) - F(u_2,x,t)),$$

for every smooth test-function $\zeta(x,t)$, vanishing on $\partial\Omega$,

$$\int_{Q_T} \left(|u_1|^{\gamma}\nabla u_1 - |u_2|^{\gamma}\nabla u_2\right)\,\nabla\zeta\,dxdt$$
$$= \int_{Q_T} \left(\nabla\left(W[u_1] - W[u_2]\right)\nabla\zeta + \left(F(u_2,x,t) - F(u_1,x,t)\right)\nabla\zeta\right)\,dxdt.$$

Integrating by parts in the first term on the right-hand side, we arrive at the following representation:

$$\int_{Q_T} \left(|u_1|^{\gamma}\nabla u_1 - |u_2|^{\gamma}\nabla u_2\right)\,\nabla\zeta\,dxdt$$

$$= \int_{Q_T} (u_1 - u_2)\,\left[-A\,\Delta\zeta + B\,\nabla\zeta\right]\,dxdt$$

with the vector-valued function B of the form

$$B = A\,\nabla\,\ln(1+\gamma) + D\,\nabla\gamma.$$

Here

$$A = \frac{W[u_1] - W[u_2]}{u_1 - u_2} = \frac{1}{1+\gamma}\,\frac{u_1|u_1|^{\gamma} - u_2|u_2|^{\gamma}}{u_1 - u_2} \geq 0,$$
$$D = \frac{F(u_1,x,t) - F(u_2,x,t)}{u_1 - u_2} = -\frac{\nabla\gamma}{2(1+\gamma)}\,\frac{u_1\,|u_1|^{\gamma(x,t)}\,\ln u_1^2 - u_2\,|u_2|^{\gamma(x,t)}\,\ln u_2^2}{u_1 - u_2}.$$

It is easy to verify that for $\gamma(x,t) \geq \gamma^- > 0$

$$\begin{aligned} & 0 \leq A \leq C, \qquad |D| \leq C\,|\nabla\gamma|, \\ & \frac{D^2}{A} = \frac{|\nabla\gamma|^2}{4(1+\gamma)}\,\frac{\left(u_1\,|u_1|^{\gamma(x,t)}\,\ln u_1^2 - u_2\,|u_2|^{\gamma(x,t)}\,\ln u_2^2\right)^2}{(u_1-u_2)\,(u_1|u_1|^{\gamma} - u_2|u_2|^{\gamma})} \leq C|\nabla\gamma|^2, \\ & \frac{D^2}{A} \leq C, \qquad \frac{B^2}{A} \leq C|\nabla\gamma|^2 \end{aligned} \tag{2.35}$$

with a constant C depending on γ^- and $\max|u_i|$. Using this notation, we rewrite identity (2.34) in the form

$$\int_{Q_T} [-\zeta_t - A\,\Delta\,\zeta + B\,\nabla\,\zeta]\;u\,dxdt = 0. \tag{2.36}$$

Let us set $\zeta(x,t) = \eta(x, T-t)$, where $\eta(x,t)$ is the solution of the parabolic problem

$$\begin{cases} \eta_t - (A+\varepsilon)\,\Delta\,\eta + B\,\nabla\,\eta = h & \text{in}\, Q_T, \\ \eta(x,0) = 0, \qquad \eta = 0 \,\text{on}\, \Gamma_T \end{cases} \tag{2.37}$$

with an arbitrary small parameter $\varepsilon > 0$ and an arbitrary $h \in L^2(Q_T)$. For every h and ε problem (2.37) has a unique continuous weak solution η such that $\eta_t,\ D^2_{ij}\eta \in L^2(Q_T)$. Instead of identity (2.36) we have now:

$$\int_{Q_T} u\;[h + \varepsilon\Delta\,\eta]\;dxdt = 0. \tag{2.38}$$

We proceed to derive a priori estimates on the derivatives of η. Let us assume that $h \in C(0,T; C^1_0(\Omega))$. Multiplying Eq. (2.37) by $\Delta\eta$ and integrating over Ω we obtain the equality

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} |\nabla\eta|^2 dx + \int_{\Omega} (A+\varepsilon)|\Delta\,\eta|^2 dx = I_1 + I_2 \tag{2.39}$$

with

$$I_1 = \int_{\Omega} B\,\nabla\eta\Delta\eta\,dx, \qquad I_2 = -\int_{\Omega} h\Delta\eta\,dx = \int_{\Omega} \nabla h\,\nabla\eta\,dx. \tag{2.40}$$

By Hölder's inequality

$$|I_1| \leq \frac{1}{2}\int_{\Omega}(A+\varepsilon)|\Delta\eta|^2\,dx + \frac{1}{2}\int_{\Omega}\frac{|B|^2}{(A+\varepsilon)}|\nabla\eta|^2\,dx, \tag{2.41}$$

$$|I_2| \leq \frac{1}{2}\int_{\Omega}|\nabla\eta|^2\,dx + \frac{1}{2}\int_{\Omega}|\nabla h|^2\,dx. \tag{2.42}$$

Gathering (2.39)–(2.42) we have:

$$\begin{aligned}\frac{d}{dt}\int_{\Omega}|\nabla\eta|^2\,dx+\int_{\Omega}(A+\varepsilon)|\Delta\eta|^2\,dx\\ \le\int_{\Omega}\left[\frac{|B|^2}{(A+\varepsilon)}+1\right]|\nabla\eta|^2\,dx+\int_{\Omega}|\nabla h|^2\,dx\\ \equiv J.\end{aligned}\tag{2.43}$$

By virtue of (2.35)

$$\frac{|B|^2}{(A+\varepsilon)}\le\frac{|B|^2}{A}\le C\sup_{\Omega}(|\nabla\gamma|)^2\tag{2.44}$$

with a constant C not depending on ε. The right-hand side of (2.43) is estimated as follows:

$$J\le\lambda(t)\int_{\Omega}|\nabla\eta|^2dx+\int_{\Omega}|\nabla h|^2dx$$

with $\lambda(t)=C\,(1+\sup_{\Omega}|\nabla\gamma|^2)$. Plugging this estimate into (2.43) and applying Gronwall's Lemma, we have:

$$\int_{\Omega}|\nabla\eta|^2dx+\int_0^t\int_{\Omega}(A+\varepsilon)|\Delta\eta|^2dxdt\le C\int_0^t\int_{\Omega}|\nabla h|^2dxdt.\tag{2.45}$$

Gathering (2.38) with (2.45) and applying Hölder's inequality we conclude that

$$\begin{aligned}\left|\int_0^T\int_{\Omega}uh\,dxdt\right|=\left|\int_0^T\int_{\Omega}\varepsilon\,\Delta\eta\,dxdt\right|\\ \le T\sqrt{\varepsilon}\sqrt{|\Omega|}\left(\int_0^T\int_{\Omega}\varepsilon|\Delta\eta|^2dxdt\right)^{\frac{1}{2}}\\ \le T\sqrt{\varepsilon}\sqrt{|\Omega|}\left(\int_0^T\int_{\Omega}(\varepsilon+A)|\Delta\eta|^2dxdt\right)^{\frac{1}{2}}\to 0\quad\text{when}\,\varepsilon\to 0,\end{aligned}$$

whence

$$\int_0^T\int_{\Omega}uh\,dxdt=0.$$

Since the last relation is true for any smooth function h, it follows that $u\equiv 0$.

Remark 2.1 We leave a gap between the conditions sufficient for the existence and uniqueness of weak solutions for problem (2.4) and (2.5). The existence is proved under the assumption $-1<\gamma(x,t)<\infty$, while the uniqueness theorem is true if

$0 < \gamma^{-} \le \gamma(x,t) < \infty$. It is known that the weak solution of the Cauchy problem for Eq. (2.4) with the constant exponent $\gamma(x,t) \equiv \gamma \in (-1,0)$ need not be unique [230, 253]. Conditions of non-uniqueness in the limit case $\gamma = -1$ are given in [162].

Similar arguments show that the solutions of problem (2.4) and (2.5) obey the comparison principle.

Lemma 2.7 *Let u_1, u_2 be the weak solutions of problem* (2.4)–(2.5) *corresponding to the initial functions u_{10}, u_{20}. If the problem data satisfy the conditions of* Theorem 2.2 *and $u_{10} \ge u_{20}$ a.e. in Ω, then $u_1 \ge u_2$ a.e. in Q_T.*

Proof Following the proof of Theorem 2.2 we find that the difference $u = u_1 - u_2$ satisfies, instead of (2.36), the equality

$$\int_{Q_T} [-\zeta_t - A\,\Delta\,\zeta + B\,\nabla\,\zeta]\,u\,dxdt = \int_{\Omega} u_0(x)\zeta(x,0)\,dx$$

with the right-hand side nonnegative by assumption. Fix an arbitrary $h \ge 0$ and take for η the solution of problem (2.37). Since $\eta \ge 0$ in Q_T by the maximum principle, this leads to the inequality

$$\int_{Q_T} u\;[h + \varepsilon\Delta\,\eta]\;dxdt = \int_{\Omega} u_0(x)\eta(x,T)\,dx \ge 0,$$

which substitutes (2.38). Then

$$\int_0^T \int_{\Omega} u\,h\,dxdt \ge -C\sqrt{\varepsilon},$$

and the assertion follows because $h \ge 0$ and $\varepsilon > 0$ are arbitrary. □

2.7 Equations with Lower-Order Terms

Let us consider the problem

$$\begin{cases} u_t = \operatorname{div}\left(a|u|^{\gamma(x,t)}\nabla u\right) + \mathbf{b}|u|^{\gamma(x,t)/2}\nabla u - c|u|^{\sigma(x,t)-2}u + d \quad \text{in } Q_T, \\ u = 0 \text{ on } \Gamma \times [0,T], \\ u(x,0) = u_0(x) \text{ in } \Omega \end{cases} \tag{2.46}$$

where the domain Ω and the boundary $\Gamma = \partial\Omega$ satisfy the foregoing assumptions and $T < \infty$. We assume that the scalar functions $a(x,t,r)$, $c(x,t,r)$ and the vector $\mathbf{b}(x,t,r)$ are given functions of the arguments $(x,t) \in Q_T = \Omega \times (0,T]$ and $r \in \mathbb{R}$.

The exponents of nonlinearity γ and σ are given measurable functions subject to the conditions

$$\forall \text{ a.e. } (x,t) \in Q_T \quad \begin{cases} \gamma(x,t) \in [\gamma^-, \gamma^+] \subset (-1, \infty), \\ \sigma(x,t) \in [\sigma^-, \sigma^+] \subseteq [1, \infty) \end{cases} \tag{2.47}$$

with some constants $\gamma^{\pm}$ and $\sigma^{\pm}$. We assume that a, $\mathbf{b}$, c and $d(x,t,r)$ are Carathèodory functions defined on $\overline{Q}_T \times \mathbb{R}$ and that there exist constants $a_0 > 0$, $c_0 \geq 0$, $D \geq 0$, a_1, b_1, c_1 such that $\forall (x,t,r) \in \overline{Q}_T \times \mathbb{R}$

$$\begin{cases} a_0 \leq a(x,t,r) \leq a_1, \qquad c_0 \leq c(x,t,r) \leq c_1 \\ |\mathbf{b}(x,t,r)| \leq b_1, \\ |d(x,t,r)| \leq D|r| + f(x,t) \quad \text{with some } f(x,t) \geq 0. \end{cases} \tag{2.48}$$

Definition 2.2 A locally integrable function $u(x,t)$ is called weak solution of problem (2.46) if:

(i) $u \in L^{\infty}(Q_T)$, $|u|^{\gamma(x,t)/2}\nabla u \in L^2(Q_T)$,
$u_t \in L^2(0,T; H^{-1}(\Omega))$,
(ii) $u = 0$ on Γ_T in the sense of traces,
(iii) for every test-function $\zeta(x,t) \in C^{\infty}(0,T; C_0^{\infty}(\Omega))$, $\zeta(x,T) = 0$ in Ω,

$$\int_{Q_T} \Big(-u\zeta_t + a\,|u|^{\gamma(x,t)}\nabla u \nabla \zeta - \zeta\,\mathbf{b}|u|^{\gamma(x,t)/2}\nabla u \\ + c\,|u|^{\sigma(x,t)-2}u\,\zeta - d\zeta \Big)\,dxdt = \int_{\Omega} u_0 \zeta(x,0)dx. \tag{2.49}$$

Theorem 2.3 *Let $\gamma(x,t)$ and $\sigma(x,t)$ be measurable in Q_T functions satisfying conditions (2.47). Assume that $\nabla\gamma \in L^2(Q_T)$. Let conditions (2.48) be fulfilled. If*

$$\|u_0\|_{\infty,\Omega} + \frac{e^{TD}-1}{D}\,\|f(x,t)\|_{\infty,Q_T}\,dt = K(T) < \infty, \tag{2.50}$$

then problem (2.46) has a weak solution in the sense of Definition 2.2. *This solution is bounded and satisfies the inequality $\|u\|_{\infty,Q_T} \leq K(T)$ with the constant $K(T)$ from condition (2.50).*

The proof of Theorem 2.3 is an adaptation of the proof of Theorem 2.1. A solution of problem (2.46) is obtained as the limit of the sequence of solutions of the regularized problems

$$\begin{cases} u_t - \operatorname{div}\left(a_{\varepsilon,M}(x,t,u))\nabla u\right) - \mathbf{b}_{\varepsilon,M}(x,t,u)\,\nabla u \\ \qquad\qquad + c_{\varepsilon,M}(x,t,u)\,u = d(x,t,u) \qquad \text{in } Q_T, \\ u(x,0) = u_0 \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T \end{cases} \tag{2.51}$$

with the coefficients

$$a_{\varepsilon,M}(x,t,u) = a(x,t,u)\left(\varepsilon^2 + \min\{u^2, M^2\}\right)^{\gamma(x,t)/2},$$
$$\mathbf{b}_{\varepsilon,M}(x,t,u) = \mathbf{b}(x,t,u)\left(\varepsilon^2 + \min\{u^2, M^2\}\right)^{\gamma(x,t)/4},$$
$$c_{\varepsilon,M}(x,t,u) = c(x,t,u)\left(\varepsilon^2 + \min\{u^2, M^2\}\right)^{(\sigma(x,t)-2)/2}$$

and positive parameters ε and M (the parameter M will be chosen later). Let denote $\mathscr{B}_R = \{v : \|v\|_{2,Q_T} < R\}$. For every fixed $v \in \mathscr{B}_R$ and $\tau \in [0,1]$ there exists a unique solution u of the linear problem

$$\begin{cases} u_t - \operatorname{div}\left(a_{\varepsilon,M}(x,t,v))\nabla u\right) - \mathbf{b}_{\varepsilon,M}(x,t,v)\,\nabla u \\ \qquad\qquad +c_{\varepsilon,M}(x,t,v)\,u = \tau\, d(x,t,v) \qquad \text{in } Q_T, \\ u(x,0) = \tau\, u_0 \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T. \end{cases} \tag{2.52}$$

The solution of problem (2.51) is sought as a fixed point of the mapping $u = \tau\,\Phi(v)$ with $\tau = 1$. The existence of a fixed point in a ball $\mathscr{B}_R = \{v : \|v\|_{2,Q_T} \le R\}$ follows from the Schauder fixed point principle and reduces to checking continuity and compactness of the mapping $\Phi(v) : \mathscr{B}_R \mapsto \mathscr{B}_R$ and to deriving suitable a priori estimates for the fixed points of the mapping $u = \tau\,\mathscr{B}(v)$ in a ball $\|v\|_{2,Q_T} \le R$. Almost all arguments used in the proof of Theorem 2.1 are applicable in the present case and can be omitted. We will present in detail only those that require modifications.

For every $v \in L^2(Q_T)$ and $\tau \in [0,1]$ problem (2.52) admits a weak solution $u \in \mathbf{V}_0$ (the space $\mathbf{V}_0$ is defined in (2.13) and (2.14). Compactness of the mapping $\tau\Phi(v)$ follows from results on the solvability of linear parabolic equations with measurable coefficients [185, Chap. 3], the mapping $\tau\Phi(v)$ is continuous in $\mathscr{B}_R$—see [185, Chap. 3, Theorem 4.5].

Lemma 2.8 *The solution of problem* (2.51) *satisfies the inequality*

$$\|u\|_{\infty,Q_T} \le \tau\,\|u_0\|_{\infty,\Omega} + \frac{e^{\tau T D} - 1}{D}\,\|f\|_{\infty,Q_T}\,. \tag{2.53}$$

Proof Multiplying Eq. (2.51) by u^{2k-1} and integrating over Ω we arrive at the equality

$$\begin{aligned} \frac{1}{2k}\frac{d}{dt}\left(\|u(\cdot,t)\|^{2k}_{2k,\Omega}\right) + (2k-1)\int_\Omega a_{\varepsilon,M}|\nabla u|^2\, u^{2(k-1)}dx + \int_\Omega c_{\varepsilon,M}|u|^{2k}dx \\ = \int_\Omega \mathbf{b}_{\varepsilon,M}\nabla u\, u^{2k-1}\,dx + \tau\int_\Omega d\, u^{2k-1}dx \equiv I_1 + I_2, \quad k = 1,2,\ldots. \end{aligned} \tag{2.54}$$

By Hölder's inequality, the terms I_1 and I_2 are estimated as follows:

$$
\begin{aligned}
|I_1| &\le \int_\Omega \frac{|\mathbf{b}|}{\sqrt{a}} |u|^k \sqrt{a_{\varepsilon,M}} |\nabla u|\, |u|^{k-1} dx \\
&\le \frac{(2k-1)}{2} \int_\Omega a_{\varepsilon,M} |\nabla u|^2 |\, u|^{2(k-1)} dx + \frac{1}{2(2k-1)} \int_\Omega \frac{|\mathbf{b}|}{a} |u|^{2k} dx \\
&\le \frac{(2k-1)}{2} \int_\Omega a_{\varepsilon,M} |\nabla u|^2 |\, u|^{2(k-1)} dx + \lambda\, \|u(\cdot,t)\|^{2k}_{2k,\Omega}\,, \qquad (2.55)
\end{aligned}
$$

where

$$
\frac{1}{2(2k-1)} \frac{|\mathbf{b}|^2}{\widetilde{a}} \le \frac{1}{2(2k-1)} \frac{|b_1|^2}{a_0} \overset{def}{\equiv} \lambda \to 0 \quad \text{for } k \to \infty,
$$

$$
|I_2| \le \tau D\, \|u(\cdot,t)\|^{2k}_{2k,\Omega} + \tau\, \|u(\cdot,t)\|^{2k-1}_{2k,\Omega}\, \|f(\cdot,t)\|_{2k,\Omega}\,. \qquad (2.56)
$$

Substituting (2.55) and (2.56) into (2.54), we come to the inequality

$$
\begin{aligned}
&\frac{1}{2k} \frac{d}{dt} \left(\|u(\cdot,t)\|^{2k}_{2k,\Omega} \right) + \frac{(2k-1)}{2} \int_\Omega a_{\varepsilon,M} |\nabla u|^2\, u^{2(k-1)} dx \\
&\qquad \le (\lambda + \tau D)\, \|u(\cdot,t)\|^{2k}_{2k,\Omega} + \tau\; \|u(\cdot,t)\|^{2k-1}_{2k,\Omega}\, \|f(\cdot,t)\|_{2k,\Omega}\,,
\end{aligned}
$$

whence

$$
\frac{d}{dt} \left(\|u(\cdot,t)\|_{2k,\Omega} \right) \le (\lambda + \tau D)\, \|u(\cdot,t)\|_{2k,\Omega} + \tau\; \|f(\cdot,t)\|_{2k,\Omega}\,. \qquad (2.57)
$$

Integrating (2.57) in t, for every $k = 1, 2, \ldots$ we get

$$
\|u(\cdot,t)\|_{2k,\Omega} \le \tau \left(\|u_0\|_{2k,\Omega} + e^{t(\lambda+\tau D)} \int_0^t e^{-s(\lambda+\tau D)}\, \|f(\cdot,s)\|_{2k,\Omega}\, ds \right).
$$

Passing in this inequality to the limit as $k \to \infty$, we obtain the needed estimate:

$$
\|u\|_{\infty,Q_T} \le \tau\, \|u_0\|_{\infty,\Omega} + \frac{e^{\tau T D} - 1}{D}\, \|f\|_{\infty,Q_T}\,.
$$

□

Corollary 2.5 *If $D = 0$, inequality (2.53) becomes*

$$
\|u\|_{\infty,Q_T} \le \tau\, \|u_0\|_{\infty,\Omega} + \tau\, T\; \|f\|_{\infty,Q_T}\,.
$$

Corollary 2.6 *Choosing $M > K(T)$, we have*

$$\min\{u^2, M^2\} = u^2,$$

which renders (2.51) a problem with the single regularization parameter ε.

Lemma 2.9 *The solution of problem (2.51) satisfies the inequalities*

$$\left\| \left(\varepsilon^2 + u^2\right)^{\gamma(x,t)/4} \nabla u \right\|_{2,Q_T} \leq C \tag{2.58}$$

with a constant C not depending on ε.

Proof For $k = 1$ equality (2.54) takes on the form

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|u(\cdot,t)\|^2_{2,\Omega}\right) + \int_\Omega a_{\varepsilon,M}|\nabla u|^2\,dx + \int_\Omega c_{\varepsilon,M}|u|^2 dx \\ = \int_\Omega \mathbf{b}_{\varepsilon,M}\nabla u\, u dx + \tau \int_\Omega d\, u\, dx \equiv I_1 + I_2. \end{aligned} \tag{2.59}$$

Applying Hölder's inequality and the already derived estimate on the maximum of the solution, we obtain the inequality

$$\begin{aligned} |I_1| &\leq \int_\Omega \frac{|\mathbf{b}|}{\sqrt{a}}|u|\sqrt{a}|\nabla u|\,dx \leq \frac{1}{2}\int_\Omega a_{\varepsilon,M}|\nabla u|^2 dx + \frac{1}{2}\int_\Omega \frac{|\mathbf{b}|^2}{a}|u|^2 dx \\ &\leq \frac{1}{2}\int_\Omega a_{\varepsilon,M}|\nabla u|^2 dx + C(K). \end{aligned}$$

Substituting this inequality into (2.59) and integrating in t, we have

$$\begin{aligned} &\frac{1}{2}\,\|u(\cdot,t)\|_{2,\Omega}\Big|_0^T \\ &+ \frac{1}{2}\int_{Q_T} a_{\varepsilon,M}|\nabla u|^2 dxdt + \int_{Q_T} c_{\varepsilon,M}|u|^2 dxdt \leq C(K, D, a_1, T), \end{aligned}$$

whence (2.58). □

Remark 2.2 The solutions of problem (2.51) satisfy the estimates

$$\left\|(\varepsilon^2 + u^2)^{\gamma^+/4}\nabla u\right\|_{2,Q_T} \leq C \tag{2.60}$$

with a constant C independent of ε.

The proof is identical to that of (2.21).

Lemma 2.10 *The solutions of problem (2.51) satisfy the inequalities*

$$\|a_{\varepsilon,M} \nabla u\|_{2,Q_T} \leq C$$

with a constant C not depending on ε constant C.

Proof Let us introduce the function

$$\psi(u) \equiv \int_0^u (\varepsilon^2 + s^2)^{\gamma^-/2} ds, \qquad \psi = 0 \text{ on } \Gamma_T,$$

which can be taken for the test-function in the integral identity (2.49). Notice that

$$\begin{aligned} I &= \int_{Q_T} u\, \psi_t \, dxdt = \int_{Q_T} u\, u_t\, (\varepsilon^2 + u^2)^{\gamma^-/2} dxdt \\ &= \int_{Q_T} \frac{\partial}{\partial t} \left(\int_0^u s\, \left(\varepsilon^2 + s^2\right)^{\gamma^-/2} ds \right) dxdt \\ &= \int_\Omega \left(\int_{\tau\, u_0(x)}^{u(x,T)} s\, \left(\varepsilon^2 + s^2\right)^{\gamma^-/2} ds \right) dx. \end{aligned}$$

By virtue of Lemma 2.8 $|I| \leq M \equiv M(\gamma^-, K)$. Multiplying (2.51) by the function $\zeta = \psi(u)$ and integrating by parts we find that

$$\begin{aligned} \int_{Q_T} a_{\varepsilon,M}\, \psi'(u)\, |\nabla u|^2 \, dxdt &\leq M + \int_{Q_T} |\psi(u)|\, |\mathbf{b}_{\varepsilon,M}| |\nabla u| \, dxdt \\ &- \int_{Q_T} c_{\varepsilon,M}\, u\, \psi(u)\, dxdt + \tau \int_{Q_T} (D\,|u| + |f|)\, \psi(u)\, dxdt \\ &\leq M + \tau \left(T \operatorname{meas} \Omega\, D\, K(T) + M_1 \int_{Q_T} |f|\, dxdt \right) \end{aligned}$$

with the constant

$$M_1 = \max \psi = 2 \int_0^K s\, \left(\varepsilon^2 + s^2\right)^{\gamma^-/2} ds.$$

Proceeding as in the proof of Lemma 2.3 we find that

$$\begin{aligned} a_{\varepsilon,M}\, \psi'(u) &= a \left(\varepsilon^2 + u^2\right)^{\gamma(x,t)/2} \psi'(u) \\ &\geq a_0 \left(\varepsilon^2 + u^2\right)^{\gamma(x,t)/2} \left(\varepsilon^2 + u^2\right)^{\gamma^-/2} \geq a_0 \left(\varepsilon^2 + u^2\right)^{\gamma(x,t)}, \end{aligned}$$

and the assertion follows. □

Corollary 2.7 *The solutions of problems* (2.51) *satisfy the estimates*

$$\left\|(\varepsilon^2 + u^2)^{\gamma^+/2}\nabla u\right\|_{2,Q_T} \le C, \qquad \left\||u|^{\gamma^+}\nabla u\right\|_{2,Q_T} \le C \tag{2.61}$$

with an independent of ε *constant* C.

Corollary 2.8 *If* $\gamma^+ \le 0$, *then* $\|\nabla u\|_{2,Q_T} \le C$.

Lemma 2.11 *The solutions of problem* (2.51) *have the weak derivatives* $u_t \in L^2(0,T;H^{-1}(\Omega))$ *and*

$$\|u_t\|_{L^2(0,T;H^{-1}(\Omega))} \le M$$

with an independent of ε *constant* M.

Proof It is sufficient to check that for every $\zeta \in C^\infty(0,T;C_0^\infty(\Omega))$, $\zeta(x,0) = \zeta(x,T) = 0$,

$$I = \left|\int_{Q_T} u\,\zeta_t\,dxdt\right| \le M\,\|\zeta\|_{L^2(0,T;H_0^1(\Omega))}$$

with a constant M not depending on ζ and u. Choosing ζ for the test-function in (2.49) and using the uniform estimates of Lemmas 2.8–2.10 we find:

$$\begin{aligned} I &\le \left\|(\varepsilon^2+u^2)^{\frac{\gamma}{2}}\nabla u\right\|_{2,Q_T}\|\nabla\zeta\|_{2,Q_T} + b_1\left\|(\varepsilon^2+u^2)^{\frac{\gamma}{4}}\nabla u\right\|_{2,Q_T}\|\zeta\|_{2,Q_T} \\ &\quad + c_1\left\|(\varepsilon^2+u^2)^{\frac{\sigma-1}{2}}\right\|_{2,Q_T}\|\zeta\|_{2,Q_T} + \big(D\|u\|_{2,Q_T} + \|f\|_{2,Q_T}\big)\|\zeta\|_{2,Q_T} \\ &\le C\|\zeta\|_{L^2(0,T;H_0^1(\Omega))} \end{aligned}$$

with a constant C depending on the norms of u_0, f and constants in the structural assumptions (2.47) and (2.48). □

Let us introduce the notation

$$Q_T^h = \{(x,t) \in Q_T : (x + \mathbf{e}_i h, t), (x, t+h) \in Q_T,\ i = 1,\dots,n\},$$

where h is a scalar parameter, $\mathbf{e}_i$ are the unit vectors of the axes x_i.

Lemma 2.12 *Let* $u(x,t)$ *be a solution of problem* (2.51) *and* $\gamma^+ \ge 0$. *Then the function*

$$z(x,t) = z[u(x,t)] \equiv \int_0^{u(x,t)} |s|^{\gamma^+}\,ds$$

satisfies the estimates

$$\|z(x+\mathbf{e}_i\,h,t)-z(x,t)\|^2_{2,Q_T^h}\leq C|h|^2,\qquad \|z(x,t+h)-z(x,t)\|^2_{2,Q_T^h}\leq C|h|. \tag{2.62}$$

Proof The inclusion $\nabla z\in L^2(Q_T)$ follows from the equality $\nabla z=|u|^{\gamma^+}\nabla u$ and estimates (2.61). The first of estimates (2.62) is then an immediate byproduct of the inequality

$$\int_{Q_T^h}\frac{|z(x+\mathbf{e}_i\,h,t)-z(x,t)|^2}{|h|^2}dx\leq C,\qquad i=1,\ldots,n.$$

For the functions $\delta u=u(x,t+h)-u(x,t)$, $\delta z=z(x,t+h)-z(x,t)$ we have

$$\begin{aligned}&\int_\Omega \delta z\delta u\,dx\\ &=\int_\Omega \delta z\int_t^{t+h}\left(\operatorname{div}(a_{\varepsilon,M}\nabla u)+\mathbf{b}_{\varepsilon,M}\nabla u-c_{\varepsilon,M}u+\tau\,d(x,\theta,u)\right)\,d\theta dx,\end{aligned} \tag{2.63}$$

$$\delta z=\int_{u_2}^{u_1}|s|^{\gamma^+}ds\leq\delta u,\qquad \|\nabla(\delta z)\|_{L^2(Q_T^h)}\leq C. \tag{2.64}$$

Let us integrate (2.63) over the interval $(0,T-h)$ and apply (2.64):

$$\begin{aligned}C\,\|\delta z\|^2_{2,Q_T^h}\leq&\int_0^{T-h}\int_\Omega|\nabla(\delta\,z)|\int_t^{t+h}|a_{\varepsilon,M}\nabla u|\,d\theta dxdt\\ &+\int_0^{T-h}\int_\Omega|\delta z|\int_t^{t+h}|\mathbf{b}_{\varepsilon,M}||\nabla u|\,d\theta dxdt\\ &-\int_0^{T-h}\int_\Omega|\delta z|\int_t^{t+h}|c_{\varepsilon,M}||u|\,d\theta dxdt\\ &+\tau\int_0^{T-h}\int_\Omega|\delta z|\int_t^{t+h}|d(x,\theta,u)|\,d\theta dxdt\\ \equiv&\, I_1+I_2+I_3+I_4.\end{aligned}$$

To estimate I_i we use Hölder's inequality, the estimates

$$|\mathbf{b}_{\varepsilon,M}|\leq b_1\left(\varepsilon^2+u^2\right)^{\gamma(x,t)/4},\quad |c_{\varepsilon,M}||u|\leq c_1\left(\varepsilon^2+u^2\right)^{(\sigma(x,t)-1)/2},$$

and Lemmas 2.8 and 2.10. □

Lemma 2.13 *Let* $\gamma^+\leq 0$. *Then*

$$\|u(x,t+h)-u(x,t)\|^2_{2,Q_T^h}\leq C\,|h|,\qquad \|z(x+\mathbf{e}_ih,t)-z(x,t)\|^2_{2,Q_T^h}\leq C|h|^2.$$

Proof It is sufficient to repeat the proof of Lemma 2.12 substituting δz by δu and taking into account the fact that $\|\nabla(\delta u)\|_{2,Q_T}$ is already estimated in Corollary 2.5. □

Let us introduce the functions $g[u] = u(\varepsilon^2 + u^2)^{\gamma/2}$ and $h[u] = u(\varepsilon^2 + u^2)^{\gamma/4}$. Arguing as in the proof of Theorem 2.1 and relying on Lemma 2.6 we extract from the sequence $\{u_\varepsilon\}$ a subsequence, for which we keep the same name, such that

$$\begin{aligned} &u_\varepsilon \to u \text{ a.e. in } Q_T, \\ &u_{\varepsilon t} \rightharpoonup u_t \text{ in } L^2(0,T; H^{-1}(\Omega)), \\ &u_\varepsilon \to u \text{ in } L^p(Q) \text{ with any } 1 < p < \infty, \\ &a\nabla u_\varepsilon \rightharpoonup \chi \text{ in } L^2(Q_T) \end{aligned} \tag{2.65}$$

and

$$\begin{aligned} &g_\varepsilon \equiv g[u_\varepsilon] = u_\varepsilon(\varepsilon^2 + u_\varepsilon^2)^{\gamma/2} \to u|u|^\gamma \quad \text{a.e. in } Q_T, \\ &h_\varepsilon \equiv h[u_\varepsilon] = u_\varepsilon(\varepsilon^2 + u_\varepsilon^2)^{\gamma/4} \to u|u|^{\gamma/2} \quad \text{a.e. in } Q_T \text{ and in } L^q(Q_T),\ 1 < q < \infty, \\ &\nabla g_\varepsilon \rightharpoonup \nabla g^*,\ \nabla h_\varepsilon \rightharpoonup \nabla h^* \text{ in } L^2(Q_T) \end{aligned}$$

with some functions u, g^*, h^*, χ. It remains to identify the limit functions ∇g^* and ∇h^*. Notice that

$$\begin{aligned} &\frac{(1+\gamma)u_\varepsilon^2 + \varepsilon^2}{\varepsilon^2 + u_\varepsilon^2} (\varepsilon^2 + u_\varepsilon^2)^{\gamma/2}\, \nabla u_\varepsilon \\ &\qquad = \nabla g_\varepsilon - \frac{1}{2} \nabla \gamma \left[u_\varepsilon\, (\varepsilon^2 + u_\varepsilon^2)^{\gamma/2} \right] \ln\,(\varepsilon^2 + u_\varepsilon^2) \end{aligned} \tag{2.66}$$

For every test-function ψ

$$\lim_{\varepsilon \to 0} \int_{Q_T} \nabla g_\varepsilon\, \psi = \int_{Q_T} \nabla g^*\, \psi = -\lim_{\varepsilon \to 0} \int_{Q_T} g_\varepsilon\, \nabla \psi = -\int_{Q_T} u\, |u|^\gamma\, \nabla \psi,$$

whence

$$\nabla g^* = \nabla\,(u|u|^\gamma) = (1+\gamma)|u|^\gamma\, \nabla u + \frac{1}{2} u|u|^\gamma (\ln\, u^2)\, \nabla \gamma \quad \text{a.e. in } Q_T. \tag{2.67}$$

In the same way we check that

$$\nabla h^* = \nabla\,(u|u|^{\gamma/2}) = \left(1 + \frac{\gamma}{2}\right) |u|^{\gamma/2}\, \nabla u + \frac{1}{4} u|u|^{\gamma/2} (\ln\, u^2)\, \nabla \gamma \quad \text{a.e. in } Q_T.$$

For every test-function from the conditions of Definition 2.2 one may pass to the limit when $\varepsilon \to 0$ in each term of the integral identity

$$\sum_{i=1}^{6} I_i^{(\varepsilon)} \equiv \int_{Q_T} \Big(- u_\varepsilon \zeta_t + a(x,t,u_\varepsilon)(\varepsilon^2 + u_\varepsilon^2)^{\gamma(x,t)/2} \nabla u_\varepsilon \nabla \zeta$$
$$- \zeta\, \mathbf{b}(x,t,u)(\varepsilon^2 + u_\varepsilon^2)^{\gamma(x,t)/4} \nabla u_\varepsilon$$
$$+ \zeta\, c(x,t,u)(\varepsilon^2 + u_\varepsilon^2)^{(\sigma(x,t)-2)/2} u_\varepsilon - f\,\zeta \Big) dxdt$$
$$- \int_\Omega u_0 \zeta(x,0)\, dx = 0.$$

By virtue of (2.65)

$$I_1^{(\varepsilon)} \to - \int_{Q_T} u\, \zeta_t\, dxdt, \qquad I_6^{(\varepsilon)} \to - \int_\Omega u_0\, \zeta(x,0)\, dx \qquad \text{as } \varepsilon \to 0,$$

while by virtue of (2.66) and (2.67)

$$I_3^{(\varepsilon)} \to \int_{Q_T} \mathbf{b}(x,t,u)|u|^{\gamma/2} \nabla u\, \zeta\, dxdt,$$
$$I_4^{(\varepsilon)} \to \int_{Q_T} c(x,t,u)|u|^{\sigma-2} u\, \zeta\, dxdt,$$

$$I_2^{(\varepsilon)} = \int_{Q_T} a(x,t,u) \frac{\varepsilon^2 + u_\varepsilon^2}{(1+\gamma)u_\varepsilon^2 + \varepsilon^2}$$
$$\times \left[\nabla g_\varepsilon - \frac{1}{2} \left[u_\varepsilon\, (\varepsilon^2 + u_\varepsilon^2)^{\gamma/2} \right] \ln\,(\varepsilon^2 + u_\varepsilon^2)\, \nabla \gamma \right] \nabla \zeta\, dxdt$$
$$\to \int_{Q_T} \frac{a(x,t,u)}{\gamma+1} \left[\nabla\,(u|u|^\gamma) - \frac{1}{2} u|u|^\gamma \left(\ln\, u^2 \right) \nabla \gamma \right] \nabla \zeta\, dxdt$$
$$= \int_{Q_T} a(x,t,u)|u|^\gamma \nabla u\, \nabla \zeta\, dxdt \qquad \text{as } \varepsilon \to 0.$$

Gathering these formulas we conclude that the limit function is a weak solution of problem (2.46).

2.8 Equations with Anisotropic Nonlinearity

A revision of the proof of Theorem 2.1 shows that the same arguments provide existence of a weak solution of problem (2.4)–(2.5) for equations with anisotropic nonlinearity. Let us consider the following problem:

$$u_t - \sum_{i=1}^{n} D_i \left(|u|^{\gamma_i(x,t)} D_i u \right) = f(x,t) \quad \text{in } Q_T, \tag{2.68}$$

$$u = 0 \text{ on } \Gamma_T = \partial\Omega \times [0,T], \qquad u(x,0) = u_0(x) \text{ in } \Omega, \tag{2.69}$$

where $\gamma_i(x,t)$ are given bounded functions defined on Q_T. It is assumed that

$$\forall \text{ a.e. } (x,t) \in Q_T \qquad \gamma_i(x,t) \in [\gamma^-, \gamma^+] \subset (-1, \infty), \quad i = 1, \ldots, n, \tag{2.70}$$

with given constants $\gamma^\pm$.

Definition 2.3 A locally integrable bounded function $u(x,t)$ is called *weak solution* of problem (2.68)–(2.69) if:

(i) $u \in L^\infty(Q_T)$, $|u|^{\gamma_i(x,t)/2} \nabla u \in L^2(Q_T)$, $u_t \in L^2(0,T; H^{-1}(\Omega))$,
(ii) $u = 0$ on Γ_T in the sense of traces,
(iii) for every test-function $\zeta(x,t) \in C^\infty(0,T; C_0^\infty(\Omega))$, $z(x,T) = 0$,

$$\int_{Q_T} (-u\zeta_t + \sum_{i=1}^{n} |u|^{\gamma_i(x,t)} D_i u \, D_i \zeta - f\zeta)\, dxdt = \int_{\Omega} u_0 \zeta(x,0)\, dx. \tag{2.71}$$

Theorem 2.4 *Let $\gamma_i(x,t)$ be measurable in Q_T functions. Assume that γ_i satisfy condition (2.70) and $\nabla \gamma_i \in L^2(Q_T)$. If $f \in L^2(Q_T) \cap L^1(0,T; L^\infty(\Omega))$ and*

$$\|u_0\|_{\infty,\Omega} + \int_0^T \|f(\cdot,t)\|_{\infty,\Omega} = K(T) < \infty, \tag{2.72}$$

then problem (2.68)–(2.69) has at least one weak solution in the sense of Definition 2.3. *The solution is bounded and satisfies the estimate $\|u\|_{\infty,Q_T} \le K(T)$ with the constant $K(T)$ from condition (2.72).*

Remark 2.3 The question of uniqueness of weak solution to the anisotropic problem (2.68)–(2.69) is left open.

2.9 Stationary Solutions

Let us consider the problem

$$\begin{cases} -\sum_i D_i(a_i(x,u)|u|^{\alpha_i(x)} D_i u) + c(x,u)|u|^{\sigma(x)-2} u = f(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.73}$$

About the domain Ω and the coefficients in Eq. (2.73) we assume the following:

1. $a_i(x, r)$ and $c(x, r)$ are Carathéodory functions,

$$\begin{aligned} &\forall x \in \Omega,\ r \in \mathbb{R} \quad 0 < a_0 \le a_i(x, r) \le A_0 < \infty, \\ &0 \le c_0 \le c(x, r) \le C_0 < \infty \end{aligned} \tag{2.74}$$

with some constants a_0, c_0, A_0, C_0.

2. $\alpha_i(x)$ and $\sigma(x)$ are continuous functions satisfying the conditions

$$\alpha_i(x) \in \left(\alpha_i^-, \alpha_i^+\right) \subseteq (\alpha^-, \alpha^+) \subset (-1, \infty), \quad \sigma(x) \in (\sigma^-, \sigma^+) \subset (1, \infty), \tag{2.75}$$

$\alpha_i^\pm$, $\alpha^\pm$, $\sigma^\pm$ are known constants,

Definition 2.4 A locally integrable in Ω function $u(x)$ is called *weak solution of problem* (2.73), if

1. $u \in L^\infty(\Omega)$, $|u|^{\alpha_i(x)/2}|D_i u| \in L^2(\Omega)$, $(i = 1, \dots, n)$,
2. $u = 0$ on Γ in the sense of traces,
3. for every test-function $\eta \in W_0^{1,2}(\Omega) \cap L^{\sigma(x)}(\Omega)$ the integral identity holds

$$\sum_i \int_\Omega a_i(x, u)|u|^{\alpha_i(x)} D_i u D_i \eta \, dx + \int_\Omega c(x, u)|u|^{\sigma(x)-2} u \eta \, dx = \int_\Omega f \eta \, dx. \tag{2.76}$$

About the function f we assume that

$$f \in \begin{cases} L^p(\Omega) \text{ with } p > n/2 & \text{if } \alpha_i(x) \ge 0 \text{ in } \Omega, \\ L^\infty(\Omega) & \text{if } \min_i \inf_\Omega \alpha_i(x) < 0. \end{cases} \tag{2.77}$$

Theorem 2.5 *Let conditions* (2.74) *and* (2.75) *be fulfilled and, additionally to these conditions,*

$$\|D_i \alpha_i(x)\|_{2,\Omega} \le C, \quad i = 1, 2, \dots, n. \tag{2.78}$$

Let us assume that either $\alpha_i(x) \ge 0$ in Ω, or $\alpha_i(x) > -1$ in Ω and $c_0 > 0$. Then for every right-hand side f, satisfying condition (2.77)*, problem* (2.73) *has a weak solution. The solution satisfies estimate*

$$\sum_i \||u|^{\alpha_i(x)/2} D_i u\|_{2,\Omega} + \sum_i \||u|^{\alpha_i(x)} D_i u\|_{2,\Omega} + c_0 \|u\|_{\sigma(\cdot),\Omega} + \|u\|_{\infty,\Omega} \le \Lambda \tag{2.79}$$

with a constant Λ depending on $\|f\|$, $|\Omega|$, n and the constants in conditions (2.74) *and* (2.75)*.*

Remark 2.4 Condition (2.78) can be relaxed in the following way:

$$\|D_i \alpha_i(x)\|_{2,\Omega_i^-} \le C, \quad \Omega_i^- = \{x \in \Omega : \alpha_i(x) \le 0\}.$$

—see Lemma 2.20 below.

A weak solution of problem (2.73) is obtained as the limit of the sequence of solutions of the regularized problems

$$\begin{cases} -\sum_i D_i(A_i(\varepsilon, M, x, u) D_i u) + C(\varepsilon, M, x, u)u = f(x) & \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma \end{cases} \tag{2.80}$$

with positive parameters ε, M and the coefficients

$$\begin{aligned} &0 < C_i'(\varepsilon, M, \alpha^\pm) \le A_i \equiv a_i(x,u)(\varepsilon^2 + \min\{u^2, M^2\})^{\alpha_i(x)/2} \\ &\quad \le C_i''(\varepsilon, M, \alpha^\pm) < \infty, \\ &0 \le C'(\varepsilon, M, \sigma^\pm) \le C \equiv c(x,u)(\varepsilon^2 + \min\{u^2, M^2\})^{(\sigma(x)-2)/2} \\ &\quad \le C''(\varepsilon, M, \sigma^\pm) < \infty. \end{aligned}$$

Lemma 2.14 *For every $\varepsilon > 0$, $\tau \in [0,1]$, $f(x) \in L^p(\Omega)$ with $p > n/2$ problem* (2.80) *has a solution $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \cap L^{\sigma(x)}(\Omega)$ satisfying the estimate*

$$c_0 \|u\|_{\sigma(\cdot)} + \|u\|_{\infty,\Omega} \le C \tag{2.81}$$

with a constant C independent of M and ε.

The existence of a weak solution of problem (2.80) is proved by means of the Schauder fixed point principle [187, Chap. 4, § 8]. Let us consider the linear problem

$$\begin{cases} -\sum_i D_i(A_i(\varepsilon, M, x, v) D_i u) + C(\varepsilon, M, x, v)u = \tau f(x) & \text{in } \Omega, \\ u = 0 \text{ on } \Gamma, \quad \tau \in [0,1], \quad v \in L^2(\Omega). \end{cases} \tag{2.82}$$

For every given $v \in L^2(\Omega)$ and every $\varepsilon > 0$, $M \ge 1$ and $\tau \in [0,1]$ problem (2.82) has a unique solution $u \in W_0^{1,2}(\Omega)$, [187, Chap. 3], satisfying the integral identity

$$\forall \eta \in W_0^{1,2}(\Omega) \quad \sum_i \int_\Omega A_i D_i u D_i \eta \, dx + \int_\Omega C u \eta \, dx = \int_\Omega f \eta \, dx. \tag{2.83}$$

Let us denote $\mathscr{B}_R = \{v : \|v\|_{L^2(\Omega)} < R\}$. Problem (2.82) defines the mapping $(v, \tau) \mapsto u$ which can be represented in the form

$$u = \tau \Phi(v) : \mathscr{B}_R \times [0,1] \mapsto L^2(\Omega)$$

because of the linearity with respect to τ. The solution of problem (2.80) is then a fixed point of the mapping $\Phi(\cdot)$. According to the Schauder principle the mapping $\Phi(v)$ has at least one fixed point in the ball $\mathscr{B}_R$, if

1. the mapping $\Phi(v) : \mathscr{B}_R \mapsto \mathscr{B}_R$ is continuous and compact,
2. for every $\tau \in [0, 1)$ all possible solutions of the equation $u = \tau\Phi(v)$ satisfy the estimate $\|u\|_{2,\Omega} < R$.

Lemma 2.15 *The mapping $\Phi(v) : \mathscr{B}_R \mapsto \mathscr{B}_R$ is continuous and compact.*

Proof Since the embedding $W_0^{1,2}(\Omega) \subset L^2(\Omega)$ is compact, so is the mapping $\Phi(v)$. The continuity of Φ follows from the continuity of the coefficients A_i and C with respect to the variable v. □

Lemma 2.16 *Let $\alpha_i(x) \geq 0$ and $f \in L^p(\Omega)$ with $p > n/2$. Then the solution of the regularized problem (2.82) with the parameter $M \geq 1$ satisfies the inequality*

$$\|u\|_{\infty,\Omega} \leq K \equiv \left(1 + \frac{C}{a_0}\|f\|_{p,\Omega}^{\frac{2}{n}-\frac{1}{p}}\right)^{\frac{1+\frac{2}{n}-\frac{1}{p}}{\frac{2}{n}-\frac{1}{p}}} \tag{2.84}$$

with a constant C not depending on M.

Proof Take an arbitrary number $k \geq 1$ and consider the function $\zeta = \max\{0, u-k\}$. The function ζ is an admissible test-function in identity (2.83). Besides,

$$\nabla\zeta = \begin{cases} 0, & \text{if } u \leq k, \\ \nabla u, & \text{if } u > k. \end{cases}$$

Let us denote $\Omega_k = \Omega \cap \{u > k\}$. Substituting ζ into the integral identity (2.83) and taking into account the inequality $u \cdot \zeta \geq 0$, we get

$$I := \sum_i \int_{\Omega_k} a_i(x, u)(\varepsilon^2 + \min\{u^2, M^2\})^{\alpha_i(x)/2}|D_i u|^2\,dx \leq \int_{\Omega_k} f(u-k)\,dx.$$

On the set Ω_k

$$\varepsilon^2 + \min\{u^2, M^2\} \geq \varepsilon^2 + \min\{k^2, M^2\} \geq 1,$$

which yields the inequality

$$(\varepsilon^2 + \min\{u^2, M^2\})^{\alpha_i(x)/2} \geq \frac{1}{k}.$$

Hence,

$$\int_{\Omega_k} |\nabla u|^2\,dx \le \frac{k}{a_0}\int_{\Omega_k} f(u-k)\,dx \equiv \frac{k}{a_0} I_3.$$

Estimating I_3 by Hölder's inequality and then applying the embedding theorem, we arrive at the estimate

$$|I_3| \le |\Omega_k|^{\frac{1}{2}-\frac{1}{p}} \|u-k\|_{2,\Omega_k} \|f\|_{p,\Omega} \le C|\Omega_k|^{\frac{1}{2}+\frac{2}{n}-\frac{1}{p}} \|f\|_{p,\Omega} \|\nabla u\|_{2,\Omega_k}.$$

Thus,

$$\int_{\Omega_k} |\nabla u|^2\,dx \le k^2 \left(\frac{C}{a_0}\right)^2 |\Omega_k|^{1+\frac{2}{n}-\frac{2}{p}} \|f\|^2_{p,\Omega}$$

with a constant C not depending on M. Using [187, Chap. 2, Lemma 5.3], we conclude that for $p > n/2$ the inequality $\|u\|_{\infty,\Omega} \le K$ holds with a constant K which depends on n, p, $1/a_0$ and $\|f\|_{p,\Omega}$ in the following way [187, Chap. 2, Lemma 5.1]: [187]

$$1 \le K \le \left(1 + \frac{C}{a_0} \|f\|^{\frac{2}{n}-\frac{1}{p}}_{p,\Omega}\right)^{\frac{1+\frac{2}{n}-\frac{1}{p}}{\frac{2}{n}-\frac{1}{p}}}.$$

□

Lemma 2.17 *Let $c_0 > 0$ and $f \in L^\infty(\Omega)$. Then the solution of the regularized problem* (2.82) *with the parameter $M \ge 1$ satisfies the inequality*

$$\|u\|_{\infty,\Omega} \le K \equiv \max\left\{1, \left(\frac{1}{c_0}\|f\|_{\infty,\Omega}\right)^{\frac{1}{\sigma^- - 1}}\right\}. \tag{2.85}$$

Proof Arguing as in the proof of Lemma 2.16 we take for the test-function $\zeta = \max\{u-k, 0\}$ with the parameter $k \ge 1$. Substituting ζ into the integral identity (2.83), we obtain the inequality

$$\sum_i \int_{\Omega_k} A_i |D_i u|^2\,dx + c_0 \int_{\Omega_k} \min\{k, M\}^{\sigma(x)-1}(u-k)\,dx \le \tau \int_{\Omega_k} f(u-k)\,dx,$$

whence the estimate

$$0 \ge (c_0 k^{\sigma^- - 1} - \tau \|f\|_{\infty,\Omega}) \int_{\Omega_k} (u-k)\,dx.$$

Increasing k, we conclude that necessarily $|\Omega_k| = 0$ for

$$k \geq k_0 \equiv (\frac{1}{c_0}\|f\|_{\infty,\Omega})^{1/(\sigma^- - 1)}.$$

The inequality $-u \leq k_0$ a. e. in Ω is proved likewise. □

The obtained estimates on the maximum of the solution of problem (2.80) do not depend on M, which allows us to choose $M = K$. Then the coefficients in equation (2.80) take on the form

$$A_i(\varepsilon, M, x, w) = (\varepsilon^2 + w^2)^{\alpha_i(x)/2}, \quad C(\varepsilon, M, x, w) = (\varepsilon^2 + w^2)^{(\sigma(x)-2)/2}$$

and (2.80) transforms into a problem with the single regularization parameter ε. Without special mentioning, in what follows we tacitly assume that $M = K$ with the constant K from (2.84) or (2.85).

Lemma 2.18 *For $c_0 > 0$ the fixed points of the mapping Φ satisfy the inequality*

$$c_0 \int_\Omega |u|^{\sigma(x)}\, dx \leq K\|f\|_{\infty,\Omega}|\Omega|$$

with the constant K from (2.85).

Proof Let us substitute u into (2.83) as the test-function and then drop the nonnegative term on the left-hand side of the appearing equality:

$$c_0 \int_\Omega (\varepsilon^2 + u^2)^{(\sigma(x)-2)/2} u^2\, dx \leq \int_\Omega |f||u|\, dx \leq K|\Omega|\|f\|_{\infty,\Omega}.$$

□

The proof of Lemma 2.14 is completed.

Lemma 2.19 *If condition* (2.77) *is fulfilled, then the solution of problem* (2.80) *satisfies the inequalities*

$$\sum_i \|\sqrt{A_i}D_i u\|^2_{2,\Omega} \leq C, \quad \sum_i \|A_i D_i u\|^2_{2,\Omega} \leq C \tag{2.86}$$

with a constant C independent of ε.

Proof The former of inequalities (2.86) follows from (2.83) with $\eta = u$. To prove the latter we make use of the following inequality, which is a byproduct of (2.84) and (2.85):

$$\left(\frac{\varepsilon^2 + u^2}{\varepsilon^2 + K^2}\right)^{\alpha_i(x)/2} \leq \left(\frac{\varepsilon^2 + u^2}{\varepsilon^2 + K^2}\right)^{\alpha^-/2} \quad \text{a. e. in } \Omega.$$

Let us introduce the function

$$\phi(u) \equiv A_0 \int_0^u (\varepsilon^2 + s^2)^{\alpha^-/2}\, ds \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega).$$

For $\alpha^- > -1$ the function $|\phi(u)|$ is uniformly with respect to ε bounded by the constant $\phi(K)$. Substituting $\phi(u)$ into (2.83) as the test-function, we obtain the inequality

$$\sum_i \int_\Omega A_i(\varepsilon^2 + u^2)^{\alpha^-/2} |D_i u|^2\, dx + c_0 \int_\Omega (\varepsilon^2 + u^2)^{(\sigma(x)-2)/2} u\phi(u)\, dx \\ \leq \int_\Omega |f||\phi(u)|\, dx.$$

Noting that $u\phi(u) \geq 0$, we have

$$\int_\Omega A_i^2 |D_i u|^2\, dx \leq \sum_i \int_\Omega a_i^2(x,u)(\varepsilon^2 + u^2)^{\alpha_i(x)} |D_i u|^2\, dx \\ \leq \phi(K) \begin{cases} |\Omega| \|f\|_{\infty,\Omega} & \text{for } f \in L^\infty(\Omega), \\ |\Omega|^{1/p'} \|f\|_{p,\Omega} & \text{for } f \in L^p(\Omega). \end{cases}$$

□

Denote by $\{u_\varepsilon\}$ the sequence of solutions of the regularized problems (2.80). Estimates (2.81) and (2.86) allow us to extract from $\{u_\varepsilon\}$ a subsequence (which will be assumed to coincide with the whole sequence), that possesses the properties:

$$\begin{aligned} &A_i D_i u_\varepsilon \rightharpoonup B_i(x) \quad \text{in } L^2(\Omega), \\ &u_\varepsilon \to u \quad \text{a.e. in } \Omega, \\ &\left(\varepsilon^2 + u_\varepsilon^2\right)^{(\sigma(x)-2)/2} u_\varepsilon \rightharpoonup |u|^{\sigma(x)-2} u \quad \text{in } L^2(\Omega) \end{aligned} \tag{2.87}$$

with some functions u and $B_i \in L^2(\Omega)$. Taking into account (2.87) and passing to the limit as $\varepsilon \to 0$ in identity (2.83), we have that for every $\eta \in W_0^{1,2}(\Omega)$

$$\sum_i \int_\Omega B_i(x) D_i \eta\, dx + \int_\Omega c(x,u)|u|^{\sigma(x)-2} u\eta\, dx = \int_\Omega f\eta\, dx.$$

It remains to show that $B_i(x) = |u|^{\alpha_i(x)} D_i u$. Consider the function

$$G_i(u) = (\varepsilon^2 + u^2)^{\alpha_i(x)} u.$$

It is easy to calculate that

$$D_iG_i(u) = \left[1 + \alpha_i \frac{u^2}{\varepsilon^2 + u^2}\right](\varepsilon^2 + u^2)^{\alpha_i/2} D_i u$$
$$+ \frac{1}{2} u(\varepsilon^2 + u^2)^{\alpha_i/2} D_i\alpha_i \ln(\varepsilon^2 + u^2),$$
$$|D_iG_i(u)|^2 \le C[|A_i|^2|D_iu|^2 + |D_i\alpha_i(x)|^2].$$

By virtue of (2.86), $\|D_iG_i(u_\varepsilon)\|_{2,\Omega} \le C$ uniformly with respect to ε. This allows us to extract a subsequence $\{u_\varepsilon\}$ such that $D_iG_i(u_\varepsilon) \rightharpoonup D_iH_i$ in $L^2(\Omega)$. Hence, for every test–function ζ

$$\lim_{\varepsilon\to 0} \int_\Omega D_iG_i(u_\varepsilon)\zeta\, dx = \int_\Omega D_iH_i\zeta\, dx = -\lim_{\varepsilon\to 0} G_i(u_\varepsilon)D_i\zeta\, dx$$
$$- \int_\Omega |u|^{\alpha_i(x)} u D_i\zeta\, dx,$$

whence the equality

$$D_iH_i = D_i(|u|^{\alpha_i(x)}u) = (1 + \alpha_i(x))|u|^{\alpha_i(x)} D_iu + \frac{1}{2}|u|^{\alpha_i(x)} u(\ln u^2) D_i\alpha_i(x).$$

Further,

$$A_i(\varepsilon, M, x, u_\varepsilon)D_iu_\varepsilon = a_i(x, u_\varepsilon)\frac{\varepsilon^2 + u_\varepsilon^2}{(1 + \alpha_i(x))u_\varepsilon^2 + \varepsilon^2}$$
$$\times \left[D_iG_i(u_\varepsilon) - \frac{1}{2}u\left(\varepsilon^2 + u_\varepsilon^2\right)^{\alpha_i(x)/2} D_i\alpha_i(x)\ln\left(\varepsilon^2 + u_\varepsilon^2\right)\right],$$

which yields that for every $\eta \in W_0^{1,2}(\Omega)$

$$\sum_i \int_\Omega A_i(\varepsilon, M, x, u_\varepsilon) D_iu_\varepsilon D_i\eta\, dx \to$$
$$\sum_i \int_\Omega a_i(x, u)|u|^{\alpha_i(x)} D_iu D_i\eta\, dx \quad \text{as } \varepsilon \to 0.$$

Lemma 2.20 *The assertion of* Theorem 2.5 *remains valid if condition* (2.78) *is removed and substituted by the following one:*

$$\|D_i\alpha_i(x)\|_{2,\Omega_i^-} \le C, \quad \Omega_i^- = \{x \in \Omega : \alpha_i(x) \le 0\}.$$

Proof Condition (2.78) was solely used to justify the limit as $\varepsilon \to 0$ in problem (2.80). Let us show that on the set $\{x \in \Omega : \alpha_i(x) > 0\}$ one may pass to the limit without this condition. Not loosing generality we assume that $\alpha^- > 0$. Let us represent the first term on the left–hand side of (2.83) as follows:

$$I := \sum_i \int_\Omega a_i(x, u_\varepsilon) \left(\varepsilon^2 + u_\varepsilon^2\right)^{\alpha_i(x)/2} D_i u_\varepsilon D_i \eta \, dx = I^{(1)}_{\delta,\varepsilon} + I^{(2)}_{\delta,\varepsilon}$$
$$\equiv \sum_i \int_{\Omega_\delta} \cdots + \sum_i \int_{\Omega\setminus\Omega_\delta} \cdots,$$

where $\Omega_\delta = \{x \in \Omega : |u_\varepsilon| > \delta\}$ and $\delta > 0$ is an arbitrary small number. It is easy to see that if (2.74), (2.75) and (2.86) are fulfilled, then

$$\mu(\delta) \sum_i \int_{\Omega_\delta} |D_i u_\varepsilon|^2 \, dx \le \sum_i \int_{\Omega_\delta} a_i(x, u_\varepsilon) \left(\varepsilon^2 + u_\varepsilon^2\right)^{\alpha_i(x)/2} |D_i u_\varepsilon|^2 \, dx \le C$$

with a constant $\mu(\delta) > 0$. Hence, the sequences $\{D_i u_\varepsilon\}$ converge weakly in $L^2(\Omega_\delta)$, while the sequences

$$\left\{a_i(x, u_\varepsilon) \left(\varepsilon^2 + u_\varepsilon^2\right)^{\alpha_i(x)/2}\right\}, \quad i = 1, \ldots, n,$$

converge almost everywhere. Therefore, for every fixed $\delta > 0$ there exists

$$\lim_{\varepsilon\to 0} I^{(1)}_{\delta,\varepsilon} = \sum_i \int_{\Omega_\delta} a(x, u)|u|^{\alpha_i(x)} D_i u D_i \eta \, dx.$$

Taking into account (2.74) and (2.86), we estimate $I^{(2)}_{\delta,\varepsilon}$ as follows:

$$\left|I^{(2)}_{\delta,\varepsilon}\right| \le \sqrt{A_0}(\delta^2 + \varepsilon^2)^{\alpha^-/2} \|\sqrt{A_i} D_i u_\varepsilon\|_{2,\Omega} \|D_i \eta\|_{2,\Omega} \le C(\delta^2 + \varepsilon^2)^{\alpha^-/2}.$$

Thus, $\lim_{\varepsilon\to 0} \left|I^{(2)}_{\varepsilon,\delta}\right| \le C\delta^{\alpha^-}$ and for every $\delta > 0$

$$\lim_{\varepsilon\to 0} I = \sum_i \int_{\Omega_\delta} a_i(x, u)|u|^{\alpha_i(x)} D_i u D_i \eta \, dx + o(\delta^{\alpha^-}).$$

Plugging estimates (2.86) and applying the Lebesgue dominated convergence theorem, we finally obtain:

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} |I| = \sum_i \int_\Omega a_i(x, u)|u|^{\alpha_i(x)} D_i u D_i \eta \, dx.$$

□

2.10 Remarks

The presentation of this chapter is based on results of [42, 43, 45, 46], see also [195] for a study of isotropic equation (2.46) with $\mathbf{b} \equiv 0$. The proof of uniqueness follows ideas of [33, 34, 37, 62, 63]. We were interested here in *energy solutions*, whose properties are studied in detail in the next chapter.

A different approach to the study of anisotropic porous medium equation with constant exponents is developed in [245, 246]. It is shown that for every nonnegative $u_0 \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ the equation

$$u_t = \sum_{i=1}^{n} \left(\left(u^{m_i}\right)_{x_i} + b_i u^{n_i} \right)_{x_i} - cu^r, \qquad m_i, n_i, r > 0, \quad c \geq 0,$$

has a nonnegative solution $u \in C^0(\overline{Q}_T)$ in a domain $Q_T = \Omega \times (0, T]$. The solution is understood in the following sense: for every subdomain $\Omega' \subset \Omega$, $T' \in (0, T)$ and every nonnegative test-function $\phi \in C^{2,1}(\Omega' \times (0, T'])$, vanishing on the lateral boundary of the cylinder $Q'_{T'}$,

$$\begin{aligned} &\int_{Q'_{T'}} \left(u\phi_t + \sum_{i=1}^{n} \left(u^{m_i} \phi_{x_i x_i} - b_i u^{n_i} \phi_{x_i} \right) - cu^r \phi \right) dx dt \\ &\qquad - \int_{\Omega'} u\phi \, dx \Big|_{t=0}^{t=T'} - \int_0^{T'} \int_{\partial\Omega'} \sum_{i=1}^{n} u^{m_i} \phi_{x_i} \cos(x_i, \boldsymbol{\nu}) \, d\Gamma dt = 0, \end{aligned}$$

where $\boldsymbol{\nu}$ denotes the unit normal vector to $\partial\Omega'$. The proof requires restrictions on the geometry of $\partial\Omega$: it is assumed that either Ω is strictly convex, or that $\partial\Omega$ satisfies the exterior ball condition. Besides, the oscillations $\max m_i - \min m_i$ should be suitably small. The existence result is extended to the Dirichlet problem in a bounded cylinder, the uniqueness theorem is proved under stronger restrictions on the data.

The Cauchy problem for the anisotropic equation

$$u_t = \sum_{i=1}^{n} \left(u^{m_i}\right)_{x_i x_i}$$

with the initial data in $L^1(\mathbb{R}^n)$ is studied in [168]. The equation is fulfilled in the sense of distributions and the solution belongs to $C([0, \infty); L^1(\mathbb{R}^n)) \cap C(\mathbb{R}^n \times (0, \infty)) \cap L^\infty(\mathbb{R}^n \times [\varepsilon, \infty))$, $\varepsilon > 0$. Another approach to the study of the anisotropic PDEs of the type

$$\partial_t(|u|^\rho u) - \sum_{i=1}^{n} D_i \left[\left(a_i |u|^{\gamma_i} + b_i |D_i u|^{p_i - 2} \right) D_i u + B_i(z, u) \right] + f(z, u) = 0$$

was proposed in [244], see also [119].

Local continuity of bounded energy solutions of the model equation (2.3) with $a \equiv 1$ and $f \equiv 0$ was established in [159, 161]. For the anisotropic equation (2.69) with constant exponents this fact was proved in [160].

A numerical study of problem (2.4) and (2.5) is performed in [121, 122]. The authors propose and justify a moving mesh algorithm and present results of numerical simulation.

http://www.springer.com/978-94-6239-111-6

Evolution PDEs with Nonstandard Growth Conditions
Existence, Uniqueness, Localization, Blow-up
Antontsev, S.; Shmarev, S.
2015, XVII, 409 p. 1 illus., Hardcover
ISBN: 978-94-6239-111-6
A product of Atlantis Press