## Contents

1 Preliminaries .................................................. 1
   1.1 Banach Spaces and Linear Operators ......................... 1
   1.2 Function Spaces ........................................... 5
   1.3 Some Theorems From Functional Analysis ...................... 7
   1.4 Ordered Banach Spaces ..................................... 8
   1.5 The Abstract Gronwall Lemma ................................ 10
   1.6 Integral Inequalities ....................................... 11
   1.7 The Kantorovich Generalized Norm .......................... 13
   1.8 Autonomous Difference Operators ............................. 15
   1.9 Time-Variant Difference Operators ........................... 24
   1.10 The Laplace Transform ..................................... 27
   1.11 Comments ................................................ 31

2 Eigenvalues and Functions of Matrices .......................... 33
   2.1 Some Definitions ........................................... 33
   2.2 Representations of Matrix Functions ........................ 35
   2.3 Norm Estimates for Resolvents ............................... 37
   2.4 Spectral Variations of Matrices .............................. 39
   2.5 Norm Estimates for Matrix Functions ........................ 41
      2.5.1 Estimates via the Resolvent ......................... 41
      2.5.2 Functions Regular on the Convex Hull of the Spectrum .. 42
      2.5.3 Proof of Theorem 2.5 ................................. 44
   2.6 Absolute Values of Elements of Matrix Functions .......... 46
      2.6.1 Statement of the Result ............................... 46
      2.6.2 Proof of Theorem 2.6 ................................. 48
   2.7 Diagonalizable Matrices .................................... 50
   2.8 Perturbations of Matrix Exponentials ........................ 52
   2.9 Functions of Matrices with Nonnegative Off-Diagonals .... 56
   2.10 Perturbations of Determinants .............................. 59
   2.11 Bounds for the Eigenvalues ................................. 63
2.11.1 Gerschgorin’s Circle Theorem ........................................ 63
2.11.2 Cassini Ovals .................................................. 64
2.11.3 The Perron Theorems .......................................... 65
2.11.4 Bounds for the Eigenvalues of Matrices “Close”
to Triangular Ones .................................................. 66
2.12 Comments .......................................................... 67

3 Difference Equations with Continuous Time. ......................... 71
  3.1 Autonomous Difference-Delay Equations .......................... 71
  3.2 Application of the Laplace Transform ............................ 76
  3.3 $L^p$-Norms of Solutions to Autonomous Equations ............ 77
      3.3.1 Equations with One Delay in $L^p, p \geq 1$ ............... 77
      3.3.2 Autonomous Difference-Delay Equations with Several Delays ...... 79
  3.4 $L^2$-Norms of Solutions to Autonomous Equations .......... 82
  3.5 Solution Estimates Via Determinants ............................ 84
  3.6 Time-Variant Equations: The General Case .................... 87
  3.7 Time-Variant Difference Equations with One Delay .......... 90
  3.8 Time-Variant Equations with Several Delays .................. 92
  3.9 Difference Equations with Commensurable Delays ............. 94
  3.10 Perturbations of Characteristic Values ....................... 97
  3.11 Perturbations of Characteristic Determinants ............... 101
  3.12 Comments ........................................................ 103

4 Linear Differential Delay Equations ................................... 105
  4.1 Homogeneous Autonomous Equations .............................. 105
  4.2 Non-homogeneous Autonomous Equations ......................... 107
  4.3 Estimates for Characteristic Matrices .......................... 110
  4.4 The Cauchy Operator of an Autonomous Equation .............. 113
      4.4.1 $L^2$-norm Estimates for the Cauchy Operator .......... 113
      4.4.2 Integrals of Characteristic Functions .................. 114
      4.4.3 Integrals of Fundamental Solutions .................... 118
      4.4.4 An Estimate for the $C$-norm of the Fundamental Solution .... 120
      4.4.5 $C$- and $L^p$-norms of the Cauchy Operator .......... 121
  4.5 Systems with Several Distributed Delays ...................... 122
  4.6 Scalar Autonomous Differential Delay Equations ............. 123
      4.6.1 The General Case ........................................ 123
      4.6.2 Equations with Positive Fundamental Solutions ........ 127
      4.6.3 Additional Lower Bounds for Quasipolynomials .......... 132
  4.7 Autonomous Systems with One Distributed Delay .............. 134
      4.7.1 The General Case ........................................ 134
      4.7.2 Application of Lemma 4.9 ............................... 135
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.7.3</td>
<td>Application of Lemma 4.13</td>
<td>137</td>
</tr>
<tr>
<td>4.7.4</td>
<td>Equations with Diagonalizable Matrices</td>
<td>138</td>
</tr>
<tr>
<td>4.8</td>
<td>Solution Estimates Via Determinants</td>
<td>139</td>
</tr>
<tr>
<td>4.9</td>
<td>Diagonally Dominant Differential Delay Systems</td>
<td>141</td>
</tr>
<tr>
<td>4.10</td>
<td>Time-Variant Equations Close to Ordinary Differential Ones</td>
<td>143</td>
</tr>
<tr>
<td>4.11</td>
<td>Equations with Slowly Varying Coefficients</td>
<td>149</td>
</tr>
<tr>
<td>4.12</td>
<td>Time-Variant Equations with Small Delays</td>
<td>151</td>
</tr>
<tr>
<td>4.13</td>
<td>Determinants of Differential Delay Systems</td>
<td>154</td>
</tr>
<tr>
<td>4.14</td>
<td>Comments</td>
<td>157</td>
</tr>
<tr>
<td>5</td>
<td>Linear Autonomous NDEs</td>
<td>159</td>
</tr>
<tr>
<td>5.1</td>
<td>Homogeneous Autonomous NDEs</td>
<td>159</td>
</tr>
<tr>
<td>5.2</td>
<td>The Fundamental Solutions to NDEs</td>
<td>164</td>
</tr>
<tr>
<td>5.3</td>
<td>Nonhomogeneous Autonomous NDEs</td>
<td>166</td>
</tr>
<tr>
<td>5.4</td>
<td>Estimates for $\theta(Q)$</td>
<td>170</td>
</tr>
<tr>
<td>5.5</td>
<td>Integrals of Characteristic Matrix Functions to NDEs</td>
<td>171</td>
</tr>
<tr>
<td>5.6</td>
<td>Integrals of Fundamental Solutions</td>
<td>176</td>
</tr>
<tr>
<td>5.7</td>
<td>Norms of the Cauchy Operator</td>
<td>178</td>
</tr>
<tr>
<td>5.8</td>
<td>Lower Estimates for Quasi-Polynomials</td>
<td>179</td>
</tr>
<tr>
<td>5.9</td>
<td>Scalar NDEs with Positive Fundamental Solutions</td>
<td>181</td>
</tr>
<tr>
<td>5.10</td>
<td>Stability Conditions via Determinants</td>
<td>182</td>
</tr>
<tr>
<td>5.11</td>
<td>NDEs with Commuting Hermitian Matrices</td>
<td>187</td>
</tr>
<tr>
<td>5.12</td>
<td>Autonomous NDEs with Small Principal Terms</td>
<td>188</td>
</tr>
<tr>
<td>5.13</td>
<td>Stability Conditions Independent of Delays</td>
<td>190</td>
</tr>
<tr>
<td>5.14</td>
<td>Perturbations of Characteristic Determinants</td>
<td>192</td>
</tr>
<tr>
<td>5.15</td>
<td>Comments</td>
<td>195</td>
</tr>
<tr>
<td>6</td>
<td>Linear Time-Variant NDEs</td>
<td>199</td>
</tr>
<tr>
<td>6.1</td>
<td>Existence Results for Linear NDEs</td>
<td>199</td>
</tr>
<tr>
<td>6.2</td>
<td>The Generalized Bohl-Perron Principle</td>
<td>204</td>
</tr>
<tr>
<td>6.3</td>
<td>Proof of Theorem 6.1</td>
<td>206</td>
</tr>
<tr>
<td>6.4</td>
<td>Time-Variant NDEs with Discrete Delays</td>
<td>210</td>
</tr>
<tr>
<td>6.5</td>
<td>Proof of Theorem 6.2</td>
<td>211</td>
</tr>
<tr>
<td>6.6</td>
<td>The $L^p$-version of the Generalized Bohl-Perron Principle</td>
<td>213</td>
</tr>
<tr>
<td>6.7</td>
<td>Proof of Theorem 6.3</td>
<td>215</td>
</tr>
<tr>
<td>6.8</td>
<td>Time-Variant NDEs Close to Autonomous Systems</td>
<td>220</td>
</tr>
<tr>
<td>6.9</td>
<td>NDEs with Small Norms of Principal Operators</td>
<td>226</td>
</tr>
<tr>
<td>6.10</td>
<td>Stability Conditions Independent of Delay</td>
<td>229</td>
</tr>
<tr>
<td>6.10.1</td>
<td>The General Case</td>
<td>229</td>
</tr>
<tr>
<td>6.10.2</td>
<td>Regular Parts with Slowly Varying Coefficients</td>
<td>231</td>
</tr>
<tr>
<td>6.11</td>
<td>Comments</td>
<td>234</td>
</tr>
</tbody>
</table>
7 Nonlinear Vector NDEs .................................................. 237
7.1 Causal Mappings ......................................................... 237
7.2 Existence of Solutions .................................................. 239
7.3 Lyapunov’s Stability ..................................................... 241
7.4 $L^2$-Absolute Stability of Nonlinear NDEs ....................... 243
7.5 Discrete Delays in Linear Parts ...................................... 246
7.6 Distributed Delays in Linear Parts .................................. 249
7.7 Exponential Stability of Nonlinear Systems ....................... 253
7.8 Proof of Theorem 7.5 .................................................... 253
7.9 Stability Conditions Via Generalized Norms ..................... 256
7.10 Systems with Diagonal Linear Parts ................................ 259
7.11 Input-to-State Stability ............................................... 260
7.12 Comments ................................................................. 261

8 Absolute Stability of Scalar NDEs ..................................... 263
8.1 Preliminaried .............................................................. 263
8.2 Absolute $L^2$ Stability ................................................ 266
8.3 The Generalized Aizerman Problem ................................ 268
8.4 The First Order Equations .......................................... 270
8.5 Higher Order Equations .............................................. 274
8.6 Additional Stability Conditions ...................................... 274
8.7 Comments ................................................................. 278

9 Bounds for Characteristic Values of NDEs ......................... 281
9.1 Sums of Moduli of Characteristic Values ......................... 281
9.2 Identities for Characteristic Values ................................ 287
9.3 Multiplicative Representations of Characteristic Matrices .... 289
9.4 Perturbations of Characteristic Values ......................... 290
9.5 Convex Functions of Characteristic Values ...................... 294
9.6 Comments ................................................................. 296

References ................................................................. 297

Index ................................................................. 303
Stability of Neutral Functional Differential Equations
Gil', M.
2014, XIII, 304 p., Hardcover
ISBN: 978-94-6239-090-4
A product of Atlantis Press