Chapter 2
Eigenvalues and Functions of Matrices

In this chapter we accumulate norm estimates for matrix-valued functions and bounds for the eigenvalues of matrices. In particular, we suggest estimates for the resolvents, powers of matrices as well as for matrix exponentials. We also present bounds for the spectral radius and investigate perturbations of eigenvalues. The material of this paper is systematically applied in the rest of the book chapters.

2.1 Some Definitions

Everywhere in this chapter, \( \|x\| \) is the Euclidean norm of \( x \in \mathbb{C}^n \): \( \|x\| = \sqrt{(x, x)} \) with a scalar product \((., .) = (., .)_{\mathbb{C}^n} \), \( I \) is the unit matrix.

For a linear operator \( A \) in \( \mathbb{C}^n \) (matrix), \( \lambda_k = \lambda_k(A) \) \( (k = 1, \ldots, n) \) are the eigenvalues of \( A \) enumerated in an arbitrary order with their multiplicities, \( \sigma(A) \) denotes the spectrum of \( A \), \( A^* \) is the adjoint to \( A \), and \( A^{-1} \) is the inverse to \( A \); \( R_s(A) = (A - \lambda I)^{-1} \) \( (\lambda \in \mathbb{C}, \lambda \not\in \sigma(A)) \) is the resolvent, \( r_s(A) \) is the spectral radius, \( \|A\| = \sup_{x \in \mathbb{C}^n} \|Ax\|/\|x\| \) is the (operator) spectral norm, \( N_2(A) \) is the Hilbert-Schmidt (Frobenius) norm of \( A \):

\[
N_2(A) = \sqrt{\text{Trace } AA^*},
\]

\( A_I = (A - A^*)/2i \) is the imaginary component, \( A_R = (A + A^*)/2 \) is the real component,

\[
\rho(A, \lambda) = \min_{k=1,\ldots,n} |\lambda - \lambda_k(A)|
\]

is the distance between \( \sigma(A) \) and a point \( \lambda \in \mathbb{C} \); \( \rho(A, C) = \min_k \inf_{s \in C} |s - \lambda_k(A)| \) is the distance between a contour \( C \) and \( \sigma(A) \). \( co(A) \) denotes the closed convex hull of \( \sigma(A) \), \( \alpha(A) = \max_k \Re \lambda_k(A), \beta(A) = \min_k \Re \lambda_k(A) \); \( r_l(A) \) is the lower spectral radius.
r_1(A) = \min_{k=1,\ldots,n} |\lambda_k(A)|.

In addition, \(C^{n \times n}\) is the set of all complex \(n \times n\)-matrices.

The following quantity plays an essential role in the sequel:

\[
g(A) = \left( N^2_2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2 \right)^{1/2}.
\]

It sometimes called the departure from normality, cf. Stewart and Sun Ji-guang (1990). Since

\[
\sum_{k=1}^{n} |\lambda_k(A)|^2 \geq |\text{Trace } A|^2,
\]

one can write

\[
g^2(A) \leq N^2_2(A) - |\text{Trace } A|^2. \quad (1.1)
\]

In Sect. 2.2 of the book Gil’ (2003) it is proved that

\[
g^2(A) \leq 2N^2_2(A) = N^2_2(\lambda - \lambda^*)/2 \quad (1.2)
\]

and

\[
g(e^{i\tau} A + zI) = g(A) \quad (1.3)
\]

for all \(\tau \in \mathbb{R}\) and \(z \in \mathbb{C}\). If \(A\) is a normal matrix: \(AA^* = A^*A\), then \(g(A) = 0\).

If \(A_1\) and \(A_2\) are commuting matrices, then

\[
g(A_1 + A_2) \leq g(A_1) + g(A_2), \quad (1.4)
\]

cf. Gil’ (2003, Sect. 2.1). By the inequality between geometric and arithmetic mean values,

\[
\left( \frac{1}{n} \sum_{k=1}^{n} |\lambda_k(A)|^2 \right)^n \leq \left( \prod_{k=1}^{n} |\lambda_k(A)| \right)^2.
\]

So

\[
g^2(A) \leq N^2_2(A) - n(\text{det } A)^2/n. \quad (1.5)
\]

By Schur’s theorem (Marcus and Minc 1964, Sect. I.4.10.2), for any operator \(A\) in \(C^n\),

there is an orthogonal normal basis (Schur’s basis) \(\{e_k\}_{k=1}^{n}\), in which \(A\) is represented

by triangular matrix. That is,
2.1 Some Definitions

\[ Ae_k = \sum_{j=1}^{k} a_{jk} e_j \text{ with } a_{jk} = (Ae_k, e_j) \ (j = 1, \ldots, n), \]

and \( a_{jj} = \lambda_j(A) \). So

\[ A = D + V \ (\sigma(A) = \sigma(D)) \quad (1.6) \]

with a normal (diagonal) operator \( D \) defined by \( De_j = \lambda_j(A)e_j \ (j = 1, \ldots, n) \) and a nilpotent (strictly upper-triangular) operator \( V \) defined by

\[ Ve_k = \sum_{j=1}^{k-1} a_{jk} e_j \ (k = 2, \ldots, n). \]

\( D \) and \( V \) will be called the diagonal part and the nilpotent part of \( A \), respectively.

As it is shown in Gil’ (2003, Lemma 2.3.2), the relation

\[ g(A) = N_2(V) \quad (1.7) \]

is true. Hence it follows that

\[ g(U^{-1}AU) = g(A) \text{ for any unitary matrix } U. \quad (1.8) \]

2.2 Representations of Matrix Functions

In this section we recall some classical representations of functions of matrices. For details see Horn and Johnson (1985, Chap. 6).

Let \( A \in \mathbb{C}^{n \times n} \) and \( M \supset \sigma(A) \) be an open simply-connected set whose boundary \( C \) consists of a finite number of rectifiable Jordan curves, oriented in the positive sense customary in the theory of complex variables. Suppose that \( M \cup C \) is contained in the domain of analyticity of a scalar-valued function \( f \). Then \( f(A) \) can be defined by the generalized integral formula of Cauchy

\[ f(A) = -\frac{1}{2\pi i} \int_{C} f(\lambda) R_\lambda(A) d\lambda. \quad (2.1) \]

If an analytic function \( f(\lambda) \) is represented by the Taylor series

\[ f(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^k \left( |\lambda| < \lim_{k \to \infty} \sqrt[k]{|c_k|} \right), \]
then one can define $f(A)$ as

$$f(A) = \sum_{k=0}^{\infty} c_k A^k$$

provided the spectral radius $r_s(A)$ of $A$ satisfies the inequality

$$r_s(A) \lim_{k \to \infty} \sqrt[k]{|c_k|} < 1.$$ 

In particular, for any matrix $A$, one has

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$ 

Consider the $n \times n$-Jordan block:

$$J_n(\lambda_0) = \begin{pmatrix}
\lambda_0 & 1 & 0 & \ldots & 0 \\
0 & \lambda_0 & 1 & \ldots & 0 \\
& \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_0 & 1 \\
0 & 0 & \ldots & 0 & \lambda_0
\end{pmatrix},$$

then

$$f(J_n(\lambda_0)) = \begin{pmatrix}
f(\lambda_0) & \frac{f(\lambda_0)}{1!} & \ldots & \frac{f^{(n-1)}(\lambda_0)}{(n-1)!} \\
0 & f(\lambda_0) & \ldots & \vdots \\
& \ddots & \ddots & \ddots \\
0 & \ldots & f(\lambda_0) & \frac{f'(\lambda_0)}{1!} \\
0 & \ldots & 0 & f(\lambda_0)
\end{pmatrix}.$$ 

Thus, if $A$ has the Jordan block-diagonal form

$$A = \text{diag} \left( J_{m_1}(\lambda_1), J_{m_2}(\lambda_2), \ldots, J_{m_{n_0}}(\lambda_{n_0}) \right),$$

where $\lambda_k, k = 1, \ldots, n_0$ are the eigenvalues whose geometric multiplicities are $m_k$, then

$$f(A) = \text{diag} \left( f(J_{m_1}(\lambda_1)), f(J_{m_2}(\lambda_2)), \ldots, f(J_{m_{n_0}}(\lambda_{n_0})) \right).$$ (2.2)
Note that in (2.2) we do not require that $f$ is regular on a neighborhood of an open set containing $\sigma(A)$; one can take an arbitrary function which has at each $\lambda_k$ derivatives up to $m_k - 1$-order.

In particular, if an $n \times n$-matrix $A$ is diagonalizable, that is its eigenvalues have the geometric multiplicities $m_k \equiv 1$, then

$$f(A) = \sum_{k=1}^{n} f(\lambda_k) Q_k,$$

(2.3)

where $Q_k$ are the eigenprojections. In the case (2.3) it is required only that $f$ is defined on the spectrum.

Now let

$$\sigma(A) = \bigcup_{k=1}^{m} \sigma_k(A) \quad (m \leq n)$$

and $\sigma_k(A) \subset M_k \ (k = 1, \ldots, m)$, where $M_k$ are open disjoint simply-connected sets: $M_k \cap M_j = \emptyset \ (j \neq k)$. Let $f_k$ be regular on $M_k$. Introduce on $M = \bigcup_{k=1}^{m} M_k$ the piece-wise analytic function by $f(z) = f_j(z) \ (z \in M_j)$. Then

$$f(A) = -\frac{1}{2\pi i} \sum_{j=1}^{m} \int_{C_j} f(\lambda) R_\lambda(A) d\lambda,$$

(2.4)

where $C_j \subset M_j$ are closed smooth contours surrounding $\sigma(A_j)$ and the integration is performed in the positive direction. For more details about representation (2.4) see Yakubovich and Starzhinskii (1975, p. 49).

For instance, let $M_1$ and $M_2$ be two disjoint disks, and

$$f(z) = \begin{cases} 
sin z & \text{if } z \in M_1, \\
cos z & \text{if } z \in M_2. \end{cases}$$

Then (2.4) holds with $m = 2$.

### 2.3 Norm Estimates for Resolvents

For a natural number $n \geq 2$, introduce the numbers

$$\gamma_{n,k} = \sqrt{\frac{(n-1)(n-2) \cdots (n-k)}{(n-1)^kk!}} \quad (k = 1, 2, \ldots, n-1), \ \gamma_{n,0} = 1.$$
Evidently, for all \( n > 2 \),

\[
\gamma_{n,k}^2 \leq \frac{1}{k!} \quad (k = 1, 2, \ldots, n - 1). \tag{3.1}
\]

Recall that \( g(A) \) is defined in Sect. 2.1.

**Theorem 2.1** Let \( A \) be a linear operator in \( \mathbb{C}^n \). Then its resolvent satisfies the inequality

\[
\|R_{\lambda}(A)\| \leq \sum_{k=0}^{n-1} \frac{g_k(A)\gamma_{n,k}}{\rho^{k+1}(A, \lambda)} \quad (\lambda \not\in \sigma(A)),
\]

where \( \rho(A, \lambda) = \min_{k=1, \ldots, n} |\lambda - \lambda_k(A)|. \)

This theorem is proved in Gil’ (2003, Theorem 2.1.1). Theorem 2.1 is sharp: if \( A \) is a normal matrix, then \( g(A) = 0 \) and Theorem 2.1 gives us the equality \( \|R_{\lambda}(A)\| = 1/\rho(A, \lambda) \). Taking into account (3.1), we get

**Corollary 2.1** Let \( A \in \mathbb{C}^{n \times n} \). Then

\[
\|R_{\lambda}(A)\| \leq \sum_{k=0}^{n-1} \frac{g_k(A)}{\sqrt{k!}\rho^{k+1}(A, \lambda)} \quad \text{for any regular } \lambda \text{ of } A.
\]

In particular, if \( A \) is invertible, then

\[
\|A^{-1}\| \leq \sum_{k=0}^{n-1} \frac{g_k(A)}{\sqrt{k!}r_1^{k+1}(A)},
\]

where \( r_1(A) = \rho(A, 0) = \min_{k=1, \ldots, n} |\lambda_k(A)|. \)

We will need also the following result.

**Theorem 2.2** Let \( A \in \mathbb{C}^{n \times n} \). Then

\[
\|R_{\lambda}(A) \det(\lambda I - A)\| \leq \left[ \frac{N_2^n(A) - 2\Re(\lambda) \, \text{Trace}(A)}{n - 1} + n|\lambda|^2 \right]^{(n-1)/2} \quad (\lambda \not\in \sigma(A)).
\]

In particular,

\[
\|A^{-1}\| \leq \frac{N_2^{n-1}(A)}{(n - 1)^{(n-1)/2} |\det(A)|}
\]

for any invertible \( A \in \mathbb{C}^{n \times n} \).
The proof of this theorem can be found in Gil’ (2003, Sect. 2.11).

We also mention the following result.

**Theorem 2.3** Let $A \in \mathbb{C}^{n \times n}$. Then

$$\| R_\lambda(A) \| \leq \frac{1}{\rho(A, \lambda)} \left[ 1 + \frac{1}{n-1} \left( 1 + \frac{g^2(A)}{\rho^2(A, \lambda)} \right) \right]^{(n-1)/2}$$

for any regular $\lambda$ of $A$.

For the proof see Gil’ (2003, Theorem 2.14.1).

**2.4 Spectral Variations of Matrices**

Let $A$ and $B$ be $n \times n$-matrices having eigenvalues

$$\lambda_1(A), \ldots, \lambda_n(A) \text{ and } \lambda_1(B), \ldots, \lambda_n(B),$$

respectively, and $q = \| A - B \|$.

The spectral variation of $B$ with respect to $A$ is

$$\text{sv}_A(B) := \max_i \min_j |\lambda_i(B) - \lambda_j(A)|,$$


The following simple lemma is proved in Gil’ 2003, Sect. 4.1.

**Lemma 2.1** Assume that

$$\| R_\lambda(A) \| \leq \phi \left( \frac{1}{\rho(A, \lambda)} \right)$$

for all regular $\lambda$ of $A$, where $\phi(x)$ is a monotonically increasing non-negative continuous function of a non-negative variable $x$, such that $\phi(0) = 0$ and $\phi(\infty) = \infty$. Then the inequality $\text{sv}_A(B) \leq z(\phi, q)$ is true, where $z(\phi, q)$ is the unique positive root of the equation $q\phi(1/z) = 1$.

This Lemma and Corollary 2.1 yield our next result.

**Theorem 2.4** Let $A$ and $B$ be $n \times n$-matrices. Then $\text{sv}_A(B) \leq z(q, A)$, where $z(q, A)$ is the unique nonnegative root of the algebraic equation

$$y^n = q \sum_{j=0}^{n-1} \frac{y^{n-j-1} g^j(A)}{\sqrt{j!}}.$$  \(\text{(4.1)}\)
This theorem is sharp: if \( A \) is normal, then \( g(A) = 0 \) and \( z(q, A) = q \). So the theorem gives us the inequality \( sv_{A}(B) \leq q \). But \( sv_{A}(B) = q \) in this case, cf. Stewart and Sun Ji-guang (1990).

Let us consider the algebraic equation

\[
z^n = p(z) \quad (n > 1), \quad \text{where} \quad p(z) = \sum_{j=0}^{n-1} c_j z^{n-j-1} \tag{4.2}
\]

with non-negative coefficients \( c_j \) (\( j = 0, \ldots, n - 1 \)).

**Lemma 2.2** The unique positive root \( z_0 \) of Eq. (4.2) is subject to the following estimates:

\[
z_0 \leq \sqrt[n]{p(1)} \text{ if } p(1) \leq 1, \tag{4.3}
\]

and

\[
1 \leq z_0 \leq p(1) \text{ if } p(1) \geq 1. \tag{4.4}
\]

**Proof** Since all the coefficients of \( p(z) \) are non-negative, it does not decrease as \( z > 0 \) increases. From this it follows that if \( p(1) \leq 1 \), then \( z_0 \leq 1 \). So \( z_0^n \leq p(1) \), as claimed.

Now let \( p(1) \geq 1 \), then due to (4.2) \( z_0 \geq 1 \) because \( p(z) \) does not decrease. It is clear that

\[
p(z_0) \leq z_0^{n-1} p(1)
\]

in this case. Substituting this inequality into (4.2), we get (4.4).

Substituting \( z = ax \) with a positive constant \( a \) into (4.2), we obtain

\[
x^n = \sum_{j=0}^{n-1} \frac{c_j}{a^{j+1}} x^{n-j-1}. \tag{4.5}
\]

Let

\[
a = 2 \max_{j=0,\ldots,n-1} \sqrt[j+1]{c_j}.
\]

Then

\[
\sum_{j=0}^{n-1} \frac{c_j}{a^{j+1}} \leq \sum_{j=0}^{n-1} 2^{-j-1} = 1 - 2^{-n} < 1.
\]

Let \( x_0 \) be the extreme right-hand root of Eq. (4.5), then by (4.3) we have \( x_0 \leq 1 \). Since \( z_0 = ax_0 \), we have derived the following result.
Corollary 2.2 The unique nonnegative root $z_0$ of Eq. (4.2) satisfies the inequality

$$z_0 \leq 2 \max_{j=0,\ldots,n-1} j+1 \sqrt{c_j}.$$ 

Now put $y = xg(A)$ into (4.1). Then we obtain the equation

$$x^n = \frac{q}{g(A)} \sum_{j=0}^{n-1} x^{n-j-1} \sqrt{j!}.$$  \hfill (4.6)

Since

$$\max_{j=0,\ldots,n-1} j+1 \sqrt{\frac{q}{g(A)}} = \frac{q}{g(A)}$$

if $q \geq g(A)$, and

$$\max_{j=0,\ldots,n-1} \frac{j+1}{\sqrt{g(A)}} = \frac{q}{\sqrt{g(A)}}$$

if $q \leq g(A)$. Applying Corollary 2.2, we get the estimate $z(q,A) \leq \delta(q)$, where

$$\delta(q) := \begin{cases} 2q & \text{if } q \geq g(A) \\ 2g^{1-1/n}(A)q^{1/n} & \text{if } q \leq g(A) \end{cases}.$$ 

Now Theorem 2.4 ensures the following result.

Corollary 2.3 One has

$$sv_A(B) \leq \delta(q).$$ \hfill (4.7)

About other estimates for $sv_A(B)$ see Comments.

2.5 Norm Estimates for Matrix Functions

2.5.1 Estimates via the Resolvent

The following result directly follows from (2.1).

Lemma 2.3 Let $f(\lambda)$ be a scalar-valued function which is regular on a neighborhood $M$ of an open simply-connected set containing the spectrum of $A \in \mathbb{C}^{n \times n}$, and $C \subset M$ be a closed smooth contour surrounding $\sigma(A)$. Then

$$\|f(A)\| \leq \frac{1}{2\pi} \int_C |f(z)||R_z(A)||dz \leq m_C(A)\|C\sup_{z \in C}|f(z)|,$$
where
\[ m_C(A) := \sup_{z \in C} \| R_z(A) \|, \quad l_C := \frac{1}{2\pi} \int_C |dz|. \]

Now we can directly apply the estimates for the resolvent from Sect. 2.3. In particular, by Corollary 2.1 we have
\[ \| R_z(A) \| \leq p(A, 1/\rho(A, z)), \quad (5.1) \]
where
\[ p(A, x) = \sum_{k=0}^{n-1} \frac{x^{k+1} s^k(A)}{\sqrt{k!}} \quad (x > 0). \]

We thus get \( m_C(A) \leq p(A, 1/\rho(A, C)) \), where \( \rho(A, C) \) is the distance between \( C \) and \( \sigma(A) \), and therefore, we arrive at our next result.

**Corollary 2.4** Let \( f(\lambda) \) be a scalar-valued function which is regular on a neighborhood \( M \) of an open simply-connected set containing the spectrum of \( A \in \mathbb{C}^{n \times n} \), and \( C \subset M \) be a closed smooth contour surrounding \( \sigma(A) \). Then
\[ \| f(A) \| \leq b(A, C) \sup_{z \in C} |f(z)|, \]
where \( b(A, C) = l_C p(A, 1/\rho(A, C)) \). In particular,
\[ \| A^m \| \leq b(A, C) \sup_{z \in C} |z|^m \quad (m = 1, 2, \ldots) \]
and
\[ \| e^{At} \| \leq b(A, C)e^{\alpha_C t} \quad (t \geq 0), \]
where \( \alpha_C = \sup_{z \in C} \text{Re } z. \)

### 2.5.2 Functions Regular on the Convex Hull of the Spectrum

In this subsection, under additional conditions, we make the results of the previous subsection sharper although obtained here estimates are less convenient than the previous corollary.

**Theorem 2.5** Let \( A \) be an \( n \times n \)-matrix and \( f \) be a function holomorphic on a neighborhood of the convex hull \( co(A) \) of \( \sigma(A) \). Then
\[ \| f(A) \| \leq \sup_{\lambda \in \sigma(A)} |f(\lambda)| + \sum_{k=1}^{n-1} \sup_{\lambda \in \sigma(A)} |f^{(k)}(\lambda)| \frac{\gamma_n k g_k^k(A)}{k!}. \]

In particular
\[ \| \exp(At) \| \leq e^{\alpha(A)t} \sum_{k=0}^{n-1} g_k^k(A) r^k(A) \frac{\gamma_n k}{k!} \] 
for \( t \geq 0 \)

where \( \alpha(A) = \max_{k=1, \ldots, n} \text{Re} \lambda_k(A) \). In addition,
\[ \| A^m \| \leq \sum_{k=0}^{n-1} \gamma_n g_k^k(A) \frac{r_s^m(A)}{(m-k)!k!} \] 
for \( m = 1, 2, \ldots \)

where \( r_s(A) \) is the spectral radius.

Recall that \( 1/(m-k)! = 0 \) if \( m < k \).

This theorem is proved in the next subsection. It is a slight improvement of Theorem 2.7.1 from Gil' (2003).

Taking into account (3.1) we get our next result.

**Corollary 2.5** Under the hypothesis of Theorem 2.5 we have
\[ \| f(A) \| \leq \sup_{\lambda \in \sigma(A)} |f(\lambda)| + \sum_{k=1}^{n-1} \sup_{\lambda \in \sigma(A)} |f^{(k)}(\lambda)| \frac{g_k^k(A)}{(k!)^{3/2}}. \]

In particular,
\[ \| \exp(At) \| \leq e^{\alpha(A)t} \sum_{k=0}^{n-1} \frac{g_k^k(A)}{(k!)^{3/2}} \] 
for \( t \geq 0 \)

and
\[ \| A^m \| \leq \sum_{k=0}^{m-1} \frac{m! g_k^k(A) r_s^{m-k}(A)}{(m-k)! (k!)^{3/2}} \] 
for \( m = 1, 2, \ldots \).

Theorem 2.5 is sharp: if \( A \) is normal, then \( g(A) = 0 \) and
\[ \| f(A) \| = \sup_{\lambda \in \sigma(A)} |f(\lambda)|. \]
2.5.3 Proof of Theorem 2.5

Let $|V|_e$ be the operator whose entries in the orthonormal basis of the triangular representation (the Schur basis) $\{e_k\}$ are the absolute values of the entries of the nilpotent part $V$ of $A$ with respect to this basis (see Sect. 2.1). That is,

$$|V|_e = \sum_{k=1}^{n} \sum_{j=1}^{k-1} |a_{jk}|(., e_k)e_j,$$

where $a_{jk} = (Ae_k, e_j)$. Put

$$I_{j_1\ldots j_{k+1}} = \frac{(-1)^{k+1}}{2\pi i} \int_C \frac{f(\lambda) d\lambda}{(\lambda_{j_1} - \lambda) \cdots (\lambda_{j_{k+1}} - \lambda)}.$$

We need the following result.

**Lemma 2.4** Let $A$ be an $n \times n$-matrix and $f$ be a holomorphic function on a Jordan domain (that is on a closed simply connected set, whose boundary is a Jordan contour), containing $\sigma(A)$. Let $D$ be the diagonal part of $A$. Then

$$\| f(A) - f(D) \| \leq \sum_{k=1}^{n-1} J_k \| |V|_e^k \|,$$

where

$$J_k = \max \{|I_{j_1\ldots j_{k+1}}| : 1 \leq j_1 < \cdots < j_{k+1} \leq n \}.$$

**Proof** From (2.1) and (1.6) we deduce that

$$f(A) - f(D) = -\frac{1}{2\pi i} \int_C f(\lambda)(R_\lambda(A) - R_\lambda(D)) d\lambda = \sum_{k=1}^{n-1} B_k,$$

where

$$B_k = (-1)^{k+1} \frac{1}{2\pi i} \int_C f(\lambda)(R_\lambda(D)V)^k R_\lambda(D)d\lambda.$$

Since $D$ is a diagonal matrix with respect to the Schur basis $\{e_k\}$ and its diagonal entries are the eigenvalues of $A$, then

$$R_\lambda(D) = \sum_{j=1}^{n} \frac{\Delta P_j}{\lambda_j(A) - \lambda},$$
where $\Delta P_k = (., e_k)e_k$. In addition, $\Delta P_j V \Delta P_k = 0$ for $j \geq k$. Consequently,

$$B_k = \sum_{j_1=1}^{j_2-1} \Delta P_{j_1} V \sum_{j_2=1}^{j_3-1} \Delta P_{j_2} V \cdots \sum_{j_k=1}^{j_{k+1}-1} V \sum_{j_{k+1}=1}^{n} \Delta P_{j_{k+1}} I_{j_1 j_2 \cdots j_{k+1}}.$$

Lemma 2.8.1 from Gil’ (2003) gives us the estimate

$$\| B_k \| \leq J_k \| \sum_{j_1=1}^{j_2-1} \Delta P_{j_1} |V| e \sum_{j_2=1}^{j_3-1} \Delta P_{j_2} |V| e \cdots \sum_{j_k=1}^{j_{k+1}-1} |V| e \sum_{j_{k+1}=1}^{n} \Delta P_{j_{k+1}} \|$$

But

$$P_{n-k} |V| e P_{n-k+1} |V| e P_{n-k+2} \cdots P_{n-1} |V| e = |V| e P_{n-k+1} |V| e P_{n-k+2} \cdots P_{n-1} |V| e$$

Thus

$$\| B_k \| \leq J_k \| |V| e^k \|. $$

This inequality and (5.2) imply the required inequality. □

Thanks to Theorem 2.5.1 from the book (Gil’ 2003), for any $n \times n$ nilpotent matrix $V_0$,

$$\| V_0^k \| \leq \gamma_{n,k} N_{1/2}^k (V_0) \ (k = 1, 2, \ldots, n - 1). \quad (5.3)$$

Since $N_2(|V| e) = N_2(V) = g(A)$, by (5.3) and the previous lemma we get the following result.

**Lemma 2.5** Under the hypothesis of Lemma 2.4 we have

$$\| f(A) - f(D) \| \leq \sum_{k=1}^{n-1} J_k \gamma_{n,k} g^k (A).$$

Let $f$ be holomorphic on a neighborhood of $\text{co}(A)$. Thanks to Lemma 1.5.1 from Gil’ (2003),

$$J_k \leq \frac{1}{k!} \sup_{\lambda \in \text{co}(A)} |f^{(k)}(\lambda)|.$$

Now the previous lemma implies
Corollary 2.6 Under the hypothesis of Theorem 2.5 we have the inequality

\[ \|f(A) - f(D)\| \leq \sum_{k=1}^{n-1} \sup_{\lambda \in co(A)} |f^{(k)}(\lambda)| \gamma_n, k g^k(A) \frac{g^k}{k!}. \]

The assertion of Theorem 2.5 directly follows from the previous corollary.

Let us point to an additional estimate, which may be more convenient than Corollary 2.6 in the case when is \( n \) rather small. Denote by \( f[a_1, a_2, \ldots, a_{k+1}] \) the \( k \)-th divided difference of \( f \) at points \( a_1, a_2, \ldots, a_{k+1} \). By the Hadamard representation (Gel’fond 1967, formula (54)), we have

\[ I_{j_1 \ldots j_{k+1}} = f[\lambda_1, \ldots, \lambda_{j_{k+1}}], \]

provided all the eigenvalues \( \lambda_j = \lambda_j(A) \) are distinct. Now Lemma 2.4 implies

Corollary 2.7 Let all the eigenvalues of an \( n \times n \)-matrix \( A \) be algebraically simple, and \( f \) be a holomorphic function in a Jordan domain containing \( \sigma(A) \). Then

\[ \|f(A) - f(D)\| \leq \sum_{k=1}^{n-1} f_k \gamma_n, k g^k(A) \leq \sum_{k=1}^{n-1} f_k \frac{g^k}{\sqrt{k!}}, \]

where

\[ f_k = \max\{|f[\lambda_1(A), \ldots, \lambda_{j_{k+1}}(A)]| : 1 < j_1 < \cdots < j_{k+1} \leq n\}. \]

2.6 Absolute Values of Elements of Matrix Functions

2.6.1 Statement of the Result

In this section we suggest bounds for the entries of a matrix function. Below we show that these bounds can be applied to the stability analysis via the generalized (vector) norms.

Everywhere in the present section, \( A = (a_{jk})_{j,k=1}^n \), \( S = \text{diag} [a_{11}, \ldots, a_{nn}] \) and the off diagonal of \( A \) is \( W = A - S \). That is, the entries \( v_{jk} \) of \( W \) are \( v_{jk} = a_{jk} \) \((j \neq k)\) and \( v_{jj} = 0 \) \((j, k = 1, 2, \ldots)\). Denote by \( co(S) \) the closed convex hull of the diagonal entries \( a_{11}, \ldots, a_{nn} \). We put \( |A| = (|a_{jk}|)_{j,k=1}^n \), i.e. \( |A| \) is the matrix whose entries are the absolute values of the entries \( A \) in the standard basis. We also write \( T \geq 0 \) if all the entries of a matrix \( T \) are nonnegative. If \( T \) and \( B \) are two matrices, then we write \( T \geq B \) if \( T - B \geq 0 \).

Thanks to the Gerschgorin bound for the eigenvalues (see Sect. 2.11) we have \( r_s(|W|) \leq \tau_W \), where
\[ \tau_W := \max_j \sum_{k=1}^n |a_{jk}|. \]

**Theorem 2.6** Let \( f(\lambda) \) be holomorphic on a neighborhood of a Jordan set, whose boundary \( C \) has the property

\[ |z - a_{jj}| > \sum_{k=1}^n |a_{jk}| \quad (6.1) \]

for all \( z \in C \) and \( j = 1, \ldots, n \). Then, with the notation

\[ \xi_k(A) := \sup_{z \in \co(S)} \frac{|f^{(k)}(z)|}{k!} \quad (k = 1, 2, \ldots), \]

the inequality

\[ |f(A) - f(S)| \leq \sum_{k=1}^{\infty} \xi_k(A) |W|^k \]

is valid, provided

\[ r_s(|W|) \lim_{k \to \infty} \sqrt[k]{\xi_k(A)} < 1. \]

This theorem is proved in the next subsection.

**Corollary 2.8** Under the hypothesis of the previous theorem with the notation

\[ \xi_0(A) := \max_k |f(a_{kk})|, \]

we have the inequality

\[ |f(A)| \leq \xi_0(A) I + \sum_{k=1}^{\infty} \xi_k(A) |W|^k = \sum_{k=1}^{\infty} \xi_k(A) |W|^k. \]

Here \( |W|^0 = I \). In particular,

\[ |A^m| \leq |A|^m \leq \sum_{k=1}^m \frac{m!}{(m-k)k!} \max_j |a_{jj}|^{m-k} |W|^k = (\max_j |a_{jj}| + |W|)^m \]

and

\[ |e^{At}| \leq e^{\alpha(S)t + |W|t} \sum_{k=1}^{\infty} \frac{t^k}{k!} |W|^k = e^{(\alpha(S)+|W|)t} \quad (t > 0) \]

where \( \alpha(S) = \max_k \Re a_{kk} \).
Let $\|A\|_l$ denote a lattice norm of $A$. That is, $\|A\|_l \leq ||A||_l$, and $\|A\|_l \leq \|\tilde{A}\|_l$ whenever $0 \leq A \leq \tilde{A}$. Now the previous theorem implies the inequality

$$\| f(A) - f(S) \|_l \leq \sum_{k=1}^{\infty} \xi_k(A) ||W|^k \|_l$$

and therefore,

$$\| f(A) \|_l \leq \sum_{k=1}^{\infty} \xi_k(A) ||W|^k \|_l.$$ 

Additional estimates for the entries of matrix functions can be found in the paper (Gil’ 2013e) and references given therein.

### 2.6.2 Proof of Theorem 2.6

By the equality $A = S + W$ we get

$$R_\lambda(A) = (S + W - \lambda I)^{-1} = (I + R_\lambda(S)W)^{-1} R_\lambda(S)$$

$$= \sum_{k=0}^{\infty} (R_\lambda(S)W)^k (-1)^k R_\lambda(S),$$

provided the spectral radius $r_0(\lambda)$ of $R_\lambda(S)W$ is less than one. The entries of this matrix are

$$\frac{a_{jk}}{a_{jj} - \lambda} \quad (\lambda \neq a_{jj}, \ j \neq k)$$

and the diagonal entries are zero. Thanks to (6.1) and the Gerschgorin bound for the eigenvalues from Sect. 2.11 of the present book, we have

$$r_0(\lambda) \leq \max_j \sum_{k=1}^{n} \frac{|a_{jk}|}{|a_{jj} - \lambda|} < 1 \ (\lambda \in C)$$

and therefore, the series

$$R_\lambda(A) - R_\lambda(S) = \sum_{k=1}^{\infty} (R_\lambda(S)W)^k (-1)^k R_\lambda(S)$$
converges. Thus
\[ f(A) - f(S) = -\frac{1}{2\pi i} \int_C f(\lambda)(R_\lambda(A) - R_\lambda(S))d\lambda = \sum_{k=1}^{\infty} M_k, \quad (6.2) \]

where
\[ M_k = (-1)^{k+1} \frac{1}{2\pi i} \int_C f(\lambda)(R_\lambda(S)W)^k R_\lambda(S)d\lambda. \]

Since \( S \) is a diagonal matrix with respect to the standard basis \( \{d_k\} \), we can write out
\[ R_\lambda(S) = \sum_{j=1}^{n} \hat{Q}_j \frac{1}{b_j - \lambda}(b_j = a_{jj}), \]

where \( \hat{Q}_k = (.,d_k)d_k \). We thus have
\[ M_k = \sum_{j_1=1}^{n} \hat{Q}_{j_1} W \sum_{j_2=1}^{n} \hat{Q}_{j_2} W \cdots W \sum_{j_{k+1}=1}^{n} \hat{Q}_{j_{k+1}} J_{j_1,j_2,\ldots,j_{k+1}}. \quad (6.3) \]

Here
\[ J_{j_1\cdots j_{k+1}} = \frac{(-1)^{k+1}}{2\pi i} \int_C \frac{f(\lambda)d\lambda}{(b_{j_1} - \lambda) \cdots (b_{j_{k+1}} - \lambda)}. \]

Lemma 1.5.1 from Gil’ (2003) gives us the inequalities
\[ |J_{j_1\cdots j_{k+1}}| \leq \xi_k(A) \quad (j_1, j_2, \ldots, j_{k+1} = 1, \ldots, n). \]

Hence, by (6.3)
\[ |M_k| \leq \xi_k(A) \sum_{j_1=1}^{n} \hat{Q}_{j_1} |W| \sum_{j_2=1}^{n} \hat{Q}_{j_2} |W| \cdots |W| \sum_{j_{k+1}=1}^{n} \hat{Q}_{j_{k+1}}. \]

But
\[ \sum_{j_1=1}^{n} \hat{Q}_{j_1} |W| \sum_{j_2=1}^{n} \hat{Q}_{j_2} |W| \cdots |W| \sum_{j_{k+1}=1}^{n} \hat{Q}_{j_{k+1}} = |W|^k. \]

Thus \( |M_k| \leq \xi_k(A)|W|^k \). Now (6.2) implies the required result. \( \square \)
2.7 Diagonalizable Matrices

Everywhere in this section it is assumed that the eigenvalues $\lambda_k = \lambda_k(A)$ ($k = 1, \ldots, n$) of $A$, taken with their algebraic multiplicities, are geometrically simple. That is, the geometric multiplicity of each eigenvalue is equal to one. As it is well-known, in this case $A$ is diagonalizable: there are biorthogonal sequences $\{u_k\}$ and $\{v_k\}$: $(v_j, u_k) = 0$ ($j \neq k$), $(v_j, u_j) = 1$ ($j, k = 1, \ldots, n$), such that

$$
A = \sum_{k=1}^{n} \lambda_k Q_k, \tag{7.1}
$$

where $Q_k = (., u_k)v_k$ ($k = 1, \ldots, n$) are one dimensional eigenprojections. Besides, there is an invertible operator $T$ and a normal operator $S$, such that

$$
TA = ST. \tag{7.2}
$$

The constant (the condition number)

$$
\kappa(A, T) = \kappa(A) := \|T\|\|T^{-1}\|
$$

is very important for various applications, cf. Stewart and Sun Ji-guang (1990). That constant is mainly numerically calculated.

Making use equality (7.2) it is not hard to prove the following lemma.

**Lemma 2.6** Let $A$ be a diagonalizable $n \times n$-matrix and $f(z)$ be a scalar function defined on the spectrum of $A$. Then $\|f(A)\| \leq \kappa(A) \max_k |f(\lambda_k)|$.

In particular, we have

$$
\|Rz(A)\| \leq \frac{\kappa(A)}{\rho(A, \lambda)},
$$

$$
\|e^{At}\| \leq \kappa(A)e^{\rho(A)t} \quad (t \geq 0) \tag{7.3}
$$

and

$$
\|A^m\| \leq \kappa(A)\rho_s^m(A) \quad (m = 1, 2, \ldots).
$$

Let $A$ and $\tilde{A}$ be complex $n \times n$-matrices whose eigenvalues $\lambda_k$ and $\tilde{\lambda}_k$, respectively, are taken with their algebraic multiplicities. Recall that

$$
sv_A(\tilde{A}) := \max_k \min_j |\tilde{\lambda}_k - \lambda_j|.
$$

**Corollary 2.9** Let $A$ be diagonalizable. Then $sv_A(\tilde{A}) \leq \kappa(A)\|A - \tilde{A}\|$.
2.7 Diagonalizable Matrices

Indeed, the operator $S = TAT^{-1}$ is normal. Put $B = T\tilde{A}T^{-1}$. Thanks to the well-known Corollary 3.4 from Stewart and Sun Ji-guang (1990), one has $sv_S(B) \leq \|S - B\|$. Hence we get the required result.

Furthermore let $\tilde{D}, V_+$ and $V_-$ be the diagonal, upper nilpotent part and lower nilpotent part of matrix $A = (a_{kk})$, respectively. Using the preceding corollary with $A_+ = \tilde{D} + V_+$, we arrive at the relations

$$\sigma(A_+) = \sigma(\tilde{D}), \text{ and } \|A - A_+\| = \|V_-\|.$$ Due to the previous corollary we get

**Corollary 2.10** Let $A = (a_{jk})_{j,k=1}^n$ be an $n \times n$-matrix, whose diagonal has the property

$$a_{jj} \neq a_{kk} \ (j \neq k; \ k = 1, \ldots, n).$$

Then for any eigenvalue $\mu$ of $A$, there is a $k = 1, \ldots, n$, such that

$$|\mu - a_{kk}| \leq \kappa(A_+)\|V_-\|,$$

and therefore the (upper) spectral radius satisfies the inequality

$$r_s(A) \leq \max_{k=1,\ldots,n} |a_{kk}| + \kappa(A_+)\|V_-\|,$$

and the lower spectral radius satisfies the inequality

$$r_s(A) \geq \min_{k=1,\ldots,n} |a_{kk}| - \kappa(A_+)\|V_-\|,$$

provided $|a_{kk}| > \kappa(A_+)\|V_-\|$ ($k = 1, \ldots, n$).

Clearly, one can exchange the places of $V_+$ and $V_-$. Let us point an estimate for the condition number $\kappa(A)$ in the case

$$\lambda_j(A) \neq \lambda_m(A) \text{ whenever } j \neq m. \quad (7.4)$$

To this end put

$$\delta := \min_{j \neq m} |\lambda_j(A) - \lambda_m(A)|, \quad \tau(A) := \sum_{k=1}^{n-2} \frac{g^{k+1}(A)}{\sqrt{k!}k^{k+1}}$$

and

$$\gamma(A) := \left(1 + \frac{\tau(A)}{\sqrt{n-1}}\right)^{2(n-1)}.$$
Theorem 2.7 Let condition (7.4) be fulfilled. Then there is an invertible matrix $T$, such that (7.2) holds with $\kappa(A) \leq \gamma(A)$.

The proof of this theorem is presented in Gil’ (2014). It is sharp: if $A$ is normal, then $g(A) = 0$ and $\gamma(A) = 1$. Thus we obtain the equality $\kappa(A) = 1$.

2.8 Perturbations of Matrix Exponentials

The matrix exponential plays an essential role in the stability analysis of NDEs whose regular parts are close to ordinary differential equations.

As it was shown in Sect. 2.5, 

$$
\|\exp(At)\| \leq e^{\alpha(A)t} \sum_{k=0}^{n-1} g^k(A) t^k \frac{\gamma_n k}{k!} \quad (t \geq 0)
$$

where $\alpha(A) = \max_{k=1,\ldots,n} \text{Re} \lambda_k(A)$. Moreover, by (3.1),

$$
\|\exp(At)\| \leq e^{\alpha(A)t} \frac{\sum_{k=0}^{n-1} g^k(A) t^k}{(k!)^{3/2}} \quad (t \geq 0).
$$

(8.1)

Taking into account that the operator $\exp(-At)$ is the inverse one to $\exp(At)$ it is not hard to show that

$$
\|\exp(At)h\| \geq \frac{e^{\beta(A)t} \|h\|}{\sum_{k=0}^{n-1} g^k(A) t^k (k!)^{-1} \gamma_n k} \quad (t \geq 0, h \in \mathbb{C}^n),
$$

where $\beta(A) = \min_{k=1,\ldots,n} \text{Re} \lambda_k(A)$. Therefore by (3.1),

$$
\|\exp(At)h\| \geq \frac{e^{\beta(A)t} \|h\|}{\sum_{k=0}^{n-1} g^k(A) (k!)^{-3/2} t^k} \quad (t \geq 0).
$$

(8.2)

Moreover, if $A$ is a diagonalizable $n \times n$-matrix, then due to (7.3) we conclude that

$$
\|e^{-At}\| \leq \kappa(A) e^{-\beta(A)t} \quad (t \geq 0).
$$

Hence,

$$
\|e^{At}h\| \geq \frac{\|h\| e^{\beta(A)t}}{\kappa(A)} \quad (t \geq 0, h \in \mathbb{C}^n).
$$

Let $A, \tilde{A} \in \mathbb{C}^{n \times n}$ and $E = \tilde{A} - A$. We will say that $A$ is stable (Hurwitzian), if $\alpha(A) < 0$. Assume that $A$ is stable and put
2.8 Perturbations of Matrix Exponentials

\[
\begin{align*}
    u(A) &= \int_0^\infty \| e^{At} \| dt. \\

\end{align*}
\]

To investigate perturbations of matrix exponentials one can use the identity

\[
    e^{\tilde{A}t} - e^{At} = \int_0^t e^{\tilde{A}(t-s)} E e^{As} ds. 
\]

Hence,

\[
    \int_0^T \| e^{\tilde{A}t} - e^{At} \| dt \leq \| E \| \int_0^T \int_0^t \| e^{\tilde{A}s} \| \| e^{A(t-s)} \| ds dt. 
\]

Consequently,

\[
    \int_0^T \| e^{\tilde{A}t} \| dt \leq u(A) \| E \| \int_0^T \int_0^s \| e^{A(t-s)} \| ds dt. 
\]

But

\[
    \int_s^T \| e^{A(t-s)} \| dt = \int_0^{T-s} \| e^{A(t_1)} \| dt_1 \leq u(A). 
\]

So

\[
    \int_0^T \| e^{\tilde{A}t} \| dt \leq u(A) + u(A) \| E \| \int_0^T \| e^{A(t)} \| ds. 
\]

We thus arrive at

**Theorem 2.8** Let A be stable, and

\[
    \| E \| u(A) < 1.
\]

Then \( \tilde{A} \) is also stable. Moreover,

\[
    u(\tilde{A}) \leq \frac{u(A)}{1 - u(A) \| E \|}. 
\]
Denote
\[
v_A = \int_0^\infty t \|e^{At}\| dt.
\]
Here we also investigate the perturbations in the case when
\[
\|\tilde{A}E - EA\|v_A < \|E\|u(A).
\]

**Theorem 2.9** Let \(A\) be stable, and
\[
\|\tilde{A}E - EA\|v_A < 1.
\]  
(8.4)

Then \(\tilde{A}\) is also stable. Moreover,
\[
u(\tilde{A}) \leq \frac{u(A) + v_A \|E\|}{1 - v_A \|\tilde{A}E - EA\|}
\]  
(8.5)

and
\[
\int_0^\infty \|e^{\tilde{A}t} - e^{At}\| dt \leq \|E\|v_A + \frac{\|\tilde{A}E - EA\|v_A(u(A) + v_A \|E\|)}{1 - v_A \|\tilde{A}E - EA\|}.
\]  
(8.6)

This theorem is proved in this section below.

For example, if \(A\) and \(\tilde{A}\) commute, then \(A\) and \(E\) commute, and \(\tilde{A}E - EA = E^2\). If, in addition \(E^2 = 0\), then \(\tilde{A}E - EA = 0\). In this case Theorem 2.9 is sharper than Theorem 2.8.

Furthermore, by (8.1) we obtain \(u(A) \leq u_0(A)\) and \(v_A \leq \hat{v}_A\), where
\[
u_0(A) := \sum_{k=0}^{n-1} \frac{g^k(A)}{|\alpha(A)|^{k+1}(k!)^{1/2}}
\]  
and
\[
\hat{v}_A := \sum_{k=0}^{n-1} \frac{(k + 1)g^k(A)}{|\alpha(A)|^{k+2}(k!)^{1/2}}.
\]

Thus, Theorem 2.9 implies

**Corollary 2.11** Let \(A\) be stable and \(\|\tilde{A}E - EA\|\hat{v}_A < 1\). Then \(\tilde{A}\) is also stable. Moreover,
\[
u(\tilde{A}) \leq \frac{u_0(A) + \hat{v}_A \|E\|}{1 - \hat{v}_A \|\tilde{A}E - EA\|}
\]
2.8 Perturbations of Matrix Exponentials

and

$$\int_0^\infty \|e^{\tilde{A}t} - e^{At}\|dt \leq \|E\|\hat{v}_A + \frac{\|	ilde{A}E - EA\|\hat{v}_A(u_0(A) + \hat{v}_A\|E\|)}{1 - \hat{v}_A\|	ilde{A}E - EA\|}. $$

Proof of Theorem 2.9: We use the following result, let $f(t), c(t)$ and $h(t)$ be matrix functions defined on $[0, b]$ $(0 < b < \infty)$. Besides, $f$ and $h$ are differentiable and $c$ is integrable. Then

$$\int_0^b f(t)c(t)h(t)dt = f(b)j(b)h(b) - \int_0^b (f'(t)j(t)h(t) + f(t)j(t)h'(t))dt \quad (8.7)$$

with

$$j(t) = \int_0^t c(s)ds.$$

Indeed, clearly,

$$\frac{d}{dt} f(t)j(t)h(t) = f'(t)j(t)h(t) + f(t)c(t)h(t) + f(t)j(t)h'(t).$$

Integrating this equality and taking into account that $j(0) = 0$, we arrive at (8.7). By (8.7)

$$e^{\tilde{A}t} - e^{At} = \int_0^t e^{A(t-s)}Ee^{As}ds$$

$$= Ete^{At} + \int_0^t e^{\tilde{A}(t-s)}[\tilde{A}E - EA]se^{As}ds.$$

Hence,

$$\int_0^\infty \|e^{\tilde{A}t} - e^{At}\|dt \leq \int_0^\infty \|Ee^{At}\|dt + \int_0^\infty \int_0^t \|e^{\tilde{A}(t-s)}\|\|	ilde{A}E - EA\||se^{As}\|ds\|dt.$$. 
But
\[
\int_0^\infty \int_0^t \|e^{\tilde{A}(t-s)}\|s\|e^{As}\|\,ds\,dt = \int_0^\infty \int_0^s \|e^{\tilde{A}(t-s)}\|s\|e^{As}\|\,ds\,dt = \int_0^\infty s\|e^{As}\|\,ds \int_0^\infty \|e^{\tilde{A}t}\|\,dt = v_Au(\tilde{A}).
\]
Thus
\[
\int_0^\infty \|e^{\tilde{A}t} - e^{At}\|\,dt \leq \|E\|v_A + \|\tilde{A}E - EA\|v_Au(\tilde{A}). \tag{8.8}
\]
Hence,
\[
u(\tilde{A}) \leq u(A) + \|E\|v_A + \|\tilde{A}E - EA\|v_Au(\tilde{A}).
\]
So according to (8.4), we get (8.5). Furthermore, due to (8.8) and (8.5) we get (8.6). As claimed. \(\square\)

2.9 Functions of Matrices with Nonnegative Off-Diagonals

In this section we establish two-sided inequalities for functions of matrices with nonnegative off-diagonals. They improve the results of Sect. 2.6 for these matrices and can be applied to the stability analysis of NDEs whose matrix coefficients have nonnegative off-diagonals.

In this section it is assumed that \(A = (a_{ij})_{j,k=1}^n\) is a real matrix with
\[
a_{ij} \geq 0 \text{ for } i \neq j. \tag{9.1}
\]
Put
\[
a = \min_{j=1,\ldots,n} a_{jj} \text{ and } b = \max_{j=1,\ldots,n} a_{jj}.
\]
For a scalar function \(f(\lambda)\) denote
\[
\alpha_k(f, A) := \inf_{a \leq x \leq b} \frac{f^{(k)}(x)}{k!}
\]
and

\[
\beta_k(f, A) := \sup_{a \leq x \leq b} \frac{f^{(k)}(x)}{k!} \quad (k = 0, 1, 2, \ldots),
\]

assuming that the derivatives exist.

Let \( W = A - \text{diag} (a_{jj}) \) be the off diagonal part of \( A \).

**Theorem 2.10** Let condition (9.1) hold and \( f(\lambda) \) be holomorphic on a neighborhood of a Jordan set, whose boundary \( C \) has the property

\[
|z - a_{jj}| > \sum_{k=1}^{n} a_{jk}
\]

for all \( z \in C \) and \( j = 1, \ldots, n \). In addition, let \( f \) be real on \([a, b]\). Then the following inequalities are valid:

\[
f(A) \geq \sum_{k=1}^{\infty} \alpha_k(f, A) W^k,
\]

provided

\[
\lim_{k \to \infty} k^{1/2} |\alpha_k(f, A)| < 1,
\]

and

\[
f(A) \leq \sum_{k=1}^{\infty} \beta_k(f, A) W^k,
\]

provided,

\[
\lim_{k \to \infty} k^{1/2} |\beta_k(f, A)| < 1.
\]

In particular, if \( \alpha_k(f, A) \geq 0 \quad (k = 0, 1, \ldots) \), then matrix \( f(A) \) has nonnegative entries.

**Proof** By (6.2) and (6.3),

\[
f(A) = f(S) + \sum_{k=1}^{\infty} M_k,
\]

where

\[
M_k = \sum_{j_1=1}^{n} \hat{Q}_{j_1} W \sum_{j_2=1}^{n} \hat{Q}_{j_2} W \ldots W \sum_{j_{k+1}=1}^{n} \hat{Q}_{j_{k+1}} J_{j_1 j_2 \ldots j_{k+1}}.
\]
Here
\[
J_{j_1 \ldots j_{k+1}} = \frac{(-1)^{k+1}}{2\pi i} \int_C \frac{f(\lambda) d\lambda}{(b_{j_1} - \lambda) \ldots (b_{j_{k+1}} - \lambda)} \quad (b_j = a_{j j}).
\]
Since \( S \) is real, Lemma 1.5.2 from Gil’ (2003) gives us the inequalities
\[
\alpha_k(f, A) \leq J_{j_1 \ldots j_{k+1}} \leq \beta_k(f, A).
\]
Hence,
\[
M_k \geq \alpha_k(f, A) \sum_{j_1=1}^n \hat{Q}_{j_1} W \sum_{j_2=1}^n \hat{Q}_{j_2} W \ldots W \sum_{j_{k+1}=1}^n \hat{Q}_{j_{k+1}} = \alpha_k(f, A) W^k.
\]
Similarly, \( M_k \leq \beta_k(f, A) W^k \). This implies the required result. □

In addition, the previous theorem implies
\[
e^{\min_j a_{j j} t} \sum_{k=1}^{\infty} \frac{t^k}{k!} W^k \leq e^{A t} \leq e^{\max_j a_{j j} t} \sum_{k=1}^{\infty} \frac{t^k}{k!} W^k \quad (t > 0).
\]
Thus we arrive at the following

**Corollary 2.12** Let condition (9.1) hold. Then the following inequalities are valid:
\[
\sum_{k=1}^m \frac{m!}{(m-k)!k!} \min_j a_{j j}^{m-k} W^k \leq A^m \leq \sum_{k=1}^m \frac{m!}{(m-k)!k!} \max_j a_{j j}^{m-k} W^k
\]
and
\[
e^{(\min_j a_{j j} + W) t} \leq e^{A t} \leq e^{(\max_j a_{j j} + W) t} \quad (t > 0).
\]
If all the diagonal elements are nonnegative then we obtain the trivial inequality.
\[
(m \min_j a_{j j} + W)^m \leq A^m \leq (m \max_j a_{j j} + W)^m
\]
If \( \max_j a_{j j} \geq 0 \), then \( A^m \leq (m \max_j a_{j j} + W)^m \).
2.10 Perturbations of Determinants

In this section \( \|A\| \) is an arbitrary matrix norm and \( \|A\| \) again denotes the spectral norm (the operator norm with respect to the Euclidean vector norm) of an \( n \times n \) matrix \( A \).

The following inequality for the determinants of two \( n \times n \)-matrices \( A \) and \( B \) is well-known (Bhatia 2007, p. 107):

\[
|\det A - \det B| \leq n M_2^{n-1}\|A - B\|, \tag{10.1}
\]

where \( M_2 := \max\{\|A\|, \|B\|\} \). The spectral norm is unitarily invariant, but often is not easy to compute that norm. Now we are going to derive the similar inequalities for non-spectral norms.

It is supposed that for a given matrix norm, there is a constant \( \alpha_n \) independent of \( A \), such that

\[
|\det A| \leq \alpha_n \|A\|^n. \tag{10.2}
\]

The aim of this paper is to prove the following result.

**Theorem 2.11** Let \( A \) and \( B \) be \( n \times n \)-matrices and condition (10.2) hold. Then

\[
|\det A - \det B| \leq \gamma_n \|A - B\| \left(\|A - B\| + \|A + B\|\right)^{n-1}, \tag{10.3}
\]

where

\[
\gamma_n := \frac{\alpha_n n^n}{2^{n-1}(n-1)^{n-1}}.
\]

The proof of this theorem is presented in this section below.

For the spectral norm inequality (10.3), can be worse than (10.1). Indeed, for the spectral norm we have \( \alpha_n = 1 \). If we take \( A = aB \) with a positive constant \( a < 1 \), then \( \|A - B\| + \|A + B\| = 2\|B\| = 2M_2 \), but

\[
\gamma_n 2^{n-1} = n(1 + \frac{1}{n-1})^{n-1} \geq n.
\]

Furthermore, let

\[
N_p(A) := (\text{Trace}(A^* A)^{p/2})^{1/p} \quad (p \geq 1)
\]

be the Schatten-von Neumann norm In particular, \( N_2(A) \) is the Frobenius (Hilbert-Schmidt) norm. Due to the inequality between the arithmetic and geometric mean values,

\[
|\det A|^p = \prod_{k=1}^n |\lambda_k(A)|^p \leq \left( \frac{1}{n} \sum_{k=1}^n |\lambda_k(A)|^p \right)^n.
\]
Thus,

$$|\det A| \leq \frac{1}{n^{n/p}} N_p^n(A).$$

So in this case $\alpha_n = \frac{1}{n^{n/p}}$ and $\gamma_n = \zeta_{n,p}$, where

$$\zeta_{n,p} := \frac{n^a}{2^{n-1} n^{n/p} (n-1)^{n-1}}.$$

Now (10.3) implies

Corollary 2.13 One has

$$|\det A - \det B| \leq \zeta_{n,p} N_p(A - B) (N_p(A - B) + N_p(A + B))^{n-1}. \quad (10.4)$$

Proof of Theorem 2.11: Let $X$ and $Y$ be complex normed spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and $F$ be a $Y$-valued function defined on $X$. Assume that $F(C + \lambda \tilde{C}) \ (\lambda \in \mathbb{C})$ is an entire function for all $C, \tilde{C} \in X$. That is, for any $\phi \in Y^*$, the functional $<\phi, F(C + \lambda \tilde{C}>$ defined on $Y$ is an entire scalar valued function of $\lambda$. In Gil’ (2008) (see also Lemma 2.14.1 from Gil’ (2010a)), the following result has been proved.

Lemma 2.7 Let $F(C + \lambda \tilde{C}) \ (\lambda \in \mathbb{C})$ be an entire function for all $C, \tilde{C} \in X$ and there be a monotone non-decreasing function $G : [0, \infty) \to [0, \infty)$, such that

$$\|F(C)\|_Y \leq G(\|C\|_X) \quad (C \in X). \quad (10.5)$$

Then

$$\|F(C) - F(\tilde{C})\|_Y \leq \|C - \tilde{C}\|_X G(1 + \frac{1}{2}\|C + \tilde{C}\|_X + \frac{1}{2}\|C - \tilde{C}\|_X) \quad (C, \tilde{C} \in X).$$

Take $C = A, \tilde{C} = B$ and $F(A + \lambda B) = \det (A + \lambda B)$. Then due to (10.2) we have (10.5) with $G(\|A\|) = \alpha_n \|A\|^n$. Now the previous lemma implies.

$$|\det A - \det B| \leq \alpha_n \|A - B\| (1 + \frac{1}{2}\|A - B\| + \frac{1}{2}\|A + B\|)^n. \quad (10.6)$$

For a constant $c > 0$ put $A_1 = cA$ and $B_1 = cB$. Then by (10.6)

$$|\det A_1 - \det B_1| \leq \|A_1 - B_1\| \alpha_n [1 + \frac{1}{2}\|A_1 - B_1\| + \frac{1}{2}\|A_1 + B_1\|]^n.$$

But

$$\|A_1\| = c\|A\|, \|B_1\| = c\|B\| \text{ and } |\det A_1 - \det B_1| = c^n |\det A - \det B|.$$
2.10 Perturbations of Determinants

Thus,

\[ c^n |\det A - \det B| \leq c \|A - B\|(1 + cb)^n, \tag{10.7} \]

where

\[ b = \frac{1}{2} \|A - B\| + \frac{1}{2} \|A + B\|. \]

Denote \( x = bc. \) Then from inequality (10.7) we obtain

\[ |\det A - \det B| \leq \alpha_n b^{n-1} \|A - B\| (1 + x)^n. \]

Minimize the function

\[ f(x) = \frac{(1 + x)^n}{x^{n-1}}, \quad x > 0. \]

Simple calculation show that

\[ \min_{x \geq 0} f(x) = \frac{n^n}{(n - 1)^{n-1}}. \]

So

\[ |\det A - \det B| \leq \frac{\alpha_n n^n}{(n - 1)^{n-1}} b^{n-1} \|A - B\|. \]

This is the assertion of the theorem. \( \square \)

The previous theorem gives us two-sided inequalities for the determinants of matrices which are “close” to triangular ones. Indeed, denote by \( \tilde{D}, V_+ \) and \( V_- \) the diagonal, strictly upper triangular and strictly lower triangular parts, respectively, of a matrix \( A = (a_{jk})_{j,k=1}^n \). Using the notation \( A_+ = \tilde{D} + V_+ \), we have

\[ \det A_+ = \prod_{k=1}^n a_{jj}, \]

since \( A_+ \) is triangular. In addition, \( A - A_+ = V_- \). Put \( \delta(A) := \|V_-\|. \) For example,

\[ N_2^2(V_-) = \sum_{j=2}^{n} \sum_{k=1}^{j-1} |a_{jk}|^2. \]
For the norm

\[ \| A \|_\infty =: \max_j \sum_{k=1}^{n} |a_{jk}|, \]

one has

\[ \| V_{-} \|_\infty = \max_{j=2, \ldots, n} \sum_{k=1}^{j-1} |a_{jk}|. \]

Now Theorem 2.11 implies

**Corollary 2.14** For a matrix \( A = (a_{jk})_{j,k=1}^{n} \) we have

\[ |det A - \prod_{k=1}^{n} a_{jj}| \leq \Delta(A), \]

where

\[ \Delta(A) := \gamma_n \delta(A) (\delta(A) + \|A + A_{+}\|)^{n-1}, \]

and therefore,

\[ \prod_{j=1}^{n} (|a_{jj}| - \Delta(A)) \leq |det A| \leq \prod_{j=1}^{n} (|a_{jj}| + \Delta(A)). \]

Consequently, \( A \) is invertible, provided

\[ \prod_{k=1}^{n} |a_{jj}| > \Delta(A). \]

Clearly, one can exchange the places of \( V_{+} \) and \( V_{-} \). The latter result is sharp: if \( A \) is a triangular matrix, then we get the equalities, since \( V_{-} = 0 \) in this case.

Now let us recall the lower bound for determinants with dominant principal diagonals established by Ostrowski (1952).

**Theorem 2.12** Let \( A = (a_{jk}) \) be an \( n \times n \)-matrix and

\[ |a_{jj}| > \sum_{m=1, m \neq j}^{n} |a_{jm}| \quad (j = 1, \ldots, n). \] (10.8)

Then

\[ |det A| \geq \prod_{j=1}^{n} (|a_{jj}| - \sum_{m=1, m \neq j}^{n} |a_{jm}|). \] (10.9)
This theorem and Theorem 2.3 imply.

**Corollary 2.15** Let condition (10.8) hold. Then $A \in \mathbb{C}^{n \times n}$ is invertible and

\[
\|A^{-1}\| \leq \frac{N_2^{n-1}(A)}{(n-1)^{(n-1)/2}} \prod_{j=1}^{n} (|a_{jj}| - \sum_{m=1, m \neq j}^{n} |a_{jm}|).
\]

### 2.11 Bounds for the Eigenvalues

#### 2.11.1 Gerschgorin’s Circle Theorem

Let $A$ be a complex $n \times n$-matrix, with entries $a_{jk}$. For $j = 1, \ldots, n$ write

\[
R_j = \sum_{k=1}^{n} |a_{jk}|.
\]

Let $\Omega(b, r)$ be the closed disc centered at $b \in \mathbb{C}$ with a radius $r$.

**Theorem 2.13** (Gerschgorin) Every eigenvalue of $A$ lies within at least one of the discs $\Omega(a_{jj}, R_j)$.

**Proof** Let $\lambda$ be an eigenvalue of $A$ and let $x = (x_j)$ be the corresponding eigenvector. Let $i$ be chosen so that $|x_i| = \max_j |x_j|$. Then $|x_i| > 0$, otherwise $x = 0$. Since $x$ is an eigenvector, $Ax = \lambda x$ or equivalent

\[
\sum_{k=1}^{n} a_{ik} = \lambda x_i
\]

so, splitting the sum, we get

\[
\sum_{k=1}^{n} a_{ik} x_k = \lambda x_i - a_{ii} x_i.
\]

We may then divide both sides by $x_i$ (choosing $i$ as we explained we can be sure that $x_i \neq 0$) and take the absolute value to obtain

\[
|\lambda - a_{ii}| \leq \sum_{k=1}^{n} |a_{ik}| \frac{|x_k|}{|x_i|} \leq R_i,
\]
where the last inequality is valid because
\[
\frac{|x_k|}{|x_l|} \leq 1.
\]
As claimed. □

Note that for a diagonal matrix the Gerschgorin discs \( \Omega(a_{jj}, R_j) \) coincide with the spectrum. Conversely, if the Gerschgorin discs coincide with the spectrum, the matrix is diagonal.

**Corollary 2.16** The spectral radius of a complex in general \( n \times n \)-matrix \( A = (a_{jk}) \) is subject to the inequality
\[
rs(A) \leq \max_j \sum_{k=1}^{n} |a_{jk}|.
\]

Moreover its lower spectral radius is subject to the inequality
\[
rl(A) \geq \min_j (|a_{jj}| - \sum_{k=1}^{n} |a_{jk}|),
\]
provided
\[
|a_{jj}| > \sum_{k=1}^{n} |a_{jk}| \quad (j = 1, \ldots, n).
\]

In addition,
\[
\alpha(A) = \max_k \Re \lambda_k(A) \leq \max_j \Re a_{jj} + \sum_{k=1}^{n} |a_{jk}|
\]
and
\[
\beta(A) = \min_k \Re \lambda_k(A) \geq \min_j \Re a_{jj} - \sum_{k=1}^{n} |a_{jk}|.
\]

### 2.11.2 Cassini Ovals

Brauer and Ostrosvki independently observed that each eigenvalue of an \( n \times n \) matrix \( A = (a_{jk}), n \geq 2, \) is contained in the Cassini ovals
\[
\Omega(m, j) := \{ \lambda \in \mathbb{C} : |\lambda - a_{mm}| |\lambda - a_{jj}| \leq T_{mj} \}
\]
where
\[ T_{mj} := \left( \sum_{k=1}^{n} |a_{mk}| \right) \left( \sum_{k=1}^{n} |a_{jk}| \right) (j, m = 1, \ldots, n; \ j \neq m) \]
(see Sect. 3.2 from the survey Marcus and Minc (1964)).

That result leads to a better but more complicated localization of the spectrum of a matrix than Gerschgorin’s theorem, cf. Varga (2004) and references therein.

From the Cassini ovals, for the eigenvalue \( \lambda \) satisfying \(|\lambda| = r_s(A)\) there are indexes \( m \) and \( j \) such that eigenvalue \( \lambda \) it follows that
\[
(r_s(A) - |a_{mm}|)(r_s(A) - |a_{jj}|) \leq |\lambda - a_{mm}||\lambda - a_{jj}| \leq T_{mj}.
\]
Solving this inequality, we obtain that \( r_s(A) \leq x_{jm} \), where
\[
x_{jm} = \frac{1}{2} (|a_{mm}| + |a_{jj}|) + \sqrt{\frac{1}{4} (|a_{mm}| + |a_{jj}|)^2 + T_{mj} - |a_{mm}a_{jj}|}.
\]
That is, \( x_{jm} \) is the larger zero of the polynomial
\[
(x - |a_{mm}|)(x - |a_{jj}|) - T_{mj}.
\]
We thus arrive at

**Corollary 2.17** The spectral radius of \( n \times n\)-matrix \( A = (a_{jk}) \) is subject to the inequality
\[
r_s(A) \leq \max_{m, j=1, \ldots, n; \ m \neq j} \left[ \frac{1}{2} (|a_{mm}| + |a_{jj}|) + \sqrt{\frac{1}{4} (|a_{mm}| - |a_{jj}|)^2 + T_{mj}} \right].
\]

### 2.11.3 The Perron Theorems

Again \( A = (a_{jk}) \) is a complex \( n \times n\)-matrix and for \( j = 1, \ldots, n \) We write
\[
R_j = \sum_{k=1}^{n} |a_{jk}|.
\]
Recall that \( \Omega(b, r) \) is the closed disc centered at \( b \in \mathbb{C} \) with radius \( r \).

Another valuable result is the following Perron theorem.

**Theorem 2.14** If \( A = (a_{jk}) \) is a non-negative matrix, then the spectral radius \( r_s(A) = \max_k |\lambda_k(A)| \) of \( A \) is its eigenvalue.
For the proof of this theorem see the book (Gantmakher 1967, Sect. 2, Chap. 13). We also state the following comparison theorem of Perron.

**Theorem 2.15** If a matrix \( A = (a_{jk})_{j,k=1}^n \) is non-negative and if in matrix \( C = (c_{jk})_{j,k=1}^n \),

\[
|c_{ji}| \leq a_{ji} \quad (i, j = 1, 2, \ldots n),
\]

then any eigenvalue \( \lambda(C) \) of \( C \) satisfies the inequality \( |\lambda(C)| \leq r_s(A) \).

**Proof** By the Gel’fand formula

\[
 r_s(A) = \lim_{n \to \infty} \sqrt[n]{\|A^n\|},
\]
where the norm is assumed to be Euclidean. But clearly, \( \|C^n\| \leq \|A^n\| \). So \( r_s(C) \leq r_s(A) \). This proves the result. \( \Box \)

### 2.11.4 Bounds for the Eigenvalues of Matrices “Close” to Triangular Ones

In the present section we improve the previous bounds for the eigenvalues in the case when the matrix \( A = (a_{jk}) \) is “close” to a triangular one.

To this end let \( \tilde{D}, V_+ \) and \( V_- \) be the diagonal, upper nilpotent part and lower nilpotent part of matrix \( A \), respectively. Using the notation \( A_+ = \tilde{D} + V_+ \), we arrive at the relations

\[
g(A_+) = N_2(V_+) \text{ and } \|A - A_+\| = \|V_-\|.
\]

Taking

\[
\delta_A := \begin{cases} 
2\|V_-\| & \text{if } \|V_-\| \geq N_2(V_+), \\
2N_2^{1-1/n}(V_+)\|V_-\|^{1/n} & \text{if } \|V_-\| \leq N_2(V_+)
\end{cases}
\]

due to Corollary 2.3 we obtain: \( s_{V_+}(A) \leq \delta_A \). But \( A_+ \) is a triangular matrix. So its diagonal entries are the eigenvalues. We therefore arrive at the following result.

**Corollary 2.18** Let \( A = (a_{jk})_{j,k=1}^n \) be an \( n \times n \)-matrix. Then for any eigenvalue \( \mu \) of \( A \), there is an index \( k = 1, \ldots, n \), such that

\[
|\mu - a_{kk}| \leq \delta_A,
\]

and therefore the (upper) spectral radius satisfies the inequality

\[
r_s(A) \leq \max_{k=1,\ldots,n} |a_{kk}| + \delta_A,
\]
and the lower spectral radius satisfies the inequality
\[ r_l(A) \geq \min_{k=1,\ldots,n} |a_{kk}| - \delta_A, \]
provided \( |a_{kk}| > \delta_A \) \((k = 1, \ldots, n)\).

Clearly, one can exchange the places of \( V_+ \) and \( V_- \).

This result is sharp: if \( A \) is a triangular matrix, we get the equalities \( \lambda_k(A) = a_{kk} \).

In the case \( a_{kk} \neq a_{jj} \) \((k \neq j)\) one can apply the results from Sect. 2.7.

### 2.12 Comments

Section 2.2. As it was above mentioned, the results contained in that section are well-known.

Sections 2.3 and 2.4 are based on Chap. 2 from Gil’ (2003). Compare our results from Sect. 2.4 with the well-known ones. To this end put
\[ w_n := n - 1 \sum_{j=0}^{n-1} \frac{1}{\sqrt{j!}}. \]
Clearly \( w_n < n \) \((n > 2)\). If
\[ qw_n \leq g(A), \quad (12.1) \]
then for the positive root \( \hat{z}(q/g(A)) \) of Eq. (4.6), Lemma 2.2 gives us the inequality
\[ \hat{z}(q/g(A)) \leq \sqrt[n]{qw_n/g(A)}. \]
Since \( z(q, A) = \hat{z}(q/g(A))g(A) \), Theorem 2.4 yields the inequality
\[ sv_A(\tilde{A}) \leq (qw_n)^{1/n} g^{1-1/n}(A). \quad (12.2) \]

The best known result of kind (12.2) or (4.7) is due to the Henrici theorem (see Stewart and Sun Ji-guang (1990, p. 172)). To compare our results with Henrici’s recall that for a given matrix \( A \) we can find a unitary matrix \( U \) such that \( UAU^* = D + V \), where \( D \) and \( V \) are the diagonal and the nilpotent parts of \( A \) (see Sect. 2.1). Such a \( V \) is not unique. If \( v \) is any matrix norm, define the \( v \)-departure from normality of \( A \) by \( \delta_v(A) = \inf V(v) \), where the infimum is taken over all \( V \), which appear as the nilpotent parts in the upper triangular forms of \( A \). For the Frobenius norm we have \( \delta_F(A) = g(A) \), cf. Stewart and Sun Ji-guang (1990, p. 172).
Theorem 2.16 (Henrici) Let \( \nu \) be a matrix norm, such that \( \nu(C) \geq \|C\| \) for all \( n \times n \)-matrices \( C \). Then for every eigenvalue \( \tilde{\lambda} \) of \( \tilde{A} \) there is an eigenvalue \( \lambda \) of \( A \), such that

\[
\frac{(|\tilde{\lambda} - \lambda| / \delta_\nu(A))^n}{1 + (|\tilde{\lambda} - \lambda| / \delta_\nu(A)) + \cdots + (|\tilde{\lambda} - \lambda| / \delta_\nu(A))^{n-1}} \leq \frac{q}{\delta_\nu(A)},
\]

where \( q = \| \tilde{A} - A \| \).

For a non-Frobenius norm the calculations of the departure from normality is a difficult task. In the case of the Frobenius norm the latter theorem it gives us the following result, cf. Stewart and Sun Ji-guang (1990, p. 173).

Corollary 2.19 If

\[
\frac{q}{g(A)} < \frac{1}{n},
\]

then

\[
v_A(\tilde{A}) \leq (qn)^{1/n} g^{1-1/n}(A).
\]

If

\[
\frac{q}{g(A)} \geq 1,
\]

then \( v_A(\tilde{A}) \leq q + g(A) \).

For instance, let the condition (12.3) hold. Since \( w_n < n \ (n > 2) \) we can assert that our inequality (12.2) improves (12.4).

Let us recall the (Elsner) inequality:

\[
v_A(B) \leq q^{1/n} (\|A\|_n + \|B\|_n)^{1-1/n},
\]

cf. Stewart and Sun Ji-guang (1990, p. 168). This inequality holds also for so called the Hausdorff distance between the spectra of \( A \) and \( B \). Assume that \( A \) is a normal matrix, then Theorem 2.4 gives us the equality \( v_A(B) = q \). So our result considerably improves (12.5) at least for matrices “close” to normal ones. For more information about perturbations of matrices, we refer the reader to the books Stewart and Sun Ji-guang (1990, p. 168) and Bhatia (2007).

The material of Sect. 2.5 is taken from Gil’ (2013e). One of the first estimates for the norm of a regular matrix-valued function was established by Gel’fand and Shilov (1958) in connection with their investigations of partial differential equations, but that estimate is not sharp; it is not attained for any matrix. The problem of obtaining a sharp estimate for the norm of a matrix-valued function has been repeatedly discussed in the literature. In the late 1970s, the author has obtained a sharp estimate for a matrix-valued function regular on the convex hull of the spectrum. It is attained in the case of normal matrices. Later, that estimate was extended to various classes of nonselfadjoint operators, such as Hilbert-Schmidt operators, quasi-Hermitian operators (i.e., linear operators with completely continuous imaginary components),
quasiunitary operators (i.e., operators represented as a sum of a unitary operator and a compact one), etc. Moreover, estimates for the norm of regular functions of infinite matrices, representing bounded operators in a Banach space with a Schauder basis also have been derived. For more details see Gil’ (2003) and (2013e).

Sections 2.6–2.10 contain the results adopted from Gil’ (2013e).

Section 2.11 contains mainly classical theorems, except Corollary 2.18, which is probably new.

For more information about the inequalities for the eigenvalues of matrices, see for instance (Marcus and Minc 1964). Many interesting results connected with Gerschgorin’s circle theorem (Theorem 1.11.1) can be found in Varga (2004).
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