Chapter 2
Differential Forms on Jet Prolongations of Fibered Manifolds

In this chapter, we present a decomposition theory of differential forms on jet prolongations of fibered manifolds; the tools inducing the decompositions are the algebraic trace decomposition theory and the canonical jet projections. Of particular interest is the structure of the *contact forms*, annihilating integrable sections of the jet prolongations. We also study decompositions of forms defined by fibered homotopy operators and state the corresponding fibered Poincare-Volterra lemma.

The theory of differential forms explained in this chapter has been developed along the lines indicated in the approach of Lepage and Dedecker to the calculus of variations (see Dedecker [D], Goldschmidt and Sternberg [GS] and Krupka [K13]). The exposition extends the theory explained in the handbook chapter Krupka [K4].

Throughout, Y is a smooth fibered manifold with base X and projection π, $n = \dim X$, $n + m = \dim Y$. $J^r Y$ is the *r-jet prolongation* of Y, and $\pi^r: J^r Y \to X$, $\pi^r: J^r Y \to X$ are the canonical jet projections. For any open set $W \subset Y$, $\Omega^r_q W$ denotes the module of q-forms on the open set $W^r = (\pi^{r,0})^{-1}(W)$ in $J^r Y$, and $\Omega^r W$ is the exterior algebra of differential forms on the set W^r. We say that a form η is *generated* by a finite family of forms μ_κ, if η is expressible as $\eta = \eta^\kappa \wedge \mu_\kappa$ for some forms η^κ; note that in this terminology, we do not require μ_κ to be 1-forms, or k-forms for a fixed integer k.

2.1 The Contact Ideal

We introduced in Sect. 1.5 a vector bundle homomorphism h between the tangent bundles $TJ^{r+1}Y$ and $TJ^r Y$ over the canonical jet projection $\pi^{r+1,r}: J^{r+1}Y \to J^r Y$, the *horizontalization*. In this section, the associated *dual* mapping between the modules of 1-forms $\Omega^r_1 W$ and $\Omega^{r+1}_1 W$ is studied. We show, in particular, that this mapping allows us to associate with any fibered chart (V, ψ) on Y and any function, defined on V^r, its *formal* (or *total*) *partial derivatives* in a geometric way and a specific basis of 1-forms on V^r, termed the *contact basis*. Then, we introduce by means of the contact

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basis a differential ideal in the exterior algebra $\Omega^r W$, characterizing the structure of forms on jet prolongations of fibered manifolds, the *contact ideal*.

Recall that the horizontalization h is defined by the formula

$$h\xi = T_x J^r\gamma \circ T\pi^{r+1} \cdot \xi, \tag{1}$$

where ξ is a tangent vector to the manifold $J^{r+1}Y$ at a point $J_x^{r+1}\gamma$. The mapping h makes the following diagram

$$\begin{array}{ccc} TJ^{r+1}Y & \xrightarrow{h} & TJ^rY \\ \downarrow & & \downarrow \\ J^{r+1}Y & \xrightarrow{\pi^{r+1,r}} & J^rY \end{array} \tag{2}$$

commutative and induces a decomposition of the projections of the tangent vectors $T\pi^{r+1,r} \cdot \xi$,

$$T\pi^{r+1,r} \cdot \xi = h\xi + p\xi. \tag{3}$$

$h\xi$ (resp. $p\xi$) is the *horizontal* (resp. *contact*) *component* of the vector ξ. Note, however, that the terminology is *not* standard: The vectors ξ and $h\xi$ do not belong to the same vector space. The horizontal and contact components satisfy

$$T\pi^r \cdot h\xi = T\pi^{r+1} \cdot \xi, \quad T\pi^r \cdot p\xi = 0. \tag{4}$$

The horizontalization h induces a mapping of modules of linear differential forms as follows. Let $J_x^{r+1}\gamma \in J^{r+1}Y$. We set for any differential 1-form ρ on W^r and any vector ξ from the tangent space $TJ^{r+1}Y$ at $J_x^{r+1}\gamma$

$$h\rho(J_x^{r+1}\gamma) \cdot \xi = \rho(J_x^r\gamma) \cdot h\xi. \tag{5}$$

The mapping $\Omega_1^r W \ni \rho \to h\rho \in \Omega_1^{r+1} W$ is called the π-*horizontalization* or just the *horizontalization* (of differential forms).

Clearly, the form $h\rho$ vanishes on π^{r+1}-vertical vectors so it is π^{r+1}-horizontal; $h\rho$ is sometimes called the *horizontal component* of ρ.

The mapping h is linear over the ring of functions $\Omega_0^r W$ along the jet projection $\pi^{r+1,r}$ in the sense that

$$h(\rho_1 + \rho_2) = h\rho_1 + h\rho_2 \quad h(f\rho) = (f \circ \pi^{r+1,r})h\rho \tag{6}$$

for all $\rho_1, \rho_2, \rho \in \Omega_1^r W$ and $f \in \Omega_0^r W$.

If in the fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, a 1-form ρ is expressed by

$$\rho = A_i dx^i + \sum_{0 \le k \le r} \sum_{j_1 \le j_2 \le \cdots \le j_k} B_\sigma^{j_1 j_2 \ldots j_k} dy^\sigma_{j_1 j_2 \ldots j_k}, \tag{7}$$

then we have from (5) at any point $J_x^{r+1}\gamma \in V^{r+1}$

$$\begin{aligned} h\rho(J_x^{r+1}\gamma) \cdot \xi &= A_i(J_x^r\gamma) dx^i (J_x^r\gamma) \cdot h\xi \\ &\quad + \sum_{0 \le k \le r} \sum_{j_1 \le j_2 \le \cdots \le j_k} B_\sigma^{j_1 j_2 \ldots j_k}(J_x^r\gamma) dy^\sigma_{j_1 j_2 \ldots j_k}(J_x^r\gamma) \cdot h\xi \\ &= \left(A_i(J_x^r\gamma) + \sum_{0 \le k \le r} \sum_{j_1 \le j_2 \le \cdots \le j_k} B_\sigma^{j_1 j_2 \ldots j_k}(J_x^r\gamma) y^\sigma_{j_1 j_2 \ldots j_k i} \right) \xi^i, \end{aligned} \tag{8}$$

thus,

$$h\rho = \left(A_i + \sum_{0 \le k \le r} \sum_{j_1 \le j_2 \le \cdots \le j_k} B_\sigma^{j_1 j_2 \ldots j_k} y^\sigma_{j_1 j_2 \ldots j_k i} \right) dx^i. \tag{9}$$

In particular, for any function $f \colon W^r \to \mathbf{R}$

$$hdf = d_i f \cdot dx^i, \tag{10}$$

where

$$d_i f = \frac{\partial f}{\partial x^i} + \sum_{j_1 \le j_2 \le \cdots \le j_k} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} y^\sigma_{j_1 j_2 \ldots j_k i}. \tag{11}$$

The function $d_i f \colon V^{r+1} \to \mathbf{R}$ is the *i-th formal derivative* of f with respect to the fibered chart (V, ψ). From (10), it follows that $d_i f$ are the components of an invariant object, the *horizontal component hdf* of the exterior derivative of f. Note that formal derivatives $d_i f$ have already been introduced in Sect. 1.5.

The following lemma summarizes basic rules for computations with the horizontalization and formal derivatives. We denote by $\bar{d}_i$ the formal derivative operator with respect to a fibered chart $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{x}^i, \bar{y}^\sigma)$.

Lemma 1 *Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y.*

(a) *The horizontalization h satisfies*

$$\begin{gathered} hdy^\sigma = y^\sigma_i dx^i, \quad hdy^\sigma_{j_1} = y^\sigma_{j_1 i} dx^i, \quad hdy^\sigma_{j_1 j_2} = y^\sigma_{j_1 j_2 i} dx^i, \\ \ldots, \; hdy^\sigma_{j_1 j_2 \ldots j_r} = y^\sigma_{j_1 j_2 \ldots j_r i} dx^i. \end{gathered} \tag{12}$$

(b) *The i-th formal derivative of the coordinate function* $y^{\nu}_{j_1 j_2 \ldots j_k}$ *is given by*

$$d_i y^{\nu}_{j_1 j_2 \ldots j_k} = y^{\nu}_{j_1 j_2 \ldots j_k i}. \tag{13}$$

(c) *If* $(\overline{V}, \overline{\psi})$, $\overline{\psi} = (\overline{x}^i, \overline{y}^{\sigma})$, *is another chart on* Y *such that* $V \cap \overline{V} \neq \emptyset$, *then for every function* $f \colon V^r \cap \overline{V}^r \to \mathbf{R}$,

$$\overline{d}_i f = d_j f \cdot \frac{\partial x^j}{\partial \overline{x}^i}. \tag{14}$$

(d) *For any two functions* $f,\ g \colon V^r \to \mathbf{R}$,

$$d_i (f \cdot g) = g \cdot d_i f + f \cdot d_i g. \tag{15}$$

(e) *For every function* $f \colon V^r \to \mathbf{R}$ *and every section* $\gamma \colon U \to V \subset Y$,

$$d_i f \circ J^{r+1}\gamma = \frac{\partial (f \circ J^r \gamma)}{\partial x^i}. \tag{16}$$

Remark 1 By (13), $\overline{y}^{\sigma}_{j_1 j_2 \ldots j_k} = \overline{d}_{j_k} \overline{y}^{\sigma}_{j_1 j_2 \ldots j_{k-1}}$. Thus, applying (14) to coordinates, we obtain the following *prolongation formula* for coordinate transformations in jet prolongations of fibered manifolds

$$\overline{y}^{\sigma}_{j_1 j_2 \ldots j_k} = d_i \overline{y}^{\sigma}_{j_1 j_2 \ldots j_{k-1}} \cdot \frac{\partial x^i}{\partial \overline{x}^{j_k}}. \tag{17}$$

Remark 2 If two functions $f, g \colon V^r \to \mathbf{R}$ coincide along a section $J^r\gamma$, that is, $f \circ J^r \gamma = g \circ J^r \gamma$, then their formal derivatives coincide along the $(r+1)$-prolongation $J^{r+1}\gamma$,

$$d_i f \circ J^{r+1}\gamma = d_i g \circ J^{r+1}\gamma. \tag{18}$$

This is an immediate consequence of formula (16).

Now, we study properties of 1-forms, belonging to the kernel of the horizontalization $\Omega^r_1 W \ni \rho \to h\rho \in \Omega^{r+1}_1 W$. We say that a 1-form $\rho \in \Omega^r_1 W$ is *contact*, if

$$h\rho = 0. \tag{19}$$

It is easy to find the chart expression of a contact 1-form. Writing ρ as in (7), condition (19) yields

$$A_i + \sum_{0 \leq k \leq r} \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} B^{j_1 j_2 \ldots j_k}_{\sigma} y^{\sigma}_{j_1 j_2 \ldots j_k i} = 0, \tag{20}$$

or, equivalently,

$$B_{\sigma}^{j_1 j_2 \dots j_r} = 0, \quad A_i = - \sum_{0 \le k \le r-1} \sum_{j_1 \le j_2 \le \dots \le j_k} B_{\sigma}^{j_1 j_2 \dots j_k} y^{\sigma}_{j_1 j_2 \dots j_k i}. \tag{21}$$

Thus, setting for all k, $0 \le k \le r - 1$,

$$\omega^{\sigma}_{j_1 j_2 \dots j_k} = dy^{\sigma}_{j_1 j_2 \dots j_k} - y^{\sigma}_{j_1 j_2 \dots j_k j} dx^j, \tag{22}$$

we see that ρ has the chart expression

$$\rho = \sum_{0 \le k \le r-1} \sum_{j_1 \le j_2 \le \dots \le j_k} B_{\sigma}^{j_1 j_2 \dots j_k} \omega^{\sigma}_{j_1 j_2 \dots j_k}. \tag{23}$$

This formula shows that any contact 1-form is expressible as a linear combination of the forms $\omega^{\sigma}_{j_1 j_2 \dots j_k}$.

The following two theorems summarize properties of the forms $\omega^{\sigma}_{j_1 j_2 \dots j_k}$.

Theorem 1

(a) *For any fibered chart* (V, ψ), $\psi = (x^i, y^{\sigma})$, *the forms*

$$dx^i, \quad \omega^{\sigma}_{j_1 j_2 \dots j_k}, \quad dy^{\sigma}_{l_1 l_2 \dots l_{r-1} l_r}, \tag{24}$$

such that $1 \le i \le n$, $1 \le \sigma \le m$, $1 \le k \le r - 1$, $1 \le j_1 \le j_2 \le \dots \le j_k \le n$, *and* $1 \le l_1 \le l_2 \le \dots \le l_r \le n$, *constitute a basis of linear forms on the set* V^r.

(b) *If* (V, ψ), $\psi = (x^i, y^{\sigma})$, *and* $(\overline{V}, \overline{\psi})$, $\overline{\psi} = (\overline{x}^i, \overline{y}^{\sigma})$, *are two fibered charts such that* $V \cap \overline{V} \neq \emptyset$, *then*

$$\omega^{\lambda}_{p_1 p_2 \dots p_k} = \sum_{0 \le m \le k} \sum_{j_1 \le j_2 \le \dots \le j_k} \frac{\partial y^{\lambda}_{p_1 p_2 \dots p_k}}{\partial \overline{y}^{\tau}_{j_1 j_2 \dots j_m}} \overline{\omega}^{\tau}_{j_1 j_2 \dots j_m}. \tag{25}$$

(c) *Let* (V, ψ), $\psi = (x^i, y^{\sigma})$, *and* $(\overline{V}, \overline{\psi})$, $\overline{\psi} = (\overline{x}^i, \overline{y}^{\sigma})$, *be two fibered charts and* α *an automorphism of* Y, *defined on* V *and such that* $\alpha(V) \subset \overline{V}$. *Then*

$$J^r \alpha^* \overline{\omega}^{\sigma}_{j_1 j_2 \dots j_k} = \sum_{i < i_2 < \dots < i_p} \frac{\partial (\overline{y}^{\sigma}_{j_1 j_2 \dots j_k} \circ J^r \alpha)}{\partial y^{\nu}_{i_1 i_2 \dots i_p}} \omega^{\nu}_{i_1 i_2 \dots i_p}. \tag{26}$$

Proof

(a) Clearly, from formula (22), we conclude that the forms (24) are expressible as linear combinations of the forms of the canonical basis dx^i, $dy^{\sigma}_{j_1 j_2 \dots j_k}$, $dy^{\sigma}_{l_1 l_2 \dots l_{r-1} l_r}$.

(b) Consider two charts (V,ψ), $\psi=(x^i,y^\sigma)$, and $(\overline{V},\overline{\psi})$, $\overline{\psi}=(\overline{x}^i,\overline{y}^\sigma)$, such that $V\cap\overline{V}\neq\emptyset$. For any function f, defined on V^r,

$$\begin{aligned}(\pi^{r+1,r})^*df &= hdf+pdf = d_if\cdot dx^i+\sum_{0\le k\le r}\sum_{l_1\le l_2\le\cdots\le l_k}\frac{\partial f}{\partial y^\nu_{l_1l_2\ldots l_k}}\omega^\nu_{l_1l_2\ldots l_k}\\ &= \overline{d}_pf\cdot d\overline{x}^p+\sum_{0\le k\le r}\sum_{j_1\le j_2\le\cdots\le j_k}\frac{\partial f}{\partial\overline{y}^\tau_{j_1j_2\ldots j_m}}\overline{\omega}^\tau_{j_1j_2\ldots j_m}\\ &= \overline{d}_pf\frac{\partial\overline{x}^p}{\partial x^i}dx^i+\sum_{0\le k\le r}\sum_{j_1\le j_2\le\cdots\le j_k}\sum_{l_1\le l_2\le\cdots\le l_k}\frac{\partial f}{\partial y^\nu_{l_1l_2\ldots l_k}}\frac{\partial y^\nu_{l_1l_2\ldots l_k}}{\partial\overline{y}^\tau_{j_1j_2\ldots j_m}}\overline{\omega}^\tau_{j_1j_2\ldots j_m}.\end{aligned}\tag{27}$$

Setting $f=y^\lambda_{p_1p_2\cdots p_k}$, where $p_1\le p_2\le\cdots\le p_k$, and using (17), we get (25).

(c) By definition,

$$J^r\alpha^*\overline{\omega}^\sigma_{j_1j_2\ldots j_k}=d(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)-(\overline{y}^\sigma_{j_1j_2\ldots j_kl}\circ J^r\alpha)d(\overline{x}^l\circ J^r\alpha).\tag{28}$$

Denote by α_0 the π-projection of α. Since from Sect. 1.6, (80)

$$\begin{aligned}&\overline{y}^\sigma_{j_1j_2\ldots j_kl}\circ J^r\alpha(J^r_x\gamma)\\ &\qquad=\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha\circ J^r\gamma\circ\varphi^{-1})}{\partial x^s}\frac{\partial(x^s\alpha_0^{-1}\overline{\varphi}^{-1})}{\partial\overline{x}^l},\end{aligned}\tag{29}$$

then

$$\begin{aligned}J^r\alpha^*\overline{\omega}^\sigma_{j_1j_2\ldots j_k} &= \frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)}{\partial x^p}dx^p+\sum_{i<i_2<\cdots<i_p}\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)}{\partial y^\nu_{i_1i_2\ldots i_p}}dy^\nu_{i_1i_2\ldots i_p}\\ &\quad-\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha\circ J^r\gamma\circ\varphi^{-1})}{\partial x^s}\frac{\partial(x^s\alpha_0^{-1}\overline{\varphi}^{-1})}{\partial\overline{x}^l}\frac{\partial(\overline{x}^l\circ J^r\alpha)}{\partial x^p}dx^p\\ &= \frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)}{\partial x^p}dx^p+\sum_{i<i_2<\cdots<i_p}\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)}{\partial y^\nu_{i_1i_2\ldots i_p}}\omega^\nu_{i_1i_2\ldots i_p}\\ &\quad+\sum_{i<i_2<\cdots<i_p}\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)}{\partial y^\nu_{i_1i_2\ldots i_p}}y^\nu_{i_1i_2\ldots i_ps}dx^s\\ &\quad-\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha\circ J^r\gamma\circ\varphi^{-1})}{\partial x^s}dx^s\\ &= \sum_{i<i_2<\cdots<i_p}\frac{\partial(\overline{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)}{\partial y^\nu_{i_1i_2\ldots i_p}}\omega^\nu_{i_1i_2\ldots i_p}.\end{aligned}\tag{30}$$

These conditions mean that the section δ is of the form $\delta=J^r(\pi^{r,0}\circ\delta)$ as required. □

The basis of 1-forms (24) on V^r is usually called the *contact basis*.

The following observations show that the contact forms $\omega^\sigma_{j_1 j_2 \dots j_k}$, defined by a fibered atlas on Y, define a (global) module of 1-forms and an ideal of the exterior algebra $\Omega^r W$ (for elementary definitions, see Appendix 7).

Corollary 1 *The contact* 1*-forms* $\omega^\sigma_{j_1 j_2 \dots j_k}$ *locally generate a submodule of the module* $\Omega^r_1 W$.

Corollary 2 *The contact* 1*-forms* $\omega^\sigma_{j_1 j_2 \dots j_k}$ *locally generate an ideal of the exterior algebra* $\Omega^r W$. *This ideal is not closed under the exterior derivative operator.*

Proof Existence of the ideal is ensured by the transformation properties of the contact 1-forms $\omega^\sigma_{j_1 j_2 \dots j_k}$ (Theorem 1, (b)). It remains to show that the ideal contains a form, which is *not* generated by the forms $\omega^\sigma_{j_1 j_2 \dots j_k}$. If ρ is a contact 1-form expressed as

$$\rho = \sum_{0 \le k \le r-1} \sum_{j_1 \le j_2 \le \dots \le j_k} B^{j_1 j_2 \dots j_k}_\sigma \omega^\sigma_{j_1 j_2 \dots j_k}, \tag{31}$$

then

$$d\rho = \sum_{0 \le k \le r-1} \sum_{j_1 \le j_2 \le \dots \le j_k} \left(dB^{j_1 j_2 \dots j_k}_\sigma \wedge \omega^\sigma_{j_1 j_2 \dots j_k} + B^{j_1 j_2 \dots j_k}_\sigma d\omega^\sigma_{j_1 j_2 \dots j_k} \right). \tag{32}$$

But in this expression,

$$d\omega^\sigma_{j_1 j_2 \dots j_k} = \begin{cases} -\omega^\sigma_{j_1 j_2 \dots j_k l} \wedge dx^l, & 0 \le k \le r-2, \\ -dy^\sigma_{j_1 j_2 \dots j_{r-1} l} \wedge dx^l, & k = r-1, \end{cases} \tag{33}$$

thus, $d\omega^\sigma_{j_1 j_2 \dots j_{r-1}}$ and in general the form ρ are *not* generated by the contact forms $\omega^\sigma_{j_1 j_2 \dots j_k}$. □

The ideal of the exterior algebra $\Omega^r W$, locally generated by the 1-forms $\omega^\sigma_{j_1 j_2 \dots j_k}$, where $0 \le k \le r-1$, is denoted by $\Theta^r_0 W$. The 1-forms $\omega^\sigma_{j_1 j_2 \dots j_k}$, where $0 \le k \le r-1$, and 2-forms $d\omega^\sigma_{j_1 j_2 \dots j_{r-1}}$ locally generate an ideal $\Theta^r W$ of the exterior algebra $\Omega^r W$, *closed* under the exterior derivative operator, that is, a *differential ideal*. This ideal is called the *contact ideal* of the exterior algebra $\Omega^r W$, and its elements are called *contact forms*. We denote

$$\Theta^r_q W = \Omega^r_q W \cap \Theta^r W. \tag{34}$$

The set $\Theta^r_q W$ of contact q-forms is a submodule of the module $\Omega^r_q W$, called the *contact submodule*.

Since the exterior derivative of a contact form is again a contact form, we have the sequence

$$0 \to \Theta_1^r W \xrightarrow{d} \Theta_2^r W \xrightarrow{d} \cdots \xrightarrow{d} \Theta_n^r W, \tag{35}$$

where the arrows denote the exterior derivative operator. If ρ is a contact form, $\rho \in \Theta_q^r W$, and f is a function on W^r, $f \in \Theta_0^r W$, then the formula

$$d(f\rho) = df \wedge \rho + fd\rho \tag{36}$$

shows that the form $d(f\rho)$ is again a contact form; however, the exterior derivative in (36) is *not* a homomorphism of $\Theta_0^r W$-modules. Restricting the multiplication in (36) to *constant* functions f, that is, to *real numbers*, the exterior derivative in (36) becomes a morphism of vector spaces.

Another consequence of Theorem 1 is concerned with sections of the fibered manifold $J^r Y$ over the base X. We say that a section δ of $J^r Y$, defined on an open set in X, is *holonomic*, or *integrable*, if there exists a section γ of Y such that

$$\delta = J^r \gamma. \tag{37}$$

Obviously, if γ exists, then applying the projection $\pi^{r.0}$ to both sides, we get $\pi^{r.0} \circ \delta = \gamma$; thus, if γ exists, it is unique and is determined by

$$\gamma = \pi^{r.0} \circ \delta. \tag{38}$$

Theorem 2 *A section* δ: $U \to J^r Y$ *is holonomic if and only if for any fibered chart* (V, ψ), $\psi = (x^i, y^\sigma)$, *such that the set* $\pi(V)$ *lies in the domain of definition of* δ,

$$\delta * \omega^\sigma_{i_1 i_2 \ldots i_k} = 0 \tag{39}$$

for all σ, k, *and* $i_1, i_2, \ldots, i_k$ *such that* $1 \le \sigma \le m$, $0 \le k \le r-1$, *and* $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$.

Proof By definition,

$$\begin{aligned} \delta * \omega^\sigma_{i_1 i_2 \ldots i_k} &= d(y^\sigma_{i_1 i_2 \ldots i_k} \circ \delta) - (y^\sigma_{i_1 i_2 \ldots i_k l} \circ \delta) dx^l \\ &= \left(\frac{\partial (y^\sigma_{i_1 i_2 \ldots i_k} \circ \delta)}{\partial x^l} - y^\sigma_{i_1 i_2 \ldots i_k l} \circ \delta \right) dx^l. \end{aligned} \tag{40}$$

Thus, condition (39) is equivalent to the conditions

$$\frac{\partial (y^\sigma_{i_1 i_2 \ldots i_k} \circ \delta)}{\partial x^l} - y^\sigma_{i_1 i_2 \ldots i_k l} \circ \delta = 0 \tag{41}$$

that can also be written as

$$\begin{aligned}
&\frac{\partial(y^\sigma \circ \delta)}{\partial x^l} - y^\sigma_l \circ \delta = 0,\\
&\frac{\partial(y^\sigma_{i_1} \circ \delta)}{\partial x^l} - y^\sigma_{i_1 l} \circ \delta = \frac{\partial^2(y^\sigma \circ \delta)}{\partial x^{i_1}\partial x^l} - y^\sigma_{i_1 l} \circ \delta = 0,\\
&\dots\\
&\frac{\partial(y^\sigma_{i_1 i_2 \dots i_{r-1}} \circ \delta)}{\partial x^l} - y^\sigma_{i_1 i_2 \dots i_{r-1} l} \circ \delta = \frac{\partial^{k+1}(y^\sigma \circ \delta)}{\partial x^{i_1}\partial x^{i_2}\dots\partial x^{i_{r-1}}\partial x^l} - y^\sigma_{i_1 i_2 \dots i_{r-1} l} \circ \delta = 0.
\end{aligned} \tag{42}$$

These conditions mean that the section δ is of the form $\delta = J^r(\pi^{r,0} \circ \delta)$ as required. □

2.2 The Trace Decomposition

Main objective in this section is the application of the trace decomposition theory of tensor spaces to differential forms defined on the r-jet prolongation J^rY of a fibered manifold Y. We decompose the components of a form, expressed in a fibered chart, by the trace operation (see Appendix 9); the resulting decomposition of differential forms will be referred to as the *trace decomposition*.

In order to study the structure of the components of a form $\rho \in \Omega^r_q W$ for *general* r, it will be convenient to introduce a *multi-index notation*. We also need a convention on the alternation and symmetrization of tensor components in a given set of indices.

Convention 1 (**Multi-indices**) We introduce a multi-index I as an ordered k-tuple $I = (i_1 i_2 \dots i_k)$, where k = 1, 2,…, r and the entries are indices such that $1 \le i_1, i_2, \dots, i_k \le n$. The number k is the *length* of I and is denoted by $|I|$. If j is any integer such that $1 \le j \le n$, we denote by Ij the multi-index $Ij = (i_1 i_2 \dots i_k j)$. In this notation, the *contact basis* of 1-forms, introduced in Sect. 2.1, Theorem 1, (a), is sometimes denoted as $(dx^i, \omega^\sigma_J, dy^\sigma_I)$, where the multi-indices satisfy $0 \le |J| \le r - 1$ and $|I| = r$; it is understood, however, that the basis includes only linearly independent 1-forms ω^σ_J, where the multi-indices $I = (i_1 i_2 \dots i_k)$ satisfy $i_1 \le i_2 \le \dots \le i_k$.

Convention 2 (**Alternation, symmetrization**) We introduce the symbol $\mathrm{Alt}(i_1 i_2 \dots i_k)$ to denote *alternation* in the indices $i_1, i_2, \dots, i_k$. If $U = U_{i_1 i_2 \dots i_k}$ is a collection of real numbers, we denote by $U_{i_1 i_2 \dots i_k}\,\mathrm{Alt}(i_1 i_2 \dots i_k)$ the *skew-symmetric* component of U. Analogously, $\mathrm{Sym}(i_1 i_2 \dots i_k)$ denotes *symmetrization* in the indices $i_1, i_2, \dots, i_k$, and the symbol $U_{i_1 i_2 \dots i_k}\,\mathrm{Sym}(i_1 i_2 \dots i_k)$ means the symmetric component of U. The operators Alt and Sym are understood as *projectors* (the coefficient $1/k!$ is included).

Note that there exists a close relationship between the trace operation on the one hand and the exterior derivative operator on the other hand. For instance, decomposing in a fibered chart the 2-form $dy^\sigma_{Jj} \wedge dx^k$ by the trace operation, we get

$$dy^\sigma_{Jj} \wedge dx^k = \frac{1}{n}\delta^k_j dy^\sigma_{Js} \wedge dx^s + dy^\sigma_{Jj} \wedge dx^k - \frac{1}{n}\delta^k_j dy^\sigma_{Js} \wedge dx^s, \tag{43}$$

where the summand, representing the *Kronecker component* of $dy^\sigma_{Jj} \wedge dx^k$, coincides, up to a constant factor, with the *exterior derivative* $d\omega^\sigma_J$, and is therefore a contact form:

$$\frac{1}{n}\delta^k_j dy^\sigma_{Js} \wedge dx^s = -\frac{1}{n} d\omega^\sigma_J. \tag{44}$$

The complementary summand in the decomposition (43), represented by the second and the third terms, is *traceless* in the indices j and k. We wish to use this observation to generalize decomposition (43) to any q-forms on J^rY.

First, we apply the trace decomposition theorem (Appendix 9, Theorem 1) to q-forms of a specific type, not containing the contact forms ω^ν_J.

Lemma 2 *Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y. Let μ be a q-form on V^r such that*

$$\begin{aligned}\mu = {} & A_{i_1 i_2 \ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q} \\ & + B^{I_1}_{\sigma_1 i_2 i_3 \ldots i_q} dy^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q} \\ & + B^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \ldots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\ & + \cdots + B^{I_1 I_2 \cdots I_{q-1}}_{\sigma_1 \sigma_2 \cdots \sigma_{q-1} i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q} \\ & + A^{I_1 I_2 \cdots I_q}_{\sigma_1 \sigma_2 \cdots \sigma_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_q}_{I_q},\end{aligned} \tag{45}$$

where the multi-indices satisfy $|I_1|, |I_2|, \ldots, |I_{q-1}| = r$. Then, μ has a decomposition

$$\mu = \mu_0 + \mu', \tag{46}$$

satisfying the following conditions:

(a) *μ_0 is generated by the forms $d\omega^\sigma_J$, where $|J| = r - 1$, that is,*

$$\mu_0 = \sum_{|J| = r-1} d\omega^\sigma_J \wedge \Phi^J_\sigma, \tag{47}$$

for some $(q-2)$-forms Φ^J_σ.

(b) *μ' has an expression*

$$\begin{aligned}\mu' &= A_{i_1 i_2 \ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q} \\ &+ A^{I_1}_{\sigma_1\, i_2 i_3 \ldots i_q} dy^{\sigma_1}_{I_1} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q} \\ &+ A^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \ldots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\ &+ \cdots + A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q} \\ &+ A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_q}_{\sigma_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_q}_{I_q},\end{aligned} \tag{48}$$

where $A^{I_1}_{\sigma_1 i_2 i_3 \ldots i_q}, A^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \ldots i_q}, \ldots, A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}$ are traceless components of the coefficients $B^{I_1}_{\sigma_1 i_2 i_3 \ldots i_q}, B^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \ldots i_q}, \ldots, B^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}$.

Proof Applying the trace decomposition theorem (Appendix 9) to the coefficients $B^{I_1}_{\sigma_1 i_2 i_3 \ldots i_q}, B^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \ldots i_q}, \ldots, B^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}$ in (45), we get

$$\begin{aligned}&B^{I_1}_{\sigma_1 i_2 i_3 \ldots i_q} = A^{I_1}_{\sigma_1\, i_2 i_3 \ldots i_q} + C^{I_1}_{\sigma_1\, i_2 i_3 \ldots i_q}, \\ &B^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \ldots i_q} = A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \ldots i_q} + C^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \ldots i_q}, \\ &\ldots \\ &B^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-2}}_{\sigma_{q-2}\, i_{q-1} i_q} = A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-2}}_{\sigma_{q-2}\, i_{q-1} i_q} + C^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-2}}_{\sigma_{q-2} i_{q-1} i_q}, \\ &B^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-2}}_{\sigma_{q-2} i_{q-1} i_q} = A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-2}}_{\sigma_{q-2} i_{q-1} i_q} + C^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-2}}_{\sigma_{q-2} i_{q-1} i_q},\end{aligned} \tag{49}$$

where the systems $A^{I_1}_{\sigma_1 i_2 i_3 \ldots i_q}, A^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \ldots i_q}, \ldots, A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}$ are traceless and $C^{I_1}_{\sigma_1 i_2 i_3 \ldots i_q}, C^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \ldots i_q}, \ldots, C^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1}\, i_q}$ are of Kronecker type. Thus, writing the multi-index I_l as $I_l = J_l j_l$, we have

$$\begin{aligned}&C^{I_1}_{\sigma_1 i_2 i_3 \ldots i_q} = \delta^{j_1}_{i_2} D^{J_1}_{\sigma_1 i_3 i_4 \ldots i_q} \quad \mathrm{Alt}(i_2 i_3 i_4 \ldots i_q) \quad \mathrm{Sym}(J_1 j_1), \\ &C^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \ldots i_q} = \delta^{j_1}_{i_3} D^{J_1 I_2}_{\sigma_1 \sigma_2 i_4 i_5 \ldots i_q} \quad \mathrm{Alt}(i_3 i_4 i_5 \ldots i_q) \quad \mathrm{Sym}(J_1 j_1) \quad \mathrm{Sym}(J_2 j_2), \\ &\ldots \\ &C^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-2}}_{\sigma_{q-2} i_{q-1} i_q} = \delta^{j_1}_{i_{q-1}} D^{J_1 I_2 I_3}_{\sigma_1 \sigma_2 \sigma_3} \cdots {}^{I_{q-2}}_{\sigma_{q-2} i_q} \quad \mathrm{Alt}(i_{q-1} i_q) \quad \mathrm{Sym}(J_1 j_1) \\ &\quad \mathrm{Sym}(J_2 j_2) \quad \ldots \quad \mathrm{Sym}(J_{q-2} j_{q-2}), \\ &C^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_{q-1}}_{\sigma_{q-1} i_q} = \delta^{j_1}_{i_q} D^{J_1 I_2 I_3}_{\sigma_1 \sigma_2 \sigma_3} \cdots {}^{I_{q-1}}_{\sigma_{q-1}} \quad \mathrm{Sym}(J_1 j_1) \quad \mathrm{Sym}(J_2 j_2) \\ &\quad \ldots \quad \mathrm{Sym}(J_{q-2} j_{q-2}).\end{aligned} \tag{50}$$

Then

$$
\begin{aligned}
\mu = {} & A_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q} \\
& + A^{I_1}_{\sigma_1 i_2 i_3 \dots i_q} dy^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q} \\
& + A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \dots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\
& + \cdots + A^{I_1 I_2 \dots I_{q-1}}_{\sigma_1 \sigma_2 \dots \sigma_{q-1} i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q} \\
& + A^{I_1 I_2 \dots I_q}_{\sigma_1 \sigma_2 \dots \sigma_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_q}_{I_q} \\
& + \delta^{j_1}_{i_2} D^{J_1}_{\sigma_1 i_3 i_4 \dots i_q} dy^{\sigma_1}_{J_1 j_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q} \\
& + \delta^{j_1}_{i_3} D^{J_1 I_2}_{\sigma_1 \sigma_2 i_4 i_5 \dots i_q} dy^{\sigma_1}_{J_1 j_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\
& + \cdots + \delta^{j_1}_{i_q} D^{J_1 I_2 I_3 \dots I_{q-1}}_{\sigma_1 \sigma_2 \sigma_3 \dots \sigma_{q-1}} dy^{\sigma_1}_{J_1 j_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q},
\end{aligned}
\tag{51}
$$

and now our assertion follows from the formula (44). □

The following theorem generalizes Lemma 2 to arbitrary forms on open sets in the r-jet prolongation $J^r Y$.

Theorem 3 (The trace decomposition theorem) *Let q be any positive integer, and let $\rho \in \Omega^r_q W$ be a q-form. Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y, such that $V \subset W$. Then, ρ has on V^r an expression*

$$\rho = \rho_0 + \rho', \tag{52}$$

with the following properties:

(a) *ρ_0 is generated by the 1-forms ω^σ_J with $0 \le |J| \le r-1$ and 2-forms $d\omega^\sigma_I$ where $|I| = r-1$.*
(b) *ρ' has an expression*

$$
\begin{aligned}
\rho' = {} & A_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q} \\
& + A^{I_1}_{\sigma_1 i_2 i_3 \dots i_q} dy^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q} \\
& + A^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \dots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\
& + \cdots + A^{I_1 I_2 \dots I_{q-1}}_{\sigma_1 \sigma_2 \dots \sigma_{q-1} i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q} \\
& + A^{I_1 I_2 \dots I_q}_{\sigma_1 \sigma_2 \dots \sigma_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_q}_{I_q},
\end{aligned}
\tag{53}
$$

where $|I_1|, |I_2|, \dots, |I_{q-1}| = r$ and all coefficients $A^{I_1}_{\sigma_1 i_2 i_3 \dots i_q}, A^{I_1 I_2}_{\sigma_1 \sigma_2 i_2 i_3 \dots i_q}, \dots, A^{I_1 I_2 \dots I_{q-1}}_{\sigma_1 \sigma_2 \dots \sigma_{q-1} i_2 i_3 \dots i_q}$ are traceless.

Proof To prove Theorem 3, we express ρ in the contact basis. Then, $\rho = \rho_1 + \mu$, where ρ_1 is generated by contact 1-forms ω_J^σ, $0 \le |J| \le r-1$, and μ does not contain any factor ω_J^σ. Thus, μ has an expression (45) and can be decomposed as in Lemma 2, (46). Using this decomposition, we get the formula (52). □

Theorem 3 is the *trace decomposition theorem* for differential forms; formula (52) is referred to as the *trace decomposition formula.* The form ρ_0 in this decomposition (43) is contact and is called the *contact component* of ρ; the form ρ' is the *traceless component* of ρ with respect to the fibered chart (V, ψ).

Lemma 3 *Let $\rho \in \Omega_q^r W$ be a q-form, and let (V, ψ), $\psi = (x^i, y^\sigma)$, and $(\overline{V}, \overline{\psi})$, $\overline{\psi} = (\overline{x}^i, \overline{y}^\sigma)$, be two fibered charts such that $V \cap \overline{V} \neq \emptyset$. Suppose that we have the trace decomposition of the form ρ with respect to (V, ψ) and $(\overline{V}, \overline{\psi})$, respectively,*

$$\rho = \rho_0 + \rho' = \overline{\rho}_0 + \overline{\rho'}. \tag{54}$$

Then, the traceless components satisfy

$$\rho' = \overline{\rho'} + \overline{\eta}, \tag{55}$$

where $\overline{\eta}$ is a contact form on the intersection $V \cap \overline{V}$.

Proof Lemma 3 can be easily verified by a direct calculation. Consider for instance the term $A^I_{\sigma\, i_2 i_3 \ldots i_q} dy^\sigma_{I_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q}$ in formula (53), and the transformation equation is

$$\frac{\partial y^\sigma_{i_1 i_2 \ldots i_r}}{\partial \overline{y}^\nu_{j_1 j_2 \ldots j_r}} = \frac{\partial y^\sigma}{\partial \overline{y}^\nu} \frac{\partial \overline{x}^{j_1}}{\partial x^{i_1}} \frac{\partial \overline{x}^{j_2}}{\partial x^{i_2}} \cdots \frac{\partial \overline{x}^{j_r}}{\partial x^{i_r}} \quad \mathrm{Sym}(j_1 j_2 \ldots j_r). \tag{56}$$

Denote $\overline{\omega}^\nu_{j_1 j_2 \ldots j_k} = d\overline{y}^\nu_{j_1 j_2 \ldots j_k} - \overline{y}^\nu_{j_1 j_2 \ldots j_k l} d\overline{x}^l$. Then, we have

$$\begin{aligned} & A_\sigma^{i_1 i_2 \ldots i_r}{}_{s_2 s_3 \ldots s_q} dy^\sigma_{i_1 i_2 \ldots i_r} \wedge dx^{s_2} \wedge dx^{s_3} \wedge \cdots \wedge dx^{s_q} \\ &\quad = A_\sigma^{i_1 i_2 \ldots i_r}{}_{s_2 s_3 \ldots s_q} \frac{\partial x^{s_2}}{\partial \overline{x}^{l_2}} \frac{\partial x^{s_3}}{\partial \overline{x}^{l_3}} \cdots \frac{\partial x^{s_q}}{\partial \overline{x}^{l_q}} \cdot \left(\left(\frac{\partial y^\sigma_{i_1 i_2 \ldots i_r}}{\partial \overline{x}^p} + \sum_{0 \le k \le r-1} \frac{\partial y^\sigma_{i_1 i_2 \ldots i_r}}{\partial \overline{y}^\nu_{j_1 j_2 \ldots j_k}} y^\nu_{j_1 j_2 \ldots j_k p} \right) d\overline{x}^p \right. \\ &\qquad \left. + \sum_{0 \le k \le r-1} \frac{\partial y^\sigma_{i_1 i_2 \ldots i_r}}{\partial \overline{y}^\nu_{j_1 j_2 \ldots j_k}} \overline{\omega}^\nu_{j_1 j_2 \ldots j_k} + \frac{\partial y^\sigma_{i_1 i_2 \ldots i_r}}{\partial \overline{y}^\nu_{j_1 j_2 \ldots j_r}} d\overline{y}^\nu_{j_1 j_2 \ldots j_r} \right) \wedge d\overline{x}^{l_2} \wedge d\overline{x}^{l_3} \wedge \cdots \wedge d\overline{x}^{l_q}. \end{aligned} \tag{57}$$

Consequently, the last summand in (57) implies

$$\overline{A}_\nu^{j_1 j_2 \ldots j_r}{}_{l_2 l_3 \ldots l_q} = A_\sigma^{i_1 i_2 \ldots i_r}{}_{s_2 s_3 \ldots s_q} \frac{\partial x^{s_2}}{\partial \overline{x}^{l_2}} \frac{\partial x^{s_3}}{\partial \overline{x}^{l_3}} \cdots \frac{\partial x^{s_q}}{\partial \overline{x}^{l_q}} \frac{\partial y^\sigma_{i_1 i_2 \ldots i_r}}{\partial \overline{y}^\nu_{j_1 j_2 \ldots j_r}}. \tag{58}$$

Substituting from (56) in this formula, we see that the trace of $\overline{A}^{j_1 j_2 \ldots j_r}_{\nu\, l_2 l_3 \ldots l_q}$ vanishes if and only if the same is true for the trace of $A^{i_1 i_2 \ldots i_r}_{\sigma\, s_2 s_3 \ldots s_q}$. Thus, the decomposition (55) is valid for the summand (56). The same applies to any other summand. □

Following Theorem 3, we can write the q-form ρ in the contact basis as $\rho = \rho_1 + \rho_2 + \rho'$, where ρ_1 is generated by the forms ω^σ_J, $0 \leq |J| \leq r-1$, ρ_2 is generated by $d\omega^\sigma_I$, $|I| = r-1$, and does not contain any factor ω^σ_J, and the form ρ' is traceless. Thus,

$$\rho_1 = \sum_{0 \leq |J| \leq r-1} \omega^\sigma_J \wedge \Phi^J_\sigma, \quad \rho_2 = \sum_{|I| = r-1} d\omega^\sigma_I \wedge \Psi^I_\sigma \tag{59}$$

for some forms Φ^J_σ and Ψ^I_σ. Then,

$$\rho = \omega^\sigma_J \wedge \Phi^J_\sigma + \omega^\sigma_I \wedge d\Psi^I_\sigma + d(\omega^\sigma_I \wedge \Psi^I_\sigma) + \rho'. \tag{60}$$

Setting

$$P\rho = \omega^\sigma_J \wedge \Phi^J_\sigma + \omega^\sigma_I \wedge d\Psi^I_\sigma, \quad Q\rho = \omega^\sigma_I \wedge \Psi^I_\sigma, \quad R\rho = \rho', \tag{61}$$

we get the following version of Theorem 3.

Theorem 4 *Let q be arbitrary, and let $\rho \in \Omega^r_q W$ be a q-form. Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y such that $V \subset W$. Then, ρ can be expressed on V^r as*

$$\rho = P\rho + dQ\rho + R\rho. \tag{62}$$

Proof This is an immediate consequence of definitions and Theorem 3. □

In the following two examples, we discuss the trace decomposition formula and the transformation equations for the *traceless* components of some differential forms on 1-jet prolongation of the fibered manifold Y. The aim is to illustrate the decomposition methods for lower-degree differential forms.

Example 1 We find the trace decomposition of a 3-form μ, written in a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, as

$$\begin{aligned}\mu = {} & A_{ijk} dx^i \wedge dx^j \wedge dx^k + B^p_{\sigma\, jk} dy^\sigma_p \wedge dx^j \wedge dx^k \\ & + B^{pq}_{\sigma\nu k} dy^\sigma_p \wedge dy^\nu_q \wedge dx^k + A^{pqr}_{\sigma\nu\tau} dy^\sigma_p \wedge dy^\nu_q \wedge dy^\tau_r.\end{aligned} \tag{63}$$

Decomposing $B^p_{\sigma jk}$, we have $B^p_{\sigma jk} = A^p_{\sigma jk} + \delta^p_j C_{\sigma k} + \delta^p_k D_{\sigma j}$, where $A^p_{\sigma jk}$ is traceless. Then, the condition $B^p_{\sigma jk} = -B^p_{\sigma kj}$ yields

$$\begin{aligned} B^p_{\sigma pk} &= \delta^p_p C_{\sigma k} + \delta^p_k D_{\sigma p} = nC_{\sigma k} + D_{\sigma k} \\ &= -B^p_{\sigma kp} = -\delta^p_k C_{\sigma p} - \delta^p_p D_{\sigma k} = -C_{\sigma k} - nD_{\sigma k}, \end{aligned} \tag{64}$$

and hence, $C_{\sigma k} = -D_{\sigma k}$. Thus,

$$B^p_{\sigma jk} = A^p_{\sigma jk} + \delta^p_j C_{\sigma k} - \delta^p_k C_{\sigma j}. \tag{65}$$

Decomposing $B^{pq}_{\sigma\nu k}$, we have $B^{pq}_{\sigma\nu k} = A^{pq}_{\sigma\nu k} + \delta^p_k C^q_{\sigma\nu} + \delta^q_k D^p_{\sigma\nu}$. Now, the condition $B^{pq}_{\sigma\nu k} = -B^{qp}_{\nu\sigma k}$ yields

$$\begin{aligned} B^{pq}_{\sigma\nu p} &= \delta^p_p C^q_{\sigma\nu} + \delta^q_p D^p_{\sigma\nu} = nC^q_{\sigma\nu} + D^q_{\sigma\nu} \\ &= -B^{qp}_{\nu\sigma p} = -\delta^q_p C^p_{\nu\sigma} - \delta^p_p D^q_{\nu\sigma} = -C^q_{\nu\sigma} - nD^q_{\nu\sigma}, \end{aligned} \tag{66}$$

and hence, $nC^q_{\sigma\nu} + C^q_{\nu\sigma} = -nD^q_{\nu\sigma} - D^q_{\sigma\nu}$. It can be easily verified that this condition implies

$$C^q_{\sigma\nu} = -D^q_{\nu\sigma}. \tag{67}$$

Indeed, symmetrization and alternation yield

$$nC^q_{\sigma\nu} + C^q_{\nu\sigma} + nC^q_{\nu\sigma} + C^q_{\sigma\nu} = -nD^q_{\nu\sigma} - D^q_{\sigma\nu} - nD^q_{\sigma\nu} - D^q_{\nu\sigma} \tag{68}$$

and

$$nC^q_{\sigma\nu} + C^q_{\nu\sigma} - nC^q_{\nu\sigma} - C^q_{\sigma\nu} = -nD^q_{\nu\sigma} - D^q_{\sigma\nu} + nD^q_{\sigma\nu} + D^q_{\nu\sigma}, \tag{69}$$

hence, $C^q_{\sigma\nu} + C^q_{\nu\sigma} = -D^q_{\nu\sigma} - D^q_{\sigma\nu}$ and $C^q_{\sigma\nu} - C^q_{\sigma\nu} = -D^q_{\nu\sigma} + D^q_{\sigma\nu}$. These equations already imply (47). Thus,

$$B^{pq}_{\sigma\nu k} = A^{pq}_{\sigma\nu k} + \delta^p_k C^q_{\sigma\nu} - \delta^q_k C^p_{\nu\sigma}. \tag{70}$$

Summarizing (65) and (70), we get

$$\begin{aligned} \mu = {} & A_{ijk}dx^i \wedge dx^j \wedge dx^k + A^p_{\sigma jk}dy^\sigma_p \wedge dx^j \wedge dx^k + A^p_{\sigma\nu k}dy^\sigma_p \wedge dy^\nu_q \wedge dx^k \\ & + \delta^p_j C_{\sigma k}dy^\sigma_p \wedge dx^j \wedge dx^k - \delta^p_k C_{\sigma j}dy^\sigma_p \wedge dx^j \wedge dx^k \\ & + \delta^p_k C^q_{\sigma\nu}dy^\sigma_p \wedge dy^\nu_q \wedge dx^k - \delta^q_k C^p_{\nu\sigma}dy^\sigma_p \wedge dy^\nu_q \wedge dx^k \\ & + A^{pqr}_{\sigma\nu\tau}dy^\sigma_p \wedge dy^\nu_q \wedge dy^\tau_r \end{aligned}$$

$$\begin{aligned}
&= A_{ijk}dx^i \wedge dx^j \wedge dx^k + A^p_{\sigma jk}dy^\sigma_p \wedge dx^j \wedge dx^k \\
&\quad + A^{pq}_{\sigma\nu k}dy^\sigma_p \wedge dy^\nu_q \wedge dx^k + A^{pqr}_{\sigma\nu\tau}dy^\sigma_p \wedge dy^\nu_q \wedge dy^\tau_r \\
&\quad + C_{\sigma k}dy^\sigma_p \wedge dx^p \wedge dx^k - C_{\sigma j}dy^\sigma_p \wedge dx^j \wedge dx^p \\
&\quad + C^q_{\sigma\nu}dy^\sigma_p \wedge dy^\nu_q \wedge dx^p - C^p_{\nu\sigma}dy^\sigma_p \wedge dy^\nu_q \wedge dx^q \\
&= A_{ijk}dx^i \wedge dx^j \wedge dx^k + A^p_{\sigma jk}dy^\sigma_p \wedge dx^j \wedge dx^k \\
&\quad + A^{pq}_{\sigma\nu k}dy^\sigma_p \wedge dy^\nu_q \wedge dx^k + A^{pqr}_{\sigma\nu\tau}dy^\sigma_p \wedge dy^\nu_q \wedge dy^\tau_r \\
&\quad - 2C_{\sigma k}d\omega^\sigma \wedge dx^k + 2C^p_{\sigma\nu}d\omega^\sigma \wedge dy^\nu_p.
\end{aligned} \tag{71}$$

Thus, applying formula (51) to any 3-form ρ on V^1, we get the decomposition

$$\rho = \rho_1 + \rho_2 + \rho', \tag{72}$$

where ρ_1 is generated by ω^σ, that is, $\rho_1 = \omega^\sigma \wedge \Phi_\sigma$, ρ_2 is generated by the contact 2-forms $d\omega^\sigma$, $\rho_2 = d\omega^\sigma \wedge \Psi_\sigma$, where the 1-forms Ψ_σ do not contain any factor ω^ν, and ρ' is traceless.

Example 2 (Transformation properties) Consider a 2-form on the 1-jet prolongation J^1Y, expressed in two fibered charts (V,ψ), $\psi = (x^i, y^\sigma)$, and $(\overline{V},\overline{\psi})$, $\overline{\psi} = (\overline{x}^i, \overline{y}^\sigma)$, as

$$\rho = \rho_1 + \rho_2 + \rho' = \overline{\rho}_1 + \overline{\rho}_2 + \overline{\rho'}, \tag{73}$$

where according to Theorem 3,

$$\begin{aligned}
&\rho_1 = \omega^\sigma \wedge P_\sigma, \quad \rho_2 = Q_\sigma d\omega^\sigma, \\
&\rho' = A_{ij}dx^i \wedge dx^j + A^i_{\nu j}dy^\nu_i \wedge dx^j + A^{ij}_{\nu\tau}dy^\nu_i \wedge dy^\tau_j,
\end{aligned} \tag{74}$$

and

$$\begin{aligned}
&\overline{\rho}_1 = \overline{\omega}^\sigma \wedge \overline{P}_\sigma, \quad \overline{\rho}_2 = \overline{Q}_\sigma d\overline{\omega}^\sigma, \\
&\overline{\rho'} = \overline{A}_{ij}d\overline{x}^i \wedge d\overline{x}^j + \overline{A}^i_{\nu j}d\overline{y}^\nu_i \wedge d\overline{x}^j + \overline{A}^{il}_{\nu\tau}d\overline{y}^\nu_i \wedge d\overline{y}^\tau_l.
\end{aligned} \tag{75}$$

We want to determine transformation formulas for the traceless components $A^{ij}_{\nu\tau}$, $A^i_{\nu j}$, and A_{ij}. Transformation equations are of the form

$$\overline{x}^i = \overline{x}^i(x^j), \quad \overline{y}^\sigma = \overline{y}^\sigma(x^j, y^\nu), \quad \overline{y}^\sigma_j = \left(\frac{\partial \overline{y}^\sigma}{\partial x^l} + \frac{\partial \overline{y}^\sigma}{\partial y^\nu}y^\nu_l\right)\frac{\partial x^l}{\partial \overline{x}^j}, \tag{76}$$

and imply

$$d\overline{y}_i^\nu = \left(\frac{\partial \overline{y}_i^\nu}{\partial x^p} + \frac{\partial \overline{y}_i^\nu}{\partial y^\kappa} y_p^\kappa\right) dx^p + \frac{\partial \overline{y}_i^\nu}{\partial y^\kappa}\omega^\kappa + \frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^i} dy_s^\kappa. \tag{77}$$

Then, a direct calculation yields

$$\begin{aligned}
\overline{A}_{\nu\tau}^{i\,l} d\overline{y}_i^\nu \wedge d\overline{y}_l^\tau = {} & \overline{A}_{\nu\tau}^{i\,l}\left(\frac{\partial \overline{y}_i^\nu}{\partial x^p} + \frac{\partial \overline{y}_i^\nu}{\partial y^\kappa} y_p^\kappa\right)\left(\frac{\partial \overline{y}_l^\tau}{\partial x^q} + \frac{\partial \overline{y}_l^\tau}{\partial y^\lambda} y_q^\lambda\right) dx^p \wedge dx^q \\
& + \overline{A}_{\nu\tau}^{i\,l}\left(\frac{\partial \overline{y}_i^\nu}{\partial x^p} + \frac{\partial \overline{y}_i^\nu}{\partial y^\kappa} y_p^\kappa\right)\frac{\partial \overline{y}_l^\tau}{\partial y^\lambda} dx^p \wedge \omega^\lambda \\
& + \overline{A}_{\nu\tau}^{il}\frac{\partial \overline{y}_i^\nu}{\partial y^\kappa}\left(\frac{\partial \overline{y}_l^\tau}{\partial x^p} + \frac{\partial \overline{y}_l^\tau}{\partial y^\lambda} y_q^\lambda\right)\omega^\kappa \wedge dx^q \\
& + \overline{A}_{\nu\tau}^{i\,l}\left(\frac{\partial \overline{y}_i^\nu}{\partial x^p} + \frac{\partial \overline{y}_i^\nu}{\partial y^\kappa} y_p^\kappa\right)\frac{\partial \overline{y}^\tau}{\partial y^\lambda}\frac{\partial x^j}{\partial \overline{x}^l} dx^p \wedge dy_j^\lambda \\
& + \overline{A}_{\nu\tau}^{i\,l}\frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^i}\left(\frac{\partial \overline{y}_l^\tau}{\partial x^p} + \frac{\partial \overline{y}_l^\tau}{\partial y^\lambda} y_q^\lambda\right) dy_s^\kappa \wedge dx^q + \overline{A}_{\nu\tau}^{i\,l}\frac{\partial \overline{y}_i^\nu}{\partial y^\kappa}\frac{\partial \overline{y}_l^\tau}{\partial y^\lambda}\omega^\kappa \wedge \omega^\lambda \\
& + \overline{A}_{\nu\tau}^{i\,l}\frac{\partial \overline{y}_i^\nu}{\partial y^\kappa}\frac{\partial \overline{y}^\tau}{\partial y^\lambda}\frac{\partial x^j}{\partial \overline{x}^l}\omega^\kappa \wedge dy_j^\lambda + \overline{A}_{\nu\tau}^{i\,l}\frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^i}\frac{\partial \overline{y}_l^\tau}{\partial y^\lambda} dy_s^\kappa \wedge \omega^\lambda \\
& + \overline{A}_{\nu\tau}^{i\,l}\frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^i}\frac{\partial \overline{y}^\tau}{\partial y^\lambda}\frac{\partial x^j}{\partial \overline{x}^l} dy_s^\kappa \wedge dy_j^\lambda.
\end{aligned} \tag{78}$$

Similarly,

$$\begin{aligned}
\overline{A}_{\nu j}^i d\overline{y}_i^\nu \wedge d\overline{x}^j = {} & \overline{A}_{\nu j}^i\frac{\partial \overline{x}^j}{\partial x^l}\left(\frac{\partial \overline{y}_i^\nu}{\partial x^p} + \frac{\partial \overline{y}_i^\nu}{\partial y^\kappa} y_p^\kappa\right) dx^p \wedge dx^l \\
& + \overline{A}_{\nu j}^i\frac{\partial \overline{x}^j}{\partial x^l}\frac{\partial \overline{y}_i^\nu}{\partial y^\kappa}\omega^\kappa \wedge dx^l + \overline{A}_{\nu j}^i\frac{\partial \overline{x}^j}{\partial x^l}\frac{\partial \overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial \overline{x}^i} dy_s^\kappa \wedge dx^l,
\end{aligned} \tag{79}$$

and

$$\overline{A}_{ij} d\overline{x}^i \wedge d\overline{x}^j = \overline{A}_{ij}\frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^l} dx^p \wedge dx^l. \tag{80}$$

To determine the traceless components $A^{ij}_{\nu\tau}$, $A^i_{\nu j}$, and A_{ij} from the formulas (78)–(80), respectively, we need the terms not containing ω^τ; we get

$$\begin{aligned}
&\overline{A}^{i\,l}_{\nu\tau}\left(\frac{\partial\overline{y}^\nu_i}{\partial x^p}+\frac{\partial\overline{y}^\nu_i}{\partial y^\kappa}y^\kappa_p\right)\left(\frac{\partial\overline{y}^\tau_l}{\partial x^q}+\frac{\partial\overline{y}^\tau_l}{\partial y^\lambda}y^\lambda_q\right)dx^p\wedge dx^q\\
&\quad+\overline{A}^{i\,l}_{\nu\tau}\left(\frac{\partial\overline{y}^\nu_i}{\partial x^p}+\frac{\partial\overline{y}^\nu_i}{\partial y^\kappa}y^\kappa_p\right)\frac{\partial\overline{y}^\tau}{\partial y^\lambda}\frac{\partial x^j}{\partial\overline{x}^l}dx^p\wedge dy^\lambda_j\\
&\quad+\overline{A}^{i\,l}_{\nu\tau}\frac{\partial\overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial\overline{x}^i}\left(\frac{\partial\overline{y}^\tau_l}{\partial x^p}+\frac{\partial\overline{y}^\tau_l}{\partial y^\lambda}y^\lambda_q\right)dy^\kappa_s\wedge dx^q\\
&\quad+\overline{A}^{i\,l}_{\nu\tau}\frac{\partial\overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial\overline{x}^i}\frac{\partial\overline{y}^\tau}{\partial y^\lambda}\frac{\partial x^j}{\partial\overline{x}^l}dy^\kappa_s\wedge dy^\lambda_j\\
&\quad+\overline{A}^i_{\nu j}\frac{\partial\overline{x}^j}{\partial x^l}\left(\frac{\partial\overline{y}^\nu_i}{\partial x^p}+\frac{\partial\overline{y}^\nu_i}{\partial y^\kappa}y^\kappa_p\right)dx^p\wedge dx^l\\
&\quad+\overline{A}^i_{\nu j}\frac{\partial\overline{x}^j}{\partial x^l}\frac{\partial\overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial\overline{x}^i}dy^\kappa_s\wedge dx^l\\
&\quad+\overline{A}_{ij}\frac{\partial\overline{x}^i}{\partial x^p}\frac{\partial\overline{x}^j}{\partial x^l}dx^p\wedge dx^l.
\end{aligned}\tag{81}$$

Now, it is immediate that

$$\begin{aligned}
A_{pq}=&\,\overline{A}^{i\,l}_{\nu\tau}\left(\frac{\partial\overline{y}^\nu_i}{\partial x^p}+\frac{\partial\overline{y}^\nu_i}{\partial y^\kappa}y^\kappa_p\right)\left(\frac{\partial\overline{y}^\tau_l}{\partial x^q}+\frac{\partial\overline{y}^\tau_l}{\partial y^\lambda}y^\lambda_q\right)\\
&+\frac{1}{2}\overline{A}^i_{\nu j}\left(\frac{\partial\overline{x}^j}{\partial x^q}\left(\frac{\partial\overline{y}^\nu_i}{\partial x^p}+\frac{\partial\overline{y}^\nu_i}{\partial y^\kappa}y^\kappa_p\right)-\frac{\partial\overline{x}^j}{\partial x^p}\left(\frac{\partial\overline{y}^\nu_i}{\partial x^q}+\frac{\partial\overline{y}^\nu_i}{\partial y^\kappa}y^\kappa_q\right)\right)\\
&+\overline{A}_{ij}\frac{\partial\overline{x}^i}{\partial x^p}\frac{\partial\overline{x}^j}{\partial x^q}
\end{aligned}\tag{82}$$

and

$$A^{sj}_{\kappa\lambda}=\frac{1}{2}\overline{A}^{i\,l}_{\nu\tau}\left(\frac{\partial\overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial\overline{x}^i}\frac{\partial\overline{y}^\tau}{\partial y^\lambda}\frac{\partial x^j}{\partial\overline{x}^l}-\frac{\partial\overline{y}^\nu}{\partial y^\lambda}\frac{\partial x^j}{\partial\overline{x}^i}\frac{\partial\overline{y}^\tau}{\partial y^\kappa}\frac{\partial x^s}{\partial\overline{x}^l}\right).\tag{83}$$

The remaining terms should determine $A^s_{\kappa q}$ as the traceless component of the expression

$$\begin{aligned}
&-\overline{A}^{i\,l}_{\nu\tau}\left(\frac{\partial\overline{y}^\nu_i}{\partial x^q}+\frac{\partial\overline{y}^\nu_i}{\partial y^\lambda}y^\lambda_q\right)\frac{\partial\overline{y}^\tau}{\partial y^\kappa}\frac{\partial x^s}{\partial\overline{x}^l}+\overline{A}^{i\,l}_{\nu\tau}\frac{\partial\overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial\overline{x}^i}\left(\frac{\partial\overline{y}^\tau_l}{\partial x^q}+\frac{\partial\overline{y}^\tau_l}{\partial y^\lambda}y^\lambda_q\right)\\
&\quad+\overline{A}^i_{\nu j}\frac{\partial\overline{x}^j}{\partial x^q}\frac{\partial\overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial\overline{x}^i}.
\end{aligned}\tag{84}$$

Recall that the traceless component W^i_k of a general system P^i_k, indexed with one contravariant and one covariant index, is defined by

$$W^s_q = P^s_q - \frac{1}{n}\delta^s_q P, \tag{85}$$

where $P = P^j_j$ is the trace of P^i_k. To apply this definition, we first calculate the trace of (84) in s and q. We get

$$\begin{aligned} &-\overline{A}^{i\,l}_{\nu\tau}\left(\frac{\partial\overline{y}^\nu_i}{\partial x^s} + \frac{\partial\overline{y}^\nu_i}{\partial y^\lambda}y^\lambda_s\right)\frac{\partial\overline{y}^\tau}{\partial y^\kappa}\frac{\partial x^s}{\partial\overline{x}^l} + \overline{A}^{i\,l}_{\nu\tau}\frac{\partial\overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial\overline{x}^i}\left(\frac{\partial\overline{y}^\tau_l}{\partial x^s} + \frac{\partial\overline{y}^\tau_l}{\partial y^\lambda}y^\lambda_s\right) \\ &+\overline{A}^i_{\nu j}\frac{\partial\overline{x}^j}{\partial x^s}\frac{\partial\overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial\overline{x}^i}. \end{aligned} \tag{86}$$

Now, we can determine the traceless component of (84). Since the resulting expression must be equal to $A^s_{\kappa q}$, we get the transformation formula

$$\begin{aligned} A^s_{\kappa q} = {} &\overline{A}^i_{\nu j}\frac{\partial\overline{x}^j}{\partial x^q}\frac{\partial\overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial\overline{x}^i} \\ &-\overline{A}^{i\,l}_{\nu\tau}\left(\frac{\partial\overline{y}^\nu_i}{\partial x^q} + \frac{\partial\overline{y}^\nu_i}{\partial y^\lambda}y^\lambda_q\right)\frac{\partial\overline{y}^\tau}{\partial y^\kappa}\frac{\partial x^s}{\partial\overline{x}^l} + \overline{A}^{i\,l}_{\nu\tau}\frac{\partial\overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^s}{\partial\overline{x}^i}\left(\frac{\partial\overline{y}^\tau_l}{\partial x^q} + \frac{\partial\overline{y}^\tau_l}{\partial y^\lambda}y^\lambda_q\right) \\ &+\frac{1}{n}\delta^s_q\overline{A}^{i\,l}_{\nu\tau}\left(\left(\frac{\partial\overline{y}^\nu_i}{\partial x^m} + \frac{\partial\overline{y}^\nu_i}{\partial y^\lambda}y^\lambda_m\right)\frac{\partial\overline{y}^\tau}{\partial y^\kappa}\frac{\partial x^m}{\partial\overline{x}^l} - \frac{\partial\overline{y}^\nu}{\partial y^\kappa}\frac{\partial x^m}{\partial\overline{x}^i}\left(\frac{\partial\overline{y}^\tau_l}{\partial x^m} + \frac{\partial\overline{y}^\tau_l}{\partial y^\lambda}y^\lambda_m\right)\right) \end{aligned} \tag{87}$$

as desired. It is straightforward to verify that the expression on the right-hand side is traceless. This completes Example 2.

2.3 The Horizontalization

We extend the horizontalization $\Omega^r_1 W \ni \rho \to h\rho \in \Omega^{r+1}_1 W$, introduced in Sect. 2.1, to a morphism h: $\Omega^r W \to \Omega^{r+1} W$ of exterior algebras.

Let $\rho \in \Omega^r_q W$ be a q-form, where $q \geq 1$, $J^{r+1}_x\gamma \in W^{r+1}$ a point. Consider the pullback $(\pi^{r+1,r})^*\rho$ and the value $(\pi^{r+1,r})^*\rho(J^{r+1}_x\gamma)(\xi_1, \xi_2, \ldots, \xi_q)$ on any tangent vectors ξ_1, ξ_2, …, ξ_q of $J^{r+1}Y$ at the point $J^{r+1}_x\gamma$. Decompose each of these vectors into the horizontal and contact components,

$$T\pi^{r+1}\cdot\xi_l = h\xi_l + p\xi_l, \tag{88}$$

and set

$$h\rho(J_x^{r+1}\gamma)(\xi_1, \xi_2, \ldots, \xi_q) = \rho(J_x^r\gamma)(h\xi_1, h\xi_2, \ldots, h\xi_q). \tag{89}$$

This formula defines a q-form $h\rho \in \Omega_q^{r+1}W$. This definition can be extended to 0-forms (functions); we set for any function f: $W^r \to \mathbf{R}$

$$hf = (\pi^{r+1,r})^* f. \tag{90}$$

It follows from the properties of the decomposition (88) that the value $h\rho(J_x^{r+1}\gamma)(\xi_1, \xi_2, \ldots, \xi_q)$ vanishes whenever at least one of the vectors ξ_1, ξ_2, …, ξ_q is π^{r+1}-vertical (cf. Sect. 1.5). Thus, the q-form $h\rho$ is π^{r+1}-*horizontal*. In particular, $h\rho = 0$ whenever $q \geq n+1$. Sometimes $h\rho$ is called the *horizontal component* of ρ.

Formulas (89) and (90) define a mapping h: $\Omega^r W \to \Omega^{r+1} W$ of exterior algebras, called the *horizontalization*. The mapping h satisfies

$$h(\rho_1 + \rho_1) = h\rho_1 + h\rho_1, \quad h(f\rho) = (\pi^{r+1,r})^* f \cdot h\rho \tag{91}$$

for all q-forms ρ_1, ρ_1, and ρ and all functions f. In particular, restricting these formulas to *constant* functions f, we see that the horizontalization h is *linear* over the field of real numbers.

Theorem 5 *The mapping h: $\Omega^r W \to \Omega^{r+1} W$ is a morphism of exterior algebras.*

Proof This assertion is a straightforward consequence of the definition of exterior product and formula (89) for the horizontal component of a form ρ. Indeed,

$$\begin{aligned}
&h(\rho \wedge \eta)(J_x^{r+1}\gamma)(\xi_1, \xi_2, \ldots, \xi_q, \xi_{p+1}, \xi_{p+2}, \ldots, \xi_{p+q}) \\
&\quad = (\rho \wedge \eta)(J_x^r\gamma)(h\xi_1, h\xi_2, \ldots, h\xi_p, h\xi_{p+1}, h\xi_{p+2}, \ldots, h\xi_{p+q}) \\
&\quad = \sum_\tau \operatorname{sgn}\tau \cdot \rho(J_x^r\gamma)(h\xi_{\tau(1)}, h\xi_{\tau(2)}, \ldots, h\xi_{\tau(p)}) \\
&\qquad \cdot \eta(J_x^r\gamma)(h\xi_{\tau(p+1)}, h\xi_{\tau(p+2)}, \ldots, h\xi_{\tau(p+q)}) \\
&\quad = \sum_\tau \operatorname{sgn}\tau \cdot h\rho(J_x^r\gamma)(\xi_{\tau(1)}, \xi_{\tau(2)}, \ldots, \xi_{\tau(p)}) \\
&\qquad \cdot h\eta(J_x^r\gamma)(\xi_{\tau(p+1)}, \xi_{\tau(p+2)}, \ldots, \xi_{\tau(p+q)}) \\
&\quad = (h\rho(J_x^{r+1}\gamma) \wedge h\eta(J_x^{r+1}\gamma))(\xi_1, \xi_2, \ldots, \xi_q, \xi_{p+1}, \xi_{p+2}, \ldots, \xi_{p+q})
\end{aligned} \tag{92}$$

(summation through all permutations τ of the set $\{1, 2, \ldots, p, p+1, \ldots, p+q\}$ such that $\tau(1) < \tau(2) < \cdots < \tau(p)$ and $\tau(p+1) < \tau(p+2) < \cdots < \tau(p+q)$). This means, however, that

$$h(\rho \wedge \eta) = h\rho \wedge h\eta. \tag{93}$$

□

The following theorem shows that the horizontalization is completely determined by its action on functions and their exterior derivatives.

Theorem 6 *Let W be an open set in the fibered manifold Y. Then, the horizontalization $\Omega^r W \ni \rho \to h\rho \in \Omega^{r+1} W$ is a unique* **R***-linear, exterior-product-preserving mapping such that for any function $f\colon W^r \to \mathbf{R}$, and any fibered chart (V, ψ), $\psi = (y^\sigma)$, with $V \subset W$,*

$$hf = f \circ \pi^{r+1,r}, \quad hdf = d_i f \cdot dx^i, \tag{94}$$

where

$$d_i f = \frac{\partial f}{\partial x^i} + \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} y^\sigma_{j_1 j_2 \ldots j_k i}. \tag{95}$$

Proof The proof that h, defined by (89) and (90), has the desired properties (94) and (95), is standard. To prove uniqueness, note that (94) and (95) imply

$$hdx^i = dx^i, \quad hdy^\sigma_{j_1 j_2 \ldots j_k} = y^\sigma_{j_1 j_2 \ldots j_k i} dx^i. \tag{96}$$

It remains to check that any two mappings h_1 and h_2 satisfying the assumptions of Theorem 6 that agree on functions and their exterior derivatives coincide. □

We determine the kernel and the image of the horizontalization h. The following are elementary consequences of the definition.

Lemma 4

(a) *A function f satisfies $hf = 0$ if and only if $f = 0$.*
(b) *If $q \geq n + 1$, then every q-form $\rho \in \Omega^r_q W$ satisfies $h\rho = 0$.*
(c) *Let $1 \leq q \leq n$, and let $\rho \in \Omega^r_q W$ be a form. Then, $h\rho = 0$ if and only if*

$$J^r \gamma^* \rho = 0 \tag{97}$$

for every C^r section γ of Y defined on an open subset of W.
(d) *If $h\rho = 0$, then also the exterior derivative $hd\rho = 0$.*

Proof

(a) This is a mere restatement of the definition.
(b) This is an immediate consequence of the definition.
(c) Choose a section γ of Y, a point x from the domain of definition of γ and any tangent vectors $\zeta_1, \zeta_2, \ldots, \zeta_q$ of X at x. Then,

$$\begin{aligned} & J^r\gamma^*\rho(x)(\zeta_1, \zeta_2, \ldots, \zeta_q) \\ & \quad = \rho(J^r_x\gamma)(T_xJ^r\gamma \cdot \zeta_1, T_xJ^r\gamma \cdot \zeta_2, \ldots, T_xJ^r\gamma \cdot \zeta_q). \end{aligned} \tag{98}$$

Since $T\pi^{r+1}$ is surjective, there exist tangent vectors ξ_l to $J^{r+1}Y$ at $J^{r+1}_x\gamma$, such that $\zeta_l = T\pi^{r+1} \cdot \xi_l$. For these tangent vectors,

$$\begin{aligned} & J^r\gamma^*\rho(x)(\zeta_1, \zeta_2, \ldots, \zeta_q) \\ & \quad = \rho(J^r_x\gamma)(T_xJ^r\gamma \cdot T\pi^{r+1} \cdot \xi_1, T_xJ^r\gamma \cdot T\pi^{r+1} \cdot \xi_2, \ldots, T_xJ^r\gamma \cdot T\pi^{r+1} \cdot \xi_q). \end{aligned} \tag{99}$$

But $h\xi = T_xJ^r\gamma \circ T\pi^{r+1} \cdot \xi$, and hence,

$$\begin{aligned} J^r\gamma^*\rho(x)(\zeta_1, \zeta_2, \ldots, \zeta_q) & = \rho(J^r_x\gamma)(h\xi_1, h\xi_2, \ldots, h\xi_q) \\ & = h\rho(J^{r+1}_x\gamma)(\xi_1, \xi_2, \ldots, \xi_q). \end{aligned} \tag{100}$$

This correspondence already proves assertion (a).
(d) This assertion (d) follows from (c). □

We are now in a position to complete the description of the kernel of the horizontalization h for q-forms such that $1 \le q \le n$.

Theorem 7 *Let $W \subset Y$ be an open set, $\rho \in \Omega^r_q W$ a form, and let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart such that $V \subset W$.*

(a) *Let $q = 1$. Then, ρ satisfies $h\rho = 0$ if and only if its chart expression is of the form*

$$\rho = \sum_{0 \le |J| \le r-1} \Phi^J_\sigma \omega^\sigma_J \tag{101}$$

for some functions $\Phi^\sigma_J \colon V^r \to \mathbf{R}$.

(b) *Let $2 \le q \le n$. Then, ρ satisfies $h\rho = 0$ if and only if its chart expression is of the form*

$$\rho = \sum_{0 \le |J| \le r-1} \omega^\sigma_J \wedge \Phi^J_\sigma + \sum_{|I| = r-1} d\omega^\sigma_I \wedge \Psi^I_\sigma, \tag{102}$$

where Φ^J_σ (resp. Ψ^I_σ) are some $(q-1)$-forms (resp. $(q-2)$-forms) on V^r.

Proof Suppose that we have a contact q-form ρ on W^r, where $1 \leq q \leq n$. Write as in Sect. 2.2, Theorem 3, $\rho = \rho_0 + \rho'$, where ρ_0 is contact and ρ' is traceless. But the horizontalization h preserves exterior product and $h\rho = 0$, so we get $h\rho' = 0$ because ρ_0 is generated by the contact forms ω_J^σ, $d\omega_I^\sigma$, which satisfy $h\omega_J^\sigma = 0$ and $hd\omega_I^\sigma = 0$. Now, using formula $hdy_I^\sigma = y_{Ii}^\sigma dx^i$, we get, expressing ρ' as in Sect. 2.2, (53)

$$\begin{aligned} h\rho' = (&A_{i_1 i_2 \dots i_q} + A^{I_1}_{\sigma_1 i_2 i_3 \dots i_q} y^{\sigma_1}_{I_1 i_1} + A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \dots i_q} y^{\sigma_1}_{I_1 i_1} y^{\sigma_3}_{I_2 i_2} \\ &+ \dots + A^{I_1 I_2 \dots I_{q-1}}_{\sigma_1 \sigma_2 \dots \sigma_{q-1} i_q} y^{\sigma_1}_{I_1 i_1} y^{\sigma_3}_{I_2 i_2} \dots y^{\sigma_{q-1}}_{I_{q-1} i_{q-1}} \\ &+ A^{I_1 I_2 \dots I_q}_{\sigma_1 \sigma_2 \dots \sigma_q} y^{\sigma_1}_{I_1 i_1} y^{\sigma_3}_{I_2 i_2} \dots y^{\sigma_q}_{I_q i_q}) dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q}, \end{aligned} \tag{103}$$

where $|I_1|, |I_2|, \dots, |I_{q-1}| = r$ and the coefficients $A^{I_1}_{\sigma_1 i_2 i_3 \dots i_q}$, $A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \dots i_q}$, $\dots$, $A^{I_1 I_2 \dots I_{q-1}}_{\sigma_1 \sigma_2 \dots \sigma_{q-1}\, i_q}$ are traceless. Then,

$$\begin{aligned} &A_{i_1 i_2 \dots i_q} + A^{I_1}_{\sigma_1 i_2 i_3 \dots i_q} y^{\sigma_1}_{I_1 i_1} + A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \dots i_q} y^{\sigma_1}_{I_1 i_1} y^{\sigma_3}_{I_2 i_2} \\ &\quad + \dots + A^{I_1 I_2 \dots I_{q-1}}_{\sigma_1 \sigma_2 \dots \sigma_{q-1}\, i_q} y^{\sigma_1}_{I_1 i_1} y^{\sigma_3}_{I_2 i_2} \dots y^{\sigma_{q-1}}_{I_{q-1} i_{q-1}} \\ &\quad + A^{I_1 I_2 \dots I_q}_{\sigma_1 \sigma_2 \dots \sigma_q} y^{\sigma_1}_{I_1 i_1} y^{\sigma_3}_{I_2 i_2} \dots y^{\sigma_q}_{I_q i_q} = 0 \quad \mathrm{Alt}(i_1 i_2 \dots i_q). \end{aligned} \tag{104}$$

But the expressions on the left-hand sides of these equations are polynomial in the variables y_K^ν with $|K| = r+1$, so the corresponding homogeneous components in (104) must vanish separately. Then, we have $A_{i_1 i_2 \dots i_q} = 0$, $A^{I_1 I_2 \dots I_q}_{\sigma_1 \sigma_2 \dots \sigma_q} = 0$, and

$$\begin{aligned} &A^{I_1}_{\sigma_1 i_2 i_3 \dots i_q} \delta^{l_1}_{i_1} = 0 \quad \mathrm{Alt}(i_1 i_2 \dots i_q) \quad \mathrm{Sym}(I_1 l_1), \\ &A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \dots i_q} \delta^{l_1}_{i_1} \delta^{l_2}_{i_2} = 0 \quad \mathrm{Alt}(i_1 i_2 \dots i_q) \quad \mathrm{Sym}(I_1 l_1) \quad \mathrm{Sym}(I_2 l_2), \\ &\dots \\ &A^{I_1 I_2 \dots I_{q-1}}_{\sigma_1 \sigma_2 \dots \sigma_{q-1} i_q} \delta^{l_1}_{i_1} \delta^{l_2}_{i_2} \dots \delta^{l_{q-1}}_{i_{q-1}} = 0 \quad \mathrm{Alt}(i_1 i_2 \dots i_q) \quad \mathrm{Sym}(I_1 l_1) \\ &\mathrm{Sym}(I_2 l_2) \quad \dots \mathrm{Sym}(I_{q-1} l_{q-1}). \end{aligned} \tag{105}$$

However, since the coefficients $A^{I_1}_{\sigma_1\, i_2 i_3 \dots i_q}, A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \dots i_q}, \dots, A^{I_1 I_2 \dots I_{q-1}}_{\sigma_1 \sigma_2 \dots \sigma_{q-1}\, i_q}$ are traceless, they must vanish identically (see Appendix 9, Theorem 4). Thus, we have in (103)

$$\begin{aligned} &A_{i_1 i_2 \dots i_q} = 0, \quad A^{I_1}_{\sigma_1 i_2 i_3 \dots i_q} = 0, \quad A^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 i_4 \dots i_q} = 0, \\ &\dots, \quad A^{I_1 I_2 \dots I_{q-1}}_{\sigma_1 \sigma_2 \dots \sigma_{q-1} i_q} = 0, \quad A^{I_1 I_2 \dots I_q}_{\sigma_1 \sigma_2 \dots \sigma_q} = 0 \end{aligned} \tag{106}$$

and hence, $h\rho' = 0$. Thus $\rho = \rho_0$, and to close the proof, we just write this result for $q = 1$ and $q > 1$ separately. □

Corollary 1 *If* $0 \le q \le n$, *then a* q-*form belongs to the kernel of the horizontalization* h *if and only if it is a contact form.*

Corollary 2 *Let* $W \subset Y$ *be an open set,* $\rho \in \Omega_q^r W$ *a* q-*form such that* $2 \le q \le n$, *and let* (V, ψ), $\psi = (x^i, y^\sigma)$, *be a fibered chart such that* $V \subset W$. *Then, the form* ρ *satisfies the condition* $h\rho = 0$ *if and only if its chart expression is of the form*

$$\rho = \sum_{0 \le |J| \le r-1} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|I|=r-1} d(\omega_I^\sigma \wedge \Psi_\sigma^I), \tag{107}$$

where Φ_σ^J *are* $(q-1)$-*forms and* Ψ_σ^I *are* $(q-2)$-*forms) on* V^r, *which do not contain* ω_J^σ, $0 \le |J| \le r-1$.

Proof We write (102) as

$$\rho = \sum_{0 \le |J| \le r-1} \omega_J^\sigma \wedge \Phi_\sigma^J - \sum_{|I|=r-1} \omega_I^\sigma \wedge d\Psi_\sigma^I + \sum_{0 \le |I| \le r-1} d(\omega_I^\sigma \wedge \Psi_\sigma^I). \tag{108}$$

□

The *image* of the horizontalization h is characterized as follows.

Lemma 5 *Let* $\rho \in \Omega_q^r W$ *be a form.*

(a) *If* $q = 0$, *then* $h\rho = (\pi^{r+1,r})^* \rho$.
(b) *If* $1 \le q \le n$, *then*

$$h\rho = h\rho'. \tag{109}$$

(c) *If* $q \ge n+1$, *then* $h\rho = h\rho' = 0$.

Proof This assertion is an immediate consequence of the definition of the horizontalization h. □

2.4 The Canonical Decomposition

Beside the horizontalization of q-forms $\Omega_q^r W$, introduced in Sects. 2.1 and 2.3, the vector bundle morphism h: $TJ^{r+1}Y \to TJ^rY$ also induces a decomposition of the modules of q-forms $\Omega_q^r W$. Let $\rho \in \Omega_q^r W$ be a q-form, where $q \ge 1$, $J_x^{r+1}\gamma \in W^{r+1}$ a point. Consider the pullback $(\pi^{r+1,r})^*\rho$ and the value $(\pi^{r+1,r})^*\rho(J_x^{r+1}\gamma)$ $(\xi_1, \xi_2, \ldots, \xi_q)$ on any tangent vectors ξ_1, ξ_2, …, ξ_q of $J^{r+1}Y$ at the point $J_x^{r+1}\gamma$. Write for each l,

$$T\pi^{r+1} \cdot \xi_l = h\xi_l + p\xi_l, \tag{110}$$

and substitute these vectors in the pullback $(\pi^{r+1,r})^*\rho$. We get

$$\begin{aligned}&(\pi^{r+1,r})^*\rho(J_x^{r+1}\gamma)(\xi_1,\xi_2,\ldots,\xi_q)\\&\quad=\rho(J_x^r\gamma)(h\xi_1+p\xi_1,h\xi_2+p\xi_2,\ldots,h\xi_q+p\xi_q).\end{aligned}\tag{111}$$

We study in this section, for each $k=0,1,2,\ldots,q$, the summands on the right-hand side, homogeneous of degree k in the contact components $p\xi_l$ of the vectors ξ_l, and describe the corresponding decomposition of the form $(\pi^{r+1,r})^*\rho$. Using properties of ρ, we set

$$\begin{aligned}&p_k\rho(J_x^{r+1}\gamma)(\xi_1,\xi_2,\ldots,\xi_q)\\&\quad=\sum\varepsilon^{j_1j_2\ldots j_kj_{k+1}\ldots j_q}\rho(J_x^r\gamma)(p\xi_{j_1},p\xi_{j_2},\ldots,p\xi_{j_k},h\xi_{j_{k+1}},h\xi_{j_{k+2}},\ldots,h\xi_{j_q}),\end{aligned}\tag{112}$$

where the summation is understood through all sequences $j_1<j_2<\cdots<j_k$ and $j_{k+1}<j_{k+2}<\cdots<j_q$. Equivalently, $p_k\rho(J_x^{r+1}\gamma)$ can also be defined by

$$\begin{aligned}&p_k\rho(J_x^{r+1}\gamma)(\xi_1,\xi_2,\ldots,\xi_q)\\&\quad=\frac{1}{k!(q-k)!}\varepsilon^{j_1j_2\ldots j_kj_{k+1}\ldots j_q}\rho(J_x^r\gamma)(p\xi_{j_1},p\xi_{j_2},\ldots,p\xi_{j_k},h\xi_{j_{k+1}},\ldots,h\xi_{j_q})\end{aligned}\tag{113}$$

(summation through *all* values of the indices $j_1,j_2,\ldots,j_k,j_{k+1},\ldots,j_q$).

Note that if $k=0$, then $p_0\rho$ coincides with the *horizontal component* of ρ, defined in Sect. 2.1, (5),

$$p_0\rho=h\rho.\tag{114}$$

We also introduce the notation

$$p\rho=p_1\rho+p_2\rho+\cdots+p_q\rho.\tag{115}$$

These definitions can be extended to 0-forms (functions). Since for a function $f\colon W^r\to\mathbf{R}$, hf was defined to be $(\pi^{r+1,r})^*f$, we set

$$pf=0.\tag{116}$$

With this notation, any q-form $\rho\in\Omega_q^rW$, where $q\geq0$, can be expressed as $(\pi^{r+1,r})^*\rho=h\rho+p\rho$, or

$$(\pi^{r+1,r})^*\rho=h\rho+p_1\rho+p_2\rho+\cdots+p_q\rho.\tag{117}$$

This formula will be referred to as the *canonical decomposition* of the form ρ (however, the decomposition concerns rather the pullback $(\pi^{r+1,r})^*\rho$ than ρ itself).

Lemma 6 *Let $q \geq 1$, and let $\rho \in \Omega_q^r W$ be a q-form. In any fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, such that $V \subset W$, $p_k\rho$ has a chart expression*

$$p_k\rho = \sum_{0 \leq |J_1|, |J_2|, \ldots, |J_k| \leq r} P^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{J_k}_{\sigma_k\, i_{k+1} i_{k+2} \ldots i_q} \omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge \cdots \wedge \omega^{\sigma_k}_{J_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_q}, \tag{118}$$

where the components $P^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{J_k}_{\sigma_k\, i_{k+1} i_{k+2} \ldots i_q}$ are real-valued functions on the set $V^r \subset W^r$.

Proof We express the pullback $(\pi^{r+1,r})^*\rho$ in the contact basis on W^{r+1}. Write in a fibered chart

$$\rho = dx^i \wedge \Phi_i + \sum_{0 < |J| < r-1} \omega^\sigma_J \wedge \Psi^J_\sigma + \sum_{|I| = r} dy^\sigma_I \wedge \Theta^I_\sigma \tag{119}$$

for some $(q-1)$-forms Φ_i, Ψ^J_σ, and Θ^I_σ. But $dy^\sigma_I = \omega^\sigma_I + y^\sigma_{Ii} dx^i$, and hence,

$$\begin{aligned}(\pi^{r+1,r})^*\rho = dx^i \wedge \left((\pi^{r+1,r})^*\Phi_i + \sum_{|I|=r} y^\sigma_{Ii} (\pi^{r+1,r})^*\Theta^I_\sigma \right) \\ + \sum_{0 < |J| < r-1} \omega^\sigma_J \wedge (\pi^{r+1,r})^*\Psi^J_\sigma + \sum_{|I|=r} \omega^\sigma_I \wedge (\pi^{r+1,r})^*\Theta^I_\sigma.\end{aligned} \tag{120}$$

Thus, the pullback $(\pi^{r+1,r})^*\rho$ is generated by the form dx^i, ω^σ_J, where $0 < |J| < r - 1$ and ω^σ_I, $|I| = r$. The same decomposition can be applied to the $(q-1)$-forms Φ_i, Ψ^J_σ, and Θ^I_σ. Consequently, $(\pi^{r+1,r})^*\rho$ has an expression

$$(\pi^{r+1,r})^*\rho = \rho_0 + \rho_1 + \rho_2 + \cdots + \rho_q, \tag{121}$$

where

$$\begin{aligned}\rho_0 &= A_{i_1 i_2 \ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}, \\ \rho_k &= \sum_{0 \leq |J_1|, |J_2|, \ldots, |J_k| \leq r} B^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{J_k}_{\sigma_k\, i_{k+1} i_{k+2} \ldots i_q} \omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge \cdots \wedge \omega^{\sigma_k}_{J_k} \\ &\quad \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_q}, \quad 1 \leq k \leq q-1, \\ \rho_q &= \sum_{0 \leq |J_1|, |J_2|, \ldots, |J_q| \leq r} B^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{J_q}_{\sigma_q i_q} \omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_3}_{J_2} \wedge \cdots \wedge \omega^{\sigma_q}_{J_q}.\end{aligned} \tag{122}$$

Theorem 1, Sect. 2.1, implies that the decomposition (121) is invariant.

We prove that $\rho_k = p_k\rho$. It is sufficient to determine the chart expression of $p_k\rho$. Let ξ be a tangent vector,

$$\xi = \xi^i \left(\frac{\partial}{\partial x^i}\right)_{J_x^{r+1}\gamma} + \sum_{k=0}^{r+1} \sum_{j_1 \le j_2 \le \cdots \le j_k} \Xi^\sigma_{j_1 j_2 \ldots j_k} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}}\right)_{J_x^{r+1}\gamma}. \tag{123}$$

From Sect. 1.5, (62)

$$h\xi = \xi^i \left(\left(\frac{\partial}{\partial x^i}\right)_{J_x^r\gamma} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} y^\sigma_{j_1 j_2 \ldots j_k i} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}}\right)_{J_x^r\gamma} \right), \tag{124}$$

and

$$p\xi = \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} (\Xi^\sigma_{j_1 j_2 \ldots j_k} - y^\sigma_{j_1 j_2 \ldots j_k i}\xi^i) \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}}\right)_{J_x^r\gamma}. \tag{125}$$

If $h\xi = 0$, then $\xi^i = 0$, and we have

$$p\xi = \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} \Xi^\sigma_{j_1 j_2 \ldots j_k} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}}\right)_{J_x^r\gamma}. \tag{126}$$

If $p\xi = 0$, then $\Xi^\sigma_{j_1 j_2 \ldots j_k} = y^\sigma_{j_1 j_2 \ldots j_k i}\xi^i$, and hence,

$$h\xi = \xi^i \left(\left(\frac{\partial}{\partial x^i}\right)_{J_x^r\gamma} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} y^\sigma_{j_1 j_2 \ldots j_k i} \left(\frac{\partial}{\partial y^\sigma_{j_1 j_2 \ldots j_k}}\right)_{J_x^r\gamma} \right). \tag{127}$$

We substitute from these formulas to expression (112). Consider the expression $p_k\rho(J_x^{r+1}\gamma)(\xi_1, \xi_2, \ldots, \xi_q)$ for $\xi_1, \xi_2, \ldots, \xi_q$ such that $h\xi_1 = 0$, $h\xi_2 = 0$, …, $h\xi_k = 0$ and $p\xi_{k+1} = 0$, $p\xi_2 = 0$, …, $p\xi_q = 0$. Then, (112) reduces to

$$\begin{aligned} &p_k\rho(J_x^{r+1}\gamma)(\xi_1, \xi_2, \ldots, \xi_q) \\ &\quad = \rho(J_x^r\gamma)(p\xi_1, p\xi_2, \ldots, p\xi_k, h\xi_{k+1}, h\xi_{k+2}, \ldots, h\xi_q). \end{aligned} \tag{128}$$

Writing

$$\begin{aligned} p\xi_l &= \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} {}^{(l)}\Xi^{\sigma}_{j_1 j_2 \ldots j_k} \left(\frac{\partial}{\partial y^{\sigma}_{j_1 j_2 \ldots j_k}} \right)_{J^r_x \gamma}, \quad 1 \le l \le k, \\ h\xi_l &= {}^{(l)}\xi^i \left(\left(\frac{\partial}{\partial x^i} \right)_{J^r_x \gamma} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \cdots \le j_k} y^{\sigma}_{j_1 j_2 \ldots j_k i} \left(\frac{\partial}{\partial y^{\sigma}_{j_1 j_2 \ldots j_k}} \right)_{J^r_x \gamma} \right), \\ &\quad k+1 \le l \le q, \end{aligned} \tag{129}$$

with l indexing the vectors ξ_l, and substituting into (128), we get

$$\begin{aligned} &p_k \rho(J^{r+1}_x \gamma)(\xi_1, \xi_2, \ldots, \xi_k, \xi_{k+1}, \xi_{k+2}, \ldots, \xi_q), \\ &= C^{I_1 I_2 \ldots I_k}_{\sigma_1 \sigma_2 \ldots \sigma_k \, i_{k+1} i_{k+2} \ldots i_q} \, {}^1\Xi^{\sigma_1}_{I_1} \, {}^2\Xi^{\sigma_2}_{I_2} \ldots {}^k\Xi^{\sigma_k}_{I_k} \, {}^{k+1}\xi^{i_{k+1}} \, {}^{k+2}\xi^{i_{k+2}} \ldots {}^q\xi^{i_q}. \end{aligned} \tag{130}$$

But

$${}^l\Xi^{\sigma}_I = \omega^{\sigma}_I(J^{r+1}_x \gamma) \cdot \xi_l, \quad {}^l\xi^i = dx^i(J^{r+1}_x \gamma) \cdot \xi_l \tag{131}$$

Therefore, $p_k \rho(J^{r+1}_x \gamma)$ must be of the form (118). □

Formula (118) implies that for any $k \ge 1$, the form $p_k \rho$ is contact; $p_k \rho$ is called the *k-contact component* of the form ρ.

If $(\pi^{r+1,r})^* \rho = p_k \rho$ or, equivalently, if $p_j \rho = 0$ for all $j \ne k$, then we say that ρ is *k-contact*, and k is the *degree of contactness* of ρ. The degree of contactness of the q-form $\rho = 0$ is equal to k for every $k = 0, 1, 2, \ldots, q$. We say that ρ is of *degree of contactness* $\ge k$, if $p_0 \rho = 0$, $p_1 \rho = 0$, $\ldots$, $p_{k-1} \rho = 0$. If $k = 0$, then the 0-contact form $p_0 \rho = h\rho$ is $\pi^{r+1,r}$*-horizontal*. The mapping $\Omega^r_q W \ni \rho \to h\rho \in \Omega^{r+1}_q W$ is called the *horizontalization.*

The following observation is immediate.

Lemma 7 *If* $q - k > n$, *then*

$$\begin{aligned} &h\rho = 0, \\ &p_1 \rho = 0, \quad p_2 \rho = 0, \quad \ldots, \quad p_{q-n-1} = 0. \end{aligned} \tag{132}$$

Proof Expression $\rho(J^r_x \gamma)(p\xi_{j_1}, p\xi_{j_2}, \ldots, p\xi_{j_k}, h\xi_{j_{k+1}}, h\xi_{j_{k+2}}, \ldots, h\xi_{j_q})$ in (113) is a $(q-k)$-linear function of vectors $\zeta_{j_{k+1}} = T\pi^{r+1} \cdot \xi_{j_{k+1}}, \zeta_{j_{k+2}} = T\pi^{r+1} \cdot \xi_{j_{k+2}}, \ldots, \zeta_{j_q} = T\pi^{r+1} \cdot \xi_{j_q}$, belonging to the tangent space $T_x X$. Consequently, if $q - k > n = \dim X$, then the skew symmetry of the form $p_k \rho(J^{r+1}_x \gamma$ implies $p_k \rho(J^{r+1}_x \gamma)(\xi_1, \xi_2, \ldots, \xi_q) = 0$. □

To complete the local description of the decomposition (117), we express the components $P^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{J_k}_{\sigma_k i_{k+1} i_{k+2} \dots i_q}$ (118) of the k-contact components $p_k \rho$ in terms of the components of ρ.

Lemma 8 *Let W be an open set in Y, q an integer, $\eta \in \Omega^r_q W$ a form, and let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y such that $V \subset W$. Assume that η has on V^r a chart expression*

$$\eta = \sum_{s=0}^{q} \frac{1}{s!(q-s)!} A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_s}_{\sigma_s}{}_{i_{s+1} i_{s+2} \dots i_q} dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \cdots \wedge dx^{i_q}, \tag{133}$$

with multi-indices $I_1, I_2, \dots, I_s$ of length r. Then, the k-contact component $p_k \eta$ of η has on V^{r+1} a chart expression

$$p_k \eta = \frac{1}{k!(q-k)!} B^{I_1 I_1}_{\sigma_1 \sigma_1} \cdots {}^{I_k}_{\sigma_k}{}_{i_{k+1} i_{k+2} \dots i_q} \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \cdots \wedge \omega^{\sigma_k}_{I_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_q}, \tag{134}$$

where

$$\begin{aligned} &B^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_k}_{\sigma_k i_{k+1} i_{k+2} \dots i_q} \\ &= \sum_{s=k}^{q} \binom{q-k}{q-s} A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_k}_{\sigma_k}{}^{I_{k+1}}_{\sigma_{k+1}}{}^{I_{k+2}}_{\sigma_{k+2}} \cdots {}^{I_s}_{\sigma_s}{}_{i_{s+1} i_{s+2} \dots i_q} y^{\sigma_{k+1}}_{I_{k+1} i_{k+1}} y^{\sigma_{k+2}}_{I_{k+2} i_{k+2}} \cdots y^{\sigma_s}_{I_s i_s} \\ &\quad \mathrm{Alt}(i_{k+1} i_{k+2} \dots i_s i_{s+1} \dots i_q). \end{aligned} \tag{135}$$

Proof To derive the formula (134), we pullback the form η to V^{r+1} and express the form $(\pi^{r+1,r})^* \Psi$ in terms of the contact basis; in the multi-index notation, the transformation equations are

$$dx^i = dx^i, \quad dy^\sigma_I = \omega^\sigma_I + y^\sigma_{Ii} dx^i, \quad |I| = r \tag{136}$$

(Sect. 2.1, Theorem 1, (a)). Thus, we set in (133) $dy^{\sigma_l}_{I_l} = \omega^{\sigma_l}_{I_l} + y^{\sigma_l}_{I_l i_l} dx^{i_l}$ and consider the terms in (133) such that $s \geq 1$. Then, the pullback of the form $dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_s}_{I_s}$ by $\pi^{r+1,r}$ is equal to

$$(\omega^{\sigma_1}_{I_1} + y^{\sigma_1}_{I_1 i_1} dx^{i_1}) \wedge (\omega^{\sigma_2}_{I_2} + y^{\sigma_2}_{I_2 i_2} dx^{i_2}) \wedge \cdots \wedge (\omega^{\sigma_s}_{I_s} + y^{\sigma_s}_{I_s i_s} dx^{i_s}). \tag{137}$$

Collecting together all terms homogeneous of degree k in the contact 1-forms $\omega^{\sigma_l}_{I_l}$, we get $\binom{s}{k}$ summands with exactly k entries the contact 1-forms $\omega^{\sigma_l}_{I_l}$. Thus, using symmetry properties of the components $A^{I_1 I_1}_{\sigma_1 \sigma_1} \cdots {}^{I_s}_{\sigma_s i_{s+1} i_{s+2} \dots i_q}$ in (133) and interchanging multi-indices, we get the terms containing k entries $\omega^{\sigma_l}_{I_l}$, for fixed s and each $k = 1, 2, \ldots, s$,

$$\frac{1}{s!(q-s)!}\binom{s}{k} A^{I_1 I_2 \ldots I_s}_{\sigma_1 \sigma_2 \ldots \sigma_s i_{s+1} i_{s+2} \ldots i_q} y^{\sigma_{k+1}}_{I_{k+1} i_{k+1}} y^{\sigma_{k+2}}_{I_{k+2} i_{k+2}} \ldots y^{\sigma_s}_{I_s i_s} \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \cdots \wedge \omega^{\sigma_k}_{I_k}$$
$$\wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \cdots \wedge dx^{i_q}. \tag{138}$$

Writing the factor as

$$\frac{1}{s!(q-s)!}\binom{s}{k} = \frac{1}{k!(q-k)!}\binom{q-k}{q-s}, \tag{139}$$

we can express (138) as

$$\frac{1}{k!(q-k)!}\binom{q-k}{q-s} A^{I_1 I_2 \ldots I_s}_{\sigma_1 \sigma_2 \ldots \sigma_s\, i_{s+1} i_{s+2} \ldots i_q} y^{\sigma_{k+1}}_{I_{k+1} i_{k+1}} y^{\sigma_{k+2}}_{I_{k+2} i_{k+2}} \ldots y^{\sigma_s}_{I_s i_s} \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2}$$
$$\wedge \cdots \wedge \omega^{\sigma_k}_{I_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \cdots \wedge dx^{i_q}. \tag{140}$$

Formula (138) is valid for each $s = 1, 2, \ldots, q$ and each $k = 1, 2, \ldots, s$ and includes summation through all these terms to get expression (133). The summation through the pairs (s, k) is given by the table

$$\begin{array}{c|cccccc} s & 1 & 2 & 3 & \ldots & q-1 & q \\ \hline k & 1 & 1,2 & 1,2,3 & \ldots & 1,2,3,\ldots,q-1 & 1,2,3,\ldots,q \end{array} \tag{141}$$

It will be convenient to pass to the summation over the same written in the opposite order. The summation through the pairs (k, s) is expressed by the table

$$\begin{array}{c|cccccc} k & 1 & 2 & 3 & \ldots & q-1 & q \\ \hline s & 1,2,3,\ldots,q & 2,3,\ldots,q & 3,4,\ldots,q & \ldots & q-1,q & q \end{array} \tag{142}$$

Now, we can substitute from (140) back to (133). We have, with multi-indices of length r,

$$\eta = \frac{1}{q!} A_{i_1 i_2 \ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}$$
$$+ \sum_{s=1}^{q} \sum_{k=1}^{s} \frac{1}{k!(q-k)!}\binom{q-k}{q-s} A^{I_1 I_2 \ldots I_s}_{\sigma_1 \sigma_2 \ldots \sigma_s\, i_{s+1} i_{s+2} \ldots i_q} y^{\sigma_{k+1}}_{I_{k+1} i_{k+1}} y^{\sigma_{k+2}}_{I_{k+2} i_{k+2}} \ldots y^{\sigma_s}_{I_s i_s}$$
$$\cdot\, \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \cdots \wedge \omega^{\sigma_k}_{I_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_s} \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_q} \tag{143}$$

hence,

$$p_k\eta = \frac{1}{q!}A_{i_1 i_2 \ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}$$
$$+ \sum_{k=1}^{q} \frac{1}{k!(q-k)!} \left(\sum_{s=k}^{q} \binom{q-k}{q-s} A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_s}_{\sigma_s\, i_{s+1} i_{s+2} \ldots i_q} y^{\sigma_{k+1}}_{I_{k+1} i_{k+1}} y^{\sigma_{k+2}}_{I_{k+2} i_{k+2}} \cdots y^{\sigma_s}_{I_s i_s} \right)$$
$$\cdot\, \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \cdots \wedge \omega^{\sigma_k}_{I_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_q}. \tag{144}$$

This proves the formulas (134) and (135). □

Remark 5 Formulas (133) and (134) are *not* invariant; the transformation properties of the components are determined in Sect. 2.1, Theorem 1, (b).

Lemma 8 can now be easily extended to general q-forms. It is sufficient to consider the case of q-forms generated by p-forms $\omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_p}_{J_p}$ with fixed p, $1 \le p \le q-p$. The proof then consists in a formal application of Lemma 8.

Theorem 8 *Let W be an open set in Y, q a positive integer, and $\rho \in \Omega^r_q W$ a q-form, and let (V,ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y such that $V \subset W$. Assume that ρ has on V^r a chart expression*

$$\rho = \sum_{s=0}^{q-p} \frac{1}{s!(q-p-s)!} A^{J_1 J_2}_{\nu_1 \nu_2} \cdots {}^{J_p I_1 I_2}_{\nu_p \sigma_1 \sigma_2} \cdots {}^{I_s}_{\sigma_s\, i_{s+1} i_{s+2} \ldots i_{q-p}} \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_p}_{J_p}$$
$$\wedge\, dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \cdots \wedge dx^{i_{q-p}}, \tag{145}$$

with multi-indices $J_1, J_2, \ldots, J_p$ of length $r-1$ and multi-indices $I_1, I_2, \ldots, I_s$ of length r. Then, the k-contact component $p_k\rho$ of ρ has on V^{r+1} the chart expression

$$p_k\rho = \frac{1}{(k-p)!(q-p-k)!} B^{J_1 J_2}_{\nu_1 \nu_2} \cdots {}^{J_p I_1 I_1}_{\nu_p \sigma_1 \sigma_1} \cdots {}^{I_{k-p}}_{\sigma_{k-p}\, i_{k-p+1} i_{k-p+2} \ldots i_{q-p}} \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2}$$
$$\wedge \cdots \wedge \omega^{\nu_p}_{J_p} \wedge \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \cdots \wedge \omega^{\sigma_{k-p}}_{I_{k-p}} \wedge dx^{i_{k-p+1}} \wedge dx^{i_{k-p+2}} \wedge \cdots \wedge dx^{i_{q-p}}, \tag{146}$$

where

$$B^{J_1 J_2}_{\nu_1 \nu_2} \cdots {}^{J_p I_1 I_1}_{\nu_p \sigma_1 \sigma_1} \cdots {}^{I_{k-p}}_{\sigma_{k-p}\, i_{k-p+1} i_{k-p+2} \ldots i_{q-p}}$$
$$= \sum_{s=k-p}^{q-p} \binom{q-k}{q-p-s} A^{J_1 J_2}_{\nu_1 \nu_2} \cdots {}^{J_p I_1 I_2}_{\nu_p \sigma_1 \sigma_2} \cdots {}^{I_{k-p} I_{k-p+1} I_{k-p+2}}_{\sigma_{k-p} \sigma_{k-p+1} \sigma_{k-p+2}} \cdots {}^{I_s}_{\sigma_s i_{s+1} i_{s+2} \ldots i_{q-p}} \tag{147}$$
$$\cdot\, y^{\sigma_{k-p+1}}_{I_{k-p+1} i_{k-p+1}} y^{\sigma_{k-p+2}}_{I_{k-p+2} i_{k-p+2}} \cdots y^{\sigma_s}_{I_s i_s} \quad \mathrm{Alt}(i_{k-p+1} i_{k-p+2} \ldots i_s i_{s+1} \ldots i_{q-p}).$$

Proof ρ can be expressed as

$$\rho = \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_p}_{J_p} \wedge \eta^{J_1 J_2 \ldots J_p}_{\nu_1 \nu_2 \ldots \nu_p}, \tag{148}$$

where

$$\begin{aligned} \eta^{J_1 J_2 \ldots J_p}_{\nu_1 \nu_2 \ldots \nu_p} = \sum_{s=0}^{q-p} \frac{1}{s!(q-p-s)!} A^{J_1 J_2 \ldots J_p I_1 I_2 \ldots I_s}_{\nu_1 \nu_2 \ldots \nu_p \sigma_1 \sigma_2 \ldots \sigma_s i_{s+1} i_{s+2} \ldots i_{q-p}} \\ \wedge dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \cdots \wedge dx^{i_{q-p}}. \end{aligned} \tag{149}$$

We can apply to $\eta^{J_1 J_2 \ldots J_p}_{\nu_1 \nu_2 \ldots \nu_p}$ formula (134). Replacing q with $q-p$ and k with $k-p$,

$$\begin{aligned} p_{k-p} \eta^{J_1 J_2 \ldots J_p}_{\nu_1 \nu_2 \ldots \nu_p} = \frac{1}{(k-p)!(q-p-k)!} B^{J_1 J_2 \ldots J_p I_1 I_1 \ldots I_{k-p}}_{\nu_1 \nu_2 \ldots \nu_p \sigma_1 \sigma_1 \ldots \sigma_{k-p} i_{k-p+1} i_{k-p+2} \ldots i_{q-p}} \\ \cdot \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \cdots \wedge \omega^{\sigma_{k-p}}_{I_{k-p}} \wedge dx^{i_{k-p+1}} \wedge dx^{i_{k-p+2}} \wedge \cdots \wedge dx^{i_{q-p}}, \end{aligned} \tag{150}$$

where

$$\begin{aligned} & B^{J_1 J_2 \ldots J_p I_1 I_1 \ldots I_{k-p}}_{\nu_1 \nu_2 \ldots \nu_p \sigma_1 \sigma_1 \ldots \sigma_{k-p}\, i_{k-p+1} i_{k-p+2} \ldots i_{q-p}} \\ & \quad = \sum_{s=k-p}^{q-p} \binom{q-k}{q-p-s} A^{J_1 J_2 \ldots J_p I_1 I_2 \ldots I_{k-p} I_{k-p+1} I_{k-p+2} \ldots I_s}_{\nu_1 \nu_2 \ldots \nu_p \sigma_1 \sigma_2 \ldots \sigma_{k-p} \sigma_{k-p+1} \sigma_{k-p+2} \ldots \sigma_s\, i_{s+1} i_{s+2} \ldots i_{q-p}} \\ & \quad \cdot y^{\sigma_{k-p+1}}_{I_{k-p+1} i_{k-p+1}} y^{\sigma_{k-p+2}}_{I_{k-p+2} i_{k-p+2}} \ldots y^{\sigma_s}_{I_s i_s} \quad \mathrm{Alt}(i_{k-p+1} i_{k-p+2} \ldots i_s i_{s+1} \ldots i_{q-p}). \end{aligned} \tag{151}$$

□

The following two corollaries are immediate consequences of Theorem 8 and Sect. 2.1, Theorem 1. The first one shows that the operators p_k behave like *projector operators* in linear algebra. The second one is a consequence of the identity $d(\pi^{r+1,r})^* \rho = (\pi^{r+1,r})^* d\rho$ for the exterior derivative operator, the canonical decomposition of forms on jet manifolds, applied to both sides, as well as the formula

$$d\omega^\nu_J = -\omega^\nu_{Jj} \wedge dx^j. \tag{152}$$

Corollary 1 *For any k and l,*

$$p_k p_l \rho = \begin{cases} (\pi^{r+2,r+1})^* p_k \rho, & k = l, \\ 0, & k \neq l. \end{cases} \tag{153}$$

Corollary 2 *For every $k \geq 1$,*

$$(\pi^{r+2,r+1})^* p_k \rho = p_k d p_{k-1} \rho + p_k d_k \rho. \tag{154}$$

Remark 6 According to Sect. 2.3, Theorem 5, the horizontalization $h\colon \Omega^r W \to \Omega^{r+1} W$ is a morphism of exterior algebras. On the other hand, if k is a *positive* integer, then the mapping $p_k\colon \Omega^r W \to \Omega^{r+1} W$ satisfies

$$p_k(\rho + \eta) = p_k\rho + p_k\eta, \quad p_k(f\rho) = (f \circ \pi^{r+1,r})p_k\rho \tag{155}$$

for all ρ, η, and f. However, $p_k\colon \Omega^r W \to \Omega^{r+1} W$ are *not* morphisms of exterior algebras.

2.5 Contact Components and Geometric Operations

In this section, we summarize some properties of the contact components and the differential-geometric operations acting on forms, such as the wedge product $\wedge$, the contraction i_ζ of a form by a vector ζ, and the Lie derivative ∂_ξ by a vector field ξ.

Theorem 9 *Let W be an open set in Y.*

(a) *For any two forms ρ and η on $W^r \subset J^r Y$,*

$$p_k(\rho \wedge \eta) = \sum_{i+j=k} p_k\rho \wedge p_k\eta. \tag{156}$$

(b) *For any form ρ and any π^{r+1}-vertical, $\pi^{r+1,r}$-projectable vector field Ξ on W^{r+1}, with $\pi^{r+1,r}$-projection ξ,*

$$i_\Xi p_k\rho = p_{k-1} i_\xi \rho. \tag{157}$$

(c) *For any form ρ and any automorphism α of Y, defined on W,*

$$p_k(J^r\alpha^*\rho) = J^{r+1}\alpha^* p_k\rho. \tag{158}$$

(d) *For any form ρ and any π-projectable vector field on Y on W*

$$p_k(\partial_{J^r\Xi}\rho) = \partial_{J^{r+1}\Xi} p_k\rho. \tag{159}$$

Proof

(a) The exterior product $(\pi^{r+1,r})^*(\rho \wedge \eta)$ commutes with the pullback, so we have $(\pi^{r+1,r})^*(\rho \wedge \eta) = (\pi^{r+1,r})^*\rho \wedge (\pi^{r+1,r})^*\eta$. Applying the trace decomposition formula (Sect. 2.2, Theorem 3) to $(\pi^{r+1,r})^*\rho$ and $(\pi^{r+1,r})^*\eta$, and comparing the k-contact components on both sides, we obtain formula (156).

(b) To prove formula (157), we use the definition of the k-contact component of a form (Sect. 2.4, (112)) and the identity $p\Xi(J_x^{r+1}\gamma) = \xi(J_x^r\gamma)$ (Sect. 1.5, Remark 2). Set $\xi_1 = \Xi(J_x^{r+1}\gamma)$. Then, $h\xi_1 = 0$ and $p\xi_1 = \xi(J_x^r\gamma)$. By definition,

$$\begin{aligned} & i_\Xi p_k\rho(J_x^{r+1}\gamma)(\xi_2,\xi_3,\ldots,\xi_q) \\ &\quad = p_k\rho(J_x^{r+1}\gamma)(\Xi(J_x^{r+1}\gamma),\xi_2,\xi_3,\ldots,\xi_q) \\ &\quad = p_k\rho(J_x^{r+1}\gamma)(\xi_1,\xi_2,\xi_3,\ldots,\xi_q) \\ &\quad = \sum \varepsilon^{j_1j_2\ldots j_kj_{k+1}\ldots j_q}\rho(J_x^r\gamma)(p\xi_{j_1},p\xi_{j_2},\ldots,p\xi_{j_k},h\xi_{j_{k+1}},h\xi_{j_{k+2}},\ldots,h\xi_{j_q}) \end{aligned} \tag{160}$$

with summation through the sequences $j_1 < j_2 < \cdots < j_k$, $j_{k+1} < j_{k+2} < \cdots < j_q$ (Sect. 2.4, (112)). On the other hand,

$$\begin{aligned} & p_{k-1}i_\xi\rho(J_x^{r+1}\gamma)(\xi_2,\xi_3,\ldots,\xi_q) \\ &\quad = \sum \varepsilon^{i_2i_3\ldots i_ki_{k+1}\ldots j_q} i_\xi\rho(J_x^r\gamma)(p\xi_{i_2},p\xi_{i_3},\ldots,p\xi_{i_k},h\xi_{i_{k+1}},h\xi_{i_{k+2}},\ldots,h\xi_{i_q}) \\ &\quad = \sum \varepsilon^{i_2i_3\ldots i_ki_{k+1}\ldots j_q}\rho(J_x^r\gamma)(p\xi_1,p\xi_{i_2},p\xi_{i_3},\ldots,p\xi_{i_k},h\xi_{i_{k+1}},h\xi_{i_{k+2}},\ldots,h\xi_{i_q}) \end{aligned} \tag{161}$$

(summation through $i_2 < i_3 < \cdots < i_k$, $i_{k+1} < i_{k+2} < \cdots < i_q$). Since $h\xi_1 = 0$, the summation in (161) can be extended to the sequences $1 < i_2 < i_3 < \cdots < i_k$ and $1 < i_{k+1} < i_{k+2} < \cdots < i_q$, and therefore, (161) coincides with (160).

(c) Formula (158) follows from the commutativity of the r-jet prolongation of automorphisms of the fibered manifold Y and the canonical jet projections, $(\pi^{r+1,r})^* J^r\alpha^*\rho = J^{r-1}\alpha^*(\pi^{r+1,r})^*\rho$, and from the property of the contact 1-forms $\omega^\nu_{i_1i_2\ldots i_p}$

$$J^r\alpha^*\overline{\omega}^\sigma_{j_1j_2\ldots j_k} = \sum_{i<i_2<\cdots<i_p} \frac{\partial(\bar{y}^\sigma_{j_1j_2\ldots j_k}\circ J^r\alpha)}{\partial y^\nu_{i_1i_2\ldots i_p}}\omega^\nu_{i_1i_2\ldots i_p} \tag{162}$$

(Sect. 2.1, Theorem 1, (c)).

(d) Formula (159) is an immediate consequence of (162). □

Remark 7 If $k = 0$, (156) reduces to the condition $h(\rho\wedge\eta) = h(\rho)\wedge h(\eta)$, stating that h is a homomorphism of exterior algebras (Sect. 2.3, Theorem 5).

2.6 Strongly Contact Forms

Let $\rho \in \Omega^r_q W$ be a q-form such that $n+1 \le q \le \dim J^rY$. Since $h\rho = 0$ and also $p_1\rho = 0, p_2\rho = 0, \ldots, p_{q-n-1}\rho = 0$ (Sect. 2.4, Theorem 8), ρ is always *contact*, and its canonical decomposition has the form

$$(\pi^{r+1,r})^*\rho = p_{q-n}\rho + p_{q-n+1}\rho + \cdots + p_q\rho. \tag{163}$$

We introduce by induction a class of q-forms, imposing a condition on the contact component $p_{q-n}\rho$. If $q = n+1$, then we say that ρ is *strongly contact*, if for every point $y_0 \in W$ there exist a fibered chart (V,ψ), $\psi = (x^i, y^\sigma)$, at y_0 and a contact n-form τ, defined on V^r, such that

$$p_1(\rho - d\tau) = 0. \tag{164}$$

If $q > n+1$, then we say that ρ is *strongly contact*, if for every $y_0 \in W$ there exist (V,ψ), $\psi = (x^i, y^\sigma)$, at y_0 and a strongly contact n-form τ, defined on V^r, such that

$$p_{q-n}(\rho - d\tau) = 0. \tag{165}$$

Lemma 9 *The following conditions are equivalent:*

(a) *ρ is strongly contact.*
(b) *There exist a q-form η and a $(q-1)$-form τ such that*

$$\rho = \eta + d\tau, \quad p_{q-n}\eta = 0, \quad p_{q-n-1}\tau = 0. \tag{166}$$

Proof If ρ is strongly contact and we have τ such that (165) holds, then we set $\eta = \rho - d\tau$. The converse is obvious. □

In view of part (b) of Lemma 9, to study the properties of strongly contact forms, we need the chart expressions of the q-forms $p_{q-n}\rho$ and $p_{q-n-1}\tau = 0$. We also need, in particular, the chart expressions of the forms ρ whose $(q-n)$-contact component vanishes,

$$p_{q-n}\rho = 0. \tag{167}$$

To this purpose, we use the contact basis. The formulas as well as the proof the subsequent theorem are based on the complete trace decomposition theory and are technically tedious because we cannot avoid extensive index notation. We write

$$\begin{aligned}\rho = \sum A^{J_1 J_2}_{\nu_1 \nu_2} \cdots {}^{J_p I_{p+1} I_{p+2}}_{\nu_p \sigma_{p+1} \sigma_{p+2}} \cdots {}^{I_{p+s}}_{\sigma_{p+s}\, i_{p+s+1} i_{p+s+2} \dots i_q}\, \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_p}_{J_p} \\ \wedge dy^{\sigma_{p+1}}_{I_{p+1}} \wedge dy^{\sigma_{p+2}}_{I_{p+2}} \wedge \cdots \wedge dy^{\sigma_{p+s}}_{I_{p+s}} \wedge dx^{i_{p+s+1}} \wedge dx^{i_{p+s+2}} \wedge \cdots \wedge dx^{i_q},\end{aligned} \tag{168}$$

where summation is taking place through the multi-indices $J_1, J_2, \dots, J_p$ of length less or equal to $r-1$ and the multi-indices $I_{p+1}, I_{p+2}, \dots, I_{p+s}$ of length equal to r.

Applying the trace decomposition theorem (Appendix 9, Theorem 1) as many times as necessary, we can write

$$\begin{aligned}\rho = \sum B^{J_1 J_2}_{\nu_1 \nu_2} \cdots {}^{J_l K_{l+1} K_{l+2}}_{\nu_l \kappa_{l+1} \kappa_{l+2}} \cdots {}^{K_{l+p} I_{l+p+1} I_{l+p+2}}_{\kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2}} \cdots {}^{I_{l+p+s}}_{\sigma_{l+p+s} i_{l+p+s+1} i_{l+p+s+2} \dots i_Q} \\ \cdot\, \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{\kappa_{l+2}}_{K_{l+2}} \wedge \cdots \wedge d\omega^{\kappa_{l+p}}_{K_{l+p}} \\ \wedge\, dy^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge dy^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge dy^{\sigma_{l+p+s}}_{I_{l+p+s}} \\ \wedge\, dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q},\end{aligned} \tag{169}$$

where

$$\begin{aligned} 0 \le |J_1|,\ |J_2|, \dots,\ |J_l|\ \le r-1, \\ |K_{l+1}|, |K_{l+2}|, \dots, |K_{l+p}| = r-1, \\ |I_{l+p+1}|, |I_{l+p+2}|, \dots, |I_{l+p+s}| = r, \end{aligned} \tag{170}$$

and the coefficients are *traceless*. The number Q in (169) is *not* the degree of ρ; it is related to the degree q by $l + 2p + s + Q - l - p - s = q$, that is,

$$p + Q = q. \tag{171}$$

Theorem 10 *Let $W \subset Y$ be an open set, q an integer such that $n+1 \le q \le \dim J^r Y$, and $\eta \in \Omega^r_q W$ a form, and let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart such that $V \subset W$. Then, $p_{q-n}\eta = 0$ if and only if*

$$\begin{aligned}\eta = \sum_{q-n+1 \le l+p} \omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge \cdots \wedge \omega^{\sigma_l}_{J_l} \wedge d\omega^{\nu_1}_{I_1} \wedge d\omega^{\nu_2}_{I_2} \wedge \cdots \wedge d\omega^{\nu_p}_{I_p} \\ \wedge\, \Phi^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{J_l I_1 I_2}_{\sigma_l \nu_1 \nu_2} \cdots {}^{I_p}_{\nu_p},\end{aligned} \tag{172}$$

where $\Phi^{J_1 J_2}_{\sigma_1 \sigma_2} \cdots {}^{J_l I_1 I_2}_{\sigma_l \nu_1 \nu_2} \cdots {}^{I_p}_{\nu_p}$ are some $(q-l-2p)$-forms on V^r and the multi-indices satisfy $0 \le |J_1|, |J_2|, \dots, |J_l| \le r-1$, $|I_1|, |I_2|, \dots, |I_p| = r-1$.

Proof Expression (169) for η can be written as V^{r+1}, where

$$\begin{aligned}\eta_0 = \sum_{l+p \ge q-n} B^{J_1 J_2}_{\nu_{11} \nu_2} \cdots {}^{J_l K_{l+1} K_{l+2}}_{\nu_l \kappa_{l+1} \kappa_{l+2}} \cdots {}^{K_{l+p} I_{l+p+1} I_{l+p+2}}_{\kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2}} \cdots {}^{I_{l+p+s}}_{\sigma_{l+p+s} i_{l+p+s+1} i_{l+p+s+2} \dots i_Q} \\ \cdot\, \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{K_{l+2}}_{\kappa_{l+2}} \wedge \cdots \wedge d\omega^{K_{l+p}}_{\kappa_{l+p}} \\ \wedge\, dy^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge dy^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge dy^{\sigma_{l+p+s}}_{I_{l+p+s}} \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q}\end{aligned} \tag{173}$$

and

$$\eta_1 = \sum_{l+p<q-n} B^{J_1 J_2}_{\nu_{11}\nu_2} \cdots {}^{J_l K_{l+1} K_{l+2}}_{\nu_l \kappa_{l+1} \kappa_{l+2}} \cdots {}^{K_{l+p} I_{l+p+1} I_{l+p+2}}_{\kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2}} \cdots {}^{I_{l+p+s}}_{\sigma_{l+p+s} i_{l+p+s+1} i_{l+p+s+2} \dots i_Q}$$
$$\cdot \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{K_{l+2}}_{\kappa_{l+2}} \wedge \cdots \wedge d\omega^{K_{l+p}}_{\kappa_{l+p}}$$
$$\wedge dy^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge dy^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge dy^{\sigma_{l+p+s}}_{I_{l+p+s}} \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q}. \tag{174}$$

We want to show that the condition $p_{q-n}\eta = 0$ implies $\eta_1 = 0$.

To determine $p_{q-n}\eta_1$, we need the pullback $(\pi^{r+1,r})^*\eta_1$; this can be obtained by replacing dy^σ_I with

$$dy^\sigma_I = \omega^\sigma_I + y^\sigma_{Ii} dx^i. \tag{175}$$

Then, the corresponding expressions on the right-hand side of the formula (174) arise by substitution

$$\begin{aligned} &dy^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge dy^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge dy^{\sigma_{l+p+s}}_{I_{l+p+s}} \\ &\quad = \left(\omega^{\sigma_{l+p+1}}_{I_{l+p+1}} + y^{\sigma_{l+p+1}}_{I_{l+p+1} i_{l+p+1}} dx^{i_{l+p+1}}\right) \wedge \left(\omega^{\sigma_{l+p+2}}_{I_{l+p+2}} + y^{\sigma_{l+p+2}}_{I_{l+p+2} i_{l+p+2}} dx^{i_{l+p+2}}\right) \\ &\qquad \wedge \cdots \wedge \left(\omega^{\sigma_{l+p+s}}_{I_{l+p+s}} + y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} dx^{i_{l+p+s}}\right). \end{aligned} \tag{176}$$

Computing the right-hand side, we obtain

$$\begin{aligned} &dy^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge dy^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge dy^{\sigma_{l+p+s}}_{I_{l+p+s}} = \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge \omega^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge \omega^{\sigma_{l+p+s}}_{I_{l+p+s}} \\ &\quad + s y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge \omega^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge \omega^{\sigma_{l+p+s-1}}_{I_{l+p+s-1}} \wedge dx^{i_{l+p+s}} \\ &\quad + \binom{s}{2} y^{\sigma_{l+p+s-1}}_{I_{l+p+s-1} i_{l+p+s-1}} y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge \omega^{\sigma_{l+p+2}}_{I_{l+p+2}} \\ &\quad \wedge \cdots \wedge \omega^{\sigma_{l+p+s-2}}_{l+p+s-2} \wedge dx^{i_{l+p+s-1}} \wedge dx^{i_{l+p+s}} \\ &\quad + \cdots + s y^{\sigma_{l+p+2}}_{I_{l+p+2} i_{l+p+2}} \cdots y^{\sigma_{l+p+s-1}}_{I_{l+p+s-1} i_{l+p+s-1}} y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \\ &\quad \wedge dx^{i_{l+p+2}} \wedge \cdots \wedge dx^{i_{l+p+s-1}} \wedge dx^{i_{l+p+s}} \\ &\quad + y^{\sigma_{l+p+1}}_{I_{l+p+1} i_{l+p+1}} \cdots y^{\sigma_{l+p+s-1}}_{I_{l+p+s-1} i_{l+p+s-1}} y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} dx^{i_{l+p+1}} \wedge \cdots \wedge dx^{i_{l+p+s-1}} \wedge dx^{i_{l+p+s}}. \end{aligned} \tag{177}$$

Now, consider a fixed summand in expression (174), with given l, p, and s,

$$\begin{aligned} &B^{J_1 J_2}_{\nu_{11}\nu_2} \cdots {}^{J_l K_{l+1} K_{l+2}}_{\nu_l \kappa_{l+1} \kappa_{l+2}} \cdots {}^{K_{k+p} I_{l+p+1} I_{l+p+2}}_{\kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2}} \cdots {}^{I_{l+p+s}}_{\sigma_{l+p+s} i_{l+p+s+1} i_{l+p+s+2} \dots i_Q} \\ &\quad \cdot \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{\kappa_{l+2}}_{K_{l+2}} \wedge \cdots \wedge d\omega^{\kappa_{l+p}}_{K_{l+p}} \\ &\quad \wedge \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge \omega^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge \omega^{\sigma_{l+p+s}}_{I_{l+p+s}} \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q}. \end{aligned} \tag{178}$$

Using (178), we get the terms

$$
\begin{aligned}
& sB^{J_1 J_2}_{\nu_1 1 \nu_2} \cdots {}^{J_l K_{l+1} K_{l+2}}_{\nu_l \kappa_{l+1} \kappa_{l+2}} \cdots {}^{K_{k+p} I_{l+p+1} I_{l+p+2}}_{\kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2}} \cdots {}^{I_{l+p+s}}_{\sigma_{l+p+s} i_{l+p+s+1} i_{l+p+s+2} \dots i_Q} \\
& \quad \cdot y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{\kappa_{l+2}}_{K_{l+2}} \wedge \cdots \wedge d\omega^{\kappa_{l+p}}_{K_{l+p}} \\
& \quad \wedge \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge \omega^{\sigma_{l+p+2}}_{I_{l+p+2}} \wedge \cdots \wedge \omega^{\sigma_{l+p+s-1}}_{I_{l+p+s-1}} \\
& \quad \wedge dx^{i_{l+p+s}} \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q}, \\
& \quad \binom{S}{2} B^{J_1 J_2}_{\nu_1 1 \nu_2} \cdots {}^{J_l K_{l+1} K_{l+2}}_{\nu_l \kappa_{l+1} \kappa_{l+2}} \cdots {}^{K_{k+p} I_{l+p+1} I_{l+p+2}}_{\kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2}} \cdots {}^{I_{l+p+s}}_{\sigma_{l+p+s} \, i_{l+p+s+1} i_{l+p+s+2} \dots i_Q} \\
& \quad \cdot y^{\sigma_{l+p+s-1}}_{I_{l+p+s-1} i_{l+p+s-1}} y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \\
& \quad \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{\kappa_{l+2}}_{K_{l+2}} \wedge \cdots \wedge d\omega^{\kappa_{l+p}}_{K_{l+p}} \wedge \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \wedge \omega^{\sigma_{l+p+2}}_{I_{l+p+2}} \\
& \quad \wedge \cdots \wedge \omega^{\sigma_{l+p+s-2}}_{l+p+s-2} \wedge dx^{i_{l+p+s-1}} \wedge dx^{i_{l+p+s}} \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q}, \\
& \quad \dots \\
& \quad sB^{J_1 J_2}_{\nu_1 1 \nu_2} \cdots {}^{J_l K_{l+1} K_{l+2}}_{\nu_l \kappa_{l+1} \kappa_{l+2}} \cdots {}^{K_{k+p} I_{l+p+1} I_{l+p+2}}_{\kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2}} \cdots {}^{I_{l+p+s}}_{\sigma_{l+p+s} \, i_{l+p+s+1} i_{l+p+s+2} \dots i_Q} \\
& \quad \cdot y^{\sigma_{l+p+2}}_{I_{l+p+2} i_{l+p+2}} \dots y^{\sigma_{l+p+s-1}}_{I_{l+p+s-1} i_{l+p+s-1}} y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \\
& \quad \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{\kappa_{l+2}}_{K_{l+2}} \wedge \cdots \wedge d\omega^{\kappa_{l+p}}_{K_{l+p}} \wedge \omega^{\sigma_{l+p+1}}_{I_{l+p+1}} \\
& \quad \wedge dx^{i_{l+p+2}} \wedge \cdots \wedge dx^{i_{l+p+s-1}} \wedge dx^{i_{l+p+s}} \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q},
\end{aligned}
\tag{179}
$$

and

$$
\begin{aligned}
& B^{J_1 J_2}_{\nu_1 1 \nu_2} \cdots {}^{J_l K_{l+1} K_{l+2}}_{\nu_l \kappa_{l+1} \kappa_{l+2}} \cdots {}^{K_{k+p} I_{l+p+1} I_{l+p+2}}_{\kappa_{l+p} \sigma_{l+p+1} \sigma_{l+p+2}} \cdots {}^{I_{l+p+s}}_{\sigma_{l+p+s} i_{l+p+s+1} i_{l+p+s+2} \dots i_Q} \\
& \quad \cdot y^{\sigma_{l+p+1}}_{I_{l+p+1} i_{l+p+1}} \dots y^{\sigma_{l+p+s-1}}_{I_{l+p+s-1} i_{l+p+s-1}} y^{\sigma_{l+p+s}}_{I_{l+p+s} i_{l+p+s}} \omega^{\nu_1}_{J_1} \wedge \omega^{\nu_2}_{J_2} \wedge \cdots \wedge \omega^{\nu_l}_{J_l} \\
& \quad \wedge d\omega^{\kappa_{l+1}}_{K_{l+1}} \wedge d\omega^{\kappa_{l+2}}_{K_{l+2}} \wedge \cdots \wedge d\omega^{\kappa_{l+p}}_{K_{l+p}} \wedge dx^{i_{l+p+1}} \wedge \cdots \wedge dx^{i_{l+p+s-1}} \wedge dx^{i_{l+p+s}} \\
& \quad \wedge dx^{i_{l+p+s+1}} \wedge dx^{i_{l+p+s+2}} \wedge \cdots \wedge dx^{i_Q}.
\end{aligned}
\tag{180}
$$

We see that the degrees of contactness of these terms are

$$
l + p + s > l + p + s - 1 > l + p + s - 2 > \cdots > l + p + 1 > l + p, \tag{181}
$$

respectively. Clearly, since we consider the terms where $l + p < q - n$, (180) does not contribute to $p_{q-n}\eta_1$. We claim that among the terms (178), there is one whose degree of contactness is $q - n$. Suppose the opposite; then $l + p + s < q - n$, but this is not possible, because the term satisfying this inequality would contain more than n factors dx^i.

Thus, the condition $p_1\eta_1 = 0$ applies to one of the expressions (179) and states that the coefficient in this expression vanishes. But the components of η_1 are traceless, and we have already seen that this is only possible when they also vanish.

This implies in turn that the forms on the left of (179) all vanish, which proves that $\eta_1 = 0$. The proof is complete. □

Corollary 1 *Let $W \subset Y$ be an open set, q an integer such that $n+1 \leq q \leq \dim J^r Y$, and $\eta \in \Omega_q^r W$ a form, and let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart such that $V \subset W$. Then, $p_{q-n}\eta = 0$ if and only if*

$$\eta = \eta_0 + d\mu, \tag{182}$$

where η_0 and μ are ω_J^σ-generated, $0 \leq |I| \leq r-1$, such that $p_{q-n}\eta_0 = 0$ and $p_{q-n-1}\mu = 0$.

Proof Write in Theorem 10 $\eta = \eta_0 + \eta'$, where η_0 includes *all* ω_J^σ-generated terms, defined by the condition $l \geq 1$, and

$$\begin{aligned}\eta' &= \sum_{q-n+1 \leq p} d\omega_{I_1}^{\nu_1} \wedge d\omega_{I_2}^{\nu_2} \wedge \cdots \wedge d\omega_{I_p}^{\nu_p} \wedge \Phi_{\sigma_1\sigma_2}^{J_1 J_2}\cdots{}_{\sigma_l \nu_1 \nu_2}^{J_l I_1 I_2}\cdots{}_{\nu_p}^{I_p} \\ &= \sum_{q-n+1 \leq p} d(\omega_{I_1}^{\nu_1} \wedge d\omega_{I_2}^{\nu_2} \wedge \cdots \wedge d\omega_{I_p}^{\nu_p} \wedge \Phi_{\sigma_1\sigma_2}^{J_1 J_2}\cdots{}_{\sigma_l \nu_1 \nu_2}^{J_l I_1 I_2}\cdots{}_{\nu_p}^{I_p}) \\ &\quad + \sum_{q-n+1 \leq p} \omega_{I_1}^{\nu_1} \wedge d\omega_{I_2}^{\nu_2} \wedge \cdots \wedge d\omega_{I_p}^{\nu_p} \wedge d(\Phi_{\sigma_1\sigma_2}^{J_1 J_2}\cdots{}_{\sigma_l \nu_1 \nu_2}^{J_l I_1 I_2}\cdots{}_{\nu_p}^{I_p}).\end{aligned} \tag{183}$$

Thus, η can also be written as $\eta = \eta_0 + d\mu$, where η_0 is ω_J^σ-generated, and μ is also ω_J^σ-generated and contains p contact factors ω_J^σ and $d\omega_I^\nu$; in particular, $p_{q-n-1}\mu = 0$. □

Remark 8 Note that the summation in Theorem 10 through the pairs (l, p) can also be defined by the inequality $q - n + 1 - p \leq l \leq q - 2p$, where the range of p is given by the conditions $p = 0, 1, 2, \ldots$ and $q - 2p \geq 0$.

Lemma 10

(a) *If ρ is a strongly contact form such that $q \geq n+2$, then for any π-vertical vector field Ξ, the form $i_{J^r\Xi}\rho$ is strongly contact.*
(b) *The exterior derivative of a strongly contact form is strongly contact.*

Proof

(a) We have $i_{J^r\Xi}\rho = i_{J^r\Xi}\eta + i_{J^r\Xi}d\tau = i_{J^r\Xi}\eta + \partial_{J^r\Xi}\tau - di_{J^r\Xi}\tau$. But by Sect. 2.5, Theorem 9 $p_{q-n-1}(i_{J^r\Xi}\eta + \partial_{J^r\Xi}\tau) = i_{J^{r+1}\Xi}p_{q-n}\eta + \partial_{J^{r+1}\Xi}p_{q-n-1}\tau$ and $p_{q-n-2}i_{J^r\Xi}\tau = i_{J^{r+1}\Xi}p_{q-n-1}\tau$; however, these expressions vanish because ρ is strongly contact. Now, we apply Lemma 9.
(b) Let the form ρ be strongly contact. Then, from (166), $d\rho = d\eta$, where $p_{q-n}\eta = 0$. We want to show that to any point y_0 from the domain of definition of ρ, there exists a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, at y_0 and a q-form τ, defined on V^r, such that $p_{q+1-n}(d\rho - d\tau) = 0$ and $p_{q-n}\tau = 0$. Taking $\tau = \eta$, we get the result.

For $n + 1 \leq q \leq \dim J^r Y$, strongly contact forms constitute an *Abelian subgroup* $\Theta^r_q W$ of the Abelian group of q-forms $\Omega^r_q W$; they do not form a submodule of $\Omega^r_q W$. It follows from Lemma 10, (b) that the subgroups $\Theta^r_q W$ together with the exterior derivative operator define a sequence

$$\Theta^r_n W \to \Theta^r_{n+1} W \to \cdots \to \Theta^r_M W \to 0. \tag{184}$$

The number M labeling the last nonzero term in this sequence is

$$M = m\binom{n+r-1}{n} + 2n - 1. \tag{185}$$

□

Remark 9 If $n + 1 \leq q \leq \dim J^r Y$, then by Lemma 1, the canonical decomposition of a contact form $\rho \in \Theta^r_q W$ is

$$(\pi^{r+1,r})^* \rho = p_{q-n} d\tau + p_{q-n+1}\rho + p_{q-n+2}\rho + \cdots + p_q \rho. \tag{186}$$

Remark 10 It is easily seen that the definition of a contact q-form $\rho \in \Omega^r_q W$ for $1 \leq q \leq n$ agrees with (165). Indeed, if $1 \leq q \leq n$, we have for any contact form $\rho' \in \Theta^r_{q-1} W$, $h(\rho - d\rho') = h\rho$ as $(\pi^{r+1})^* h d\rho' = hdh\rho' = 0$ (Corollary 2). Thus, if $h\rho = 0$, then $h(\rho - d\rho') = 0$ for any $\rho' \in \Theta^r_{q-1} W$.

2.7 Fibered Homotopy Operators on Jet Prolongations of Fibered Manifolds

In this section, we introduce the fibered homotopy operators for differential forms on jet prolongations of fibered manifolds. We study their relations with the canonical decomposition of forms and the exactness problem for contact and strongly contact forms. The general theory of fibered homotopy operators is summarized in Appendix 6.

The relevant underlying structure we need is a trivial fibered manifold $W = U \times V$, where U is an open set in $\mathbf{R}^n$ and V an open ball in $\mathbf{R}^m$ with center at the origin; the projection is the first Cartesian projection of $U \times V$ onto U, denoted by π. The r-jet prolongation $J^r W$ is also denoted by W^r. By definition

$$W^r = U \times V \times L(\mathbf{R}^n, \mathbf{R}^m) \times L^2_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^m) \times \cdots \times L^r_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^m), \tag{187}$$

where $L^k_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^m)$ is the vector space of k-linear symmetric mappings from $\mathbf{R}^n$ to $\mathbf{R}^m$. The canonical coordinates on W are denoted by (x^i, y^σ), and the associated coordinates on W^r are $(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$. Any Cartesian projections

$\pi^{r,s}: W^r \to W^s$, with $0 \le s < r$, define in an obvious way a homotopy $\chi^{r,s}$ and the *fibered homotopy operator* $I^{r,s}$ (see Appendix 6, (27)), so the Volterra-Poincare lemma holds in these cases.

In this section, we consider the fibered homotopy operator $I = I^{r,0}$. Recall that the homotopy $\chi = \chi^{r,s}$ is a mapping from $[0,1] \times W^r$ to W^r, defined by

$$\chi(s,(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})) = (x^i, sy^\sigma, sy^\sigma_{j_1}, sy^\sigma_{j_1 j_2}, \ldots, sy^\sigma_{j_1 j_2 \ldots j_r}). \tag{188}$$

It is immediately verified that the pullback by χ satisfies

$$\begin{gathered} \chi^* dx^i = dx^i, \quad \chi^* dy^\sigma_{j_1 j_2 \ldots j_k} = y^\sigma_{j_1 j_2 \ldots j_k} ds + s dy^\sigma_{j_1 j_2 \ldots j_k}, \\ \chi^* \omega^\sigma_{j_1 j_2 \ldots j_k} = y^\sigma_{j_1 j_2 \ldots j_k} ds + s\omega^\sigma_{j_1 j_2 \ldots j_k}. \end{gathered} \tag{189}$$

In accordance with the general theory, these formulas lead to explicit description of the operator I. For any q-form ρ on W^r, $\chi^*\rho$ has a unique decomposition

$$\chi^*\rho = ds \wedge \rho^{(0)}(s) + \rho'(s) \tag{190}$$

such that the $(q-1)$-form $\rho^{(0)}(s)$ and the q-form $\rho'(s)$ do not contain ds. Then,

$$I\rho = \int_0^1 \rho^{(0)}(s), \tag{191}$$

where the expression on the right-hand side denotes the integration of the coefficients in the form $\rho^{(0)}(s)$ over s from 0 to 1.

The following is a version of a general theorem on fibered homotopy operators on fibered manifolds. ζ stands for the *zero section* of W^r over U.

Theorem 11

(a) *For every differentiable function* $f: W^r \to \mathbf{R}$,

$$f = Idf + (\pi^r)^*\zeta^* f. \tag{192}$$

(b) *Let* $q \ge 1$. *Then, for every differential* q-*form* ρ *on* W^r,

$$\rho = Id\rho + dI\rho + (\pi^r)^*\zeta^*\rho. \tag{193}$$

Proof Slight modification of Theorem 1, Appendix 6. □

Theorem 12 *Let ρ be a contact q-form on W^r.*

(a) *The contact components of ρ satisfy*

$$Ih\rho = 0, \quad Ip_k\rho = p_{k-1}I\rho, \quad 1 \le k \le q. \tag{194}$$

(b) *If ρ is strongly contact, then $I\rho$ is strongly contact.*

Proof

(a) Expressing the forms ρ and $(\pi^{r+1,r})^*\rho$ in the basis of 1-forms (dx^i, dy^σ_J), $0 \le |J| \le r$, we have

$$(\pi^{r+1,r})^* I\rho = I(\pi^{r+1,r})^*\rho. \tag{195}$$

The canonical decomposition of the form ρ yields

$$(\pi^{r+1,r})^* I\rho = I(\pi^{r+1,r})^*\rho = I\left(\sum_{0 \le l \le q} p_l\rho\right) = \sum_{0 \le l \le q} Ip_l\rho. \tag{196}$$

But by (191), $Ip_l\rho$ is $(l-1)$-contact; thus, applying p_k to both sides of (195) and comparing k-contact components, we get (194).

(b) Let $q \ge n+1$ and suppose we have a strongly contact q-form ρ on W^r. Then, $\rho = \eta + d\tau$ for some q-form η and $(q-1)$-form τ such that $p_{q-n}\eta = 0$ and $p_{q-n-1}\tau = 0$; hence, $I\rho = I\eta + Id\tau = I\eta + \tau - dI\tau - \tau_0$, where τ_0 is a $(q-1)$-form on U. If $q > n+1$, then always $\tau_0 = 0$. If $q = n+1$, then always $d\tau_0 = 0$, and we may replace τ with $\tau - \tau_0$; then, $I\rho = I\eta + \tau - dI\tau$. The $(q-1)$-form $I\eta + \tau$ satisfies

$$p_{q-n-1}(I\eta + \tau) = Ip_{q-n}\eta + p_{q-n-1}\tau = p_{q-n-1}\tau = 0. \tag{197}$$

If $q \ge n+2$, then $q - n - 2 \ge 0$ and $p_{q-n-2}I\tau = Ip_{q-n-1}\tau = 0$; consequently, $I\rho$ is strongly contact. If $q = n+1$, then from (195), $h\tau = 0$ as required. $\square$

Corollary 1 (The fibered Volterra–Poincare lemma) *If $d\rho = 0$, then there exists a $(q-1)$-form η such that $\rho = d\eta$.*

The following two theorems extend the fibered Volterra-Poincare lemma to contact and strongly contact forms. Their proofs are based on the trace decomposition theorem (Sect. 2.2, Theorem 3), Appendix 9, Theorem 4, and on the fibered Volterra-Poincare lemma.

Theorem 13 *Let $1 \le q \le n$ and let ρ be a contact q-form such that $d\rho = 0$. Then $\rho = d\eta$ for some contact $(q-1)$-form η.*

Proof

1. Let ρ be a contact 1-form, expressed as

$$\rho = \sum_{0 \le |J| \le r-1} \Phi_\nu^J \omega_J^\nu. \tag{198}$$

Then,

$$d\rho = \sum_{0 \le |J| \le r-1} (d\Phi_\nu^J \wedge \omega_J^\nu - \Phi_\nu^J dy_{Jj}^\nu \wedge dx^j). \tag{199}$$

Condition $d\rho = 0$ implies, for $|J| = r - 1$, $\Phi_\nu^J \delta_j^k = 0$ $\mathrm{Sym}(Jk)$, and the trace operation yields, up to the factor $(n + r - 1)/r$,

$$\Phi_\nu^J = 0. \tag{200}$$

Thus, ρ must be of the form

$$\rho = \sum_{0 \le |J| \le r-2} \Phi_\nu^J \omega_J^\nu. \tag{201}$$

Repeating the same procedure, we get $\rho = 0$.

2. Let $2 \le q \le n$. We show in several steps that if ρ is a contact q-form such that $d\rho = 0$, then there exist a contact q-form τ and a contact $(q-1)$-form κ such that

$$\rho = \tau + d\kappa, \quad p_1\tau = 0. \tag{202}$$

First, we find a decomposition

$$\rho = \rho_0 + \tau_0 + d\kappa_0, \tag{203}$$

with the following properties:

(a) ρ_0 is generated by the forms ω_J^σ such that $0 \le |J| \le r - 1$,

$$\rho_0 = \sum_{0 \le |J| \le r-2} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge \Delta_\sigma^J, \tag{204}$$

where the $(q-1)$-forms Δ_σ^J are traceless.

(b) τ_0 is generated by $\omega_J^\sigma \wedge \omega_I^\nu$ and $\omega_J^\sigma \wedge d\omega_L^\nu$, where $|J| = r - 1$, $0 \le |I| \le r - 1$, $|L| = r - 1$.

(c) κ_0 is a contact $(q-1)$-form.

Expressing ρ as in Sect. 2.3, Corollary 2, we have

$$\rho = \sum_{0 \le |J| \le r-2} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge \Phi_\sigma^J + d\kappa_0, \tag{205}$$

where κ_0 is a contact $(q-1)$-form. Decompose the $(q-1)$-forms Φ_ν^J, indexed with multi-indices J of length $r-1$, by the trace operation. We get a decomposition

$$\Phi_\nu^J = \Delta_\nu^J + Z_\nu^J, \tag{206}$$

where the expression Δ_ν^J is the traceless and Z_ν^J is the contact component. Then,

$$\rho = \sum_{0 \le |J| \le r-2} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge \Delta_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge Z_\sigma^J + d\kappa_0. \tag{207}$$

Setting

$$\begin{aligned} \rho_0 &= \sum_{0 \le |J| \le r-2} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge \Delta_\sigma^J, \\ \tau_0 &= \sum_{|J|=r-1} \omega_J^\sigma \wedge Z_\sigma^J, \end{aligned} \tag{208}$$

we get (203).

Second, we show that ρ has a decomposition

$$\rho = \rho_1 + \tau_1 + d\kappa_1 \tag{209}$$

with the following properties:

(a) The form ρ_1 is generated by the contact forms ω_J^σ, such that $0 \le |J| \le r-2$, that is,

$$\rho_1 = \sum_{0 \le |J| \le r-3} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-2} \omega_J^\sigma \wedge \Delta_\sigma^J, \tag{210}$$

where the $(q-1)$-forms Δ_σ^J are traceless.

(b) τ_1 is generated by $\omega_J^\sigma \wedge \omega_I^\nu$ and $\omega_J^\sigma \wedge d\omega_L^\nu$, where $|J| = r-1$, $0 \le |I| \le r-1$, $|L| = r-1$.

(c) κ_1 is a contact $(q-1)$-form.

Indeed, we apply condition $d\rho = 0$ to expression (203). We have, since $d\omega_J^\sigma = -dy_{Jj}^\sigma \wedge dx^j$,

$$\sum_{0 \le |J| \le r-2} d(\omega_J^\sigma \wedge \Phi_\sigma^J) - \sum_{|J|=r-1} (dy_{Jj}^\sigma \wedge dx^j \wedge \Delta_\sigma^J + \omega_J^\sigma \wedge d\Delta_\sigma^J) + d\tau_0 = 0. \tag{211}$$

But the terms $dy_{Jj}^\nu \wedge dx^j \wedge \Delta_\nu^J$ in this expression do not contain any form ω_J^ν or $d\omega_J^\nu$ and must vanish separately. Thus,

$$\sum_{|J|=r-1} dy_{Jj}^\nu \wedge dx^j \wedge \Delta_\nu^J = 0. \tag{212}$$

The 1-contact component gives

$$\sum_{|J|=r-1} \omega_{Jj}^\nu \wedge h(dx^j \wedge \Delta_\nu^J) = 0 \tag{213}$$

hence

$$h(dx^j \wedge \Delta_\nu^J) = 0 \quad \mathrm{Sym}(Jj). \tag{214}$$

The traceless form Δ_ν^J can be expressed as

$$\begin{aligned}\Delta_\nu^J = {} & A_{\nu i_2 i_3 \dots i_q}^J dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_q} \\ & + A_{\nu\sigma_2 i_3 i_4 \dots i_q}^{JI_2} dy_{I_2}^{\sigma_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_q} \\ & + A_{\nu\sigma_2\sigma_3 i_4 i_5 \dots i_q}^{JI_2 I_3} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge dx^{i_4} \wedge dx^{i_5} \wedge \dots \wedge dx^{i_q} \\ & + \dots + A_{\nu\sigma_2\sigma_3 \dots \sigma_{q-1} i_q}^{JI_2 I_3 \dots I_{q-1}} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge \dots \wedge dy_{I_{q-1}}^{\sigma_{q-1}} \wedge dx^{i_q} \\ & + A_{\nu\sigma_2\sigma_3 \dots \sigma_q}^{JI_2 I_3 \dots I_q} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge \dots \wedge dy_{I_q}^{\sigma_q},\end{aligned} \tag{215}$$

where the multi-indices $I_2, I_3, \dots, I_q$ satisfy $|I_2|, |I_3|, \dots, |I_q| = r$ and all coefficients $A_{\nu\sigma_2 i_3 i_4 \dots i_q}^{JI_2}, A_{\nu\sigma_2\sigma_3 i_4 i_5 \dots i_q}^{JI_2 I_3}, \dots, A_{\nu\sigma_2\sigma_3 \dots \sigma_{q-1} i_q}^{JI_2 I_3 \dots I_{q-1}}$ are traceless in the indices $i_3, i_4, \dots, i_q$ and the multi-indices $I_2, I_3, \dots, I_{q-1}$. Then, Eq. (214) reads

$$\begin{aligned}(A_{\nu i_2 i_3 \dots i_q}^J & + A_{\nu\sigma_2 i_3 i_4 \dots i_q}^{JI_2} y_{I_2 i_2}^{\sigma_2} + A_{\nu\sigma_2\sigma_3 i_4 i_5 \dots i_q}^{JI_2 I_3} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \\ & + \dots + A_{\nu\sigma_2\sigma_3 \dots \sigma_{q-1} i_q}^{JI_2 I_3 \dots I_{q-1}} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_{q-1} i_{q-1}}^{\sigma_{q-1}} \\ & + A_{\nu\sigma_2\sigma_3 \dots \sigma_q}^{JI_2 I_3 \dots I_q} y_{I_2 i_2}^{\sigma_2} y_{I_3 i_3}^{\sigma_3} \dots y_{I_q i_q}^{\sigma_q}) \\ & \cdot \delta_{i_1}^l dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_q} = 0 \quad \mathrm{Sym}(Jj).\end{aligned} \tag{216}$$

Setting

$$\begin{aligned}
&B^{Jl}_{\nu i_1 i_2 i_3 \ldots i_q} = A^{J}_{\nu i_2 i_3 \ldots i_q} \delta^l_{i_1} \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_2 i_3 \ldots i_q),\\
&B^{JlI_2}_{\nu \sigma_2 i_1 i_3 i_4 \ldots i_q} = A^{JI_2}_{\sigma_2 i_3 i_4 \ldots i_q \nu} \delta^l_{i_1} \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_3 i_4 \ldots i_q),\\
&B^{JlI_2I_3}_{\nu \sigma_2 \sigma_3 i_1 i_4 i_5 \ldots i_q} = A^{JI_2I_3}_{\nu \sigma_2 \sigma_3 i_4 i_5 \ldots i_q} \delta^l_{i_1} \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_4 i_5 \ldots i_q),\\
&\ldots\\
&B^{JI_2I_3 \ldots I_{q-1}}_{\nu \sigma_2 \sigma_3 \ldots \sigma_{q-1} i_1 i_q} = A^{JI_2I_3 \ldots I_{q-1}}_{\nu \sigma_2 \sigma_3 \ldots \sigma_{q-1} i_q} \delta^l_{i_1} \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_q),\\
&B^{JlI_2I_3 \ldots I_q}_{\nu \sigma_2 \sigma_3 \ldots \sigma_q i_1} = A^{JI_2I_3 \ldots I_q}_{\nu \sigma_2 \sigma_3 \ldots \sigma_q} \delta^l_{i_1} \quad \mathrm{Sym}(Jl),
\end{aligned} \tag{217}$$

we get the system

$$\begin{aligned}
&B^{Jl}_{\nu i_1 i_2 i_3 \ldots i_q} = 0,\\
&B^{JlI_2}_{\nu \sigma_2 i_1 i_3 i_4 \ldots i_q} \delta^{j_2}_{i_2} = 0 \quad \mathrm{Sym}(I_2 j_2) \quad \mathrm{Alt}(i_1 i_2 i_3 \ldots i_q),\\
&B^{JlI_2I_3}_{\nu \sigma_2 \sigma_3 i_1 i_4 i_5 \ldots i_q} \delta^{j_2}_{i_2} \delta^{j_3}_{i_3} = 0 \quad \mathrm{Sym}(I_2 j_2) \quad \mathrm{Sym}(I_3 j_3) \quad \mathrm{Alt}(i_1 i_2 i_3 \ldots i_q),\\
&\ldots\\
&B^{JlI_2I_3 \ldots I_{q-1}}_{\nu \sigma_2 \sigma_3 \ldots \sigma_{q-1} i_1 i_q} \delta^{j_2}_{i_2} \delta^{j_3}_{i_3} \ldots \delta^{j_{q-1}}_{i_{q-1}} = 0 \quad \mathrm{Sym}(I_2 j_2) \quad \mathrm{Sym}(I_3 j_3)\\
&\quad \ldots \quad \mathrm{Sym}(I_{q-1} j_{q-1}) \quad \mathrm{Alt}(i_1 i_2 i_3 \ldots i_q),\\
&B^{JlI_2I_3 \ldots I_q}_{\nu \sigma_2 \sigma_3 \ldots \sigma_q i_1} \delta^{j_2}_{i_2} \delta^{j_3}_{i_3} \ldots \delta^{j_q}_{i_q} = 0 \quad \mathrm{Sym}(I_2 j_2) \quad \mathrm{Sym}(I_3 j_3)\\
&\quad \ldots \quad \mathrm{Sym}(I_q j_q) \quad \mathrm{Alt}(i_1 i_2 i_3 \ldots i_q).
\end{aligned} \tag{218}$$

Since the unknown functions, $B^{Jl\,I_2}_{\nu\,\sigma_2\,i_1 i_3 i_4 \ldots i_q}$, $B^{JI_2lI_3}_{\nu \sigma_2 \sigma_3 i_1 i_4 i_5 \ldots i_q}$, …, $B^{Jl\,I_2\,I_3 \ldots I_{q-1}}_{\nu\,\sigma_2\,\sigma_3 \ldots \sigma_{q-1}\,i_1 i_q}$, $B^{Jl\,I_2\,I_3 \ldots I_q}_{\nu\,\sigma_2\,\sigma_3 \ldots \sigma_q\,i_1}$, are traceless, for each fixed multi-index $I = Jl$ and each index ν, this system has only the trivial solution (see Appendix 9), and we have from (217)

$$\begin{aligned}
&A^{J}_{\nu i_2 i_3 \ldots i_q} \delta^l_{i_1} = 0 \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_2 i_3 \ldots i_q),\\
&A^{JI_2}_{\nu \sigma_{2 i_3 i_4 \ldots i_q}} \delta^l_{i_1} = 0 \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_3 i_4 \ldots i_q),\\
&A^{JI_2I_3}_{\nu \sigma_2 \sigma_3 i_4 i_5 \ldots i_q} \delta^l_{i_1} = 0 \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_4 i_5 \ldots i_q),\\
&\ldots\\
&A^{JI_2I_3 \ldots I_{q-1}}_{\nu \sigma_2 \sigma_3 \ldots \sigma_{q-1} i_q} \delta^l_{i_1} = 0 \quad \mathrm{Sym}(Jl) \quad \mathrm{Alt}(i_1 i_q),\\
&A^{JI_2I_3 \ldots I_q}_{\sigma_2 \nu \sigma_3 \ldots \sigma_q} \delta^l_{i_1} = 0 \quad \mathrm{Sym}(Jl).
\end{aligned} \tag{219}$$

The solutions of this system are of *Kronecker type*; we have, denoting the multi-index J as $J = Kk$,

$$\begin{aligned}
&A^{Kk}_{\nu i_2 i_3 \ldots i_q} = C^{K}_{\nu i_3 i_4 \ldots i_q} \delta^k_{i_2} \quad \mathrm{Sym}(Kk) \quad \mathrm{Alt}(i_2 i_3 i_4 \ldots i_q),\\
&A^{KkI_2}_{\nu \sigma_2 i_3 i_4 \ldots i_q} = C^{KI_2}_{\nu \sigma_2 i_4 i_5 \ldots i_q} \delta^k_{i_3} \quad \mathrm{Sym}(Kk) \quad \mathrm{Alt}(i_3 i_4 i_5 \ldots i_q),\\
&A^{KkI_2I_3}_{\nu \sigma_2 \sigma_3 i_4 i_5 \ldots i_q} = C^{KI_2I_3}_{\nu \sigma_2 \sigma_3 i_5 i_6 \ldots i_q} \delta^k_{i_4} \quad \mathrm{Sym}(Kk) \quad \mathrm{Alt}(i_4 i_5 i_6 \ldots i_q),\\
&\ldots\\
&A^{KkI_2I_3 \ldots I_{q-1}}_{\nu \sigma_2 \sigma_3 \ldots \sigma_{q-1} i_q} = C^{KI_2I_3 \ldots I_{q-1}}_{\nu \sigma_2 \sigma_3 \ldots \sigma_{q-1}} \delta^k_{i_q} \quad \mathrm{Sym}(Jl),\\
&A^{KkI_2I_3 \ldots I_q}_{\nu \sigma_2 \sigma_3 \ldots \sigma_q} = 0.
\end{aligned} \tag{220}$$

Consequently,

$$
\begin{aligned}
\sum_{|J|=r-1} \omega_J^\sigma \wedge \Delta_\nu^J = \omega_{Kk}^\sigma \wedge (& C_{\nu i_3 i_4 \ldots i_q}^{K} \delta_{i_2}^k dx^{i_2} \wedge dx^{i_3} \wedge \ldots \wedge dx^{i_q} \\
& + C_{\nu\,\sigma_2 i_4 i_5 \ldots i_q}^{K\,I_2} \delta_{i_3}^k dy_{I_2}^{\sigma_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\
& + C_{\nu\sigma_2\sigma_3 i_5 i_6 \ldots i_q}^{KI_2 I_3} \delta_{i_4}^k dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge dx^{i_4} \wedge dx^{i_5} \wedge \cdots \wedge dx^{i_q} \\
& + \cdots + C_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1}}^{KI_2 I_3 \ldots I_{q-1}} \delta_{i_q}^k dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge \cdots \wedge dy_{I_{q-1}}^{\sigma_{q-1}} \wedge dx^{i_q}) \\
= d\omega_K^\sigma \wedge (& -C_{\nu i_3 i_4 \ldots i_q}^{K} dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\
& + C_{\nu\sigma_2 i_4 i_5 \ldots i_q}^{KI_2} dy_{I_2}^{\sigma_2} \wedge dx^{i_4} \wedge dx^{i_5} \wedge \cdots \wedge dx^{i_q} \\
& - C_{\nu\sigma_2\sigma_3 i_5 i_6 \ldots i_q}^{KI_2 I_3} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge dx^{i_5} \wedge dx^{i_6} \wedge \cdots \wedge dx^{i_q} \\
& + \cdots + (-1)^{q-1} C_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1}}^{KI_2 I_3 \ldots I_{q-1}} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge \cdots \wedge dy_{I_{q-1}}^{\sigma_{q-1}}).
\end{aligned}
\tag{221}
$$

This expression splits in two terms,

$$
\begin{aligned}
& d(\omega_K^\sigma \wedge (-C_{\nu i_3 i_4 \ldots i_q}^{K} dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\
& \quad + C_{\nu\sigma_2 i_4 i_5 \ldots i_q}^{KI_2} dy_{I_2}^{\sigma_2} \wedge dx^{i_4} \wedge dx^{i_5} \wedge \cdots \wedge dx^{i_q} \\
& \quad - C_{\nu\sigma_2\sigma_3 i_5 i_6 \ldots i_q}^{KI_2 I_3} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge dx^{i_5} \wedge dx^{i_6} \wedge \cdots \wedge dx^{i_q} \\
& \quad + \cdots + (-1)^{q-1} C_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1}}^{KI_2 I_3 \ldots I_{q-1}} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge \cdots \wedge dy_{I_{q-1}}^{\sigma_{q-1}})),
\end{aligned}
\tag{222}
$$

and

$$
\begin{aligned}
& -\omega_K^\sigma \wedge d(-C_{\nu i_3 i_4 \ldots i_q}^{K} dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\
& \quad + C_{\nu\sigma_2 i_4 i_5 \ldots i_q}^{KI_2} dy_{I_2}^{\sigma_2} \wedge dx^{i_4} \wedge dx^{i_5} \wedge \cdots \wedge dx^{i_q} \\
& \quad - C_{\nu\sigma_2\sigma_3 i_5 i_6 \ldots i_q}^{KI_2 I_3} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge dx^{i_5} \wedge dx^{i_6} \wedge \cdots \wedge dx^{i_q} \\
& \quad + \cdots + (-1)^{q-1} C_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1}}^{KI_2 I_3 \ldots I_{q-1}} dy_{I_2}^{\sigma_2} \wedge dy_{I_3}^{\sigma_3} \wedge \cdots \wedge dy_{I_{q-1}}^{\sigma_{q-1}}),
\end{aligned}
\tag{223}
$$

which can be distributed to the terms $d\kappa_0$ and ρ_0 in the decomposition (207). Therefore, ρ can be written as

$$
\begin{aligned}
\rho &= \sum_{0 \le |J| \le r-2} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge \Delta_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge Z_\sigma^J + d\kappa_0 \\
&= \sum_{0 \le |J| \le r-2} \omega_J^\sigma \wedge \tilde{\Phi}_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge Z_\sigma^J + d\kappa_1 \\
&= \sum_{0 \le |J| \le r-3} \omega_J^\sigma \wedge \tilde{\Phi}_\sigma^J + \sum_{|J|=r-2} \omega_J^\sigma \wedge \tilde{\Phi}_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge Z_\sigma^J + d\kappa_1
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{0 \le |J| \le r-3} \omega_J^\sigma \wedge \tilde{\Phi}_\sigma^J + \sum_{|J| = r-2} \omega_J^\sigma \wedge \tilde{\Phi}_\sigma^J + \sum_{|J| = r-1} \omega_J^\sigma \wedge \mathrm{Z}_\sigma^J + d\kappa_1 \\
&= \sum_{0 \le |J| \le r-3} \omega_J^\sigma \wedge \tilde{\Phi}_\sigma^J + \sum_{|J| = r-2} \omega_J^\sigma \wedge \Delta_\sigma^J + \sum_{|J| = r-2} \omega_J^\sigma \wedge \mathrm{Z}_\sigma^J \\
&\quad + \sum_{|J| = r-1} \omega_J^\sigma \wedge \mathrm{Z}_\sigma^J + d\kappa_1
\end{aligned} \tag{224}$$

where we use the trace decomposition $\tilde{\Phi}_\sigma^J = \Delta_\sigma^J + \mathrm{Z}_\sigma^J$ for $|J| = r-1$. Summarizing and replacing for simplicity of notation $\tilde{\Phi}_\sigma^J$ with Φ_σ^J, we get the decomposition (209).

Third, we construct as in the second step the decompositions

$$\begin{aligned}
&\rho_0 = \sum_{0 \le |J| \le r-2} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-1} \omega_J^\sigma \wedge \Delta_\sigma^J, \\
&\rho_1 = \sum_{0 \le |J| \le r-3} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|J|=r-2} \omega_J^\sigma \wedge \Delta_\sigma^J, \\
&\dots \\
&\rho_{r-2} = \omega^\sigma \wedge \Phi_\sigma + \sum_j \omega_j^\sigma \wedge \Delta_\sigma^j, \\
&\rho_{r-1} = \omega^\sigma \wedge \Delta_\sigma,
\end{aligned} \tag{225}$$

and

$$\begin{aligned}
&\rho = \rho_0 + \tau_0 + d\kappa_0 = \rho_1 + \tau_1 + d\kappa_1 = \rho_2 + \tau_2 + d\kappa_2 \\
&\dots = \rho_{r-2} + \tau_{r-2} + d\kappa_{r-2} = \rho_{r-1} + \tau_{r-1} + d\kappa_{r-1}.
\end{aligned} \tag{226}$$

Note, however, the different meaning of the symbols Φ_σ^J and Δ_σ^J in the lines of expressions (225), which are defined in the construction.

Finally, we show that ρ has a decomposition

$$\rho = \tau_{r-1} + d\kappa_{r-1}, \tag{227}$$

where τ_{r-1} is generated by the contact forms $\omega_J^\sigma \wedge \omega_I^\nu$ and $\omega_J^\sigma \wedge d\omega_L^\nu$, $|J| = r-1$, $0 \le |I| \le r-1$, $|L| = r-1$ and κ_{r-1} is a contact $(q-1)$-form. It is sufficient to show that in the decomposition $\rho = \rho_{r-1} + \tau_{r-1} + d\kappa_{r-1}$ (226), the form ρ_{r-1} vanishes. Condition $d\rho = 0$ implies

$$d\omega^\sigma \wedge \Delta_\sigma - \omega^\sigma \wedge d\Delta_\sigma + d\tau_{r-1} = 0. \tag{228}$$

The 1-contact component yields $-\omega_l^\sigma \wedge dx^l \wedge h\Delta_\sigma - \omega^\sigma \wedge hd\Delta_\sigma = 0$; hence,

$$h(dx^l \wedge \Delta_\sigma) = 0. \tag{229}$$

Writing the traceless form Δ_ν as

$$\begin{aligned}\Delta_\nu = {} & A_{\nu i_2 i_3 \ldots i_q} dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q} \\ & + A^{I_2}_{\nu\sigma_2 i_3 i_4 \ldots i_q} dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\ & + A^{I_2 I_3}_{\nu\sigma_2\sigma_3 i_4 i_5 \ldots i_q} dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge dx^{i_4} \wedge dx^{i_5} \wedge \cdots \wedge dx^{i_q} \\ & + \cdots + A^{I_2 I_3 \ldots I_{q-1}}_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1} i_q} dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \cdots \wedge dy^{\sigma_{q-1}}_{I_{q-1}} \wedge dx^{i_q} \\ & + A^{I_2 I_3 \ldots I_q}_{\nu\sigma_2\sigma_3 \ldots \sigma_q} dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \cdots \wedge dy^{\sigma_q}_{I_q},\end{aligned} \tag{230}$$

we have

$$\begin{aligned} h(dx^l \wedge \Delta_\nu) = {} & \Big(A_{\nu i_2 i_3 \ldots i_q} + A^{I_2}_{\nu\sigma_2 i_3 i_4 \ldots i_q} y^{\sigma_2}_{I_2 i_2} + A^{I_2 I_3}_{\nu\sigma_2\sigma_3 i_4 i_5 \ldots i_q} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \\ & + \cdots + A^{I_2 I_3 \ldots I_{q-1}}_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1} i_q} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \ldots y^{\sigma_{q-1}}_{I_{q-1} i_{q-1}} + A^{I_2 I_3 \ldots I_q}_{\nu\sigma_2\sigma_3 \ldots \sigma_q} y^{\sigma_2}_{I_2 i_2} y^{\sigma_3}_{I_3 i_3} \ldots y^{\sigma_q}_{I_q i_q} \Big) \\ & \cdot dx^l \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q} = 0, \end{aligned} \tag{231}$$

which implies, because the coefficients are traceless,

$$\begin{aligned} & A_{\nu i_2 i_3 \ldots i_q} = 0, \quad A^{I_2}_{\nu\sigma_2 i_3 i_4 \ldots i_q} = 0, \quad A^{I_2 I_3}_{\nu\sigma_2\sigma_3 i_4 i_5 \ldots i_q} = 0, \\ & \ldots \quad A^{I_2 I_3 \ldots I_{q-1}}_{\nu\sigma_2\sigma_3 \ldots \sigma_{q-1} i_q} = 0, \quad A^{I_2 I_3 \ldots I_q}_{\nu\sigma_2\sigma_3 \ldots \sigma_q} = 0. \end{aligned} \tag{232}$$

Consequently, $\rho_{r-1} = 0$ proving (227).

3. To conclude the proof, we apply the contact homotopy decomposition to the form τ_{r-1} (Theorem 11). We have $\tau_{r-1} = Id\tau_{r-1} + dI\tau_{r-1}$. But $d\tau_{r-1} = 0$, and thus, $\tau_{r-1} = dI\tau_{r-1}$, and since the order of contactness of τ_{r-1} is ≥ 2, we have $hI\tau_{r-1} = Ihp_1\tau_{r-1} = 0$, so $I\tau_{r-1}$ is contact. Then, however,

$$\rho = Id\tau_{r-1} + dI\tau_{r-1} + d\kappa_{r-1} = d(I\tau_{r-1} + d\kappa_{r-1}). \tag{233}$$

Setting $\eta = I\tau_{r-1} + d\kappa_{r-1}$, we complete the proof. □

Theorem 14 *If ρ is strongly contact and $d\rho = 0$, then there exists a strongly contact $(q-1)$-form η such that $\rho = d\eta$.*

Proof We express ρ as $\rho = Id\rho + dI\rho$. But by hypothesis $d\rho = 0$, thus setting $\eta = I\rho$, we have $\rho = d\eta$; now, our assertion follows from Theorem 12, (b). □

Remark 11 The concept of a strongly contact form, used in Theorem 14, has been introduced by means of the exterior derivative d and the pullback operation by the canonical jet projection $\pi^{r+1,r}: J^{r+1}Y \to J^rY$. The decompositions of the forms on J^rY, related to this concept, represent a basic tool in the higher-order variational theory on the jet spaces J^rY. A broader concept of a strongly contact form is considered in Chap. 8.

References

[D] P. Dedecker, On the generalization of symplectic geometry to multiple integrals in the calculus of variations, in: *Lecture notes in Math.* 570, Springer, Berlin, 1977, 395-456

[GS] H. Goldschmidt, S. Sternberg, The Hamilton-Cartan formalism in the calculus of variations, Ann. Inst. H. Poincare 23 (1973) 203-267

[K13] D. Krupka, *Some Geometric Aspects of Variational Problems in Fibered Manifolds*, Folia Fac. Sci. Nat. UJEP Brunensis, Physica 14, Brno, Czech Republic, 1973, 65 pp.; arXiv:math-ph/0110005

[K4] D. Krupka, Global variational theory in fibred spaces, in: D. Krupka, D. Saunders, Eds., *Handbook of Global Analysis*, Elsevier, 2008, 773-836

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