In this chapter, we present a decomposition theory of differential forms on jet prolongations of fibered manifolds; the tools inducing the decompositions are the algebraic trace decomposition theory and the canonical jet projections. Of particular interest is the structure of the contact forms, annihilating integrable sections of the jet prolongations. We also study decompositions of forms defined by fibered homotopy operators and state the corresponding fibered Poincare-Volterra lemma.

The theory of differential forms explained in this chapter has been developed along the lines indicated in the approach of Lepage and Dedecker to the calculus of variations (see Dedecker [D], Goldschmidt and Sternberg [GS] and Krupka [K13]). The exposition extends the theory explained in the handbook chapter Krupka [K4].

Throughout, Y is a smooth fibered manifold with base X and projection π, \( n = \text{dim } X, \ n + m = \text{dim } Y \). \( J^rY \) is the r-jet prolongation of Y, and \( \pi^r: J^rY \to X, \pi^r: J^rY \to X \) are the canonical jet projections. For any open set \( W \subset Y, \Omega^q_W \) denotes the module of q-forms on the open set \( W \), \( \Omega^r_{J^rW} \) is the exterior algebra of differential forms on the set \( W \). We say that a form \( \eta \) is generated by a finite family of forms \( \mu_k, \) if \( \eta \) is expressible as \( \eta = \eta^k \wedge \mu_k \) for some forms \( \eta^k; \) note that in this terminology, we do not require \( \mu_k \) to be 1-forms, or \( k \)-forms for a fixed integer \( k \).

2.1 The Contact Ideal

We introduced in Sect. 1.5 a vector bundle homomorphism \( h \) between the tangent bundles \( TJ^{r+1}Y \) and \( T(J^rY) \) over the canonical jet projection \( \pi^{r+1}: J^{r+1}Y \to J^rY \), the horizontalization. In this section, the associated dual mapping between the modules of 1-forms \( \Omega^1_W \) and \( \Omega^{r+1}_W \) is studied. We show, in particular, that this mapping allows us to associate with any fibered chart \( (V, \psi) \) on Y and any function, defined on \( V^r \), its formal (or total) partial derivatives in a geometric way and a specific basis of 1-forms on \( V^r \), termed the contact basis. Then, we introduce by means of the contact
basis a differential ideal in the exterior algebra $\Omega^r W$, characterizing the structure of forms on jet prolongations of fibered manifolds, the contact ideal.

Recall that the horizontalization $h$ is defined by the formula

$$h\xi = T_x J^{r+1} \gamma \circ T \pi^{r+1} \cdot \xi,$$

where $\xi$ is a tangent vector to the manifold $J^{r+1} Y$ at a point $J^{r+1} \gamma$. The mapping $h$ makes the following diagram

$$
\begin{array}{ccc}
TJ^{r+1} Y & \xrightarrow{h} & TJ^r Y \\
\downarrow & & \downarrow \\
J^{r+1} Y & \xrightarrow{\pi^{r+1} r} & J' Y
\end{array}
$$

commutative and induces a decomposition of the projections of the tangent vectors $T \pi^{r+1} r \cdot \xi$,

$$T \pi^{r+1} r \cdot \xi = h\xi + p\xi. \quad (3)$$

$h\xi$ (resp. $p\xi$) is the horizontal (resp. contact) component of the vector $\xi$. Note, however, that the terminology is not standard: The vectors $\xi$ and $h\xi$ do not belong to the same vector space. The horizontal and contact components satisfy

$$T \pi^r \cdot h\xi = T \pi^{r+1} \cdot \xi, \quad T \pi^r \cdot p\xi = 0. \quad (4)$$

The horizontalization $h$ induces a mapping of modules of linear differential forms as follows. Let $J^{r+1} \gamma X r W$. We set for any differential 1-form $\rho$ on $W'$ and any vector $\xi$ from the tangent space $TJ^{r+1} Y$ at $J^{r+1} \gamma$,

$$h\rho(J^{r+1} \gamma) \cdot \xi = \rho(J_x \gamma) \cdot h\xi. \quad (5)$$

The mapping $\Omega^{r+1} W \ni \rho \rightarrow h\rho \in \Omega^{r+1} W$ is called the $\pi$-horizontalization or just the horizontalization (of differential forms).

Clearly, the form $h\rho$ vanishes on $\pi^{r+1}$-vertical vectors so it is $\pi^{r+1}$-horizontal; $h\rho$ is sometimes called the horizontal component of $\rho$.

The mapping $h$ is linear over the ring of functions $\Omega^r W$ along the jet projection $\pi^{r+1} r$ in the sense that

$$h(\rho_1 + \rho_2) = h\rho_1 + h\rho_2 \quad h(f \rho) = (f \circ \pi^{r+1} r) h\rho \quad (6)$$

for all $\rho_1, \rho_2, \rho \in \Omega^{r+1} W$ and $f \in \Omega^r W$. 

36 2 Differential Forms on Jet Prolongations of Fibered Manifolds
If in the fibered chart \((V, \psi)\), \(\psi = (x^i, y^\sigma)\), a 1-form \(\rho\) is expressed by

\[
\rho = A_i dx^i + \sum_{0 \leq k \leq r_1 \leq j_2 \leq \cdots \leq j_k} B^{j_2 \ldots j_k}_{\sigma} dy^\sigma_{j_1 j_2 \ldots j_k},
\]

then we have from (5) at any point \(J^r_{\alpha} + 1 \in V^{r+1}\)

\[
hp(J^r_{\alpha} + 1) \cdot \zeta = A_i (J^r_{\alpha}) dx^i(J^r_{\alpha}) \cdot h\zeta
+ \sum_{0 \leq k \leq r_1 \leq j_2 \leq \cdots \leq j_k} B^{j_2 \ldots j_k}_{\sigma} dy^\sigma_{j_1 j_2 \ldots j_k} (J^r_{\alpha}) \cdot h\zeta
= \left( A_i (J^r_{\alpha}) + \sum_{0 \leq k \leq r_1 \leq j_2 \leq \cdots \leq j_k} B^{j_2 \ldots j_k}_{\sigma} (J^r_{\alpha}) y^\sigma_{j_1 j_2 \ldots j_k} \right) \zeta^i,
\]

thus,

\[
hp = \left( A_i + \sum_{0 \leq k \leq r_1 \leq j_2 \leq \cdots \leq j_k} B^{j_2 \ldots j_k}_{\sigma} y^\sigma_{j_1 j_2 \ldots j_k} \right) dx^i.
\]

In particular, for any function \(f: W^r \to \mathbb{R}\)

\[
hdf = df \cdot dx^i,
\]

where

\[
df = \frac{\partial f}{\partial x^i} + \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} y^\sigma_{j_1 j_2 \ldots j_k}.
\]

The function \(df : V^{r+1} \to \mathbb{R}\) is the \(i\)-th formal derivative of \(f\) with respect to the fibered chart \((V, \psi)\). From (10), it follows that \(df\) are the components of an invariant object, the horizontal component \(hdf\) of the exterior derivative of \(f\). Note that formal derivatives \(df\) have already been introduced in Sect. 1.5.

The following lemma summarizes basic rules for computations with the horizontalization and formal derivatives. We denote by \(\overline{\partial}_i\) the formal derivative operator with respect to a fibered chart \((\overline{V}, \overline{\psi})\), \(\overline{\psi} = (\overline{x}^i, \overline{y}^\sigma)\).

**Lemma 1** Let \((V, \psi)\), \(\psi = (x^i, y^\sigma)\), be a fibered chart on \(Y\).

(a) The horizontalization \(h\) satisfies

\[
hd_{y^\sigma} = y^\sigma_i dx^i, \quad hd_{y^\sigma_{j_1}} = y^\sigma_{j_1 i} dx^i, \quad hd_{y^\sigma_{j_1 j_2}} = y^\sigma_{j_1 j_2 i} dx^i, \\
\ldots, \quad hd_{y^\sigma_{j_1 j_2 \ldots j_k}} = y^\sigma_{j_1 j_2 \ldots j_k i} dx^i.
\]
(b) The i-th formal derivative of the coordinate function $y_{j_1j_2...j_k}^r$ is given by
\[ d_i y_{j_1j_2...j_k}^r = y_{j_1j_2...j_k}^r. \]  

(c) If $(\bar{V}, \bar{y}), \bar{y} = (\bar{x}, \bar{y})$, is another chart on $Y$ such that $V \cap \bar{V} \neq \emptyset$, then for every function $f: V' \cap \bar{V} \rightarrow \mathbb{R}$,
\[ \bar{d}f = df \frac{\partial x_i}{\partial \bar{x}}. \]  

(d) For any two functions $f, g: V' \rightarrow \mathbb{R}$,
\[ d_i (f \cdot g) = g \cdot df + f \cdot dg. \]  

(e) For every function $f: V' \rightarrow \mathbb{R}$ and every section $\gamma: U \rightarrow V \subset Y$,
\[ df \circ J^{r+1} \gamma = \frac{\partial (f \circ J^{r} \gamma)}{\partial x_i}. \]  

Remark 1 By (13), $\bar{y}_{j_1j_2...j_k}^r = \bar{d}_k y_{j_1j_2...j_k-1}^r$. Thus, applying (14) to coordinates, we obtain the following prolongation formula for coordinate transformations in jet prolongations of fibered manifolds
\[ \bar{y}_{j_1j_2...j_k}^r = d_i \bar{y}_{j_1j_2...j_k-1}^r \cdot \frac{\partial x_i}{\partial \bar{x}}. \]  

Remark 2 If two functions $f, g: V' \rightarrow \mathbb{R}$ coincide along a section $J^r \gamma$, that is, $f \circ J^r \gamma = g \circ J^r \gamma$, then their formal derivatives coincide along the $(r + 1)$-prolongation $J^{r+1} \gamma$,
\[ df \circ J^{r+1} \gamma = d_i g \circ J^{r+1} \gamma. \]  

This is an immediate consequence of formula (16).

Now, we study properties of 1-forms, belonging to the kernel of the horizontization $\Omega^1_i W \ni \rho \rightarrow h\rho \in \Omega^{r+1}_1 W$. We say that a 1-form $\rho \in \Omega^1_i W$ is contact, if
\[ h\rho = 0. \]  

It is easy to find the chart expression of a contact 1-form. Writing $\rho$ as in (7), condition (19) yields
\[ A_i + \sum_{0 \leq k \leq \rho_1 \leq j_2 \leq \cdots \leq j_k} \sum B_{j_1j_2...j_k}^i y_{j_1j_2...j_k}^r = 0, \]
or, equivalently,
\[ B_{\sigma}^{ij_1...j_2} = 0, \quad A_i = - \sum_{0 \leq k \leq r-1} \sum_{j_1 \leq j_2 \leq \ldots \leq j_k} B_{\sigma}^{ij_1...j_k} y_{ij_1...j_k}. \] (21)

Thus, setting for all \( k, 0 \leq k \leq r - 1 \),
\[ \omega_{ij_1...j_k} = dy_{ij_1...j_k} - y_{ij_1...j_k} dx^i, \] (22)
we see that \( \rho \) has the chart expression
\[ \rho = \sum_{0 \leq k \leq r-1} \sum_{j_1 \leq j_2 \leq \ldots \leq j_k} B_{\sigma}^{ij_1...j_k} \omega_{ij_1...j_k}. \] (23)

This formula shows that any contact 1-form is expressible as a linear combination of the forms \( \omega_{ij_1...j_k} \).

The following two theorems summarize properties of the forms \( \omega_{ij_1...j_k} \).

**Theorem 1**

(a) For any fibered chart \((V, \psi), \psi = (x^i, y^\sigma)\), the forms
\[ dx^i, \quad \omega_{ij_1...j_k}, \quad dy_{l_1l_2...l_m}, \] (24)
such that \( 1 \leq i \leq n, \ 1 \leq \sigma \leq m, \ 1 \leq k \leq r - 1, \ 1 \leq j_1 \leq j_2 \leq \ldots \leq j_k \leq n, \) and \( 1 \leq l_1 \leq l_2 \leq \ldots \leq l_m \leq n \), constitute a basis of linear forms on the set \( V \).

(b) If \((V, \psi), \psi = (x^i, y^\sigma)\), and \((\overline{V}, \overline{\psi}), \overline{\psi} = (\overline{x}^i, \overline{y}^\sigma)\), are two fibered charts such that \( V \cap \overline{V} \neq \emptyset \), then
\[ \omega_{{p_1p_2...p_k}} = \sum_{0 \leq m \leq k} \sum_{j_1 \leq j_2 \leq \ldots \leq j_k} \frac{\partial y_{{p_1p_2...p_k}}}{\partial y_{j_1j_2...j_m}} \overline{\omega_{j_1j_2...j_m}}. \] (25)

(c) Let \((V, \psi), \psi = (x^i, y^\sigma)\), and \((\overline{V}, \overline{\psi}), \overline{\psi} = (\overline{x}^i, \overline{y}^\sigma)\), be two fibered charts and \( \varepsilon \) an automorphism of \( Y \), defined on \( V \) and such that \( \varepsilon(V) \subset \overline{V} \). Then
\[ J^r x^\varepsilon \overline{\omega}_{j_1j_2...j_k} = \sum_{i_1 = i_2 = \ldots = i_p} \frac{\partial (y_{{j_1j_2...j_k}} \circ J^r x^\varepsilon)}{\partial y_{i_1i_2...i_p}} \omega_{i_1i_2...i_p}. \] (26)

**Proof**

(a) Clearly, from formula (22), we conclude that the forms (24) are expressible as linear combinations of the forms of the canonical basis \( dx^i, \ dy_{ij_1...j_k}, \ dy_{l_1l_2...l_m} \).
(b) Consider two charts \((V, \psi), \psi = (x^i, y^\sigma),\) and \((\overline{V}, \overline{\psi}), \overline{\psi} = (\overline{x}^i, \overline{y}^\sigma),\) such that \(V \cap \overline{V} \neq \emptyset\). For any function \(f\), defined on \(V^r\),

\[
(p^{r+1,r} \star df = hdf + pdf = df \cdot dx^i + \sum_{0 \leq k \leq r, 1 \leq l \leq \cdots \leq l' \leq h} \frac{\partial f}{\partial y_{l1\ldots l'}} \omega_{l1\ldots l'},
\]

\[
= \overline{df} \cdot dx^\sigma + \sum_{0 \leq k \leq r, 1 \leq l \leq \cdots \leq h} \frac{\partial f}{\partial \overline{y}_{l1\ldots l}} \overline{\omega}_{l1\ldots l},
\]

\[
= \overline{df} \overline{\partial x} \cdot dx^\sigma + \sum_{0 \leq k \leq r, 1 \leq l \leq \cdots \leq h} \frac{\partial f}{\partial \overline{y}_{l1\ldots l}} \frac{\partial \overline{y}_{l1\ldots l}}{\partial x^i} \overline{\omega}_{l1\ldots l}.
\]  

(27)

Setting \(f = y^\sigma_{p_1 p_2 \ldots p_k}\), where \(p_1 \leq p_2 \leq \cdots \leq p_k\), and using (17), we get (25).

(c) By definition,

\[
J' x^\gamma \overline{\omega}_{l1\ldots l} = d(\overline{y}^\sigma_{l1\ldots l} \circ J' x^\gamma) - (\overline{y}^\sigma_{l1\ldots l} \circ J' x^\gamma) d(\overline{x} \circ J' x^\gamma).
\]  

(28)

Denote by \(z_0\) the \(\pi\)-projection of \(x\). Since from Sect. 1.6, (80)

\[
\overline{y}^\sigma_{l1\ldots l} \circ J' x^\gamma = \frac{\partial (\overline{y}^\sigma_{l1\ldots l} \circ J' x^\gamma \circ J' x^\gamma) \circ z_0^{-1} \overline{\psi}^{-1}}{\partial \overline{x}^\sigma},
\]  

(29)

then

\[
J' x^\gamma \overline{\omega}_{l1\ldots l} = \frac{\partial (\overline{y}^\sigma_{l1\ldots l} \circ J' x^\gamma)}{\partial \overline{x}^\sigma} dx^\sigma + \sum_{i < l_2 < \ldots < l_p} \frac{\partial (\overline{y}^\sigma_{l1\ldots l} \circ J' x^\gamma)}{\partial y_{l1\ldots l}} \omega_{l1\ldots l} y_{l1\ldots l} dx^\sigma
\]

\[
- \frac{\partial (\overline{y}^\sigma_{l1\ldots l} \circ J' x^\gamma \circ J' x^\gamma) \circ z_0^{-1} \overline{\psi}^{-1}}{\partial \overline{x}^\sigma} \frac{\partial (\overline{x} \circ J' x^\gamma)}{\partial \overline{x}^\sigma} dx^\sigma
\]

\[
\frac{\partial (\overline{y}^\sigma_{l1\ldots l} \circ J' x^\gamma)}{\partial \overline{x}^\sigma} dx^\sigma + \sum_{i < l_2 < \ldots < l_p} \frac{\partial (\overline{y}^\sigma_{l1\ldots l} \circ J' x^\gamma)}{\partial y_{l1\ldots l}} \omega_{l1\ldots l} y_{l1\ldots l} dx^\sigma
\]

\[
+ \sum_{i < l_2 < \ldots < l_p} \frac{\partial (\overline{y}^\sigma_{l1\ldots l} \circ J' x^\gamma)}{\partial y_{l1\ldots l}} \omega_{l1\ldots l} y_{l1\ldots l} dx^\sigma
\]

\[
\frac{\partial (\overline{y}^\sigma_{l1\ldots l} \circ J' x^\gamma \circ J' x^\gamma) \circ z_0^{-1} \overline{\psi}^{-1}}{\partial \overline{x}^\sigma} \frac{\partial (\overline{x} \circ J' x^\gamma)}{\partial \overline{x}^\sigma} dx^\sigma
\]

\[
= \sum_{i < l_2 < \ldots < l_p} \frac{\partial (\overline{y}^\sigma_{l1\ldots l} \circ J' x^\gamma)}{\partial y_{l1\ldots l}} \omega_{l1\ldots l} y_{l1\ldots l} dx^\sigma.
\]  

(30)

These conditions mean that the section \(\delta\) is of the form \(\delta = J'(\pi^{r,0} \circ \delta)\) as required. □
The basis of 1-forms (24) on $V^r$ is usually called the contact basis.

The following observations show that the contact forms $\omega^\sigma_{j_1j_2...j_k}$, defined by a fibered atlas on $Y$, define a (global) module of 1-forms and an ideal of the exterior algebra $\Omega' W$ (for elementary definitions, see Appendix 7).

**Corollary 1** The contact 1-forms $\omega^\sigma_{j_1j_2...j_k}$ locally generate a submodule of the module $\Omega'_1 W$.

**Corollary 2** The contact 1-forms $\omega^\sigma_{j_1j_2...j_k}$ locally generate an ideal of the exterior algebra $\Omega' W$. This ideal is not closed under the exterior derivative operator.

**Proof** Existence of the ideal is ensured by the transformation properties of the contact 1-forms $\omega^\sigma_{j_1j_2...j_k}$ (Theorem 1, (b)). It remains to show that the ideal contains a form, which is not generated by the forms $\omega^\sigma_{j_1j_2...j_k}$. If $\rho$ is a contact 1-form expressed as

$$\rho = \sum_{0 \leq k \leq r-1} \sum_{j_1 \leq j_2 \leq \ldots \leq j_k} B^j_{\sigma} \omega^\sigma_{j_1j_2...j_k},$$

then

$$d\rho = \sum_{0 \leq k \leq r-1} \sum_{j_1 \leq j_2 \leq \ldots \leq j_k} \left( dB^j_{\sigma} \wedge \omega^\sigma_{j_1j_2...j_k} + B^j_{\sigma} d\omega^\sigma_{j_1j_2...j_k} \right).$$

But in this expression,

$$d\omega^\sigma_{j_1j_2...j_k} = \begin{cases} -\omega^\sigma_{j_1j_2...j_k} \wedge dx^l, & 0 \leq k \leq r-2, \\ -dy^\sigma_{j_1j_2...j_{k-1}} \wedge dx^l, & k = r-1, \end{cases}$$

thus, $d\omega^\sigma_{j_1j_2...j_{k-1}}$ and in general the form $\rho$ are not generated by the contact forms $\omega^\sigma_{j_1j_2...j_k}$. \qed

The ideal of the exterior algebra $\Omega' W$, locally generated by the 1-forms $\omega^\sigma_{j_1j_2...j_k}$, where $0 \leq k \leq r-1$, is denoted by $\Theta'_0 W$. The 1-forms $\omega^\sigma_{j_1j_2...j_k}$, where $0 \leq k \leq r-1$, and 2-forms $d\omega^\sigma_{j_1j_2...j_{k-1}}$ locally generate an ideal $\Theta' W$ of the exterior algebra $\Omega' W$, closed under the exterior derivative operator, that is, a differential ideal. This ideal is called the contact ideal of the exterior algebra $\Omega' W$, and its elements are called contact forms. We denote

$$\Theta'_q W = \Omega'_q W \cap \Theta' W.$$  

The set $\Theta'_q W$ of contact $q$-forms is a submodule of the module $\Omega'_q W$, called the contact submodule.
Since the exterior derivative of a contact form is again a contact form, we have the sequence

$$0 \rightarrow \Theta^r_1 W \xrightarrow{d} \Theta^r_2 W \xrightarrow{d} \cdots \xrightarrow{d} \Theta^r_n W,$$

where the arrows denote the exterior derivative operator. If $\rho$ is a contact form, $\rho \in \Theta^r_q W$, and $f$ is a function on $W$, $f \in \Theta^r_0 W$, then the formula

$$d(f \rho) = df \wedge \rho + fd\rho$$

shows that the form $d(f \rho)$ is again a contact form; however, the exterior derivative in (36) is not a homomorphism of $\Theta^r_0 W$-modules. Restricting the multiplication in (36) to constant functions $f$, that is, to real numbers, the exterior derivative in (36) becomes a morphism of vector spaces.

Another consequence of Theorem 1 is concerned with sections of the fibered manifold $J^r Y$ over the base $X$. We say that a section $d$ of $J^r Y$, defined on an open set in $X$, is holonomic, or integrable, if there exists a section $\gamma$ of $Y$ such that

$$\delta = J^r \gamma.$$ 

Obviously, if $\gamma$ exists, then applying the projection $\pi^{r,0}$ to both sides, we get $\pi^{r,0} \circ \delta = \gamma$; thus, if $\gamma$ exists, it is unique and is determined by

$$\gamma = \pi^{r,0} \circ \delta.$$ 

**Theorem 2** A section $\delta: U \to J^r Y$ is holonomic if and only if for any fibered chart $(V, \psi), \psi = (x^l, y^r)$, such that the set $\pi(V)$ lies in the domain of definition of $\delta$,

$$\delta^* \omega_{i_1 i_2 \ldots i_k}^\sigma = 0$$

for all $\sigma$, $k$, and $i_1, i_2, \ldots, i_k$ such that $1 \leq \sigma \leq m$, $0 \leq k \leq r - 1$, and $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$.

**Proof** By definition,

$$\delta^* \omega_{i_1 i_2 \ldots i_k}^\sigma = d(y_{i_1 i_2 \ldots i_k}^\sigma \circ \delta) - (y_{i_1 i_2 \ldots i_k}^\sigma \circ \delta) dx^l$$

$$= \left( \frac{\partial(y_{i_1 i_2 \ldots i_k}^\sigma \circ \delta)}{\partial x^l} - y_{i_1 i_2 \ldots i_k}^\sigma \circ \delta \right) dx^l.$$ 

Thus, condition (39) is equivalent to the conditions

$$\frac{\partial(y_{i_1 i_2 \ldots i_k}^\sigma \circ \delta)}{\partial x^l} - y_{i_1 i_2 \ldots i_k}^\sigma \circ \delta = 0$$
that can also be written as
\[
\frac{\partial (y^\sigma \circ \delta)}{\partial x^l} - y^\sigma_i \circ \delta = 0,
\]
\[
\frac{\partial (y^\sigma_i \circ \delta)}{\partial x^l} - y^\sigma_{i,l} \circ \delta = \frac{\partial^2 (y^\sigma \circ \delta)}{\partial x^l \partial x^i} - y^\sigma_j \circ \delta = 0,
\]
\[
\ldots
\]
\[
\frac{\partial (y^\sigma_{i_1 i_2 \ldots i_r} \circ \delta)}{\partial x^l} - y^\sigma_{i_1 i_2 \ldots i_r l} \circ \delta = \frac{\partial^{k+1} (y^\sigma \circ \delta)}{\partial x^{i_1} \partial x^{i_2} \ldots \partial x^{i_r-1} \partial x^l} - y^\sigma_{i_1 i_2 \ldots i_r l} \circ \delta = 0.
\]

These conditions mean that the section \( \delta \) is of the form \( \delta = J' (\pi^r)^0 \circ \delta \) as required. \( \square \)

2.2 The Trace Decomposition

Main objective in this section is the application of the trace decomposition theory of tensor spaces to differential forms defined on the \( r \)-jet prolongation \( JY \) of a fibered manifold \( Y \). We decompose the components of a form, expressed in a fibered chart, by the trace operation (see Appendix 9); the resulting decomposition of differential forms will be referred to as the trace decomposition.

In order to study the structure of the components of a form \( \rho \in \Omega^q W \) for general \( r \), it will be convenient to introduce a multi-index notation. We also need a convention on the alternation and symmetrization of tensor components in a given set of indices.

**Convention 1 (Multi-indices)** We introduce a multi-index \( I \) as an ordered \( k \)-tuple \( I = (i_1, i_2, \ldots, i_k) \), where \( k = 1, 2, \ldots, r \) and the entries are indices such that \( 1 \leq i_1, i_2, \ldots, i_k \leq n \). The number \( k \) is the length of \( I \) and is denoted by \( |I| \). If \( j \) is any integer such that \( 1 \leq j \leq n \), we denote by \( I_j \) the multi-index \( I_j = (i_1, i_2, \ldots, i_k) \). In this notation, the contact basis of \( 1 \)-forms, introduced in Sect. 2.1, Theorem 1, (a), is sometimes denoted as \( (dx^i, \omega^i_\sigma, dy^i_\sigma) \), where the multi-indices satisfy \( 0 \leq |J| \leq r - 1 \) and \( |I| = r \); it is understood, however, that the basis includes only linearly independent \( 1 \)-forms \( \omega^i_\sigma \), where the multi-indices \( I = (i_1, i_2, \ldots, i_k) \) satisfy \( i_1 \leq i_2 \leq \cdots \leq i_k \).

**Convention 2 (Alternation, symmetrization)** We introduce the symbol \( \text{Alt}(i_1, i_2, \ldots, i_k) \) to denote alternation in the indices \( i_1, i_2, \ldots, i_k \). If \( U = [U_{i_1 i_2 \ldots i_k}] \) is a collection of real numbers, we denote by \( U_{i_1 i_2 \ldots i_k} \) \( \text{Alt}(i_1, i_2, \ldots, i_k) \) the skew-symmetric component of \( U \). Analogously, \( \text{Sym}(i_1, i_2, \ldots, i_k) \) denotes symmetrization in the indices \( i_1, i_2, \ldots, i_k \), and the symbol \( U_{i_1 i_2 \ldots i_k} \) \( \text{Sym}(i_1, i_2, \ldots, i_k) \) means the symmetric component of \( U \). The operators \( \text{Alt} \) and \( \text{Sym} \) are understood as projectors (the coefficient \( 1/k! \) is included).
Note that there exists a close relationship between the trace operation on the one hand and the exterior derivative operator on the other hand. For instance, decomposing in a fibered chart the 2-form $\text{dy}^\sigma_{j_j} \wedge dx^k$ by the trace operation, we get

$$\text{dy}^\sigma_{j_j} \wedge dx^k = \frac{1}{n} \delta^k_j \text{dy}^\sigma_{j_s} \wedge dx^s + \frac{1}{n} \delta^k_j \text{dy}^\sigma_{j_s} \wedge dx^s,$$

(43)

where the summand, representing the Kronecker component of $\text{dy}^\sigma_{j_j} \wedge dx^k$, coincides, up to a constant factor, with the exterior derivative $d\omega^\sigma_j$, and is therefore a contact form:

$$\frac{1}{n} \delta^k_j \text{dy}^\sigma_{j_s} \wedge dx^s = -\frac{1}{n} d\omega^\sigma_j,$$

(44)

The complementary summand in the decomposition (43), represented by the second and the third terms, is traceless in the indices $j$ and $k$. We wish to use this observation to generalize decomposition (43) to any $q$-forms on $J^k Y$.

First, we apply the trace decomposition theorem (Appendix 9, Theorem 1) to $q$-forms of a specific type, not containing the contact forms $\omega^\sigma_j$.

**Lemma 2** Let $(V, \psi), \psi = (x^i, y^\sigma)$, be a fibered chart on $Y$. Let $\mu$ be a $q$-form on $V$ such that

$$\mu = A_{i_1i_2...i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}$$

$$+ B^{i_1}_{\sigma_1i_2...i_q} dy^{\sigma_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q}$$

$$+ B^{i_2} \sigma_1 \sigma_2...i_q dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q}$$

$$+ \cdots + B^{i_q} \sigma_1 \sigma_2...\sigma_{q-1}i_q dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge \cdots \wedge dy^{\sigma_{q-1}} \wedge dx^{i_q}$$

$$+ A^{i_1i_2...i_q} dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge \cdots \wedge dy^{\sigma_q},$$

(45)

where the multi-indices satisfy $|I_1|, |I_2|, \ldots, |I_{q-1}| = r$. Then, $\mu$ has a decomposition

$$\mu = \mu_0 + \mu',$n

(46)

satisfying the following conditions:

(a) $\mu_0$ is generated by the forms $d\omega^\sigma_j$, where $|J| = r - 1$, that is,

$$\mu_0 = \sum_{|J| = r-1} d\omega^\sigma_j \wedge \Phi^J_{\sigma},$$

(47)

for some $(q-2)$-forms $\Phi^J_{\sigma}$.
2.2 The Trace Decomposition

(b) $\mu'$ has an expression

$$
\mu' = A_{i_1i_2...i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q} \\
+ A^{l_1}_{\sigma_1i_3...i_q} dy^{\sigma_1}_{l_1} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q} \\
+ A^{l_2}_{\sigma_1\sigma_2i_3...i_q} dy^{\sigma_1}_{l_2} \wedge dy^{\sigma_2}_{l_3} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\
+ \cdots + A^{l_{q-1}}_{\sigma_1\sigma_2...\sigma_{q-1}i_q} dy^{\sigma_1}_{l_1} \wedge dy^{\sigma_2}_{l_2} \wedge \cdots \wedge dy^{\sigma_{q-1}}_{l_{q-1}} \wedge dx^{i_q} \\
+ A^{l_q}_{\sigma_1\sigma_2...\sigma_{q-1}i_q} dy^{\sigma_1}_{l_1} \wedge dy^{\sigma_2}_{l_2} \wedge \cdots \wedge dy^{\sigma_q}_{l_q},
$$

where $A^{l_1}_{\sigma_1i_3...i_q}, A^{l_2}_{\sigma_1\sigma_2i_3...i_q}, \ldots, A^{l_{q-1}}_{\sigma_1\sigma_2...\sigma_{q-1}i_q}$ are traceless components of the coefficients $B^{l_1}_{\sigma_1i_3...i_q}, B^{l_2}_{\sigma_1\sigma_2i_3...i_q}, \ldots, B^{l_{q-1}}_{\sigma_1\sigma_2...\sigma_{q-1}i_q}$.

**Proof** Applying the trace decomposition theorem (Appendix 9) to the coefficients $B^{l_1}_{\sigma_1i_3...i_q}, B^{l_2}_{\sigma_1\sigma_2i_3...i_q}, \ldots, B^{l_{q-1}}_{\sigma_1\sigma_2...\sigma_{q-1}i_q}$ in (45), we get

$$
B^{l_1}_{\sigma_1i_3...i_q} = A^{l_1}_{\sigma_1i_3...i_q} + C^{l_1}_{\sigma_1i_1i_3...i_q},
$$

$$
B^{l_2}_{\sigma_1\sigma_2i_3...i_q} = A^{l_2}_{\sigma_1\sigma_2i_3...i_q} + C^{l_2}_{\sigma_1\sigma_2i_3i_4...i_q},
$$

$$
\vdots
$$

$$
B^{l_{q-1}}_{\sigma_1\sigma_2...\sigma_{q-1}i_q} = A^{l_{q-1}}_{\sigma_1\sigma_2...\sigma_{q-1}i_q} + C^{l_{q-1}}_{\sigma_1\sigma_2...\sigma_{q-1}i_q},
$$

where the systems $A^{l_1}_{\sigma_1i_3...i_q}, A^{l_2}_{\sigma_1\sigma_2i_3...i_q}, \ldots, A^{l_{q-1}}_{\sigma_1\sigma_2...\sigma_{q-1}i_q}$ are traceless and $C^{l_1}_{\sigma_1i_3...i_q}, C^{l_2}_{\sigma_1\sigma_2i_3...i_q}, \ldots, C^{l_{q-1}}_{\sigma_1\sigma_2...\sigma_{q-1}i_q}$ are of Kronecker type. Thus, writing the multi-index $I_l$ as $I_l = J_l^{ij}$, we have

$$
C^{l_1}_{\sigma_1i_3...i_q} = d^{l_1}_{ij} D^{l_1}_{ij13i4...i_q} \text{ Alt}(i_1i_2i_3i_4...) \text{ Sym}(J_1j_1),
$$

$$
C^{l_2}_{\sigma_1\sigma_2i_3i_4...i_q} = d^{l_2}_{ij} D^{l_2}_{ij} D^{l_1}_{ij13i4...i_q} \text{ Alt}(i_1i_2i_3i_4...) \text{ Sym}(J_1j_1) \text{ Sym}(J_2j_2),
$$

$$
\vdots
$$

$$
C^{l_{q-1}}_{\sigma_1\sigma_2...\sigma_{q-2}i_{q-1}i_q} = d^{l_{q-1}}_{ij} D^{l_{q-1}}_{ij} D^{l_{q-1}}_{ij13i4...i_q} \text{ Alt}(i_1...i_{q-1}) \text{ Sym}(J_1j_1),
$$

$$
C^{l_q}_{\sigma_1\sigma_2...\sigma_{q-1}i_q} = d^{l_q}_{ij} D^{l_q}_{ij} D^{l_{q-1}}_{ij13i4...i_q} \text{ Alt}(i_1...i_{q-1}) \text{ Sym}(J_1j_1) \text{ Sym}(J_2j_2),
$$

$$
\vdots \text{ Sym}(J_{q-2}j_{q-2}),
$$

$$
\text{ Sym}(J_{q-2}j_{q-2}).
$$
Then

\[
\mu = A_{i_1i_2\ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q} \\
+ A_{i_1i_2\ldots i_q}^1 dy_{i_1}^{\sigma_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q} \\
+ A_{i_1i_2\ldots i_q}^{01} dy_{i_1}^{\sigma_1} \wedge dy_{i_2}^{\sigma_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\
+ \cdots + A^1_{\sigma_1\sigma_2\ldots \sigma_{q-1}i_q} dy_{i_1}^{\sigma_1} \wedge dy_{i_2}^{\sigma_2} \wedge \cdots \wedge dy_{i_{q-1}}^{\sigma_{q-1}} \wedge dx^{i_q} \\
+ A^{01}_{\sigma_1\sigma_2\ldots \sigma_{q-1}i_q} dy_{i_1}^{\sigma_1} \wedge \cdots \wedge dy_{i_{q-1}}^{\sigma_{q-1}} \wedge dx^{i_q} \\
+ \cdots + A^1_{\sigma_2\sigma_3\ldots \sigma_{q-1}i_q} dy_{i_1}^{\sigma_2} \wedge \cdots \wedge dy_{i_{q-1}}^{\sigma_{q-1}} \wedge dx^{i_q} \\
+ \cdots + A^{01}_{\sigma_2\sigma_3\ldots \sigma_{q-1}i_q} dy_{i_1}^{\sigma_2} \wedge \cdots \wedge dy_{i_{q-1}}^{\sigma_{q-1}} \wedge dx^{i_q},
\]

(51)

and now our assertion follows from the formula (44).

The following theorem generalizes Lemma 2 to arbitrary forms on open sets in the \( r \)-jet prolongation \( J^rY \).

**Theorem 3** (The trace decomposition theorem) Let \( q \) be any positive integer, and let \( \rho \in \Omega_q^r W \) be a \( q \)-form. Let \( (V, \phi), \phi = (x^i, y^\sigma) \), be a fibered chart on \( Y \), such that \( V \subset W \). Then, \( \rho \) has an expression

\[
\rho = \rho_0 + \rho',
\]

with the following properties:

(a) \( \rho_0 \) is generated by the 1-forms \( \omega_J \) with \( 0 \leq |J| \leq r - 1 \) and 2-forms \( d\omega_J \) where \( |J| = r - 1 \).

(b) \( \rho' \) has an expression

\[
\rho' = A_{i_1i_2\ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q} \\
+ A_{i_1i_2\ldots i_q}^1 dy_{i_1}^{\sigma_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q} \\
+ A_{i_1i_2\ldots i_q}^{01} dy_{i_1}^{\sigma_1} \wedge dy_{i_2}^{\sigma_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q} \\
+ \cdots + A^1_{\sigma_1\sigma_2\ldots \sigma_{q-1}i_q} dy_{i_1}^{\sigma_1} \wedge dy_{i_2}^{\sigma_2} \wedge \cdots \wedge dy_{i_{q-1}}^{\sigma_{q-1}} \wedge dx^{i_q} \\
+ A^{01}_{\sigma_1\sigma_2\ldots \sigma_{q-1}i_q} dy_{i_1}^{\sigma_1} \wedge \cdots \wedge dy_{i_{q-1}}^{\sigma_{q-1}} \wedge dx^{i_q} \\
+ \cdots + A^1_{\sigma_2\sigma_3\ldots \sigma_{q-1}i_q} dy_{i_1}^{\sigma_2} \wedge \cdots \wedge dy_{i_{q-1}}^{\sigma_{q-1}} \wedge dx^{i_q} \\
+ \cdots + A^{01}_{\sigma_2\sigma_3\ldots \sigma_{q-1}i_q} dy_{i_1}^{\sigma_2} \wedge \cdots \wedge dy_{i_{q-1}}^{\sigma_{q-1}} \wedge dx^{i_q},
\]

where \( |I_1|, |I_2|, \ldots, |I_{q-1}| = r \) and all coefficients \( A_{\sigma_1\sigma_2\ldots \sigma_{q-1}i_q}^1, A_{\sigma_1\sigma_2\ldots \sigma_{q-1}i_q}^{01}, \ldots, A_{\sigma_1\sigma_2\ldots \sigma_{q-1}i_q}^{1}, A_{\sigma_1\sigma_2\ldots \sigma_{q-1}i_q}^{01} \) are traceless.
Proof To prove Theorem 3, we express $\rho = \rho_1 + \mu$, where $\rho_1$ is generated by contact 1-forms $\omega_j$, $0 \leq |J| \leq r - 1$, and $\mu$ does not contain any factor $\omega_j$. Thus, $\mu$ has an expression (45) and can be decomposed as in Lemma 2, (46). Using this decomposition, we get the formula (52).

Theorem 3 is the trace decomposition theorem for differential forms; formula (52) is referred to as the trace decomposition formula. The form $\rho_0$ in this decomposition (43) is contact and is called the contact component of $\rho$; the form $\rho'$ is the traceless component of $\rho$ with respect to the fibered chart $(V, \psi)$.

Lemma 3 Let $\rho \in \Omega^q W$ be a $q$-form, and let $(V, \psi) = (x^i, y^\sigma)$, and $(\overline{V}, \overline{\psi})$, be two fibered charts such that $V \cap \overline{V} \neq \emptyset$. Suppose that we have the trace decomposition of the form $\rho$ with respect to $(V, \psi)$ and $(\overline{V}, \overline{\psi})$, respectively,

$$\rho = \rho_0 + \rho' = \overline{\rho_0} + \overline{\rho'}.$$  

Then, the traceless components satisfy

$$\rho' = \overline{\rho'} + \overline{\eta},$$  

where $\overline{\eta}$ is a contact form on the intersection $V \cap \overline{V}$.

Proof Lemma 3 can be easily verified by a direct calculation. Consider for instance the term $A^i_{\alpha_1 \alpha_2 \ldots \alpha_q} d\gamma_{i_1 i_2} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k}$ in formula (53), and the transformation equation is

$$\frac{\partial y_{i_1 i_2 \ldots i_k}}{\partial y^\sigma_{j_1 j_2 \ldots j_k}} = \frac{\partial y^\sigma}{\partial x^{i_1}} \frac{\partial x^{i_1}}{\partial x^{j_1}} \frac{\partial x^{j_1}}{\partial x^{j_2}} \cdots \frac{\partial x^{j_k}}{\partial x^{j_k}} \mathrm{Sym}(j_1 j_2 \ldots j_r).$$  

Denote $\overline{\alpha}_{\alpha_1 \alpha_2 \ldots \alpha_q} = d\gamma^y_{j_1 j_2 \ldots j_k} - \overline{\gamma}^y_{j_1 j_2 \ldots j_k} d\overline{x}$. Then, we have

$$A^i_{\alpha_1 \alpha_2 \ldots \alpha_q} d\gamma_{i_1 i_2} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k} = A^i_{\sigma} d\gamma^\sigma_{i_1 i_2} \wedge \frac{\partial x^{j_1}}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial x^{j_k}}{\partial x^{i_k}} \cdot \left( \frac{\partial y_{i_1 i_2 \ldots i_k}}{\partial x^{j_1}} + \sum_{0 \leq k \leq r-1} \frac{\partial y_{i_1 i_2 \ldots i_k}}{\partial x^{j_1}} y_{j_1 j_2 \ldots j_k} \right) d\overline{x}^\sigma$$

$$+ \sum_{0 \leq k \leq r-1} \frac{\partial y_{i_1 i_2 \ldots i_k}}{\partial x^{j_1}} \overline{\alpha}_{j_1 j_2 \ldots j_k} + \frac{\partial y_{i_1 i_2 \ldots i_k}}{\partial x^{j_1}} d\overline{\gamma}^y_{j_1 j_2 \ldots j_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k}.$$  

Consequently, the last summand in (57) implies

$$\overline{\alpha}_{j_1 j_2 \ldots j_k} = A^i_{\sigma} d\gamma^\sigma_{i_1 i_2} \frac{\partial x^{j_1}}{\partial x^{i_1}} \frac{\partial x^{j_2}}{\partial x^{i_2}} \cdots \frac{\partial x^{j_k}}{\partial x^{i_k}} \frac{\partial y_{i_1 i_2 \ldots i_k}}{\partial x^{j_1}}.$$  

Proof
Substituting from (56) in this formula, we see that the trace of $A_{i_1 i_2 \ldots i_r}$ vanishes if and only if the same is true for the trace of $A_{j_1 j_2 \ldots j_r}$. Thus, the decomposition (55) is valid for the summand (56). The same applies to any other summand.

Following Theorem 3, we can write the q-form $\rho$ in the contact basis as

$$\rho = \rho_1 + \rho_2 + \rho'$$

where $\rho_1$ is generated by the forms $\omega^q_f$, $|J| \leq r - 1$, $\rho_2$ is generated by $d\omega^q_f$, $|I| = r - 1$, and does not contain any factor $\omega^q_f$, and the form $\rho'$ is traceless. Thus,

$$\rho_1 = \sum_{0 \leq |J| \leq r-1} \omega^q_f \wedge \Phi^J_f, \quad \rho_2 = \sum_{|I|=r-1} d\omega^q_f \wedge \Psi^I_f$$

(59)

for some forms $\Phi^J_f$ and $\Psi^I_f$. Then,

$$\rho = \omega^q_f \wedge \Phi^J_f + \omega^q_I \wedge d\Psi^I_f + d(\omega^q_f \wedge \Psi^I_f) + \rho'.$$

(60)

Setting

$$P\rho = \omega^q_f \wedge \Phi^J_f + \omega^q_I \wedge d\Psi^I_f, \quad Q\rho = \omega^q_I \wedge \Psi^I_f, \quad R\rho = \rho',$$

(61)

we get the following version of Theorem 3.

**Theorem 4** Let $q$ be arbitrary, and let $\rho \in \Omega^q_Y$ be a q-form. Let $(V, \psi), \psi = (x^i, y^\sigma)$, be a fibered chart on $Y$ such that $V \subset W$. Then, $\rho$ can be expressed on $V^r$ as

$$\rho = P\rho + dQ\rho + R\rho.$$  

(62)

**Proof** This is an immediate consequence of definitions and Theorem 3.

In the following two examples, we discuss the trace decomposition formula and the transformation equations for the traceless components of some differential forms on 1-jet prolongation of the fibered manifold $Y$. The aim is to illustrate the decomposition methods for lower-degree differential forms.

**Example 1** We find the trace decomposition of a 3-form $\mu$, written in a fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, as

$$\mu = A_{ijk} dx^i \wedge dx^j \wedge dx^k + B^p_{\sigma jk} dy^\sigma_p \wedge dx^j \wedge dx^k$$

$$+ B^p_{\sigma vk} dy^\sigma_p \wedge dy^\sigma_q \wedge dx^k + A^{pq}_{\sigma rv} dy^\sigma_p \wedge dy^\sigma_r \wedge dy^\sigma_v.$$  

(63)
Decomposing $B^p_{jk}$, we have $B^p_{jk} = A^p_{ijk} + \delta^p_j C_{ak} + \delta^p_k D_{aj}$, where $A^p_{ijk}$ is traceless. Then, the condition $B^p_{jk} = -B^p_{kj}$ yields

$$B^p_{jk} = \delta^p_j C_{ak} + \delta^p_k D_{aj} = nC_{ok} + D_{ok}$$

and hence, $C_{ok} = -D_{ok}$. Thus,

$$B^p_{jk} = A^p_{ijk} + \delta^p_j C_{ak} - \delta^p_k C_{aj}. \quad (65)$$

Decomposing $B^q_{vk}$, we have $B^q_{vk} = A^q_{vsk} + \delta^q_v C^l_{av} + \delta^q_l D^p_{av}$. Now, the condition $B^q_{vk} = -B^q_{vk}$ yields

$$B^q_{vk} = \delta^q_v C^l_{av} + \delta^q_l D^p_{av} = nC^q_{av} + D^q_{av}$$

and hence, $C^q_{av} = -D^q_{av}$. It can be easily verified that this condition implies

$$nC^q_{av} + C^q_{va} = -nD^q_{va} - D^q_{av}. \quad (67)$$

Indeed, symmetrization and alternation yield

$$nC^q_{av} + C^q_{va} + nC^q_{va} + C^q_{av} = -nD^q_{va} - D^q_{av} - nD^q_{av} - D^q_{va} \quad (68)$$

and

$$nC^q_{av} + C^q_{va} - nC^q_{va} - C^q_{av} = -nD^q_{va} - D^q_{av} + nD^q_{av} + D^q_{va} \quad (69)$$

hence, $C^q_{av} + C^q_{va} = -D^q_{va} - D^q_{av}$ and $C^q_{av} - C^q_{va} = -D^q_{va} + D^q_{av}$. These equations already imply (47). Thus,

$$B^q_{vk} = A^q_{vsk} + \delta^q_v C^l_{av} - \delta^q_l C^p_{av}. \quad (70)$$

Summarizing (65) and (70), we get

$$\mu = A_{ijk} dx^i \wedge dx^j \wedge dx^k + A^p_{ijk} dy^p_\sigma \wedge dx^j \wedge dx^k + A^p_{vsk} dy^p_\sigma \wedge dy^q_\tau \wedge dx^k$$

$$+ \delta^p_j C_{ak} dy^p_\sigma \wedge dx^j \wedge dx^k - \delta^p_k C_{aj} dy^p_\sigma \wedge dx^j \wedge dx^k$$

$$+ \delta^q_v C^l_{av} dy^p_\sigma \wedge dy^q_\tau \wedge dx^k - \delta^q_l C^p_{av} dy^p_\sigma \wedge dy^q_\tau \wedge dx^k$$

$$+ A^p_{vsk} dy^p_\sigma \wedge dy^q_\tau \wedge dy^r_\sigma.$$
\[
A_{ijk} dx^i \wedge dx^j \wedge dx^k + A^p_{ijk} dy^p \wedge dx^j \wedge dx^k \\
+ A^p_{qij} dy^p \wedge dy^q \wedge dx^k + A^p_{qir} dy^p \wedge dy^r \wedge dy^i \\
+ C_{apk} dy^p \wedge dx^k \wedge dx^k - C_{apk} dy^p \wedge dx^j \wedge dx^k \\
+ C^q_{ap} dy^q \wedge dx^q \wedge dx^q - C^q_{apr} dy^q \wedge dy^q \wedge dx^q \\
+ C_{apq} d\omega^q \wedge dx^q \wedge dx^q - 2C_{apq} d\omega^q \wedge dx^j \wedge dx^q
\]

(71)

Thus, applying formula (51) to any 3-form \( \rho \) on \( V^1 \), we get the decomposition

\[
\rho = \rho_1 + \rho_2 + \rho',
\]

(72)

where \( \rho_1 \) is generated by \( \omega^\sigma \), that is, \( \rho_1 = \omega^\sigma \wedge \Phi_\sigma \), \( \rho_2 \) is generated by the contact 2-forms \( d\omega^\sigma \), \( \rho_2 = d\omega^\sigma \wedge \Psi_\sigma \), where the 1-forms \( \Psi_\sigma \) do not contain any factor \( \omega^y \), and \( \rho' \) is traceless.

**Example 2** (Transformation properties) Consider a 2-form on the 1-jet prolongation \( J^1 Y \), expressed in two fibered charts \( (V, \psi) \), \( \psi = (x^i, y^\sigma) \), and \( (V, \bar{\psi}) \), \( \bar{\psi} = (\bar{x}^i, \bar{y}^\sigma) \), as

\[
\rho = \rho_1 + \rho_2 + \rho' = \bar{\rho}_1 + \bar{\rho}_2 + \bar{\rho}',
\]

(73)

where according to Theorem 3,

\[
\rho_1 = \omega^\sigma \wedge P_\sigma, \quad \rho_2 = Q_\sigma d\omega^\sigma, \\
\rho' = A_{ij} dx^i \wedge dx^j + A^i_{ij} dy^i \wedge dx^j + A^i_{ij} dy^i \wedge dy^j,
\]

(74)

and

\[
\bar{\rho}_1 = \bar{\omega}^\sigma \wedge \bar{P}_\sigma, \quad \bar{\rho}_2 = \bar{Q}_\sigma d\bar{\omega}^\sigma, \\
\bar{\rho}' = \bar{A}_{ij} d\bar{x}^i \wedge d\bar{x}^j + \bar{A}^i_{ij} \bar{d}y^i \wedge d\bar{x}^j + \bar{A}^i_{ij} \bar{d}y^i \wedge \bar{d}y^j.
\]

(75)

We want to determine transformation formulas for the traceless components \( A^i_{ij}, \bar{A}^i_{ij}, \) and \( A_{ij} \). Transformation equations are of the form

\[
\bar{x}^i = \bar{x}^i(x^j), \quad \bar{y}^\sigma = \bar{y}^\sigma(x^j, y^v), \quad \bar{y}^i = \left( \frac{\partial \bar{y}^\sigma}{\partial x^j} + \frac{\partial \bar{y}^\sigma}{\partial y^v} y^v_i \right) \frac{\partial x^j}{\partial \bar{x}^i},
\]

(76)
and imply
\[
dy^i_j = \left( \frac{\partial y^i_j}{\partial \theta^p} + \frac{\partial y^i_j}{\partial y^k} y^k_p \right) dx^p + \frac{\partial y^i_j}{\partial y^k} \omega^k + \frac{\partial y^i_j}{\partial y^k} \frac{\partial x^k}{\partial x^l} dy^i_j. \tag{77}
\]

Then, a direct calculation yields
\[
\begin{align*}
\overline{A}^i_{vi} dy^j_v \wedge dy^i_v &= \overline{A}^i_{vi} \left( \frac{\partial y^i_j}{\partial \theta^p} + \frac{\partial y^i_j}{\partial y^k} y^k_p \right) \left( \frac{\partial y^i_j}{\partial \theta^q} + \frac{\partial y^i_j}{\partial y^k} y^k_q \right) dx^p \wedge dx^q \\
&+ \overline{A}^i_{vi} \left( \frac{\partial y^i_j}{\partial \theta^p} + \frac{\partial y^i_j}{\partial y^k} y^k_p \right) \frac{\partial y^i_j}{\partial y^k} dx^p \wedge \omega^q \\
&+ \overline{A}^i_{vi} \left( \frac{\partial y^i_j}{\partial \theta^p} + \frac{\partial y^i_j}{\partial y^k} y^k_p \right) \omega^k \wedge dx^q \\
&+ \overline{A}^i_{vi} \left( \frac{\partial y^i_j}{\partial \theta^p} + \frac{\partial y^i_j}{\partial y^k} y^k_p \right) \frac{\partial y^i_j}{\partial y^k} \frac{\partial x^k}{\partial x^l} dx^p \wedge dy^j_v \\
&+ \overline{A}^i_{vi} \frac{\partial y^i_j}{\partial \theta^p} \frac{\partial y^i_j}{\partial \theta^q} \frac{\partial x^k}{\partial \theta^l} \frac{\partial x^l}{\partial \theta^k} \omega^k \wedge \omega^l \\
&+ \overline{A}^i_{vi} \frac{\partial y^i_j}{\partial \theta^p} \frac{\partial y^i_j}{\partial \theta^q} \frac{\partial x^k}{\partial \theta^l} \frac{\partial x^l}{\partial \theta^k} dy^k_v \wedge \omega^l \\
&+ \overline{A}^i_{vi} \frac{\partial y^i_j}{\partial \theta^p} \frac{\partial y^i_j}{\partial \theta^q} \frac{\partial x^k}{\partial \theta^l} \frac{\partial x^l}{\partial \theta^k} dy^k_v \wedge dy^j_v. \tag{78}
\end{align*}
\]

Similarly,
\[
\begin{align*}
\overline{A}^i_{vj} dy^j_v \wedge dx^i &= \overline{A}^i_{vj} \frac{\partial x^i}{\partial \theta^p} \left( \frac{\partial y^i_j}{\partial \theta^p} + \frac{\partial y^i_j}{\partial y^k} y^k_p \right) dx^p \wedge dx^l \\
&+ \overline{A}^i_{vj} \frac{\partial x^i}{\partial \theta^p} \frac{\partial y^i_j}{\partial \theta^p} \omega^k \wedge dx^l + \overline{A}^i_{vj} \frac{\partial x^i}{\partial \theta^p} \frac{\partial y^i_j}{\partial \theta^p} \frac{\partial x^k}{\partial x^l} dy^k_v \wedge dx^l, \tag{79}
\end{align*}
\]
and
\[
\overline{A}^i_{ji} dx^i \wedge dx^j = \overline{A}^i_{ji} \frac{\partial x^i}{\partial \theta^p} \frac{\partial x^j}{\partial \theta^q} dx^p \wedge dx^q. \tag{80}
\]
To determine the traceless components $A_{ij}^t$, $A_{ij}^s$, and $A_{ij}$ from the formulas (78)–(80), respectively, we need the terms not containing $\omega^s$; we get

\[
\begin{align*}
&\mathcal{A}_{ij}^t \left( \frac{\partial \gamma_i^p}{\partial x^p} + \frac{\partial \gamma_i^p}{\partial y^k x_p^k} \right) \left( \frac{\partial \gamma_j^q}{\partial x^q} + \frac{\partial \gamma_j^q}{\partial y^k x_q^k} \right) dx^p \wedge dx^q \\
&\quad + \mathcal{A}_{ij}^t \left( \frac{\partial \gamma_i^p}{\partial x^p} + \frac{\partial \gamma_i^p}{\partial y^k x_p^k} \right) \frac{\partial \gamma_j^q}{\partial y^k} \partial x_j^k dx^p \wedge dy^q_j \\
&\quad + \mathcal{A}_{ij}^s \left( \frac{\partial \gamma_i^p}{\partial y^k} \frac{\partial \gamma_j^q}{\partial x^p} \right) dx^p \wedge dx^q \\
&\quad + \mathcal{A}_{ij}^s \left( \frac{\partial \gamma_i^p}{\partial y^k} \frac{\partial \gamma_j^q}{\partial x^p} \right) dy^k_s \wedge dx^l \\
&\quad + \mathcal{A}_{ij}^s \left( \frac{\partial \gamma_i^p}{\partial x^p} \frac{\partial \gamma_j^q}{\partial x^p} \right) dx^p \wedge dx^l. \\
\end{align*}
\]

(81)

Now, it is immediate that

\[
\begin{align*}
A_{pq} &= \mathcal{A}_{ij}^t \left( \frac{\partial \gamma_i^p}{\partial x^p} + \frac{\partial \gamma_i^p}{\partial y^k x_p^k} \right) \left( \frac{\partial \gamma_j^q}{\partial x^q} + \frac{\partial \gamma_j^q}{\partial y^k x_q^k} \right) \\
&\quad + \frac{1}{2} \mathcal{A}_{ij}^t \left( \frac{\partial \gamma_i^p}{\partial x^p} + \frac{\partial \gamma_i^p}{\partial y^k x_p^k} \right) \left( \frac{\partial \gamma_j^q}{\partial x^q} + \frac{\partial \gamma_j^q}{\partial y^k x_q^k} \right) \\
&\quad + \mathcal{A}_{ij}^t \frac{\partial \gamma_i^p}{\partial x^p} \frac{\partial \gamma_j^q}{\partial x^p} dx^p \wedge dx^q. \\
\end{align*}
\]

(82)

and

\[
\begin{align*}
A_{ij}^{k/l} &= \frac{1}{2} \mathcal{A}_{ij}^t \left( \frac{\partial \gamma_i^p}{\partial y^k} \frac{\partial \gamma_j^q}{\partial x^p} \partial x^k \partial x^l + \frac{\partial \gamma_j^q}{\partial y^k} \frac{\partial \gamma_i^p}{\partial x^p} \partial x^k \partial x^l - \frac{\partial \gamma_i^p}{\partial y^k} \frac{\partial \gamma_j^q}{\partial x^p} \partial x^k \partial x^l \right). \\
\end{align*}
\]

(83)

The remaining terms should determine $A_{kj}^s$ as the traceless component of the expression

\[
\begin{align*}
- \mathcal{A}_{ij}^t \left( \frac{\partial \gamma_i^p}{\partial x^p} + \frac{\partial \gamma_i^p}{\partial y^k x_p^k} \right) \frac{\partial \gamma_i^q}{\partial y^k} \partial x_i^k + \mathcal{A}_{ij}^t \frac{\partial \gamma_j^q}{\partial x^p} \partial x_i^p + \mathcal{A}_{ij}^s \left( \frac{\partial \gamma_i^q}{\partial x^p} + \frac{\partial \gamma_i^q}{\partial y^k x_q^k} \right) \\
+ \mathcal{A}_{ij}^t \frac{\partial \gamma_j^q}{\partial x^p} \partial x_i^p. \\
\end{align*}
\]

(84)
Recall that the traceless component $W^i_j$ of a general system $P^i_k$, indexed with one contravariant and one covariant index, is defined by

$$W^i_q = P^i_q - \frac{1}{n} \delta^i_q P,$$  \hspace{1cm} (85)

where $P = P^i_j$ is the trace of $P^i_k$. To apply this definition, we first calculate the trace of (84) in $s$ and $q$. We get

$$-A^i_{\nu r} \left( \frac{\partial y^i}{\partial x^s} + \frac{\partial y^i}{\partial y^s} y^s \right) \frac{\partial y^s}{\partial x^r} + A^i_{\nu r} \frac{\partial y^i}{\partial x^s} \frac{\partial x^s}{\partial x^r} + A^i_{\nu r} \frac{\partial y^i}{\partial x^s} \frac{\partial x^s}{\partial x^r} \left( \frac{\partial y^i}{\partial x^s} + \frac{\partial y^i}{\partial y^s} y^s \right)$$

$$+ A^i_{\nu r} \frac{\partial y^i}{\partial x^s} \frac{\partial y^i}{\partial x^s} \frac{\partial x^s}{\partial x^r},$$  \hspace{1cm} (86)

Now, we can determine the traceless component of (84). Since the resulting expression must be equal to $A^i_{\kappa q}$, we get the transformation formula

$$A^i_{\kappa q} = A^i_{\nu r} \frac{\partial y^i}{\partial x^s} \frac{\partial y^i}{\partial x^s} \frac{\partial x^s}{\partial x^r} - A^i_{\nu r} \left( \frac{\partial y^i}{\partial x^s} + \frac{\partial y^i}{\partial y^s} y^s \right) \frac{\partial y^s}{\partial x^r} + A^i_{\nu r} \frac{\partial y^i}{\partial x^s} \frac{\partial x^s}{\partial x^r} \left( \frac{\partial y^i}{\partial x^s} + \frac{\partial y^i}{\partial y^s} y^s \right)$$

$$+ A^i_{\nu r} \frac{\partial y^i}{\partial x^s} \frac{\partial y^i}{\partial x^s} \frac{\partial x^s}{\partial x^r} \left( \frac{\partial y^i}{\partial x^s} + \frac{\partial y^i}{\partial y^s} y^s \right)$$

$$+ \frac{1}{n} \delta^i_q A^i_{\nu r} \left( \frac{\partial y^i}{\partial x^s} + \frac{\partial y^i}{\partial y^s} y^s \right) \frac{\partial y^s}{\partial x^r} - \frac{\partial y^i}{\partial x^s} \frac{\partial x^s}{\partial x^r} \left( \frac{\partial y^i}{\partial x^s} + \frac{\partial y^i}{\partial y^s} y^s \right)$$

as desired. It is straightforward to verify that the expression on the right-hand side is traceless. This completes Example 2.

### 2.3 The Horizontalization

We extend the horizontalization $\Omega^i_W \ni \rho \rightarrow h \rho \in \Omega^{r+1}_W$, introduced in Sect. 2.1, to a morphism $h: \Omega^i_W \rightarrow \Omega^{r+1}_W$ of exterior algebras.

Let $\rho \in \Omega^i_W$ be a $q$-form, where $q \geq 1$, $J^{r+1}_x \in W^{r+1}$ a point. Consider the pullback $(\pi^{r+1, r})^* \rho$ and the value $(\pi^{r+1, r})^* \rho(J^{r+1}_x)(\xi_1, \xi_2, \ldots, \xi_q)$ on any tangent vectors $\xi_1, \xi_2, \ldots, \xi_q$ of $J^{r+1}_x$ at the point $J^{r+1}_x$. Decompose each of these vectors into the horizontal and contact components,

$$T^{\pi^{r+1}, \xi}_l = h \xi_l + p \xi_l,$$  \hspace{1cm} (88)
and set
\[ h_{\nu}(J_{\chi}^{r+1})_{\nu}(\xi_1, \xi_2, \ldots, \xi_q) = \rho(J_{\chi}^{r+1})(h\xi_1, h\xi_2, \ldots, h\xi_q). \] (89)

This formula defines a \( q \)-form \( h\nu \in \Omega_{\nu}^{r+1}W \). This definition can be extended to 0-forms (functions); we set for any function \( f: W^r \to \mathbb{R} \)
\[ hf = (\pi^{r+1,J})^*f. \] (90)

It follows from the properties of the decomposition (88) that the value \( h_{\nu}(J_{\chi}^{r+1})_{\nu}(\xi_1, \xi_2, \ldots, \xi_q) \) vanishes whenever at least one of the vectors \( \xi_1, \xi_2, \ldots, \xi_q \) is \( \pi^{r+1} \)-vertical (cf. Sect. 1.5). Thus, the \( q \)-form \( h\nu \) is \( \pi^{r+1} \)-horizontal. In particular, \( h\nu = 0 \) whenever \( q \geq n + 1 \). Sometimes \( h\nu \) is called the horizontal component of \( \rho \).

Formulas (89) and (90) define a mapping \( h: \Omega^rW \to \Omega_{\nu}^{r+1}W \) of exterior algebras, called the horizontalization. The mapping \( h \) satisfies
\[ h(\rho_1 + \rho_1) = h\rho_1 + h\rho_1, \quad h(f \rho) = (\pi^{r+1,J})^*f \cdot h\rho \] (91)
for all \( q \)-forms \( \rho_1, \rho_1, \) and \( \rho \) and all functions \( f \). In particular, restricting these formulas to constant functions \( f \), we see that the horizontalization \( h \) is linear over the field of real numbers.

**Theorem 5** The mapping \( h: \Omega^rW \to \Omega_{\nu}^{r+1}W \) is a morphism of exterior algebras.

**Proof** This assertion is a straightforward consequence of the definition of exterior product and formula (89) for the horizontal component of a form \( \rho \). Indeed,
\[
\begin{align*}
\h(\rho \wedge \eta)(J_{\chi}^{r+1})_{\nu}(\xi_1, \xi_2, \ldots, \xi_q, \xi_{p+1}, \xi_{p+2}, \ldots, \xi_{p+q}) \\
= (\rho \wedge \eta)(J_{\chi}^{r+1})(h\xi_1, h\xi_2, \ldots, h\xi_{p+1}, h\xi_{p+2}, \ldots, h\xi_{p+q}) \\
= \sum_{\tau} \text{sgn}(\rho)(J_{\chi}^{r+1})(h\xi_{\tau(1)}, h\xi_{\tau(2)}, \ldots, h\xi_{\tau(p)}) \\
\cdot \eta(J_{\chi}^{r+1})(h\eta(\xi_{\tau(p+1)}), h\eta(\xi_{\tau(p+2)}), \ldots, h\eta(\xi_{\tau(p+q)})) \\
= \sum_{\tau} \text{sgn}(\rho)h\eta(J_{\chi}^{r+1})(\xi_{\tau(1)}, \xi_{\tau(2)}, \ldots, \xi_{\tau(p)}) \\
\cdot h\eta(J_{\chi}^{r+1})(\xi_{\tau(p+1)}, \xi_{\tau(p+2)}, \ldots, \xi_{\tau(p+q)})) \\
= (h\rho(J_{\chi}^{r+1}) \wedge h\eta(J_{\chi}^{r+1}))(\xi_1, \xi_2, \ldots, \xi_q, \xi_{p+1}, \xi_{p+2}, \ldots, \xi_{p+q})
\end{align*}
\]
(summation through all permutations $\tau$ of the set $\{1, 2, \ldots, p, p+1, \ldots, p+q\}$ such that $\tau(1) < \tau(2) < \cdots < \tau(p)$ and $\tau(p+1) < \tau(p+2) < \cdots < \tau(p+q)$). This means, however, that

$$h(\rho \land \eta) = \rho \land h\eta. \quad (93)$$

The following theorem shows that the horizontalization is completely determined by its action on functions and their exterior derivatives.

**Theorem 6** Let $W$ be an open set in the fibered manifold $Y$. Then, the horizontalization $\Omega^r W \ni \rho \rightarrow h\rho \in \Omega^{r+1} W$ is a unique $\mathbb{R}$-linear, exterior-product-preserving mapping such that for any function $f: W' \rightarrow \mathbb{R}$, and any fibered chart $(V, \psi), \psi = (y^\alpha)$, with $V \subset W$,

$$hf = f \circ \pi^{r+1}, \quad hdf = df \cdot dx^i, \quad (94)$$

where

$$df = \frac{\partial f}{\partial x^i} + \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} \frac{\partial f}{\partial y^\alpha_{j_1j_2\cdots j_k}} y^\alpha_{j_1j_2\cdots j_k} dx^i. \quad (95)$$

**Proof** The proof that $h$, defined by (89) and (90), has the desired properties (94) and (95), is standard. To prove uniqueness, note that (94) and (95) imply

$$hdx^i = dx^i, \quad hd \eta_{j_1j_2\cdots j_k} = y^\alpha_{j_1j_2\cdots j_k} dx^i. \quad (96)$$

It remains to check that any two mappings $h_1$ and $h_2$ satisfying the assumptions of Theorem 6 that agree on functions and their exterior derivatives coincide. \qed

We determine the kernel and the image of the horizontalization $h$. The following are elementary consequences of the definition.

**Lemma 4**

(a) A function $f$ satisfies $hf = 0$ if and only if $f = 0$.
(b) If $q \geq n + 1$, then every $q$-form $\rho \in \Omega^r_q W$ satisfies $h\rho = 0$.
(c) Let $1 \leq q \leq n$, and let $\rho \in \Omega^r_q W$ be a form. Then, $h\rho = 0$ if and only if

$$J^r \gamma^* \rho = 0 \quad (97)$$

for every $C^r$ section $\gamma$ of $Y$ defined on an open subset of $W$.
(d) If $h\rho = 0$, then also the exterior derivative $hd\rho = 0$. 
Proof

(a) This is a mere restatement of the definition.
(b) This is an immediate consequence of the definition.
(c) Choose a section \( \gamma \) of \( Y \), a point \( x \) from the domain of definition of \( \gamma \) and any tangent vectors \( \xi_1, \xi_2, \ldots, \xi_q \) of \( X \) at \( x \). Then,

\[
J^r \gamma^* \rho(x)(\xi_1, \xi_2, \ldots, \xi_q) = \rho(J^r \gamma)(T_x J^r \gamma \cdot \xi_1, T_x J^r \gamma \cdot \xi_2, \ldots, T_x J^r \gamma \cdot \xi_q).
\] (98)

Since \( T_p r + 1 \) is surjective, there exist tangent vectors \( n_l \) to \( J^r + 1 Y \) at \( J^r + 1 x \), such that \( f_l = T_p r + 1 / C^1 q n_l \). For these tangent vectors,

\[
J^r \gamma^* \rho(x)(\xi_1, \xi_2, \ldots, \xi_q) = \rho(J^r \gamma_h)(h_\xi_1, h_\xi_2, \ldots, h_\xi_q) = h_\rho(J^r \gamma_h)(h_\xi_1, h_\xi_2, \ldots, h_\xi_q).
\] (99)

But \( h_\xi = T_p J^r \gamma \circ T_p r + 1 / C^1 n_\xi \), and hence,

\[
J^r \gamma^* \rho(x)(\xi_1, \xi_2, \ldots, \xi_q) = \rho(J^r \gamma_h)(h_\xi_1, h_\xi_2, \ldots, h_\xi_q).
\] (99)

This correspondence already proves assertion (a).

(d) This assertion (d) follows from (c).

Theorem 7 Let \( W \subset Y \) be an open set, \( \rho \in \Omega^q W \) a form, and let \((V, \psi), \psi = (x^i, y^\sigma)\), be a fibered chart such that \( V \subset W \).

(a) Let \( q = 1 \). Then, \( \rho \) satisfies \( h_\rho = 0 \) if and only if its chart expression is of the form

\[
\rho = \sum_{0 \leq |\sigma| \leq r-1} \Phi^r_\sigma \omega^\sigma_j
\] (101)

for some functions \( \Phi^r_\sigma : V^r \to \mathbb{R} \).

(b) Let \( 2 \leq q \leq n \). Then, \( \rho \) satisfies \( h_\rho = 0 \) if and only if its chart expression is of the form

\[
\rho = \sum_{0 \leq |\sigma| \leq r-1} \omega^\sigma_j \wedge \Phi^r_\sigma + \sum_{|\sigma| = r-1} d \omega^\sigma_j \wedge \Psi^r_\sigma,
\] (102)

where \( \Phi^r_\sigma \) (resp. \( \Psi^r_\sigma \)) are some \((q-1)\)-forms (resp. \((q-2)\)-forms) on \( V^r \).
Proof Suppose that we have a contact \( q \)-form \( \rho \) on \( W' \), where \( 1 \leq q \leq n \). Write as in Sect. 2.2, Theorem 3, \( \rho = \rho_0 + \rho' \), where \( \rho_0 \) is contact and \( \rho' \) is traceless. But the horizontalization \( h \) preserves exterior product and \( h\rho = 0 \), so we get \( h\rho' = 0 \) because \( \rho_0 \) is generated by the contact forms \( \omega_0^q, d\omega_0^q \), which satisfy \( h\omega_0^q = 0 \) and \( h d\omega_0^q = 0 \). Now, using formula \( hdy_j^q = y_j^q dx' \), we get, expressing \( \rho' \) as in Sect. 2.2, (53)

\[
h\rho' = (A_{l_1l_2...l_q} + A^H_{l_1} \sigma_{l_1l_2}...l_q y^\sigma_{l_1l_2} + A^H_{l_2} \sigma_{l_1l_2l_3}...l_q y^\sigma_{l_1l_2l_3} + ... + A^H_{l_2} \sigma_{l_1l_2l_3}...l_q y^\sigma_{l_1l_2l_3})dx_{l_1} \wedge dx_{l_2} \wedge ... \wedge dx_{l_q},
\]

where \( |I_1|, |I_2|, ..., |I_{q-1}| = r \) and the coefficients \( A^H_{l_1l_2}, A^H_{l_1l_2l_3}...l_q \) are traceless. Then,

\[
A_{l_1l_2...l_q} + h^H_{l_1} \sigma_{l_1l_2}...l_q y^\sigma_{l_1l_2} + h^H_{l_2} \sigma_{l_1l_2l_3}...l_q y^\sigma_{l_1l_2l_3} + ... + h^H_{l_2} \sigma_{l_1l_2l_3}...l_q y^\sigma_{l_1l_2l_3} = 0 \quad \text{Alt}(i_1l_2...i_q).
\]

But the expressions on the left-hand sides of these equations are polynomial in the variables \( y_k^q \) with \( |K| = r + 1 \), so the corresponding homogeneous components in (104) must vanish separately. Then, we have \( A_{i_1l_2...l_q} = 0, A^H_{l_1l_2} ... l_q = 0, \) and

\[
A^H_{l_1} \sigma_{l_1l_2...l_q} y^\sigma_{l_1} = 0 \quad \text{Alt}(i_1l_2...i_q) \quad \text{Sym}(I_1l_1),
\]

\[
A^H_{l_2} \sigma_{l_1l_2l_3}...l_q y^\sigma_{l_1l_2l_3} = 0 \quad \text{Alt}(i_1l_2...i_q) \quad \text{Sym}(I_1l_1) \quad \text{Sym}(I_2l_2),
\]

\[
A^H_{l_3} \sigma_{l_1l_2l_3}...l_q y^\sigma_{l_1l_2l_3} = 0 \quad \text{Alt}(i_1l_2...i_q) \quad \text{Sym}(I_1l_1) \quad \text{Sym}(I_2l_2) \quad \text{Sym}(I_3l_3),
\]

\[
A^H_{l_2} \sigma_{l_1l_2l_3}...l_q y^\sigma_{l_1l_2l_3} ... y^\sigma_{l_q} = 0 \quad \text{Alt}(i_1l_2...i_q) \quad \text{Sym}(I_1l_1) \quad \text{Sym}(I_2l_2) \quad \text{Sym}(I_3l_3) \quad \text{Sym}(I_ql_q-1).
\]

However, since the coefficients \( A^H_{l_1} \sigma_{l_1l_2...l_q} A^H_{l_2} \sigma_{l_1l_2l_3}...l_q \) are traceless, they must vanish identically (see Appendix 9, Theorem 4). Thus, we have in (103)

\[
A_{i_1l_2...l_q} = 0, \quad A^H_{l_1} \sigma_{l_1l_2...l_q} y^\sigma_{l_1} = 0, \quad A^H_{l_2} \sigma_{l_1l_2l_3}...l_q = 0,
\]

\[
A^H_{l_3} \sigma_{l_1l_2l_3}...l_q y^\sigma_{l_1l_2l_3} ... y^\sigma_{l_q} = 0 \quad \text{Alt}(i_1l_2...i_q) \quad \text{Sym}(I_1l_1) \quad \text{Sym}(I_2l_2) \quad \text{Sym}(I_3l_3) \quad \text{Sym}(I_ql_q-1).
\]

and hence, \( h\rho' = 0 \). Thus \( \rho = \rho_0 \), and to close the proof, we just write this result for \( q = 1 \) and \( q > 1 \) separately. \( \square \)
Corollary 1 If $0 \leq q \leq n$, then a $q$-form belongs to the kernel of the horizontalization $h$ if and only if it is a contact form.

Corollary 2 Let $W \subset Y$ be an open set, $\rho \in \Omega^r_q W$ a $q$-form such that $2 \leq q \leq n$, and let $(V, \psi)$, $\psi = (x^i, y^\sigma)$, be a fibered chart such that $V \subset W$. Then, the form $\rho$ satisfies the condition $h \rho = 0$ if and only if its chart expression is of the form

$$
\rho = \sum_{0 \leq |J| \leq r-1} \omega_J^g \wedge \Phi_J^g + \sum_{|J|=r-1} d(\omega_J^g \wedge \Psi_J^g),
$$

where $\Phi_J^g$ are $(q-1)$-forms and $\Psi_J^g$ are $(q-2)$-forms) on $V'$, which do not contain $\omega_J^g$, $0 \leq |J| \leq r-1$.

Proof We write (102) as

$$
\rho = \sum_{0 \leq |J| \leq r-1} \omega_J^g \wedge \Phi_J^g - \sum_{|J|=r-1} \omega_J^g \wedge d\Psi_J^g + \sum_{0 \leq |J| \leq r-1} d(\omega_J^g \wedge \Psi_J^g). \tag{108}
$$

The image of the horizontalization $h$ is characterized as follows.

Lemma 5 Let $\rho \in \Omega^r_q W$ be a form.

(a) If $q = 0$, then $h \rho = (\pi^{r+1,*})^* \rho$.

(b) If $1 \leq q \leq n$, then

$$
h \rho = h \rho'. \tag{109}
$$

(c) If $q \geq n + 1$, then $h \rho = h \rho' = 0$.

Proof This assertion is an immediate consequence of the definition of the horizontalization $h$. \qed

2.4 The Canonical Decomposition

Beside the horizontalization of $q$-forms $\Omega^r_q W$, introduced in Sects. 2.1 and 2.3, the vector bundle morphism $h: TJ^{r+1} Y \to T J^r Y$ also induces a decomposition of the modules of $q$-forms $\Omega^r_q W$. Let $\rho \in \Omega^r_q W$ be a $q$-form, where $q \geq 1$, $J^{r+1}_x \gamma \in W^{r+1}$ a point. Consider the pullback $(\pi^{r+1,*})^* \rho$ and the value $(\pi^{r+1,*})^* \rho(J^{r+1}_x \gamma)$ $(\xi_1, \xi_2, \ldots, \xi_q)$ on any tangent vectors $\xi_1, \xi_2, \ldots, \xi_q$ of $J^{r+1} Y$ at the point $J^{r+1}_x \gamma$. Write for each $l$,

$$
T \pi^{r+1} \cdot \xi_l = h \xi_l + p \xi_l, \tag{110}
$$
and substitute these vectors in the pullback $(\pi^{r+1,r})^*\rho$. We get

\[
(\pi^{r+1,r})^*\rho(J_x^r \gamma)(\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_q) = \rho(J_x^r \gamma)(h\tilde{\xi}_1 + p\tilde{\xi}_1, h\tilde{\xi}_2 + p\tilde{\xi}_2, \ldots, h\tilde{\xi}_q + p\tilde{\xi}_q).
\]

We study in this section, for each $k = 0, 1, 2, \ldots, q$, the summands on the right-hand side, homogeneous of degree $k$ in the contact components $p\tilde{\xi}_i$ of the vectors $\tilde{\xi}_i$, and describe the corresponding decomposition of the form $(\pi^{r+1,r})^*\rho$. Using properties of $\rho$, we set

\[
p_k \rho(J_x^r \gamma)(\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_q) = \sum_{\ell}^{j_1 < j_2 < \cdots < j_k} \rho(J_x^r \gamma)(p\tilde{\xi}_{j_1}, p\tilde{\xi}_{j_2}, \ldots, p\tilde{\xi}_{j_k}, h\tilde{\xi}_{j_{k+1}}, h\tilde{\xi}_{j_{k+2}}, \ldots, h\tilde{\xi}_{j_q}),
\]

where the summation is understood through all sequences $j_1 < j_2 < \cdots < j_k$ and $j_{k+1} < j_{k+2} < \cdots < j_q$. Equivalently, $p_k \rho(J_x^r \gamma)$ can also be defined by

\[
p_k \rho(J_x^r \gamma)(\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_q) = \frac{1}{k!(q-k)!} \rho(J_x^r \gamma)(p\tilde{\xi}_{j_1}, p\tilde{\xi}_{j_2}, \ldots, p\tilde{\xi}_{j_k}, h\tilde{\xi}_{j_{k+1}}, \ldots, h\tilde{\xi}_{j_q})
\]

(summation through all values of the indices $j_1, j_2, \ldots, j_k, j_{k+1}, \ldots, j_q$).

Note that if $k = 0$, then $p_0 \rho$ coincides with the horizontal component of $\rho$, defined in Sect. 2.1, (5),

\[
p_0 \rho = h \rho.
\]

We also introduce the notation

\[
p \rho = p_1 \rho + p_2 \rho + \cdots + p_q \rho.
\]

These definitions can be extended to 0-forms (functions). Since for a function $f : W' \to R$, $hf$ was defined to be $(\pi^{r+1,r})^*f$, we set

\[
pf = 0.
\]

With this notation, any $q$-form $\rho \in \Omega_q^r W$, where $q \geq 0$, can be expressed as $(\pi^{r+1,r})^*\rho = h\rho + p\rho$, or

\[
(\pi^{r+1,r})^*\rho = h\rho + p_1 \rho + p_2 \rho + \cdots + p_q \rho.
\]

This formula will be referred to as the canonical decomposition of the form $\rho$ (however, the decomposition concerns rather the pullback $(\pi^{r+1,r})^*\rho$ than $\rho$ itself).
Lemma 6 Let \( q \geq 1 \), and let \( \rho \in \Omega^r_w \) be a \( q \)-form. In any fibered chart \((V, \psi)\), \( \psi = (x^i, y^\sigma) \), such that \( V \subset W \), \( p_k \rho \) has a chart expression
\[
p_k \rho = \sum_{0 \leq |i_1|, |i_2|, \ldots, |i_q| \leq r} p_{i_1 i_2 \ldots i_q}^k \omega_{j_1}^{\sigma_1} \wedge \omega_{j_2}^{\sigma_2} \wedge \cdots \wedge \omega_{j_q}^{\sigma_q} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q},
\]
where the components \( p_{i_1 i_2 \ldots i_q}^k \) are real-valued functions on the set \( V' \subset W' \).

Proof We express the pullback \((\pi^{r+1,r})^* \rho\) in the contact basis on \( W^{r+1} \). Write in a fibered chart
\[
\rho = dx^i \wedge \Phi_i + \sum_{0 < |j| < r-1} \omega^j_\sigma \wedge \Psi^I_\sigma + \sum_{|I| = r} dy^I_\sigma \wedge \Theta^I_\sigma
\]
for some \((q-1)\)-forms \( \Phi_i \), \( \Psi^I_\sigma \), and \( \Theta^I_\sigma \). But \( dy^I_\sigma = \omega^I_\sigma + y^\sigma_i dx^i \), and hence,
\[
(\pi^{r+1,r})^* \rho = dx^i \wedge \left((\pi^{r+1,r})^* \Phi_i + \sum_{|I| = r} y^I_\sigma (\pi^{r+1,r})^* \Theta^I_\sigma\right) + \sum_{0 < |j| < r-1} \omega^j_\sigma \wedge (\pi^{r+1,r})^* \Psi^I_\sigma + \sum_{|I| = r} \omega^I_\sigma \wedge (\pi^{r+1,r})^* \Theta^I_\sigma.
\]
Thus, the pullback \((\pi^{r+1,r})^* \rho\) is generated by the form \( dx^i \), \( \omega^j_\sigma \), where \( 0 < |j| < r - 1 \) and \( \omega^j_\sigma \), \( |I| = r \). The same decomposition can be applied to the \((q-1)\)-forms \( \Phi_i \), \( \Psi^I_\sigma \), and \( \Theta^I_\sigma \). Consequently, \((\pi^{r+1,r})^* \rho\) has an expression
\[
(\pi^{r+1,r})^* \rho = \rho_0 + \rho_1 + \rho_2 + \cdots + \rho_q,
\]
where
\[
\rho_0 = A_{i_1 i_2 \ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q},
\]
\[
\rho_k = \sum_{0 \leq |i_1|, |i_2|, \ldots, |i_k| \leq r} p_{i_1 i_2 \ldots i_k}^k \omega_{j_1}^{\sigma_1} \wedge \omega_{j_2}^{\sigma_2} \wedge \cdots \wedge \omega_{j_k}^{\sigma_k} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}, \quad 1 \leq k \leq q - 1,
\]
\[
\rho_q = \sum_{0 \leq |i_1|, |i_2|, \ldots, |i_q| \leq r} p_{i_1 i_2 \ldots i_q}^q \omega_{j_1}^{\sigma_1} \wedge \omega_{j_2}^{\sigma_2} \wedge \cdots \wedge \omega_{j_q}^{\sigma_q}.
\]
Theorem 1, Sect. 2.1, implies that the decomposition (121) is invariant.
We prove that \( q_k = p_k \rho \). It is sufficient to determine the chart expression of \( p_k \rho \). Let \( \zeta \) be a tangent vector,

\[
\zeta = \zeta^i \left( \frac{\partial}{\partial x^i} \right)_{\gamma} + \sum_{k=0}^{r+1} \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} \Xi_{j_1 j_2 \ldots j_k}^\sigma \left( \frac{\partial}{\partial y_{j_1 j_2 \ldots j_k}^\sigma} \right)_{\gamma}.
\]  

From Sect. 1.5, (62)

\[
h_\zeta = \zeta^i \left( \frac{\partial}{\partial x^i} \right)_{\gamma} + \sum_{k=0}^{r} \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} y_{j_1 j_2 \ldots j_k}^\sigma \left( \frac{\partial}{\partial y_{j_1 j_2 \ldots j_k}^\sigma} \right)_{\gamma},
\]

and

\[
p_\zeta = \sum_{k=0}^{r} \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} (\Xi_{j_1 j_2 \ldots j_k}^\sigma - y_{j_1 j_2 \ldots j_k}^\sigma \zeta^i) \left( \frac{\partial}{\partial y_{j_1 j_2 \ldots j_k}^\sigma} \right)_{\gamma}.
\]

If \( h_\zeta = 0 \), then \( \zeta^i = 0 \), and we have

\[
p_\zeta = \sum_{k=0}^{r} \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} \Xi_{j_1 j_2 \ldots j_k}^\sigma \left( \frac{\partial}{\partial y_{j_1 j_2 \ldots j_k}^\sigma} \right)_{\gamma}.
\]

If \( p_\zeta = 0 \), then \( \Xi_{j_1 j_2 \ldots j_k}^\sigma = y_{j_1 j_2 \ldots j_k}^\sigma \zeta^i \), and hence,

\[
h_\zeta = \zeta^i \left( \frac{\partial}{\partial x^i} \right)_{\gamma} + \sum_{k=0}^{r} \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} y_{j_1 j_2 \ldots j_k}^\sigma \left( \frac{\partial}{\partial y_{j_1 j_2 \ldots j_k}^\sigma} \right)_{\gamma}.
\]

We substitute from these formulas to expression (112). Consider the expression \( p_k \rho(J_x^{r+1} \gamma)(\bar{\xi}, \bar{\eta}, \ldots, \bar{\zeta}) \) for \( \bar{\xi}, \bar{\eta}, \ldots, \bar{\zeta} \) such that \( h_\bar{\xi} = 0, h_\bar{\eta} = 0, \ldots, h_\bar{\zeta} = 0 \) and \( p_\bar{\zeta}_{k+1} = 0, p_\bar{\zeta}_2 = 0, \ldots, p_\bar{\zeta}_q = 0 \). Then, (112) reduces to

\[
p_k \rho(J_x^{r+1} \gamma)(\bar{\xi}, \bar{\eta}, \ldots, \bar{\zeta}) = \rho(J_x^{r} \gamma)(p_\bar{\zeta}_1, p_\bar{\zeta}_2, \ldots, p_\bar{\zeta}_k, h_\bar{\zeta}_{k+1}, h_\bar{\zeta}_{k+2}, \ldots, h_\bar{\zeta}_q).
\]
Writing
\[
p^i_\xi_l = \sum_{k=0}^{r} \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} (l) \Xi^\sigma_{\xi j_1 j_2 \cdots j_k} \left( \frac{\partial}{\partial y^\sigma_{\xi j_1 j_2 \cdots j_k}} \right)_{y^\gamma}, \quad 1 \leq l \leq k,
\]
\[
h^i_\xi_l = (l) \Xi^\sigma_{\xi} \left( \frac{\partial}{\partial x^i} \right) + \sum_{k=0}^{r} \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} y^\sigma_{j_1 j_2 \cdots j_k} \left( \frac{\partial}{\partial y^\sigma_{j_1 j_2 \cdots j_k}} \right)_{y^\gamma}, \quad k + 1 \leq l \leq q,
\]
with \( l \) indexing the vectors \( \xi_l \), and substituting into (128), we get
\[
p_k \rho(J_x^{r+1} \gamma)(\xi_1, \xi_2, \ldots, \xi_k, \xi_{k+1}, \xi_{k+2}, \ldots, \xi_q),
\]
\[
= C^{l_1 l_2 \cdots l_k}_{\sigma_1 \sigma_2 \cdots \sigma_k} \xi_{l_1} \Xi_{l_2}^{\sigma_2} \cdots \Xi_{l_k}^{\sigma_k} \xi_{k+1} \xi_{k+2} \cdots \xi_q,
\]
But
\[
^l \Xi^\sigma_{\xi} = \omega^\sigma_l (J_x^{r+1} \gamma) \cdot \xi_l, \quad ^l \xi_l = dx^l (J_x^{r+1} \gamma) \cdot \xi_l
\]
(131)
Therefore, \( p_k \rho(J_x^{r+1} \gamma) \) must be of the form (118).

Formula (118) implies that for any \( k \geq 1 \), the form \( p_k \rho \) is contact; \( p_k \rho \) is called the \( k \)-contact component of the form \( \rho \).

If \( (\pi^{r+1}_q)^* \rho = p_k \rho \) or, equivalently, if \( p_j \rho = 0 \) for all \( j \neq k \), then we say that \( \rho \) is \( k \)-contact, and \( k \) is the degree of contactness of \( \rho \). The degree of contactness of the \( q \)-form \( \rho = 0 \) is equal to \( k \) for every \( k = 0, 1, 2, \ldots, q \). We say that \( \rho \) is of degree of contactness \( \geq k \), if \( p_0 \rho = 0, p_1 \rho = 0, \ldots, p_k \rho = 0 \). If \( k = 0 \), then the 0-contact form \( p_0 \rho = h \rho \) is \( \pi^{r+1}_q \)-horizontal. The mapping \( \Omega_q^r W \ni \rho \rightarrow h \rho \in \Omega_q^{r+1} W \) is called the horizontalization.

The following observation is immediate.

**Lemma 7** If \( q - k > n \), then
\[
h \rho = 0, \quad p_{r} \rho = 0, \quad p_{r+1} \rho = 0, \quad \ldots, \quad p_{q-n-1} \rho = 0.
\]
(132)

**Proof** Expression \( \rho(J_x^{r+1} \gamma)(p_{\xi_1}, p_{\xi_2}, \ldots, p_{\xi_k}, h_{\xi_{k+1}}, h_{\xi_{k+2}}, \ldots, h_{\xi_q}) \) in (113) is a \( (q-k) \)-linear function of vectors \( \xi_{k+1} = T \pi^{r+1} \xi_{k+1}, \xi_{k+2} = T \pi^{r+1} \xi_{k+2}, \ldots, \xi_q = T \pi^{r+1} \xi_q \), belonging to the tangent space \( T_x X \). Consequently, if \( q-k > n = \dim X \), then the skew symmetry of the form \( p_k \rho(J_x^{r+1} \gamma) \) implies \( p_k \rho(J_x^{r+1} \gamma)(\xi_1, \xi_2, \ldots, \xi_q) = 0 \).
To complete the local description of the decomposition (117), we express the components $P^{l_1,l_2}_{\sigma_1,\sigma_2,\ldots,\sigma_{i_k+1}\sigma_{i_k+2}\ldots\sigma_q}$ (118) of the $k$-contact components $p_k\rho$ in terms of the components of $\rho$.

**Lemma 8** Let $W$ be an open set in $Y$, $q$ an integer, $\eta \in \Omega^q Y$ a form, and let $(V, \psi)$, $\psi = (x^1, y^q)$, be a fibered chart on $Y$ such that $V \subset W$. Assume that $\eta$ has on $V$ a chart expression

$$
\eta = \sum_{i=0}^{q} \frac{1}{s!(q-s)!} A_{\sigma_1,\sigma_2,\ldots,\sigma_{i}i_{i+1}i_{i+2}\ldots i_q}^\eta dy^\eta_{l_1} \wedge dy^\eta_{l_2} \wedge \cdots \wedge dy^\eta_{l_k} \\
\wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \cdots \wedge dx^{i_q},
$$

(133)

with multi-indices $I_1, I_2, \ldots, I_s$ of length $r$. Then, the $k$-contact component $p_k\eta$ of $\eta$ has on $V^{r+1}$ a chart expression

$$
p_k\eta = \frac{1}{k!(q-k)!} B^{l_1,l_2}_{\sigma_1,\sigma_2,\ldots,\sigma_{i_{k+1}}i_{k+2}\ldots i_q} \omega^\eta_{l_1} \wedge \omega^\eta_{l_2} \wedge \cdots \wedge \omega^\eta_{l_k} \\
\wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_q},
$$

(134)

where

$$
B^{l_1,l_2}_{\sigma_1,\sigma_2,\ldots,\sigma_{i_{k+1}}i_{k+2}i_{k+3}\ldots i_q} = \sum_{s=k}^{q} \frac{(q-k)}{(q-s)} A^{l_1,l_2}_{\sigma_1,\sigma_2,\ldots,\sigma_{i_s},i_{i+1}i_{i+2}\ldots i_q} \omega^\eta_{l_1} \wedge \omega^\eta_{l_2} \wedge \cdots \wedge \omega^\eta_{l_s} \wedge \operatorname{Alt}(i_{k+1}i_{k+2}\ldots i_{k+s+1}\ldots i_q).
$$

(135)

**Proof** To derive the formula (134), we pullback the form $\eta$ to $V^{r+1}$ and express the form $(\pi^{r+1})^*\Psi$ in terms of the contact basis; in the multi-index notation, the transformation equations are

$$
dx^i = dx^i, \quad dy^\sigma_i = \omega^\sigma_i + y^\sigma_i dx^i, \quad |I| = r
$$

(136)

(Sect. 2.1, Theorem 1, (a)). Thus, we set in (133) $dy^\sigma_i = \omega^\sigma_i + y^\sigma_i dx^i$ and consider the terms in (133) such that $s \geq 1$. Then, the pullback of the form $dy^\sigma_{l_1} \wedge dy^\sigma_{l_2} \wedge \cdots \wedge dy^\sigma_{l_k}$ by $\pi^{r+1}$ is equal to

$$
(\omega^\sigma_{l_1} + y^\sigma_{l_1} dx^i) \wedge (\omega^\sigma_{l_2} + y^\sigma_{l_2} dx^i) \wedge \cdots \wedge (\omega^\sigma_{l_k} + y^\sigma_{l_k} dx^i).
$$

(137)

Collecting together all terms homogeneous of degree $k$ in the contact 1-forms $\omega^\sigma_{l_i}$, we get \binom{k}{r} summands with exactly $k$ entries the contact 1-forms $\omega^\sigma_{l_i}$. Thus, using symmetry properties of the components $A^{l_1,l_2}_{\sigma_1,\sigma_2,\ldots,\sigma_{i_k+1}i_{k+2}i_{k+3}\ldots i_q}$ in (133) and...
interchanging multi-indices, we get the terms containing \( k \) entries \( \omega^e_{i_1} \), for fixed \( s \) and each \( k = 1, 2, \ldots, s \),

\[
\frac{1}{s!(q-s)!} \binom{q}{s} A^{I_1, I_2, \ldots, I_k}_{\sigma_1, \sigma_2, \ldots, \sigma_k} \cdot l_{i_1, i_2, \ldots, i_k} y^{\sigma_{k+1}}_{i_1, i_{k+1}} y^{\sigma_{k+2}}_{i_1, i_{k+2}} \cdots y^{\sigma_s}_{i_1, i_s} \cdot \omega_{I_1}^{e_1} \wedge \omega_{I_2}^{e_2} \wedge \cdots \wedge \omega_{I_k}^{e_k} \\
\wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_s} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_q}.
\]  

(138)

Writing the factor as

\[
\frac{1}{s!(q-s)!} \binom{q}{s} = \frac{1}{k!(q-k)!} \binom{q-k}{q-s},
\]

(139)

we can express (138) as

\[
\frac{1}{k!(q-k)!} \binom{q-k}{q-s} A^{I_1, I_2, \ldots, I_k}_{\sigma_1, \sigma_2, \ldots, \sigma_k} \cdot l_{i_1, i_2, \ldots, i_k} y^{\sigma_{k+1}}_{i_1, i_{k+1}} y^{\sigma_{k+2}}_{i_1, i_{k+2}} \cdots y^{\sigma_s}_{i_1, i_s} \cdot \omega_{I_1}^{e_1} \wedge \omega_{I_2}^{e_2} \wedge \cdots \wedge \omega_{I_k}^{e_k} \\
\wedge \cdots \wedge \omega_{I_k}^{e_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_s} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_q}.
\]

(140)

Formula (138) is valid for each \( s = 1, 2, \ldots, q \) and each \( k = 1, 2, \ldots, s \) and includes summation through all these terms to get expression (133). The summation through the pairs \((s, k)\) is given by the table

\[
\begin{array}{cccccc}
  s & 1 & 2 & 3 & \ldots & q-1 & q \\
  k & 1, 2 & 1, 2, 3 & \ldots & 1, 2, 3, \ldots, q-1 & 1, 2, 3, \ldots, q
\end{array}
\]

(141)

It will be convenient to pass to the summation over the same written in the opposite order. The summation through the pairs \((k, s)\) is expressed by the table

\[
\begin{array}{cccccc}
  k & 1 & 2 & 3 & \ldots & q-1 & q \\
  s & 1, 2, 3, \ldots, q & 2, 3, \ldots, q & 3, 4, \ldots, q & \ldots & q-1, q & q
\end{array}
\]

(142)

Now, we can substitute from (140) back to (133). We have, with multi-indices of length \( r \),

\[
\eta = \frac{1}{q!} A^{I_1, I_2, \ldots, I_q}_{i_1, i_2, \ldots, i_q} \cdot dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}
\]

\[
+ \sum_{s=1}^{q} \sum_{k=1}^{s} \frac{1}{k!(q-k)!} \binom{q-k}{q-s} A^{I_1, I_2, \ldots, I_k}_{\sigma_1, \sigma_2, \ldots, \sigma_k} \cdot l_{i_1, i_2, \ldots, i_k} y^{\sigma_{k+1}}_{i_1, i_{k+1}} y^{\sigma_{k+2}}_{i_1, i_{k+2}} \cdots y^{\sigma_s}_{i_1, i_s} \cdot \omega_{I_1}^{e_1} \wedge \omega_{I_2}^{e_2} \wedge \cdots \wedge \omega_{I_k}^{e_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_s} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_q}
\]

(143)
hence,

\[ p_k \eta = \frac{1}{q!} A_{i_1 i_2 \ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q} \]

\[ + \sum_{k=1}^{q} \frac{1}{k!(q - k)!} \left( \sum_{s=k}^{q} \frac{(q - k)}{q - s} A_{i_1 i_2 \ldots i_s}^{J_1 J_2 \ldots J_{s}} \sigma_{i_{s+1}i_{s+2} \ldots i_{q}}^{J_{s+1}i_{s+2} \ldots i_{q}} \right) \]

\[ \cdot \omega_{i_1}^{J_1} \wedge \omega_{i_2}^{J_2} \wedge \cdots \wedge \omega_{i_s}^{J_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \cdots \wedge dx^{i_q}. \]  

(144)

This proves the formulas (134) and (135).

Remark 5 Formulas (133) and (134) are not invariant; the transformation properties of the components are determined in Sect. 2.1, Theorem 1, (b).

Lemma 8 can now be easily extended to general $q$-forms. It is sufficient to consider the case of $q$-forms generated by $p$-forms $\omega_{i_1}^{J_1} \wedge \omega_{i_2}^{J_2} \wedge \cdots \wedge \omega_{i_p}^{J_p}$ with fixed $p$, $1 \leq p \leq q - p$. The proof then consists in a formal application of Lemma 8.

Theorem 8 Let $W$ be an open set in $Y$, $q$ a positive integer, and $\rho \in \Omega_q^{\rho} W$ a $q$-form, and let $(V, \psi)$, $\psi = (x^1, y^m)$, be a fibered chart on $Y$ such that $V \subset W$. Assume that $\rho$ has on $V^r$ a chart expression

\[ \rho = \sum_{s=0}^{q-p} \frac{1}{s!(q - p - s)!} A_{i_1 i_2 \ldots i_{s}}^{J_1 J_2 \ldots J_{s}} \sigma_{i_{s+1}i_{s+2} \ldots i_{q-p}}^{J_{s+1}i_{s+2} \ldots i_{q-p}} \omega_{i_1}^{J_1} \wedge \omega_{i_2}^{J_2} \wedge \cdots \wedge \omega_{i_s}^{J_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \cdots \wedge dx^{i_{q-p}}, \]  

(145)

with multi-indices $J_1, J_2, \ldots, J_s$ of length $r - 1$ and multi-indices $I_1, I_2, \ldots, I_s$ of length $r$. Then, the $k$-contact component $p_k \rho$ of $\rho$ has on $V^{r+1}$ the chart expression

\[ p_k \rho = \frac{1}{(k - p)!(q - p - k)!} B_{i_1 i_2 \ldots i_{s}}^{J_1 J_2 \ldots J_{s}} \sigma_{i_{s+1}i_{s+2} \ldots i_{q-p}}^{J_{s+1}i_{s+2} \ldots i_{q-p}} \omega_{i_1}^{J_1} \wedge \omega_{i_2}^{J_2} \wedge \cdots \wedge \omega_{i_s}^{J_s} \wedge dx^{i_{s+1}} \wedge dx^{i_{s+2}} \wedge \cdots \wedge dx^{i_{q-p}}, \]  

(146)

where

\[ B_{i_1 i_2 \ldots i_{s}}^{J_1 J_2 \ldots J_{s}} = \sum_{s=0}^{q-p} \left( \frac{q - k}{q - s} \right) A_{i_1 i_2 \ldots i_{s}}^{J_1 J_2 \ldots J_{s}} \sigma_{i_{s+1}i_{s+2} \ldots i_{q-p}}^{J_{s+1}i_{s+2} \ldots i_{q-p}} \right) \]

\[ \cdot y_{i_{s+1}i_{s+2} \ldots i_{q-p}}^{J_{s+1}i_{s+2} \ldots i_{q-p}} y_{i_{s+1}i_{s+2} \ldots i_{q-p}}^{J_{s+1}i_{s+2} \ldots i_{q-p}} \cdot \text{Alt}(i_{k-p+1}i_{k-p+2} \ldots i_{k-s}i_{k-s+1} \ldots i_{q-p}). \]  

(147)
Proof ρ can be expressed as
\[
\rho = \omega_{J_1}^{v_1} \wedge \omega_{J_2}^{v_2} \wedge \cdots \wedge \omega_{J_p}^{v_p} \wedge \eta_{v_1 \ldots v_p}^{J_1 \ldots J_p},
\]
where
\[
\eta_{v_1 \ldots v_p}^{J_1 \ldots J_p} = \sum_{s=0}^{q-p} \frac{1}{s!(q-p-s)!} A_{v_1 \ldots v_p}^{J_1 \ldots J_p} \cdot I_s \wedge dy_{t_1}^{\sigma_1} \wedge \cdots \wedge dy_{t_q}^{\sigma_q} \wedge dx_{i_{p+1}}^{\sigma_{p+1}} \wedge dx_{i_{p+2}}^{\sigma_{p+2}} \wedge \cdots \wedge dx_{i_{q-p}}^{\sigma_{q-p}}.
\]

We can apply to $\eta_{v_1 \ldots v_p}^{J_1 \ldots J_p}$ formula (134). Replacing $q$ with $q - p$ and $k$ with $k - p$,
\[
p_{k-p} \eta_{v_1 \ldots v_p}^{J_1 \ldots J_p} = \frac{1}{(k-p)!(q-p-k)!} B_{v_1 \ldots v_p}^{J_1 \ldots J_p} \cdot I_k \wedge \omega_{t_1}^{\sigma_1} \wedge \cdots \wedge \omega_{t_q}^{\sigma_q} \wedge dx_{i_{p+1}}^{\sigma_{p+1}} \wedge dx_{i_{p+2}}^{\sigma_{p+2}} \wedge \cdots \wedge dx_{i_{q-p}}^{\sigma_{q-p}},
\]
where
\[
B_{v_1 \ldots v_p}^{J_1 \ldots J_p} = \sum_{s=k-p}^{q-p} \frac{(q-k)}{(q-p-s)} A_{v_1 \ldots v_p}^{J_1 \ldots J_p} \cdot I_s \wedge y_{t_{k+1}}^{\sigma_{k+1}} \wedge y_{t_{k+2}}^{\sigma_{k+2}} \wedge y_{t_{k+s}}^{\sigma_{k+s}} \wedge \operatorname{Alt}(i_{k+1} i_{k+2} \ldots i_{k+s} i_{k+s+1} \ldots i_{q-p}).
\]

The following two corollaries are immediate consequences of Theorem 8 and Sect. 2.1, Theorem 1. The first one shows that the operators $p_k$ behave like projector operators in linear algebra. The second one is a consequence of the identity $d(\pi^{r+1,1})^\ast \rho = (\pi^{r+1,1})^\ast d\rho$ for the exterior derivative operator, the canonical decomposition of forms on jet manifolds, applied to both sides, as well as the formula
\[
d\omega_{\varepsilon}^j = -\omega_{\varepsilon}^j \wedge dx^j.
\]

Corollary 1 For any $k$ and $l$,
\[
p_k p_l \rho = \begin{cases} (\pi^{r+2,r+1})^\ast p_k \rho, & k = l, \\ 0, & k \neq l. \end{cases}
\]

Corollary 2 For every $k \geq 1$,
\[
(\pi^{r+2,r+1})^\ast p_k \rho = p_k dp_{k-1} \rho + p_k d_k \rho.
\]
Remark 6 According to Sect. 2.3, Theorem 5, the horizontalization
\[ h: \Omega' W \to \Omega^{r+1} W \]
is a morphism of exterior algebras. On the other hand, if \( k \) is a positive
integer, then the mapping \( p_k: \Omega' W \to \Omega^{r+1} W \) satisfies
\[
p_k(\rho + \eta) = p_k\rho + p_k\eta, \quad p_k(f\rho) = (f \circ \pi^{r+1,r})p_k\rho
\]
for all \( \rho, \eta \), and \( f \). However, \( p_k: \Omega' W \to \Omega^{r+1} W \) are not morphisms of exterior
algebras.

2.5 Contact Components and Geometric Operations

In this section, we summarize some properties of the contact components and the
differential-geometric operations acting on forms, such as the wedge product \( \wedge \), the
contraction \( i_\zeta \) of a form by a vector \( \zeta \), and the Lie derivative \( \partial_\xi \) by a vector field \( \xi \).

**Theorem 9** Let \( W \) be an open set in \( Y \).

(a) For any two forms \( \rho \) and \( \eta \) on \( W^r \subset J'^r Y \),
\[
p_k(\rho \wedge \eta) = \sum_{i+j=k} p_k\rho \wedge p_k\eta.
\]

(b) For any form \( \rho \) and any \( \pi^{r+1,-r} \)-vertical, \( \pi^{r+1,r} \)-projectable vector field \( \Xi \) on
\( W^{r+1} \), with \( \pi^{r+1,r} \)-projection \( \xi \),
\[
i_\Xi p_k\rho = p_{k-1}i_\xi \rho.
\]

(c) For any form \( \rho \) and any automorphism \( \alpha \) of \( Y \), defined on \( W \),
\[
p_k(J' \alpha^* \rho) = J^{r+1} \alpha^* p_k\rho.
\]

(d) For any form \( \rho \) and any \( \pi \)-projectable vector field on \( Y \) on \( W \)
\[
p_k(\partial_\rho \Xi \rho) = \partial_{J^{r+1} \Xi} p_k\rho.
\]

**Proof**

(a) The exterior product \( (\pi^{r+1,r})^* (\rho \wedge \eta) \) commutes with the pullback, so we have
\( (\pi^{r+1,r})^* (\rho \wedge \eta) = (\pi^{r+1,r})^* \rho \wedge (\pi^{r+1,r})^* \eta \). Applying the trace decomposition
formula (Sect. 2.2, Theorem 3) to \( (\pi^{r+1,r})^* \rho \) and \( (\pi^{r+1,r})^* \eta \), and comparing the
\( k \)-contact components on both sides, we obtain formula (156).
Let $\rho \in \Omega_q^Y W$ be a $q$-form such that $n + 1 \leq q \leq \dim J^r Y$. Since $h \rho = 0$ and also $p_1 \rho = 0, p_2 \rho = 0, \ldots, p_{q-n-1} \rho = 0$ (Sect. 2.4, Theorem 8), $\rho$ is always contact, and its canonical decomposition has the form

\[
\begin{align*}
(i) p_k &\phi(J^r_q Y)(\xi_2, \xi_3, \ldots, \xi_q) \\
&= \rho(\Xi(J^r_q Y), \xi_2, \xi_3, \ldots, \xi_q) \\
&= \sum c^i_{j_1 \ldots j_{k+1}} \rho(J^r_q Y)(p_{\xi^i_{j_1}}, p_{\xi^i_{j_2}}, \ldots, p_{\xi^i_{j_k}}, h_{\xi_{j_k+1}}, h_{\xi_{j_k+2}}, \ldots, h_{\xi_{j_q}}) \\
\end{align*}
\]
We introduce by induction a class of $q$-forms, imposing a condition on the contact component $p_{q-n}\rho$. If $q = n + 1$, then we say that $\rho$ is strongly contact, if for every point $y_0 \in W$ there exist a fibered chart $(V, \psi)$, $\psi = (x^i, y^\sigma)$, at $y_0$ and a contact $n$-form $\tau$, defined on $V'$, such that

$$p_1(\rho - d\tau) = 0. \quad (164)$$

If $q > n + 1$, then we say that $\rho$ is strongly contact, if for every $y_0 \in W$ there exist $(V, \psi)$, $\psi = (x^i, y^\sigma)$, at $y_0$ and a strongly contact $n$-form $\tau$, defined on $V'$, such that

$$p_{q-n}(\rho - d\tau) = 0. \quad (165)$$

**Lemma 9** The following conditions are equivalent:

(a) $\rho$ is strongly contact.

(b) There exist a $q$-form $\eta$ and a $(q - 1)$-form $\tau$ such that

$$\rho = \eta + d\tau, \quad p_{q-n}\eta = 0, \quad p_{q-n-1}\tau = 0. \quad (166)$$

**Proof** If $\rho$ is strongly contact and we have $\tau$ such that (165) holds, then we set $\eta = \rho - d\tau$. The converse is obvious. \qed

In view of part (b) of Lemma 9, to study the properties of strongly contact forms, we need the chart expressions of the $q$-forms $p_{q-n}\rho$ and $p_{q-n-1}\tau = 0$. We also need, in particular, the chart expressions of the forms $\rho$ whose $(q - n)$-contact component vanishes,

$$p_{q-n}\rho = 0. \quad (167)$$

To this purpose, we use the contact basis. The formulas as well as the proof the subsequent theorem are based on the complete trace decomposition theory and are technically tedious because we cannot avoid extensive index notation. We write

$$\rho = \sum A_{J_1J_2 \ldots I_p}^{J_1J_2 \ldots I_p} \sigma_{p+1}^{\sigma_{p+1}} \sigma_{p+2}^{\sigma_{p+2}} \ldots \sigma_{p+s}^{\sigma_{p+s}} \omega_{J_1}^{y_1} \wedge \omega_{J_2}^{y_2} \wedge \ldots \wedge \omega_{J_p}^{y_p} \wedge dy_{I_{p+1}}^{i_{p+1}} \wedge dy_{I_{p+2}}^{i_{p+2}} \wedge \ldots \wedge dy_{I_{p+s}}^{i_{p+s}} \wedge dx^{I_{p+s+1}} \wedge dx^{I_{p+s+2}} \wedge \ldots \wedge dx^{I_{r}}, \quad (168)$$

where summation is taking place through the multi-indices $J_1, J_2, \ldots, J_p$ of length less or equal to $r - 1$ and the multi-indices $I_{p+1}, I_{p+2}, \ldots, I_{p+s}$ of length equal to $r$. 
Applying the trace decomposition theorem (Appendix 9, Theorem 1) as many times as necessary, we can write

\[
\rho = \sum B_{I_1J_1}^I J_{K_1L_1, \ldots, K_pL_p+1}^{I_1I_2 \ldots I_{l+p+q}} \cdot \omega_{J_1}^{I_1} \wedge \omega_{J_2}^{I_2} \wedge \cdots \wedge \omega_{J_p}^{I_p} \wedge d\omega_{K_1}^{I_1+1} \wedge d\omega_{K_2}^{I_2+2} \wedge \cdots \wedge d\omega_{K_p}^{I_p+q} \\
\wedge dy_{I_{l+p+1}}^{\sigma_{I_{l+p+2}}} \wedge \cdots \wedge dy_{I_{l+p+p}}^{\sigma_{I_{l+p+p+2}}} \wedge \cdots \wedge dx^{J_{l+p+p+2}}
\]

where

\[
0 \leq |J_1|, |J_2|, \ldots, |J_p| \leq r - 1,
\]
\[
|K_{l+1}|, |K_{l+2}|, \ldots, |K_{l+p}| = r - 1,
\]
\[
|I_{l+p+1}|, |I_{l+p+2}|, \ldots, |I_{l+p+q}| = r,
\]

and the coefficients are traceless. The number \(Q\) in (169) is not the degree of \(\rho\); it is related to the degree \(q\) by \(l + 2p + s + Q - l - p - s = q\), that is,

\[
p + Q = q.
\]

**Theorem 10** Let \(W \subseteq Y\) be an open set, \(q\) an integer such that \(n + 1 \leq q \leq \dim J^r Y\), and \(\eta \in \Omega_q^r W\) a form, and let \((V, \psi)\), \(\psi = (x^i, y^\sigma)\), be a fibered chart such that \(V \subseteq W\). Then, \(p_{q-n-1} \eta = 0\) if and only if

\[
\eta = \sum_{q-n+1 \leq l+p} \omega_{J_1}^{\sigma_{J_1}} \wedge \cdots \wedge \omega_{J_p}^{\sigma_{J_p}} \wedge d\omega_{I_1}^{y_{I_1}} \wedge d\omega_{I_2}^{y_{I_2}} \wedge \cdots \wedge d\omega_{I_q}^{y_{I_q}} \\
\end{equation}

where \(\Phi_{J_1J_2}^{I_1I_2} \cdots I_p^{y_{I_p}}\) are some \((q - l - 2p)\)-forms on \(V^r\) and the multi-indices satisfy \(0 \leq |J_1|, |J_2|, \ldots, |J_p| \leq r - 1\), \(|I_1|, |I_2|, \ldots, |I_q| = r - 1\).

**Proof** Expression (169) for \(\eta\) can be written as \(V^{r+1}\), where

\[
\end{equation}

\[
\eta_0 = \sum_{l+p+q \geq n} B_{J_1J_2}^{I_1I_2} J_{K_1L_1, \ldots, K_pL_q}^{I_1I_2 \ldots I_{l+p+q}} \cdot \omega_{J_1}^{I_1} \wedge \omega_{J_2}^{I_2} \wedge \cdots \wedge \omega_{J_q}^{I_q} \wedge d\omega_{K_1}^{I_1+1} \wedge d\omega_{K_2}^{I_2+2} \wedge \cdots \wedge d\omega_{K_p}^{I_p+q} \\
\wedge dy_{I_{l+p+1}}^{\sigma_{I_{l+p+2}}} \wedge \cdots \wedge dy_{I_{l+p+p}}^{\sigma_{I_{l+p+p+2}}} \wedge \cdots \wedge dx^{J_{l+p+p+2}}
\]

(173)
2.6 Strongly Contact Forms

Now, consider a

\[ \eta_1 = \sum_{l+p < q-n} B_{Y_{l+1}^p} \cdots J_{K_{l+1}^p} K_{l+1}^p L_{l+p+1} L_{l+p+2} \cdots I_{l+p+q} \]

\[ \times \omega_{J_1}^Y \wedge \omega_{J_2}^Y \wedge \cdots \wedge \omega_{J_p}^Y \wedge d\omega_{K_1}^{K_1} \wedge d\omega_{K_2}^{K_2} \wedge \cdots \wedge d\omega_{K_p}^{K_p} \]

\[ \wedge dy_{l+p+1}^{\sigma_{l+p+1}} \wedge dy_{l+p+2}^{\sigma_{l+p+2}} \wedge \cdots \wedge dy_{l+p+q}^{\sigma_{l+p+q}} \wedge dx_{l+p+1}^{\delta_{l+p+1}} \wedge dx_{l+p+2}^{\delta_{l+p+2}} \wedge \cdots \wedge dx^0. \]

(174)

We want to show that the condition \( p_{q-n} \eta = 0 \) implies \( \eta_1 = 0 \).

To determine \( p_{q-n} \eta \), we need the pullback \((\pi^{r+1,r})^* \eta_1\); this can be obtained by replacing \( dy_i^\sigma \) with

\[ dy_i^\sigma = \omega_i^\sigma + y_i^\sigma dx^i. \]

(175)

Then, the corresponding expressions on the right-hand side of the formula (174) arise by substitution

\[ dy_{l+p+1}^{\sigma_{l+p+1}} \wedge dy_{l+p+2}^{\sigma_{l+p+2}} \wedge \cdots \wedge dy_{l+p+q}^{\sigma_{l+p+q}} = \left( \omega_{l+p+1}^{\sigma_{l+p+1}} + y_{l+p+1}^{\sigma_{l+p+1}} dx_{l+p+1} \right) \wedge \left( \omega_{l+p+2}^{\sigma_{l+p+2}} + y_{l+p+2}^{\sigma_{l+p+2}} dx_{l+p+2} \right) \wedge \cdots \wedge \left( \omega_{l+p+q}^{\sigma_{l+p+q}} + y_{l+p+q}^{\sigma_{l+p+q}} dx_{l+p+q} \right). \]

(176)

Computing the right-hand side, we obtain

\[ dy_{l+p+1}^{\sigma_{l+p+1}} \wedge dy_{l+p+2}^{\sigma_{l+p+2}} \wedge \cdots \wedge dy_{l+p+q}^{\sigma_{l+p+q}} = \omega_{l+p+1}^{\sigma_{l+p+1}} \wedge \omega_{l+p+2}^{\sigma_{l+p+2}} \wedge \cdots \wedge \omega_{l+p+q}^{\sigma_{l+p+q}} \]

\[ + \left( \frac{1}{2} \right) y_{l+p+1}^{\sigma_{l+p+1}} \omega_{l+p+1}^{\sigma_{l+p+1}} \wedge \omega_{l+p+2}^{\sigma_{l+p+2}} \wedge \cdots \wedge \omega_{l+p+q}^{\sigma_{l+p+q}} \wedge d\omega_{l+p+1} \]

\[ + \cdots + \left( \frac{1}{2} \right) y_{l+p+1}^{\sigma_{l+p+1}} \omega_{l+p+1}^{\sigma_{l+p+1}} \wedge \omega_{l+p+2}^{\sigma_{l+p+2}} \wedge \cdots \wedge \omega_{l+p+q}^{\sigma_{l+p+q}} \wedge d\omega_{l+p+1} \]

\[ \wedge dx_{l+p+1}^{\delta_{l+p+1}} \wedge \cdots \wedge dx_{l+p+q}^{\delta_{l+p+q}}. \]

(177)

Now, consider a fixed summand in expression (174), with given \( l, p, \) and \( s, \)

\[ B_{Y_{l+1}^p} \cdots J_{K_{l+1}^p} K_{l+1}^p L_{l+p+1} L_{l+p+2} \cdots I_{l+p+q} \]

\[ \times \omega_{J_1}^Y \wedge \omega_{J_2}^Y \wedge \cdots \wedge \omega_{J_p}^Y \wedge d\omega_{K_1}^{K_1} \wedge d\omega_{K_2}^{K_2} \wedge \cdots \wedge d\omega_{K_p}^{K_p} \]

\[ \wedge \omega_{l+p+1}^{\sigma_{l+p+1}} \wedge \omega_{l+p+2}^{\sigma_{l+p+2}} \wedge \cdots \wedge \omega_{l+p+q}^{\sigma_{l+p+q}} \wedge dx_{l+p+1}^{\delta_{l+p+1}} \wedge dx_{l+p+2}^{\delta_{l+p+2}} \wedge \cdots \wedge dx^0. \]

(178)
Using (178), we get the terms

\[ sB_{i_1j_2}^{l_1} \cdots sB_{i_{p+1}j_{p+2}}^{l_{p+2}} \omega_{i_1}^{j_1} \wedge \cdots \wedge \omega_{i_{p+1}}^{j_{p+1}} \wedge \omega_{i_{p+2}}^{j_{p+2}} \wedge \cdots \wedge \omega_{i_{p+q}}^{j_{p+q}}, \]

\[ B_{i_1j_2}^{l_1} \cdots B_{i_{p+1}j_{p+2}}^{l_{p+2}} \omega_{i_1}^{j_1} \wedge \cdots \wedge \omega_{i_{p+1}}^{j_{p+1}} \wedge \omega_{i_{p+2}}^{j_{p+2}} \wedge \cdots \wedge \omega_{i_{p+q}}^{j_{p+q}}. \]

and

\[ \frac{1}{2} \left( B_{i_1j_2}^{l_1} \cdots B_{i_{p+1}j_{p+2}}^{l_{p+2}} \omega_{i_1}^{j_1} \wedge \cdots \wedge \omega_{i_{p+1}}^{j_{p+1}} \wedge \omega_{i_{p+2}}^{j_{p+2}} \wedge \cdots \wedge \omega_{i_{p+q}}^{j_{p+q}} \right) \]

\[ \left( \frac{1}{2} B_{i_1j_2}^{l_1} \cdots B_{i_{p+1}j_{p+2}}^{l_{p+2}} \omega_{i_1}^{j_1} \wedge \cdots \wedge \omega_{i_{p+1}}^{j_{p+1}} \wedge \omega_{i_{p+2}}^{j_{p+2}} \wedge \cdots \wedge \omega_{i_{p+q}}^{j_{p+q}} \right) \]

\[ \left( \frac{1}{2} B_{i_1j_2}^{l_1} \cdots B_{i_{p+1}j_{p+2}}^{l_{p+2}} \omega_{i_1}^{j_1} \wedge \cdots \wedge \omega_{i_{p+1}}^{j_{p+1}} \wedge \omega_{i_{p+2}}^{j_{p+2}} \wedge \cdots \wedge \omega_{i_{p+q}}^{j_{p+q}} \right) \]

\[ \left( \frac{1}{2} B_{i_1j_2}^{l_1} \cdots B_{i_{p+1}j_{p+2}}^{l_{p+2}} \omega_{i_1}^{j_1} \wedge \cdots \wedge \omega_{i_{p+1}}^{j_{p+1}} \wedge \omega_{i_{p+2}}^{j_{p+2}} \wedge \cdots \wedge \omega_{i_{p+q}}^{j_{p+q}} \right) \]

We see that the degrees of contactness of these terms are

\[ l + p + s > l + p + s - 1 > l + p + s - 2 > \cdots > l + p + 1 > l + p, \]

respectively. Clearly, since we consider the terms where \( l + p < q - n \), (180) does not contribute to \( p_{q-n} \eta_1 \). We claim that among the terms (178), there is one whose degree of contactness is \( q - n \). Suppose the opposite; then \( l + p + s < q - n \), but this is not possible, because the term satisfying this inequality would contain more than \( n \) factors \( dx^l \).

Thus, the condition \( p_{1} \eta_1 = 0 \) applies to one of the expressions (179) and states that the coefficient in this expression vanishes. But the components of \( \eta_1 \) are traceless, and we have already seen that this is only possible when they vanish.
This implies in turn that the forms on the left of (179) all vanish, which proves that $\eta_1 = 0$. The proof is complete.

**Corollary 1** Let $W \subset Y$ be an open set, $q$ an integer such that $n + 1 \leq q \leq \dim J'Y$, and $\eta \in \Omega_q^W$ a form, and let $(V, \psi)$, $\psi = (x^i, y^\sigma)$, be a fibered chart such that $V \subset W$. Then, $p_{q-n} \eta = 0$ if and only if

$$\eta = \eta_0 + d\mu,$$

(182)

where $\eta_0$ and $\mu$ are $\omega^q_J$-generated, $0 \leq |l| \leq r - 1$, such that $p_{q-n} \eta_0 = 0$ and $p_{q-n-1} \mu = 0$.

**Proof** Write in Theorem 10 $\eta = \eta_0 + \eta'$, where $\eta_0$ includes all $\omega^q_J$-generated terms, defined by the condition $l \geq 1$, and

$$\eta' = \sum_{q-n+1 \leq l \leq p} d\omega^{\gamma_1}_{l_1} \wedge d\omega^{\gamma_2}_{l_2} \wedge \cdots \wedge d\omega^{\gamma_p}_{l_p} \wedge \Phi^{J_1 J_2 \cdots J_l I_1 I_2 \cdots I_p}_{\sigma_1 \sigma_2 \cdots \sigma_l \gamma_1 \gamma_2 \cdots \gamma_p}.$$

(183)

Thus, $\eta$ can also be written as $\eta = \eta_0 + d\mu$, where $\eta_0$ is $\omega^q_J$-generated, and $\mu$ is also $\omega^q_J$-generated and contains $p$ contact factors $\omega^q_J$ and $d\omega^q_I$; in particular, $p_{q-n-1} \mu = 0$.

**Remark 8** Note that the summation in Theorem 10 through the pairs $(l, p)$ can also be defined by the inequality $q - n + 1 - p \leq l \leq q - 2p$, where the range of $p$ is given by the conditions $p = 0, 1, 2, \ldots$ and $q - 2p \geq 0$.

**Lemma 10**

(a) If $\rho$ is a strongly contact form such that $q \geq n + 2$, then for any $\pi$-vertical vector field $\Xi$, the form $i\rho \partial \cdot \rho$ is strongly contact.

(b) The exterior derivative of a strongly contact form is strongly contact.

**Proof**

(a) We have $i\rho \partial \cdot \rho = i\rho \partial \cdot \rho d\tau = i\rho \partial \cdot \rho d\tau - di\rho \partial \cdot \rho$. But by Sect. 2.5, Theorem 9 $p_{q-n-1} (i\rho \partial \cdot \rho d\tau) = i\rho \partial \cdot \rho d\rho = p_{q-n} \rho + \partial \rho \partial \cdot \rho d\rho = 0$; however, these expressions vanish because $\rho$ is strongly contact. Now, we apply Lemma 9.

(b) Let the form $\rho$ be strongly contact. Then, from (166), $d\rho = d\rho \cdot \rho$, where $p_{q-n} \rho = 0$. We want to show that to any point $y_0$ from the domain of definition of $\rho$, there exists a fibered chart $(V, \psi)$, $\psi = (x^i, y^\sigma)$, at $y_0$ and a $q$-form $\tau$, defined on $V^*$, such that $p_{q+1-n} (d\rho - d\tau) = 0$ and $p_{q-n} \tau = 0$. Taking $\tau = \rho$, we get the result.
For $n + 1 \leq q \leq \dim J^r Y$, strongly contact forms constitute an Abelian subgroup $\Theta_q^r W$ of the Abelian group of $q$-forms $\Omega_q^r W$; they do not form a submodule of $\Omega_q^r W$.

It follows from Lemma 10, (b) that the subgroups $\Theta_q^r W$ together with the exterior derivative operator define a sequence

$$
\Theta_n^r W \rightarrow \Theta_{n+1}^r W \rightarrow \cdots \rightarrow \Theta_M^r W \rightarrow 0. \quad (184)
$$

The number $M$ labeling the last nonzero term in this sequence is

$$
M = m\left(\frac{n + r - 1}{n}\right) + 2n - 1. \quad (185)
$$

**Remark 9** If $n + 1 \leq q \leq \dim J^r Y$, then by Lemma 1, the canonical decomposition of a contact form $\rho \in \Theta_q^r W$ is

$$
(p^{r+1})^\ast \rho = p_{q-n}d\tau + p_{q-n+1}\rho + p_{q-n+2}\rho + \cdots + p_q\rho. \quad (186)
$$

**Remark 10** It is easily seen that the definition of a contact $q$-form $\rho \in \Omega_q^r W$ for $1 \leq q \leq n$ agrees with (165). Indeed, if $1 \leq q \leq n$, we have for any contact form $\rho' \in \Theta_{q-1}^r W$, $h(\rho - d\rho') = h\rho$ as $(p^{r+1})^\ast hd\rho' = hdh\rho' = 0$ (Corollary 2). Thus, if $h\rho = 0$, then $h(\rho - d\rho') = 0$ for any $\rho' \in \Theta_{q-1}^r W$.

### 2.7 Fibered Homotopy Operators on Jet Prolongations of Fibered Manifolds

In this section, we introduce the fibered homotopy operators for differential forms on jet prolongations of fibered manifolds. We study their relations with the canonical decomposition of forms and the exactness problem for contact and strongly contact forms. The general theory of fibered homotopy operators is summarized in Appendix 6.

The relevant underlying structure we need is a trivial fibered manifold $W = U \times V$, where $U$ is an open set in $\mathbb{R}^n$ and $V$ an open ball in $\mathbb{R}^m$ with center at the origin; the projection is the first Cartesian projection of $U \times V$ onto $U$, denoted by $\pi$. The $r$-jet prolongation $J^r W$ is also denoted by $W^r$. By definition

$$
W^r = U \times V \times L(\mathbb{R}^n, \mathbb{R}^m) \times L_{sym}^2(\mathbb{R}^n, \mathbb{R}^m) \times \cdots \times L_{sym}^r(\mathbb{R}^n, \mathbb{R}^m), \quad (187)
$$

where $L_{sym}^k(\mathbb{R}^n, \mathbb{R}^m)$ is the vector space of $k$-linear symmetric mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$. The canonical coordinates on $W$ are denoted by $(x^i, y^\sigma)$, and the associated coordinates on $W^r$ are $(x^i, y^\sigma, y_{j_1}^\sigma, y_{j_2}^\sigma, \ldots, y_{j_1j_2\ldots j_r}^\sigma)$. Any Cartesian projections
2.7 Fibered Homotopy Operators on Jet Prolongations of Fibered Manifolds

\[ \pi^*: W^r \to W^s, \text{ with } 0 \leq s < r, \text{ define in an obvious way a homotopy } \chi^r \text{ and the fibered homotopy operator } I^{r,s} \text{ (see Appendix 6, (27))}, \text{ so the Volterra-Poincare lemma holds in these cases.} \]

In this section, we consider the fibered homotopy operator \( I = I^{r,0} \). Recall that the homotopy \( \chi = \chi^r \) is a mapping from \([0, 1] \times W^r \) to \( W^r \), defined by

\[
\chi(s, (x^i, y^j_{j_1}, \ldots, y^j_{j_k})) = (x^i, sy^j_{j_1}, sy^j_{j_2}, \ldots, sy^j_{j_k}). \tag{188}
\]

It is immediately verified that the pullback by \( \chi \) satisfies

\[
\chi^* dx^i = dx^i, \quad \chi^* dy^j_{j_1 \ldots j_k} = y^j_{j_1 \ldots j_k} ds + sy^j_{j_1 \ldots j_k},
\]

\[
\chi^* \omega_{j_1 \ldots j_k} = y^j_{j_1 \ldots j_k} ds + s\omega_{j_1 \ldots j_k}. \tag{189}
\]

In accordance with the general theory, these formulas lead to explicit description of the operator \( I \). For any \( q \)-form \( \rho \) on \( W^r \), \( \chi^* \rho \) has a unique decomposition

\[
\chi^* \rho = ds \wedge \rho^{(0)}(s) + \rho'(s) \tag{190}
\]

such that the \((q - 1)\)-form \( \rho^{(0)}(s) \) and the \( q \)-form \( \rho'(s) \) do not contain \( ds \). Then,

\[
I \rho = \int_0^1 \rho^{(0)}(s), \tag{191}
\]

where the expression on the right-hand side denotes the integration of the coefficients in the form \( \rho^{(0)}(s) \) over \( s \) from 0 to 1.

The following is a version of a general theorem on fibered homotopy operators on fibered manifolds. \( \zeta \) stands for the zero section of \( W^r \) over \( U \).

**Theorem 11**

(a) For every differentiable function \( f: W^r \to \mathbb{R} \),

\[
f = \text{Id}f + (\pi')^*\zeta^*f. \tag{192}
\]

(b) Let \( q \geq 1 \). Then, for every differential \( q \)-form \( \rho \) on \( W^r \),

\[
\rho = \text{Id} \rho + d \rho + (\pi')^*\zeta^*\rho. \tag{193}
\]

**Proof** Slight modification of Theorem 1, Appendix 6. \( \square \)
Theorem 12 Let \( \rho \) be a contact \( q \)-form on \( W' \).

(a) The contact components of \( \rho \) satisfy

\[
Ih \rho = 0, \quad Ip_k \rho = p_{k-1}I \rho, \quad 1 \leq k \leq q. \tag{194}
\]

(b) If \( \rho \) is strongly contact, then \( I \rho \) is strongly contact.

Proof

(a) Expressing the forms \( \rho \) and \((\pi^{r+1,r}')^* \rho\) in the basis of 1-forms \((dx^j, dy^j)\), \(0 \leq |J| \leq r\), we have

\[
(\pi^{r+1,r}')^*I \rho = I(\pi^{r+1,r}')^* \rho. \tag{195}
\]

The canonical decomposition of the form \( \rho \) yields

\[
(\pi^{r+1,r}')^*I \rho = I(\pi^{r+1,r}')^* \rho = I \left( \sum_{0 \leq l \leq q} p_l I \rho \right) = \sum_{0 \leq l \leq q} Ip_l I \rho. \tag{196}
\]

But by (191), \( Ip_l I \rho \) is \((l-1)\)-contact; thus, applying \( p_k \) to both sides of (195) and comparing \( k \)-contact components, we get (194).

(b) Let \( q \geq n + 1 \) and suppose we have a strongly contact \( q \)-form \( \rho \) on \( W' \). Then, \( \rho = \eta + dt \tau \) for some \( q \)-form \( \eta \) and \((q-1)\)-form \( \tau \) such that \( p_{q-n} \eta = 0 \) and \( p_{q-n-1} \tau = 0 \); hence, \( I \rho = I \eta + Id \tau = I \eta + \tau - dI \tau - \tau_0 \), where \( \tau_0 \) is a \((q-1)\)-form on \( U \). If \( q > n + 1 \), then always \( \tau_0 = 0 \). If \( q = n + 1 \), then always \( dt \tau_0 = 0 \), and we may replace \( \tau \) with \( \tau - \tau_0 \); then, \( I \rho = I \eta + \tau - dI \tau \). The \((q-1)\)-form \( I \eta + \tau \) satisfies

\[
p_{q-n-1}(I \eta + \tau) = Ip_{q-n} \eta + p_{q-n-1} \tau = p_{q-n-1} \tau = 0. \tag{197}
\]

If \( q \geq n + 2 \), then \( q - n - 2 \geq 0 \) and \( p_{q-n-2} I \tau = Ip_{q-n-1} \tau = 0 \); consequently, \( I \rho \) is strongly contact. If \( q = n + 1 \), then from (195), \( h \tau = 0 \) as required. \( \square \)

Corollary 1 (The fibered Volterra–Poincare lemma) If \( d \rho = 0 \), then there exists a \((q-1)\)-form \( \eta \) such that \( \rho = d \eta \).

The following two theorems extend the fibered Volterra-Poincare lemma to contact and strongly contact forms. Their proofs are based on the trace decomposition theorem (Sect. 2.2, Theorem 3), Appendix 9, Theorem 4, and on the fibered Volterra-Poincare lemma.

Theorem 13 Let \( 1 \leq q \leq n \) and let \( \rho \) be a contact \( q \)-form such that \( d \rho = 0 \). Then \( \rho = d \eta \) for some contact \((q-1)\)-form \( \eta \).
Proof

1. Let $\rho$ be a contact 1-form, expressed as

$$\rho = \sum_{0 \leq |J| \leq r-1} \Phi^I_j \omega^j_J. \quad (198)$$

Then,

$$d\rho = \sum_{0 \leq |J| \leq r-1} (d\Phi^I_j \wedge \omega^j_J - \Phi^I_j dy^j_J \wedge dx^I). \quad (199)$$

Condition $d\rho = 0$ implies, for $|J| = r-1$, $\Phi^I_j \delta^j_I = 0$ Sym($Jk$), and the trace operation yields, up to the factor $(n + r - 1)/r$,

$$\Phi^I_j = 0. \quad (200)$$

Thus, $\rho$ must be of the form

$$\rho = \sum_{0 \leq |J| \leq r-2} \Phi^I_j \omega^j_J. \quad (201)$$

Repeating the same procedure, we get $\rho = 0$.

2. Let $2 \leq q \leq n$. We show in several steps that if $\rho$ is a contact $q$-form such that $d\rho = 0$, then there exist a contact $q$-form $\tau$ and a contact $(q-1)$-form $\kappa$ such that

$$\rho = \tau + d\kappa, \quad p_1 \tau = 0. \quad (202)$$

First, we find a decomposition

$$\rho = \rho_0 + \tau_0 + d\kappa_0, \quad (203)$$

with the following properties:

(a) $\rho_0$ is generated by the forms $\omega^I_j$ such that $0 \leq |J| \leq r-1$,

$$\rho_0 = \sum_{0 \leq |J| \leq r-2} \omega^I_j \wedge \Phi^I_j + \sum_{|I|=r-1} \omega^I_j \wedge \Delta^I_j, \quad (204)$$

where the $(q-1)$-forms $\Delta^I_j$ are traceless.

(b) $\tau_0$ is generated by $\omega^I_j \wedge \omega^j_I$ and $\omega^I_j \wedge d\omega^I_L$, where $|J| = r-1$, $0 \leq |I| \leq r-1$, $|L| = r-1$.

(c) $\kappa_0$ is a contact $(q-1)$-form.
Expressing \( \rho \) as in Sect. 2.3, Corollary 2, we have

\[
\rho = \sum_{0 \leq |J| \leq r-2} \omega^\sigma_j \wedge \Phi^J_\sigma + \sum_{|J|=r-1} \omega^\sigma_j \wedge \Phi^J_\sigma + d\kappa_0, \tag{205}
\]

where \( \kappa_0 \) is a contact \((q-1)\)-form. Decompose the \((q-1)\)-forms \( \Phi^J_\sigma \), indexed with multi-indices \( J \) of length \( r-1 \), by the trace operation. We get a decomposition

\[
\Phi^J_\sigma = \Delta^J_\sigma + Z^J_\sigma, \tag{206}
\]

where the expression \( \Delta^J_\sigma \) is the traceless and \( Z^J_\sigma \) is the contact component. Then,

\[
\rho = \sum_{0 \leq |J| \leq r-2} \omega^\sigma_j \wedge \Phi^J_\sigma + \sum_{|J|=r-1} \omega^\sigma_j \wedge \Delta^J_\sigma + \sum_{|J|=r-1} \omega^\sigma_j \wedge Z^J_\sigma + d\kappa_0. \tag{207}
\]

Setting

\[
\rho_0 = \sum_{0 \leq |J| \leq r-2} \omega^\sigma_j \wedge \Phi^J_\sigma + \sum_{|J|=r-1} \omega^\sigma_j \wedge \Delta^J_\sigma, \quad \tau_0 = \sum_{|J|=r-1} \omega^\sigma_j \wedge Z^J_\sigma, \tag{208}
\]

we get (203).

Second, we show that \( \rho \) has a decomposition

\[
\rho = \rho_1 + \tau_1 + d\kappa_1 \tag{209}
\]

with the following properties:

(a) The form \( \rho_1 \) is generated by the contact forms \( \omega^\sigma_j \), such that \( 0 \leq |J| \leq r-2 \), that is,

\[
\rho_1 = \sum_{0 \leq |J| \leq r-3} \omega^\sigma_j \wedge \Phi^J_\sigma + \sum_{|J|=r-2} \omega^\sigma_j \wedge \Delta^J_\sigma, \tag{210}
\]

where the \((q-1)\)-forms \( \Delta^J_\sigma \) are traceless.

(b) \( \tau_1 \) is generated by \( \omega^\sigma_j \wedge \omega^\sigma_I \) and \( \omega^\sigma_j \wedge d\omega^\sigma_L \), where \( |J| = r-1 \), \( 0 \leq |I| \leq r-1 \), \( |L| = r-1 \).

(c) \( \kappa_1 \) is a contact \((q-1)\)-form.
Indeed, we apply condition \( d\rho = 0 \) to expression (203). We have, since \( d\omega_J^y = -dy_J^y \wedge dx^l \),

\[
\sum_{0 \leq |I| \leq r-2} d(\omega_J^y \wedge \Phi^I_J) - \sum_{|J|=r-1} (dy_J^y \wedge dx^l \wedge \Delta^I_J + \omega_J^y \wedge d\Delta^I_J) + d\tau_0 = 0.
\] (211)

But the terms \( dy_J^y \wedge dx^l \wedge \Delta^I_J \) in this expression do not contain any form \( \omega_J^y \) or \( d\omega_J^y \) and must vanish separately. Thus,

\[
\sum_{|J|=r-1} dy_J^y \wedge dx^l \wedge \Delta^I_J = 0.
\] (212)

The 1-contact component gives

\[
\sum_{|J|=r-1} \omega_J^y \wedge h(dx^l \wedge \Delta^I_J) = 0
\] (213)

hence

\[
h(dx^l \wedge \Delta^I_J) = 0 \quad \text{Sym}(Jf).
\] (214)

The traceless form \( \Delta^I_J \) can be expressed as

\[
\Delta^I_J = A^I_{ij} dx^i \wedge dx^j \wedge \cdots \wedge dx^l
\]

\[
+ A^I_{i2} dy_{\sigma_2}^i \wedge dx^j \wedge dx^l \wedge \cdots \wedge dx^{\sigma_q}
\]

\[
+ A^I_{i3} dy_{\sigma_2}^i \wedge dy_{\sigma_3} dx^j \wedge dx^l \wedge \cdots \wedge dx^{\sigma_q}
\]

\[
+ \cdots + A^I_{i4} dy_{\sigma_2}^i \wedge dy_{\sigma_3} \cdots dy_{\sigma_q} dx^j \wedge dx^l \wedge \cdots \wedge dx^{\sigma_q}
\]

\[
+ \cdots + A^I_{i5} dy_{\sigma_2}^i \wedge dy_{\sigma_3} \cdots dy_{\sigma_q} \cdots dy_{\sigma_{q-1}} dx^j \wedge dx^l \wedge \cdots \wedge dx^{\sigma_q}
\]

(215)

where the multi-indices \( I_2, I_3, \ldots, I_q \) satisfy \( |I_2|, |I_3|, \ldots, |I_q| = r \) and all coefficients \( A^I_{i2}, A^I_{i3}, \ldots, A^I_{i4}, \ldots, \) \( A^I_{i5}, \ldots, \) are traceless in the indices \( i_3, i_4, \ldots, i_q \) and the multi-indices \( I_2, I_3, \ldots, I_q-1 \). Then, Eq. (214) reads

\[
(A^I_{ij} + A^I_{i2} y_{\sigma_2}^i \cdots y_{\sigma_q}^i) dx^j \wedge dx^l \wedge \cdots \wedge dx^{\sigma_q}
\]

\[
+ \cdots + A^I_{i4} y_{\sigma_2}^i \cdots y_{\sigma_q}^i \cdots y_{\sigma_{q-1}} dx^j \wedge dx^l \wedge \cdots \wedge dx^{\sigma_q-1}
\]

\[
+ A^I_{i5} y_{\sigma_2}^i \cdots y_{\sigma_q}^i \cdots y_{\sigma_{q-1}} \cdots y_{\sigma_{q-1}} dy_{\sigma_{q-1}}
\]

\[
\cdot \delta^I_{ij} dx^j \wedge dx^l \wedge \cdots \wedge dx^{\sigma_q} = 0 \quad \text{Sym}(Jf).
\] (216)
Setting

\[
\begin{align*}
B_{j=1}^{H} & = A_{i=1}^{J} \delta_{i}^{j} \text{ Sym}(J) \quad \text{Alt}(i_1 i_2 i_3 \ldots i_q), \\
B_{v=1}^{H_2} & = A_{v=1}^{H_2} \delta_{v}^{j} \text{ Sym}(J) \quad \text{Alt}(i_1 i_2 i_4 \ldots i_q), \\
B_{\sigma=1}^{H_3} & = A_{\sigma=1}^{H_3} \delta_{\sigma}^{j} \text{ Sym}(J) \quad \text{Alt}(i_1 i_4 i_5 \ldots i_q), \\
& \quad \vdots \\
B_{\sigma_1}^{H_l} & = A_{\sigma_1}^{H_l} \delta_{\sigma_1}^{j} \text{ Sym}(J) \quad \text{Alt}(i_1 i_q), \\
B_{\sigma_1}^{H_l} & = A_{\sigma_1}^{H_l} \delta_{\sigma_1}^{j} \text{ Sym}(J), \\
& \quad \vdots \\
& \quad \vdots \\
& \quad \vdots
\end{align*}
\]  

(217)

we get the system

\[
\begin{align*}
B_{j=1}^{H} & = 0, \\
B_{v=1}^{H_2} & = 0 \text{ Sym}(I_2) \quad \text{Alt}(i_1 i_2 i_3 \ldots i_q), \\
B_{\sigma=1}^{H_3} & = 0 \text{ Sym}(I_2) \quad \text{Alt}(i_1 i_2 i_3 \ldots i_q), \\
& \quad \vdots \\
B_{\sigma_1}^{H_l} & = 0 \text{ Sym}(I_2 \ldots i_q) \quad \text{Alt}(i_1 i_2 i_3 \ldots i_q). \\
& \quad \vdots \\
& \quad \vdots
\end{align*}
\]  

(218)

Since the unknown functions, \(B_{v=1}^{H_2}, B_{\sigma=1}^{H_3}, \ldots, B_{\sigma_1}^{H_l}, \ldots \), are traceless, for each fixed multi-index \(I = Jl\) and each index \(v\), this system has only the trivial solution (see Appendix 9), and we have from (217)

\[
\begin{align*}
A_{j=1}^{J} & = 0 \text{ Sym}(J) \quad \text{Alt}(i_1 i_2 i_3 \ldots i_q), \\
A_{v=1}^{H_2} & = 0 \text{ Sym}(J) \quad \text{Alt}(i_1 i_3 i_4 \ldots i_q), \\
A_{\sigma=1}^{H_3} & = 0 \text{ Sym}(J) \quad \text{Alt}(i_1 i_4 i_5 \ldots i_q), \\
& \quad \vdots \\
A_{\sigma_1}^{H_l} & = 0 \text{ Sym}(J) \quad \text{Alt}(i_1 i_q), \\
A_{\sigma_1}^{H_l} & = 0 \text{ Sym}(J). \\
& \quad \vdots \\
& \quad \vdots
\end{align*}
\]  

(219)

The solutions of this system are of Kronecker type; we have, denoting the multi-index \(J = Kk\),

\[
\begin{align*}
A_{j=1}^{K} & = C_{j=1}^{K} \delta_{j}^{k} \text{ Sym}(Kk) \quad \text{Alt}(i_2 i_3 i_4 \ldots i_q), \\
A_{v=1}^{K} & = C_{v=1}^{K} \delta_{v}^{k} \text{ Sym}(Kk) \quad \text{Alt}(i_3 i_4 i_5 \ldots i_q), \\
A_{\sigma=1}^{K} & = C_{\sigma=1}^{K} \delta_{\sigma}^{k} \text{ Sym}(Kk) \quad \text{Alt}(i_4 i_5 i_6 \ldots i_q), \\
& \quad \vdots \\
A_{\sigma_1}^{K} & = C_{\sigma_1}^{K} \delta_{\sigma_1}^{k} \text{ Sym}(Kk), \\
A_{\sigma_1}^{K} & = 0.
\end{align*}
\]  

(220)
Consequently,

\[
\sum_{|J|=r-1} \omega^\sigma_J \wedge \Delta^J = \omega^\sigma_K \wedge (C^K_{\nu\lambda\cdots\mu} \delta^k_{J_1} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q})
+ C^{KL}_{\nu\rho\cdots\sigma_{q+1}} \delta^k_{J_1} \frac{dy^\sigma_{J_2}}{L_{J_2}} \wedge \frac{dy^\sigma_{J_3}}{L_{J_3}} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}
+ \cdots + C^{KL}_{\nu\rho\cdots\sigma_{q+1}} \frac{L_{J_2}^{q-1}}{L_{J_2}} \delta^k_{J_1} \frac{dy^\sigma_{J_2}}{L_{J_2}} \wedge \frac{dy^\sigma_{J_3}}{L_{J_3}} \wedge \cdots \wedge \frac{dy^\sigma_{q+1}}{L_{J_2}} \wedge dx^{i_q})
\]

\[
= d\omega^\sigma_K \wedge (C^K_{\nu\lambda\cdots\mu} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q})
+ C^{KL}_{\nu\rho\cdots\sigma_{q+1}} \frac{dy^\sigma_{J_2}}{L_{J_2}} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}
- C^{KL}_{\nu\rho\cdots\sigma_{q+1}} \frac{dy^\sigma_{J_2}}{L_{J_2}} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}
+ \cdots + (-1)^{q-1} C^{KL}_{\nu\rho\cdots\sigma_{q+1}} \frac{dy^\sigma_{J_2}}{L_{J_2}} \wedge \frac{dy^\sigma_{J_3}}{L_{J_3}} \wedge \cdots \wedge \frac{dy^\sigma_{q+1}}{L_{J_2}}).
\]

This expression splits in two terms,

\[
d(\omega^\sigma_K \wedge (C^K_{\nu\lambda\cdots\mu} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q})
+ C^{KL}_{\nu\rho\cdots\sigma_{q+1}} \frac{dy^\sigma_{J_2}}{L_{J_2}} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}
- C^{KL}_{\nu\rho\cdots\sigma_{q+1}} \frac{dy^\sigma_{J_2}}{L_{J_2}} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}
+ \cdots + (-1)^{q-1} C^{KL}_{\nu\rho\cdots\sigma_{q+1}} \frac{dy^\sigma_{J_2}}{L_{J_2}} \wedge \frac{dy^\sigma_{J_3}}{L_{J_3}} \wedge \cdots \wedge \frac{dy^\sigma_{q+1}}{L_{J_2}}).
\]

and

\[
- \omega^\sigma_K \wedge d(-C^K_{\nu\lambda\cdots\mu} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q})
+ C^{KL}_{\nu\rho\cdots\sigma_{q+1}} \frac{dy^\sigma_{J_2}}{L_{J_2}} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}
- C^{KL}_{\nu\rho\cdots\sigma_{q+1}} \frac{dy^\sigma_{J_2}}{L_{J_2}} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}
+ \cdots + (-1)^{q-1} C^{KL}_{\nu\rho\cdots\sigma_{q+1}} \frac{dy^\sigma_{J_2}}{L_{J_2}} \wedge \frac{dy^\sigma_{J_3}}{L_{J_3}} \wedge \cdots \wedge \frac{dy^\sigma_{q+1}}{L_{J_2}})
\]

which can be distributed to the terms \(dk_0\) and \(\rho_0\) in the decomposition (207). Therefore, \(\rho\) can be written as

\[
\rho = \sum_{0 \leq |J| \leq r-2} \omega^\sigma_J \wedge \Phi^J + \sum_{|J|=r-1} \omega^\sigma_J \wedge \Delta^J + \sum_{|J|=r-1} \omega^\sigma_J \wedge Z^J + d\kappa_0
\]

\[
= \sum_{0 \leq |J| \leq r-2} \omega^\sigma_J \wedge \Phi^J + \sum_{|J|=r-1} \omega^\sigma_J \wedge Z^J + d\kappa_1
\]

\[
= \sum_{0 \leq |J| \leq r-3} \omega^\sigma_J \wedge \Phi^J + \sum_{|J|=r-2} \omega^\sigma_J \wedge \Phi^J + \sum_{|J|=r-1} \omega^\sigma_J \wedge Z^J + d\kappa_1
\]
\[ \begin{align*}
&= \sum_{0 \leq |j| \leq r-3} \omega_j^\sigma \wedge \Phi^j_\sigma + \sum_{|j| = r-2} \omega_j^\sigma \wedge \Phi^j_\sigma + \sum_{|j| = r-1} \omega_j^\sigma \wedge Z^j_\sigma + d\kappa_1 \\
&= \sum_{0 \leq |j| \leq r-3} \omega_j^\sigma \wedge \Phi^j_\sigma + \sum_{|j| = r-2} \omega_j^\sigma \wedge \Delta^j_\sigma + \sum_{|j| = r-2} \omega_j^\sigma \wedge Z^j_\sigma + d\kappa_1 \\
&+ \sum_{|j| = r-1} \omega_j^\sigma \wedge Z^j_\sigma + d\kappa_1 \\
\end{align*} \] (224)

where we use the trace decomposition \( \Phi^j_\sigma = \Delta^j_\sigma + Z^j_\sigma \) for \( |j| = r-1 \).

Summarizing and replacing for simplicity of notation \( \Phi^j_\sigma \) with \( \Phi^j_\sigma \), we get the decomposition (209).

Third, we construct as in the second step the decompositions

\[ \rho_0 = \sum_{0 \leq |j| \leq r-2} \omega_j^\sigma \wedge \Phi^j_\sigma + \sum_{|j| = r-1} \omega_j^\sigma \wedge \Delta^j_\sigma, \]

\[ \rho_1 = \sum_{0 \leq |j| \leq r-3} \omega_j^\sigma \wedge \Phi^j_\sigma + \sum_{|j| = r-2} \omega_j^\sigma \wedge \Delta^j_\sigma, \]

\[ \vdots \]

\[ \rho_{r-2} = \omega^\sigma \wedge \Phi_\sigma + \sum_j \omega_j^\sigma \wedge \Delta^j_\sigma, \]

\[ \rho_{r-1} = \omega^\sigma \wedge \Delta_\sigma, \]

and

\[ \rho = \rho_0 + \tau_0 + d\kappa_0 = \rho_1 + \tau_1 + d\kappa_1 = \rho_2 + \tau_2 + d\kappa_2 \]

\[ \vdots = \rho_{r-2} + \tau_{r-2} + d\kappa_{r-2} = \rho_{r-1} + \tau_{r-1} + d\kappa_{r-1}. \] (226)

Note, however, the different meaning of the symbols \( \Phi^j_\sigma \) and \( \Delta^j_\sigma \) in the lines of expressions (225), which are defined in the construction.

Finally, we show that \( \rho \) has a decomposition

\[ \rho = \tau_{r-1} + d\kappa_{r-1}, \] (227)

where \( \tau_{r-1} \) is generated by the contact forms \( \omega_j^\sigma \wedge \omega_j^\sigma \) and \( \omega_j^\sigma \wedge d\omega_j^\sigma \), \( |j| = r-1, 0 \leq |j| \leq r-1, |L| = r-1 \) and \( \kappa_{r-1} \) is a contact \((q-1)\)-form.

It is sufficient to show that in the decomposition \( \rho = \rho_{r-1} + \tau_{r-1} + d\kappa_{r-1} \) (226), the form \( \rho_{r-1} \) vanishes. Condition \( d\rho = 0 \) implies

\[ d\omega^\sigma \wedge \Delta_\sigma - \omega^\sigma \wedge d\Delta_\sigma + d\tau_{r-1} = 0. \] (228)

The 1-contact component yields \( -\omega_j^\sigma \wedge dx^j \wedge h\Delta_\sigma - \omega^\sigma \wedge hd\Delta_\sigma = 0 \); hence,

\[ h(dx^j \wedge \Delta_\sigma) = 0. \] (229)
Writing the traceless form $\Delta_{\nu}$ as
\[
\Delta_{\nu} = A_{\nu i_1 \ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}
+ A^I_{\nu i_1 \ldots i_q} dy^I_{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}
+ A^I_{\nu i_1 \ldots i_q} dy^I_{i_2} \wedge dy^I_{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_q}
+ \cdots + A^I_{\nu i_1 \ldots i_q} dy^I_{i_{q-1}} \wedge \cdots \wedge dy^I_{i_q} \wedge dx^{i_q}
\]
we have
\[
h(dx^I \wedge \Delta_{\nu}) = \left(A_{\nu i_1 \ldots i_q} + A^I_{\nu i_1 \ldots i_q} y^I_{i_1} + A^{I^I}_{\nu i_1 \ldots i_q} y^I_{i_1} y^I_{i_2} y^I_{i_3}
+ \cdots + A^{I^I}_{\nu i_1 \ldots i_q} y^I_{i_1} y^I_{i_2} \cdots y^I_{i_{q-1}} y^I_{i_q} + A^{I^I}_{\nu i_1 \ldots i_q} y^I_{i_1} y^I_{i_2} \cdots y^I_{i_{q-1}} y^I_{i_q}ight)
\cdot dx^I \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_q} = 0,
\]
which implies, because the coefficients are traceless,
\[
A_{\nu i_1 \ldots i_q} = 0, \quad A^I_{\nu i_1 \ldots i_q} = 0, \quad A^{I^I}_{\nu i_1 \ldots i_q} = 0,
\]
\[
\cdots A^{I^I}_{\nu i_1 \ldots i_q} = 0, \quad A^{I^I}_{\nu i_1 \ldots i_q} = 0.
\]
Consequently, $\rho_{r-1} = 0$ proving (227).

3. To conclude the proof, we apply the contact homotopy decomposition to the form $\tau_{r-1}$ (Theorem 11). We have $\tau_{r-1} =Id\tau_{r-1} + dI\tau_{r-1}$. But $d\tau_{r-1} = 0$, and thus, $\tau_{r-1} = dI\tau_{r-1}$, and since the order of contactness of $\tau_{r-1}$ is $\geq 2$, we have $hI\tau_{r-1} = I\rho_1\tau_{r-1} = 0$, so $I\tau_{r-1}$ is contact. Then, however,
\[
\rho = Id\tau_{r-1} + dI\tau_{r-1} + d\kappa_{r-1} = d(I\tau_{r-1} + d\kappa_{r-1}).
\]
Setting $\eta = I\tau_{r-1} + d\kappa_{r-1}$, we complete the proof. $\square$

**Theorem 14** If $\rho$ is strongly contact and $d\rho = 0$, then there exists a strongly contact $(q-1)$-form $\eta$ such that $\rho = d\eta$.

**Proof** We express $\rho$ as $\rho = Id\rho + dI\rho$. But by hypothesis $d\rho = 0$, thus setting $\eta = I\rho$, we have $\rho = d\eta$; now, our assertion follows from Theorem 12, (b). $\square$

**Remark 11** The concept of a strongly contact form, used in Theorem 14, has been introduced by means of the exterior derivative $d$ and the pullback operation by the canonical jet projection $\pi^{r+1,r}: J^{r+1}Y \to J^rY$. The decompositions of the forms on $J^rY$, related to this concept, represent a basic tool in the higher-order variational theory on the jet spaces $J^rY$. A broader concept of a strongly contact form is considered in Chap. 8.
References


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