Chapter 2
Perturbation and Variational Methods

2.1 Introduction

In this chapter and Chap. 3, we will use mathematical methods (analytical and numerical methods) for solving strongly nonlinear systems in field dynamics and vibration. More of these methods are mathematics methods that have been introduced by Chinese scientists, especially Professor Ji-Huan He.

In the course of a professional career spanning 2 decades, He has made countless contributions in variational theory, asymptotic techniques, nanotechnology, life science, and high-energy particle physics as well.

In 1997, in his Ph.D. Thesis, He proposed a novel method called the *semi-inverse* method to search for variational formulations directly from field equations and boundary conditions without a Lagrange multiplier (He 1997b, c). In 1998, the well-known variational iteration method (VIM) was suggested by using general Lagrange multipliers and restricted variations (He 1998a, b). Many asymptotic techniques, including the homotopy perturbation method (HPM) (He 1999a), the energy method (He 2006a), modifications of the Lindstedt–Poincaré method (He 2001c), the bookkeeping parameter method (He 2001a), the parametrized perturbation method (He 1999b), the iteration perturbation method (He 2001b), and other methods and their applications were suggested by He during 1999–2006 (He 2000a, b, c, d, 1997a, 2004a, b, c, 2005a, b, c, 2006b, d, 2003a, b, 1998c, 1999c, 2002a, b, c, 2007, 2008; He and Wu 2006a, b; Shou and He 2008; He and Abdou 2007). For a relatively comprehensive survey on the method and its applications, the reader is referred to He's review article (He 2006a) and monograph (He 2006c).

In the following, we introduce some early important methods, with their applications in nonlinear systems and nonlinear equations in dynamics and vibrations, that are today being widely used in engineering and applied sciences.

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2.2 The Basic Ideas of Perturbation Analysis

Perturbation theory has its roots in early celestial mechanics, where the theory of epicycles was used to make small corrections to the predicted paths of planets. Curiously, it was the need for more and more epicycles that eventually led to the sixteenth century Copernican revolution in the understanding of planetary orbits. The development of basic perturbation theory for differential equations was fairly complete by the middle of the nineteenth century. It was at that time that Charles-Eugène Delaunay was studying the perturbative expansion for the Earth–Moon–Sun system and discovered the so-called "problem of small denominators." Here, the denominator appearing in the nth term of the perturbative expansion could become arbitrarily small, causing the nth correction to be as large as or larger than the first-order correction. At the turn of the twentieth century, this problem led Henri Poincaré to make one of the first deductions of the existence of chaos, or what is prosaically called the "butterfly effect": that even a very small perturbation can have a very large effect on a system.

Perturbation theory saw a particularly dramatic expansion and evolution with the arrival of quantum mechanics. Although perturbation theory was used in the semiclassical theory of the Bohr atom, the calculations were monstrously complicated and subject to somewhat ambiguous interpretation. The discovery of Heisenberg's matrix mechanics allowed a vast simplification of the application of perturbation theory. Notable examples are the Stark effect and the Zeeman effect, which have a simple enough theory to be included in standard undergraduate textbooks in quantum mechanics. Other early applications include the fine structure and the hyperfine structure in the hydrogen atom.

In modern times, perturbation theory underlies much of quantum chemistry and quantum field theory. In chemistry, perturbation theory was used to obtain the first solutions for the helium atom.

In the middle of the twentieth century, Richard Feynman realized that the perturbative expansion could be given a dramatic and beautiful graphical representation in terms of what are now called Feynman diagrams. Although originally applied only in quantum field theory, such diagrams now find increasing use in any area where perturbative expansions are studied.

A partial resolution of the small-divisor problem was given by the statement of the KAM theorem in 1954. Developed by Andrey Kolmogorov, Vladimir Arnold, and Jürgen Moser, this theorem stated the conditions under which a system of partial differential equations will have only mildly chaotic behavior under small perturbations.

In the late twentieth century, broad dissatisfaction with perturbation theory in the quantum physics community, including not only the difficulty of going beyond second order in the expansion, but also questions about whether the perturbative expansion is even convergent has led to a strong interest in the area of nonperturbative analysis—that is, the study of exactly solvable models. The prototypical model is the Korteweg–de Vries equation, a highly nonlinear equation for which the interesting solutions, the solitons, cannot be reached by perturbation theory, even if the perturbations were carried out to infinite order. Much of the theoretical work in nonperturbative analysis goes under the name of quantum groups and noncommutative geometry.

2.2.1 Variation of Free Constants and Systems in Standard Form

The main idea of perturbation methods is to consider systems being close to an unperturbed one. It is supposed that the solutions of the unperturbed system are easy to find. In other words, it is supposed that the unperturbed system can be integrated in a closed form (see Ref. Fidlin 2006).

Consider a system

$$\dot{z} = Z(z, t, \varepsilon). \tag{2.1}$$

Here, $\varepsilon \ll 1$ is a small parameter. Consider the corresponding unperturbed system as follows:

$$\dot{z}_0 = Z(z_0, t, 0). \tag{2.2}$$

We suppose that its general solution is known as

$$z_0 = f(t, C) \Leftrightarrow \frac{\partial f}{\partial t} = Z(f(t, C), t, 0). \tag{2.3}$$

Here, C is a vector of arbitrary constants. Taking C as a set of new variables—that is, considering Eq. 2.3 as a transformation for the perturbed system Eq. 2.1—the following equation can be easily obtained:

$$\frac{\partial f}{\partial C}\dot{C} + \frac{\partial f}{\partial t} = Z(f(t, C), t, \varepsilon) = Z(f(t, C), t, 0) + \varepsilon \frac{\partial Z}{\partial \varepsilon}|_{\varepsilon=0} + \cdots. \tag{2.4}$$

Here, the symbol $\cdots$ stands for the terms $O(\varepsilon^2)$. Taking Eq. 2.3 into account and supposing the matrix $(\partial f/\partial C)$ not to be degenerated—that is, $\det(\partial f/\partial C) \neq 0$—the following equation for the new variables C can be obtained:

$$\dot{C} = \varepsilon \left(\frac{\partial f}{\partial C}\right)^{-1} + \frac{\partial Z(f(t, C), t, \varepsilon)}{\partial \varepsilon}|_{\varepsilon=0} + \cdots. \tag{2.5}$$

A system in the form of Eq. 2.5—that is, a system in which the right-hand side is multiplied by a small parameter—is called a "system in the standard from" for the averaging method. Usually, it is written as

$$\dot{x} = \varepsilon X(x, t, \varepsilon). \tag{2.6}$$

Here, x is the n-dimensional vector of the state variables, X is the n-dimensional vector function depending on the state variables, time, and, perhaps, the small parameter ε.

Actually, the statement "the system (Eq. 2.6) is in the standard form because the small parameter stays as a factor in front of its right-hand side" is too simplified. The functions at the right-hand side of Eq. 2.6 have to be, in addition, bounded and smooth, and the time average of the right-hand sides must exist (see below). These additional conditions are not always easy to satisfy. Even if the unperturbed system is a linear excited and damped oscillator (Eq. 2.7) and the excitation and damping are not small, it cannot be directly transformed to the standard form:

$$m\ddot{x} + \beta\dot{x} + cx = a \sin \omega t, \;\; a = O(1), \;\; \beta = O(1). \tag{2.7}$$

Nevertheless, there are two large and very important classes of systems suitable for perturbation analysis.

The quasi-conservative, especially quasi-linear, systems belong to the first class. Systems with strong excitation—that is, systems with dominating external and inertial forces—belong to the second class. It is usual to write the governing equations for the problems of this class in a slightly different form:

$$\ddot{x} = F(x, \dot{x}, t) + \omega\Phi(x, t, \tau), \quad \tau = \omega t, \; \omega \gg 1. \tag{2.8}$$

This form expresses better the fact that the term $\omega\Phi(x, t, \tau)$, containing the high-frequency excitation, is dominating here (because ω is the large parameter).

There are many different methods for an asymptotic analysis of perturbed dynamical systems. First of all, these methods differ according to the type of solutions they deal with. There are numerous methods considering only periodic solutions and their stability. Most of them are based on the ideas of Poincaré and Liapunov.

Another group of methods also considers transient solutions of dynamical systems; that is, these methods allow analyzing an infinitesimal vicinity of periodic solutions and their attraction area.

$$\ddot{x} = F(x, \dot{x}, t) + \omega\Phi(x, t, \tau), \quad \tau = \omega t, \; \omega \gg 1. \tag{2.9}$$

Three of these methods are most popular today. We are going to start with the standard averaging.

2.2.2 *Standard Averaging as an Almost Identical Transformation*

Initial value problems in the standard form are investigated by averaging:

$$\dot{x} = \varepsilon X(x, \dot{x}, t), \qquad x(0) = x_0. \tag{2.10}$$

Let us first introduce the time average of the function X:

$$\langle X(x,t,\varepsilon)\rangle_t = \lim_{T\to\infty}\frac{1}{T}\int_0^T X(x,t,\varepsilon)\,\mathrm{d}t. \tag{2.11}$$

The integration here has to be performed with respect to the explicit time (variables x are considered as constant parameters).

If the function X is periodic with respect to the explicit time t, the definition of the time average can be significantly simplified:

$$\begin{aligned} &\forall t : X(x,t+2\pi,\varepsilon) = X(x,t,\varepsilon) \\ &\Rightarrow \langle X(x,t,\varepsilon)\rangle_t = \frac{1}{2\pi}\int_0^{2\pi} X(x,t,\varepsilon)\,\mathrm{d}t \end{aligned} \tag{2.12}$$

The main idea of the averaging method is not to try to solve the system (Eq. 2.10) but to try to find another system, simpler than the original one, in which solutions are close to the solutions of the original system for a sufficiently long time interval. The simplification that can be achieved using averaging is to eliminate the independent variable t from the considered equations—that is, to reduce the effective order of the system by one.

In order to do it formally (without mathematical proof) for the simplest periodic case, the following approximate identical transformation can be applied:

$$x = \xi + \varepsilon u(\xi,t,\varepsilon) + O\left(\varepsilon^2\right). \tag{2.13}$$

It is very important to understand the sense of this transformation in order to comprehend the physical meaning of the method. It splits the solution to Eq. 2.10 into two parts: the large slowly varying part ξ, describing the evolution of the system, and the small fast oscillating part u, which is responsible for the oscillations of the solution around the slow component.

Consider that the new variable ξ is governed by the autonomous equation

$$\dot{\xi} = \varepsilon\,\Xi(\xi,\varepsilon) + O\left(\varepsilon^2\right). \tag{2.14}$$

Both the unknown functions, $u(\xi,t,\varepsilon)$, which have to be periodic functions of time, and $\Xi(\xi,\varepsilon)$ have to be determined by the following procedure. Applying Eqs. 2.13 and 2.14 to Eq. 2.10, the following equation can be obtained:

$$\varepsilon\,\Xi(\xi,\varepsilon) + \varepsilon\frac{\partial u}{\partial t} = \varepsilon X(\xi,t,\varepsilon) + O\left(\varepsilon^2\right). \tag{2.15}$$

Balancing the terms $O(\varepsilon)$ and considering that u has to be a bounded periodic function, we obtain that this condition can be fulfilled only if Ξ is the time average of X:

$$\varepsilon\,\Xi(\xi,\varepsilon) = \langle X(\xi,t,\varepsilon)\rangle_t$$
$$u(\xi,t,\varepsilon) = \int_0^t (X(\xi,\vartheta,\varepsilon) - \Xi(\xi,\varepsilon))\mathrm{d}\vartheta + u_0(\xi,\varepsilon). \tag{2.16}$$

It is usual to choose the free functions $u_0(\xi,\varepsilon)$ in order to guarantee that the time average of the functions u is equal to zero—that is, $\langle u(\xi,t,\varepsilon)\rangle_t = 0$.

It is not the unique possible choice of the functions u_0. Another one is convenient if the Eq. 2.10 have the Hamiltonian form. Then the free functions can be chosen in order to guarantee that the averaged equations also have the Hamiltonian form.

Higher order approximations can be obtained in a similar way. For the second-order approximation, we apply the following transformation:

$$x = \xi + \varepsilon u_1(\xi,t,\varepsilon) + \varepsilon^2 u_2(\xi,t,\varepsilon) + \cdots. \tag{2.17}$$

We require further that the new variable ξ be governed by an autonomous equation:

$$\dot{\xi} = \varepsilon\,\Xi_1(\xi,\varepsilon) + \varepsilon^2\,\Xi_2(\xi,\varepsilon) + \cdots. \tag{2.18}$$

Balancing the terms $O(\varepsilon)$ we obtain, as above,

$$\begin{aligned} &\frac{\partial u_1}{\partial t} = X(\xi,t,\varepsilon) - \Xi_1(\xi,\varepsilon) \Rightarrow \\ &\Xi_1(\xi,\varepsilon) = \langle X(\xi,t,\varepsilon)\rangle_t, \quad u_1 = \int [X(\xi,t,\varepsilon) - \Xi_1(\xi,\varepsilon)]\mathrm{d}t, \\ &\langle u_1\rangle = 0 \end{aligned} \tag{2.19}$$

Balancing the terms $O(\varepsilon^2)$, we obtain, as above,

$$\begin{aligned} &\frac{\partial u_2}{\partial t} = \frac{\partial X}{\partial x}\Big|_{x=\xi}\,u_1 - \Xi_2 - \frac{\partial u_1}{\partial \xi}\,\Xi_1 \Rightarrow \\ &\Xi_2 = \left\langle \frac{\partial X}{\partial x}\Big|_{x=\xi}\,u_1 - \frac{\partial u_1}{\partial \xi}\,\Xi_1 \right\rangle_t = \left\langle \frac{\partial X}{\partial x}\Big|_{x=\xi}\,u_1 \right\rangle_t. \end{aligned} \tag{2.20}$$

Finally, the equation of the second-order approximation is

$$\dot{\xi} = \varepsilon\langle X(\xi,t,\varepsilon)\rangle + \varepsilon^2\left\langle \frac{\partial X}{\partial x}\Big|_{x=\xi}\,u_1 \right\rangle_t + O\left(\varepsilon^3\right). \tag{2.21}$$

The above procedure is purely formal, because it does not explain whether one can shorten the equations for ξ neglecting the small terms $O(\varepsilon^2)$ or $O(\varepsilon^3)$ in Eqs. 2.14 and 2.21, respectively. Neither does it explain why and for how long a time the solutions of the original system (Eq. 2.10) and those of the shortened averaged systems (Eqs. 2.14 or 2.21) are close to each other.

Answers to these questions were given by Bogoliubov in his first theorem. Consider the system (2.10) and assume:

1. X is a measurable with respect to t vector function.
2. It is bounded and satisfies the Lipschitz condition with respect to the vector argument x:

$$\begin{aligned} \|X(x,t,\varepsilon)\| &\leq M \\ \|X(x_1,t,\varepsilon) - X(x_2,t,\varepsilon)\| &\leq \lambda\|x_1 - x_2\| \end{aligned} \tag{2.22}$$

3. The time average of the function X exists uniformly with respect to x.

Consider the averaged system satisfying the same initial conditions:

$$\dot{\xi} = \varepsilon\,\Xi(\xi,\varepsilon), \quad \xi(0) = x_0. \tag{2.23}$$

Under these conditions, the mistake made by using the system (Eq. 2.23) with functions Ξ determined by the relationships (Eq. 2.16), instead of the original one, has the magnitude order of the small parameter ε for the asymptotically long time interval $t = O(1/\varepsilon)$.

The proof of this theorem is not very complex. Readers interested in the mathematical background can find it in Fidlin (2006).

If the averaged system (Eq. 2.23) has an asymptotically stable singular point in the linear approximation and the function X is continuously differentiable with respect to it, then the original system (Eq. 2.10) has a solution that remains in the vicinity of this point for infinite time.

For more details in this section refer to Fidlin (2006).

2.2.3 Method of Multiple Scales

In mathematics and physics, the method of multiple scales comprises techniques used to construct uniformly valid approximations to the solutions of perturbation problems, for both small and large values of the independent variables. This is done by introducing fast-scale and slow-scale variables for an independent variable and subsequently treating these variables, fast and slow, as if they are independent. In the solution process of the perturbation problem thereafter, the resulting additional freedom—introduced by the new independent variables—is used to remove (unwanted) secular terms. The latter puts constraints on the approximate solution and are called *solvability conditions* (see Ref. de Sterke and Sipe 1988).

The basic logic of the method of multiple scales can be easily illustrated by considering a system in the standard form (Eq. 2.10). The first step of the solution is to convert to two independent variables $\theta = t$ and $\tau = \varepsilon t$, supposing that

$x = \varphi(\theta, \tau)$—that is, to convert from the system of ordinary differential equations (Eq. 2.10) to a system with partial derivatives:

$$\frac{\partial \varphi}{\partial \theta} + \varepsilon \frac{\partial \varphi}{\partial \tau} = \varepsilon X(\varphi(\theta, \tau), \varepsilon). \tag{2.24}$$

The relationship between Eqs. 2.10 and 2.24 is determined by the condition that, if $\varphi(\theta, \tau)$ is a solution to Eq. 2.24, then $x = \varphi(t, \varepsilon t)$ is a solution to Eq. 2.10. In other words, the system (Eq. 2.24) is more general than the original equations (Eq. 2.10) and any solution (Eq. 2.24) taken along the straight line $\theta = t, \tau = \varepsilon t$ satisfies the equation (Eq. 2.10).

We require $\varphi(\theta, \tau)$ to be a 2π—periodic function of τ and try to find φ as a formal asymptotic expansion in terms of ε:

$$\varphi(\theta, \tau) = \psi_0(\theta, \tau) + \varepsilon \psi_1(\theta, \tau) + \cdots. \tag{2.25}$$

All the functions here have to be bounded functions of the fast time θ. Substituting this expression into Eq. 2.24 and balancing the terms with equal powers of ε, the following relationships can be obtained:

$$\begin{aligned} \varepsilon^0: &\quad \frac{\partial \psi_0}{\partial \theta} = 0 \\ \varepsilon^1: &\quad \frac{\partial \psi_0}{\partial \tau} + \frac{\partial \psi_1}{\partial \theta} = X(\psi_0(\theta, \tau), \theta, \varepsilon). \end{aligned} \tag{2.26}$$

The first of these equations means that ψ_0 depends only on the slow time τ:

$$\psi_0 = \xi(\tau). \tag{2.27}$$

Substituting Eq. 2.27 into the second relationship from Eq. 2.26, a simple equation for $\psi_1(\theta, \tau)$ can be obtained:

$$\frac{\partial \psi_1}{\partial t} = X(\xi(\tau), t, \varepsilon) - \frac{\partial \xi}{\partial \tau}. \tag{2.28}$$

The function $\psi_1(\theta, \tau)$ has to be a bounded function of θ; that is, its derivative can contain only oscillating components. It is possible if the function $\frac{\partial \xi}{\partial \tau}$ annihilates the constant component of X. It means that

$$\frac{\partial \xi}{\partial \tau} = \langle X(\xi, \theta, \varepsilon) \rangle_\theta. \tag{2.29}$$

Here, the average is calculated with respect to the fast time θ. Returning to the straight line $\theta = t, \tau = \varepsilon t$, we find that the slow component of the solution is governed by the equation

$$\frac{\partial \xi}{\partial \tau} = \langle X(\xi, t, \varepsilon) \rangle_t. \tag{2.30}$$

The fast oscillating small correction ψ_1 can be calculated as

$$\psi_1 = \int_0^t \left(X(\xi, \theta, \varepsilon) - \langle X(\xi, \theta, \varepsilon) \rangle_\theta \right) \mathrm{d}\theta + \psi_1^0(\xi, \varepsilon). \tag{2.31}$$

Comparing Eqs. 2.25 and 2.29–2.31 with the corresponding relationships from the previous subsection describing the averaging method (Eqs. 2.13, and 2.16) Eq. 2.23, it is easy to notice that they are identical ($\psi_1 \equiv u$). Considering the higher order terms in the expansion (Eq. 2.25), the higher order approximations to Eq. 2.10 can be obtained by the multiple scales technique. They are the same as those obtained by the averaging method.

2.2.4 Direct Separation of Motions

Direct separation of motions was formulated originally for systems of second-order differential equations, but it can be easily reformulated as follows.

Consider the system of ordinary differential equations (see Fidlin 2006):

$$\dot{x} = F(x, \varepsilon t) + \Phi(x, \varepsilon t, t). \tag{2.32}$$

The basic idea of the direct separation of motions is to consider only solutions that are a superposition of slow evolution and fast oscillations. The object of main interest is the slow component:

$$\begin{gathered} x(t) = \xi(\tau) + \psi(\tau, t) \\ \tau = \varepsilon t; \quad \langle \psi(\tau, t) \rangle_t = \frac{1}{2\pi} \int_0^{2\pi} \psi(\tau, t) \mathrm{d}t = 0 \end{gathered} \tag{2.33}$$

The next step is to go over from the system of n differential equations (Eq. 2.32) to a system of $2n$ integral–differential equations:

$$\begin{aligned} \dot{\xi} &= F(\xi, \tau) + \langle F(\xi + \psi(\xi, \tau), \tau) - F(\xi, \tau) \rangle_t + \langle \Phi(\xi + \psi(\xi, \tau), \tau, t) \rangle_t \\ \dot{\psi} &= F(\xi + \psi(\xi, \tau), \tau) - F(\xi, \tau) - \langle F(\xi + \psi(\xi, \tau), \tau) - F(\xi, \tau) \rangle_t \\ &\quad + \Phi(\xi + \psi(\xi, \tau), \tau, t) - \langle \Phi(\xi + \psi(\xi, \tau), \tau, t) \rangle_t \end{aligned} \tag{2.34}$$

The relationship between systems 2.32 and 2.34 is as follows: If a pair (ξ, ψ) is a solution to Eq. 2.24, then $x(t)$ determined according to Eq. 2.33 is automatically a solution to Eq. 2.32. It means that the system (2.34) is more general than the original one. This system is not more general, but it is at first sight more complex. Nevertheless, in many important cases, it is easy to solve with the assumption that the variable ξ in the second equation (2.34) is constant.

The system in the standard form (Eq. 2.10) can be considered as an example. In this case, we have

$$F(x, \varepsilon t) = 0; \quad \Phi(x, \varepsilon t, t) = \varepsilon X(x, t). \tag{2.35}$$

Substituting Eq. 2.35 into Eq. 2.34, the following equations can easily be obtained:

$$\begin{aligned} \dot{\xi} &= \varepsilon \langle X(\xi + \psi, t) \rangle_t \\ \dot{\psi} &= \varepsilon X(\xi + \psi, t) - \varepsilon \langle X(\xi + \psi, t) \rangle_t \end{aligned} \tag{2.36}$$

Solving the second equation of the system (Eq. 2.36) asymptotically is obvious:

$$\psi = \psi_0 + \varepsilon \psi_1 + \cdots. \tag{2.37}$$

Inserting this expression into the second equation (2.36) and balancing terms with the equal powers of the small parameter, we will have

$$\begin{aligned} \psi_0 &= 0 \\ \dot{\xi} &= \varepsilon \langle X(\xi, t) \rangle \\ \dot{\psi}_1 &= X(\xi, t) - \langle X(\xi, t) \rangle \end{aligned} \tag{2.38}$$

Equation 2.38 does not differ from the equations of the first-order approximation (2.14, 2.16, or 2.30) (Eq. 2.31).

Unfortunately, neither the method of multiple scales nor the method of the direct separation of motions has a mathematical proof differing from that for the standard averaging.

2.2.5 Relationship Between These Methods

All the considered methods are very useful and efficient in the analysis of engineering systems such as oscillating systems. All of them applied to the system in standard form give the same result. (It is actually the necessary condition for such a procedure to be called a method.) So selecting one of them in any special case is mainly a personal preference. From the author's point of view, the multiple scales and, especially, the direct separation of motions are slightly easier for practical use, as compared with the standard averaging method. Their main advantage is the straightforward algorithm used to solve the problem, which does not require an initial transformation of the system to the standard form. This transformation may sometimes be rather difficult.

But this statement is correct only for systems that are in standard form or can be transformed to it. The situation becomes much more interesting if it is impossible to transform a system to the standard form or one of the conditions (2.22 or 2.23) is not fulfilled. In such a case, one can try any of the described methods. The problem is how to make sure that the obtained results are correct. The main advantage of the averaging method becomes clear in these cases. There is a clear way, based on

Gronwall's lemma, to prove the accuracy of the averaging procedure. Thus, the method contains an instrument to generalize itself. This situation enables researchers to move away from pure physical intuition (being the most effective in many cases) and to take the path of rigorous mathematical analysis.

In different chapters, we will use these perturbation techniques.

2.2.6 Application

Example 2.1

First, consider the forced oscillation of an undamped pendulum,

$$x^2 + \frac{y^2}{\omega^2} = \text{constant}, \tag{2.39}$$

in which, without loss of generality, we can assume that $\omega_0 > 0, \omega > 0$ and $F > 0$ (since $F < 0$ implies a phase difference that can be eliminated by a change of time origin and corresponding modification of initial conditions).

Consider the equation

$$\sin x \approx x - \frac{1}{6}x^3. \tag{2.40}$$

Use Eq. 2.40 to allow for moderately large swings, which is accurate to 1 % for $|x| < 1$ radian (57°). Then Eq. 2.39 becomes approximately

$$\ddot{x} + \omega_0^2 x - \frac{1}{6}\omega_0^2 x^3 = F\cos\omega t. \tag{2.41}$$

Standardize the form of Eq. 2.41 considering the correlation

$$\tau = \omega t, \quad \Omega^2 = \omega_0^2/\omega^2 \ \ (\Omega > 0), \quad \Gamma = F/\omega^2. \tag{2.42}$$

We obtain

$$x'' + \Omega^2 x - \frac{1}{6}\Omega^2 x^3 = \Gamma\cos\tau, \tag{2.43}$$

where dashes represent differentiation with respect to τ. This is a special case of Duffing's equation, which is characterized by a cubic nonlinear term. If Eq. 2.43 actually arises by consideration of a pendulum, the coefficients and variables are all dimensionless.

The methods to be described depend on how small the nonlinear term is. Here, we assume that $\frac{1}{6}\Omega^2$ is small, and then

$$\frac{1}{6}\Omega^2 = \varepsilon_0. \tag{2.44}$$

Then Eq. 2.43 becomes

$$x'' + \Omega^2 x - \varepsilon_0 x^3 = \Gamma \cos \tau. \tag{2.45}$$

Instead of taking Eq. 2.45 as it stands, with $\Omega, \Gamma, \varepsilon_0$ as constants, we consider the family of differential equations

$$x'' + \Omega^2 x - \varepsilon x^3 = \Gamma \cos \tau, \tag{2.46}$$

where ε is a parameter occupying an interval I_ε that includes $\varepsilon = 0$. When $\varepsilon = \varepsilon_0$, we recover Eq. 2.45, and when $\varepsilon = 0$, we obtain the linearized equation corresponding to the family 2.46:

$$x'' + \Omega^2 x = \Gamma \cos \tau. \tag{2.47}$$

The solutions of (2.46) are now thought of as functions of both ε and τ, and we will write $x(\varepsilon, \tau)$.

The most elementary version of the perturbation method is to attempt a representation of the solutions of (2.46) in the form of a power series in ε:

$$x(\varepsilon, \tau) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \cdots, \tag{2.48}$$

whose coefficients $x_i(\tau)$ are only functions of τ. To form equations for $x_i(\tau), i = 0, 1, 2, \ldots$, substitute the series (Eq. 2.48) into Eq. 2.46:

$$(x_0'' + \varepsilon x_1'' + \cdots) + \Omega^2(x_0 + \varepsilon x_1 + \cdots) - \varepsilon(x_0 + \varepsilon x_1 + \cdots)^3 = \Gamma \cos \tau. \tag{2.49}$$

Since this is assumed to hold for every member of the family Eq. 2.46—that is, for every ε on I_ε—coefficients of the powers of ε must balance, and we obtain

$$x_0'' + \Omega^2 x_0 = \Gamma \cos \tau, \tag{2.50a}$$

$$x_1'' + \Omega^2 x_1 = x_0^3, \tag{2.50b}$$

$$x_2'' + \Omega^2 x_2 = 3x_0^2 x_1, \tag{2.50c}$$

and so on.

We shall be concerned only with periodic solutions having the period, 2π, of the forcing term. Then, for all ε on I_ε and for all τ,

$$x(\varepsilon, \tau + 2\pi) = x(\varepsilon, \tau). \tag{2.51}$$

By Eq. 2.51, it is sufficient that for all τ,

$$x_i(\tau + 2\pi) = x_i(\tau), \quad i = 0, 1, 2, \ldots. \tag{2.52}$$

Equation 2.51, together with the condition 2.52, is sufficient to provide the solutions required. For the present, note that Eq. 2.50a is the same as the "linearized equation" (2.47), necessarily, since putting $\varepsilon_0 = 0$ in Eq. 2.45 implies putting $\varepsilon = 0$ in Eq. 2.48. The major term in Eq. 2.48 is therefore a periodic solution of the linearized equation (2.47). It is therefore clear that this process restricts us to finding the solutions of the nonlinear equations that are close to (or branch from, or bifurcate from) the solution of the linearized equation. The method will not expose any other periodic solutions. The zero-order solution $x_0(\tau)$ is known as a generating solution for the family of Eq. 2.46.

Example 2.2

2.2.7 Introduction

This example is concerned with responses of systems with two degrees of freedom and cubic nonlinearities to multifrequency parametric excitations governed by the following equations:

$$\begin{aligned}
&\ddot{X}_1 + \omega_1^2 X_1 + \varepsilon\Big[2\mu_1\dot{X}_1 + \alpha_1 X_1^3 + 3\alpha_2 X_1^2 X_2 + \alpha_3 X_1 X_2^2 + \alpha_4 X_2^3 \\
&\qquad\qquad + 2\sum_{m=1}^{M}\{X_1 f_{1m} + X_2 f_{2m}\}\cos(\Omega_m t + \tau_{1m})\Big] = 0, \\
&\ddot{X}_2 + \omega_2^2 X_2 + \varepsilon\Big[2\mu_2\dot{X}_2 + \alpha_2 X_1^3 + 3\alpha_3 X_1^2 X_2 + \alpha_4 X_1 X_2^2 + \alpha_5 X_2^3 \\
&\qquad\qquad + 2\sum_{n=1}^{M}\{X_1 g_{1n} + X_2 g_{2n}\}\cos(\Omega_n t + \gamma_{1n})\Big] = 0.
\end{aligned} \tag{2.53}$$

where $\omega_n, \mu_n, \alpha_n, f_{mn}, g_{mn}, \Omega_m, \Omega_n, \tau_{1m}$, and γ_{1n} are constants, ε is a small dimensionless parameter, and dots indicate differentiation with respect to the time t. These equations when quadratic terms are included model the responses of ships and bowed structural elements.

2.2.8 The Method of Multiple Scales

To determine a first-order uniform solution of Eq. 2.53, we use the method of multiple scales and let

$$X(t;\varepsilon) = X_{n0}(T_0, T_1) + \varepsilon X_{n1}(T_0, T_1) + \cdots, \tag{2.54}$$

where $T_0 = t$ is a fast scale, which is associated with changes occurring at the frequencies ω_n, Ω_m, and Ω_n and $T_1 = \varepsilon t$ is a slow scale, which is associated with modulations in the amplitudes and phases resulting from the nonlinearities and parametric resonances.

In terms of T_0 and T_1, the time derivative becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \varepsilon D_1 + \cdots, \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots, \tag{2.55}$$

where $D_n = \partial/\partial T_n$. Substituting Eqs. 2.54 and 2.55 into Eq. 2.53 and equating coefficients of power ε, we obtain

$$D_0^2X_{10} + \omega_1^2X_{10} = 0, \qquad D_0X_{20}^2 + \omega_1X_{20} = 0, \tag{2.56}$$

$$\begin{aligned} D_0^2X_{11} + \omega_1^2X_{11} = & -2D_0D_1X_{10} - 2\mu_1(D_0X_{10}) - \alpha_1X_{10}^3 - 3\alpha_2X_{10}^2X_{20} - 3\alpha_3X_{10}X_{20}^2 \\ & - \alpha_4X_{20}^3 + 2\sum_{m=1}^{M}\{X_{10}f_{1m} + X_{20}f_{2m}\}\cos(\Omega_mt + \tau_{1m}), \end{aligned} \tag{2.57}$$

$$\begin{aligned} D_0^2X_{21} + \omega_2^2X_{21} = & -2D_0D_1X_{20} - 2\mu_2(D_0X_{20}) - \alpha_2X_{10}^3 - 3\alpha_3X_{10}^2X_{20} - 3\alpha_4X_{10}X_{20}^2 \\ & - \alpha_5X_{20}^3 + 2\sum_{n=1}^{M}\{X_{10}g_{1n} + X_{20}g_{2n}\}\cos(\Omega_nt + \gamma_{1n}). \end{aligned} \tag{2.58}$$

The solution of Eq. 2.56 can be expressed as

$$X_{10} = A_1\exp(i\omega_1T_0) + \mathrm{cc}, \qquad X_{20} = A_2\exp(i\omega_1T_0) + \mathrm{cc}, \tag{2.59}$$

where cc denotes the complex conjugate of the preceding terms. Inserting Eq. 2.59 into Eq. 2.58 yields

$$\begin{aligned} D_0^2X_{11} + \omega_1^2X_{11} = & [-2i\omega_1(A' + \mu_1A_1) + 3\alpha_1A_{10}^2\overline{A}_{10} + 6\alpha_3A_1A_2^2\overline{A}_2]\exp(i\omega_1T_0) \\ & - [6\alpha_2A_1\overline{A}_1A_2 + 3\alpha_4A_2^2\overline{A}_2]\exp(2i\omega_2T_0) \\ & - \alpha_1A_1^3\exp(3i\omega_1T_0) - \alpha_4A_2^3\exp(3i\omega_2T_0) \\ & - 3\alpha_2[A_1^2A_2\exp i(2\omega_1 + \omega_2)T_0 + \overline{A}_1^2A_2\exp i(\omega_2 - 2\omega_1)T_0] \\ & - 3\alpha_3[A_1A_2^2\exp i(\omega_1 + 2\omega_2)T_0 + A_1\overline{A}_2^2\exp i(\omega_1 - 2\omega_2)T_0 \\ & \qquad + \overline{A}_1A_2^2\exp i(2\omega_2 - \omega_1)T_0] \\ & - A_1\sum_{m=1}^{M}f_{1m}\exp((\Omega_m{+}\omega_1)T_0 + \tau_{1m}) - \overline{A}_1\sum_{m=1}^{M}f_{1m}\exp((\Omega_m{-}\omega_1)T_0 + \tau_{1m}) \\ & - A_2\sum_{m=1}^{M}f_{2m}\exp((\Omega_m{+}\omega_2)T_0 + \tau_{1m}) - \overline{A}_2\sum_{m=1}^{M}f_{2m}\exp((\Omega_m{-}\omega_2)T_0 + \tau_{1m}), \end{aligned} \tag{2.60}$$

$$\begin{aligned} D_0^2X_{21} + \omega_2^2X_{21} = & -[2i\omega_2(A_2' + \mu_2A_2) + 6\alpha_3A_1\overline{A}_1A_2 + 3\alpha_5\overline{A}_2^2A_2]\exp(i\omega_2T_0) \\ & - [3\alpha_2A_1^2\overline{A}_1 + 6\alpha_4A_2\overline{A}_2]\exp(i\omega_1T_0) \\ & - \alpha_2A_1^3\exp(3i\omega_1T_0) - \alpha_5A_2^3\exp(3i\omega_2T_0) \\ & - 3\alpha_4[A_1A_2^2\exp i(\omega_1 + 2\omega_2)T_0 + A_1\overline{A}_2^2\exp i(\omega_1 - 2\omega_2)T_0] \\ & - 3\alpha_3[A_1A_2^2\exp i(\omega_1 + 2\omega_2)T_0 + A_1\overline{A}_2^2\exp i(\omega_1 - 2\omega_2)T_0 \\ & \qquad + \overline{A}_1^2A_2^2\exp i(\omega_2 + 2\omega_1)T_0] \\ & - A_1\sum_{n=1}^{N}g_{1n}\exp((\Omega_n{+}\omega_1)T_0 + \gamma_{1n}) + \overline{A}_1\sum_{n=1}^{N}g_{1n}\exp((\Omega_n{-}\omega_1)T_0 + \gamma_{1n}) \\ & - A_2\sum_{n=1}^{N}g_{2n}\exp((\Omega_n{+}\omega_2)T_0 + \gamma_{1n}) - \bar{A}_2\sum_{n=1}^{N}g_{2n}\exp((\Omega_n{-}\omega_2)T_0 + \gamma_{1n}), \end{aligned}$$

where the over term indicates the complex conjugate and the prime indicates differentiation with respect to T_1. Any particular solution of Eq. 2.60 contains secular or small divisor terms depending on the resonant conditions (1) $\omega_2 \cong 2\omega_1$, internal resonance, and (2) $\Omega_r \cong 2\omega_1$, principal parametric resonance of the first mode. To treat this case, we introduce detuning parameters σ_1 and σ_2 to convert the small divisor terms into secular terms, defined according to the correlation

$$\omega_2 = 2\omega_1 + \varepsilon\sigma_1, \qquad \Omega_r = 2\omega_1 + \varepsilon\sigma_2. \tag{2.61}$$

Substituting Eq. 2.61 into Eq. 2.60 and eliminating the secular terms from X_{11} and X_{21}, we obtain

$$\begin{aligned}
&2i\omega_1(A_1' + \mu_1 A_1) + 3\alpha_1 A_1^2\overline{A}_1 + 6\alpha_3 A_1 A_2 \overline{A}_2 \\
&\quad + 3\alpha_2 \overline{A}{}_1^2 A_2 \exp(i\sigma_1 T_1) + \overline{A}_1 f_{1r} \exp i(\sigma_2 T_1 + \tau_{1r}) = 0, \\
&2i\omega_2(A_2' + \mu_2 A_2) + 6\alpha_3 A_1 \overline{A}_1 A_2 + 3\alpha_5 A_2^2 A_2 \overline{A}_2 + \alpha_2 A_1^3 \exp(-i\sigma_2 T_1) = 0.
\end{aligned} \tag{2.62}$$

Consequently, the particular solutions of Eq. 2.60 are

$$\begin{aligned}
U_{11} = &-\left[\frac{6\alpha_2 A_1 \overline{A}_1 A_2 + 3\alpha_4 A_2^2 \overline{A}_2}{\omega_1^2 - \omega_2^2}\right]\exp(i\omega_2 T_0) \\
&+ \left[\frac{\alpha_1 A_1^3}{8\omega_1^2}\right]\exp(3i\omega_1 T_0) - \left[\frac{\alpha_4 A_2^3}{\omega_1^2 - 9\omega_2^2}\right]\exp(3i\omega_2 T_0) \\
&- \left[\frac{3\alpha_2 A_1^2 A_2}{\{\omega_1^2 - (2\omega_1 + \omega_2)^2\}}\right]\exp i(2\omega_1 + \omega_2)T_0 \\
&- \left[\frac{3\alpha_2 \overline{A}{}_1^2 A_2}{\{\omega_1^2 - (\omega_2 - 2\omega_1)^2\}}\right]\exp i(\omega_2 - 2\omega_1)T_0 \\
&- \left[\frac{3\alpha_3 A_1 A_2^2}{\{\omega_1^2 - (\omega_1 + 2\omega_2)^2\}}\right]\exp i(\omega_1 + 2\omega_2)T_0 \\
&- \left[\frac{3\alpha_3 A_1 \overline{A}{}_2^2}{\{\omega_1^2 - (\omega_1 - 2\omega_2)^2\}}\right]\exp i(\omega_1 - 2\omega_2)T_0 \\
&- \left[\frac{3\alpha_3 \overline{A}_1 A_2^2}{\{\omega_1^2 - (2\omega_2 - \omega_1)^2\}}\right]\exp i(2\omega_2 - \omega_1)T_0 \\
&- \left[\frac{f_{1r} A_1}{\{\omega_1^2 - (\Omega_r + \omega_1 + \tau_{1r})^2\}}\right]\exp i((\Omega_r + \omega_1)T_0 + \tau_{1r})
\end{aligned}$$

$$
\begin{aligned}
& - \left[\frac{f_{2r} A_2}{\{\omega_1^2 - (\Omega_r + \omega_2 + \tau_{2r})^2\}} \right] \exp i((\Omega_r + \omega_2) T_0 + \tau_{1r}) \\
& - \left[\frac{f_{2r} \overline{A}_2}{\{\omega_1^2 - (\Omega_r - \omega_2 + \tau_{2r})^2\}} \right] \exp i((\Omega_r - \omega_2) T_0 + \tau_{1r}).
\end{aligned}
$$

$$
\begin{aligned}
U_{21} = & - \left[\frac{3\alpha_2 A_1^2 \overline{A}_1 + 6\alpha_4 A_1 A_2 \overline{A}_2}{\omega_2^2 - \omega_1^2} \right] \exp(i\omega_1 T_0) \\
& - \left[\frac{\alpha_2 A_1^3}{\omega_2^2 - 9\omega_1^2} \right] \exp(3i\omega_1 T_0) + \left[\frac{\alpha_5 A_2^3}{8\omega_2^2} \right] \exp(3i\omega_2 T_0) \\
& - \left[\frac{3\alpha_3 A_1^2 A_2}{\{\omega_1^2 - (2\omega_1 - \omega_2)^2\}} \right] \exp i(2\omega_1 + \omega_2) T_0 \\
& - \left[\frac{3\alpha_3 \overline{A}_1^2 A_2}{\{\omega_2^2 - (\omega_2 - 2\omega_1)^2\}} \right] \exp i(\omega_2 - 2\omega_1) T_0 \\
& - \left[\frac{3\alpha_4 A_1 A_2^2}{\{\omega_2^2 - (\omega_1 + 2\omega_2)^2\}} \right] \exp i(\omega_1 + 2\omega_2) T_0 \\
& - \left[\frac{3\alpha_4 A_1 \overline{A}_2^2}{\{\omega_2^2 - (\omega_1 - 2\omega_2)^2\}} \right] \exp i(\omega_1 - 2\omega_2) T_0 \\
& - \left[\frac{3\alpha_4 \overline{A}_1 A_2^2}{\{\omega_2^2 - (2\omega_2 - \omega_1)^2\}} \right] \exp i(2\omega_2 - \omega_1) T_0 \\
& - \left[\frac{g_{1s} A_1}{\{\omega_2^2 - (\Omega_s + \omega_1 + \gamma_{1s})^2\}} \right] \exp i((\Omega_s + \omega_1) T_0 + \gamma_{1s}) \\
& - \left[\frac{g_{1s} \overline{A}_1}{\{\omega_2^2 - (\Omega_s - \omega_1 + \gamma_{1s})^2\}} \right] \exp i((\Omega_s + \omega_1) T_0 + \gamma_{1s}) \\
& - \left[\frac{g_{2r} A_2}{\{\omega_2^2 - (\Omega_s + \omega_2 + \gamma_{2s})^2\}} \right] \exp i((\Omega_s + \omega_2) T_0 + \gamma_{1s}).
\end{aligned}
\tag{2.63}
$$

Expressing A_n in the polar notation, we have

$$
A_n = \frac{1}{2} a_n \exp(i\beta_n), \tag{2.64}
$$

and separating the real and imaginary parts of Eq. 2.62, we obtain

$$
\begin{aligned}
a_1' &= -\mu_1 a_1 - \frac{3\alpha_2}{8\omega_1} a_1^2 a_2 \sin\theta_1 - \frac{1}{2\omega_1} f_{1r} a_1 \sin\theta_2, \\
a_1\beta_1' &= \frac{3\alpha_1}{8\omega_1} a_1^3 + \frac{3\alpha_3}{4\omega_1} a_1 a_2^2 + \frac{3\alpha_2}{8\omega_1} a_1^2 a_2 \cos\theta_1 + \frac{1}{2\omega_1} f_{1r} a_1 \cos\theta_2, \\
a_2' &= -\mu_2 a_2 + \frac{\alpha_2}{8\omega_2} a_1^3 \sin\theta_1, \\
a_2\beta_2' &= \frac{3\alpha_3}{4\omega_2} a_1^2 a_2 + \frac{3\alpha_5}{8\omega_2} a_2^3 + \frac{\alpha_2}{8\omega_2} a_1^3 \cos\theta_1.
\end{aligned}
\tag{2.65}
$$

where

$$
\theta_1 = \sigma_1 T_1 + \beta_2 - 3\beta_1, \qquad \theta_2 = \sigma_2 T_1 - 2\beta_1 + \tau_{1r}. \tag{2.66}
$$

Inserting Eqs. 2.58 and 2.63 into Eq. 2.54 yields the approximate solutions

$$
\begin{aligned}
U_1 = {} & a_1\cos(\omega_1 t + \beta_1) + \varepsilon\Bigg[-\left[\frac{6\alpha_2 a_1^2 a_2 + 3\alpha_4 a_2^2}{4\omega_1^2 - 4\omega_2^2}\right]\cos\psi_2 + \left[\frac{\alpha_1 a_1^3}{32\omega_1^2}\right]\cos 3\psi_1 - \left[\frac{\alpha_4 a_2^3}{4(\omega_1^2 - 9\omega_2^2)}\right]\cos 3\psi_2 \\
& - \left[\frac{3\alpha_2 a_1^2 a_2}{\{4\omega_1^2 - 4(2\omega_1 + \omega_2)^2\}}\right]\cos(2\psi_1 + \psi_2) - \left[\frac{3\alpha_2 a_1^2 a_2}{\{4\omega_1^2 - 4(2\omega_1 + \omega_2)^2\}}\right]\cos(\psi_2 - 2\psi_1) \\
& - \left[\frac{3\alpha_3 a_1 a_2^2}{\{4\omega_1^2 - 4(\omega_1 + 2\omega_2)^2\}}\right]\cos(\psi_1 + 2\psi_2) - \left[\frac{3\alpha_3 a_1 a_2^2}{\{4\omega_2^2 - 4(\omega_1 - 2\omega_2)^2\}}\right]\cos(\psi_1 - 2\psi_2) \\
& - \left[\frac{3\alpha_3 a_1 a_2^2}{\{4\omega_2^2 - 4(2\omega_2 - \omega_1)^2\}}\right]\cos(2\psi_2 - \psi_1) - \left[\frac{f_{1m} a_1}{\{\omega_1^2 - (\Omega_r + \omega_1 + \psi_{1r})^2\}}\right]\cos(\psi_3 + \psi_1) \\
& - \left[\frac{f_{2r} a_2}{\{\omega_1^2 - (\Omega_r + \omega_2 + \psi_{2r})^2\}}\right]\cos(\psi_3 + \psi_2) - \left[\frac{f_{2r} a_2}{\{\omega_1^2 - (\Omega_r - \omega_2 + \psi_{2r})^2\}}\right]\cos(\psi_4 - \psi_2)\Bigg] + o(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
U_2 = {} & a_2\cos(\omega_2 t + \beta_2) + \varepsilon\Bigg[\left[\frac{3\alpha_2 a_1^3 a_2 + 6\alpha_4 a_1 a_2^2}{4\omega_2^2 - 4\omega_1^2}\right]\cos\psi_1 - \left[\frac{\alpha_2 a_1^3}{\omega_2^2 - 9\omega_1^2}\right]\cos 3\psi_2 + \left[\frac{\alpha_5 a_2^3}{32\omega_2^2}\right]\cos 3\psi_2 \\
& - \left[\frac{3\alpha_3 a_1^2 a_2}{\{4\omega_2^2 - 4(\omega_2 - 2\omega_1)^2\}}\right]\cos(\psi_1 + 2\psi_2) - \left[\frac{3\alpha_3 a_1^2 a_2}{\{4\omega_2^2 - 4(\omega_2 - 2\omega_1)^2\}}\right]\cos(\psi_2 - 2\psi_1) \\
& - \left[\frac{3\alpha_4 a_1 a_2^2}{\{4\omega_2^2 - 4(\omega_1 + 2\omega_2)^2\}}\right]\cos(\psi_1 + 2\psi_2) - \left[\frac{3\alpha_4 a_1 a_2^2}{\{4\omega_2^2 - 4(\omega_1 - 2\omega_2)^2\}}\right]\cos(\psi_1 - 2\psi_2) \\
& - \left[\frac{3\alpha_4 a_1 a_2^2}{\{4\omega_2^2 - 4(2\omega_2 - \omega_1)^2\}}\right]\cos(2\psi_2 - \psi_1) - \left[\frac{g_{1s} a_1}{\{\omega_2^2 - (\Omega_s + \omega_1 + \gamma_{1s})^2\}}\right]\cos(\psi_4 + \psi_1) \\
& - \left[\frac{g_{1s} a_1}{\{\omega_2^2 - (\Omega_s - \omega_1 + \gamma_{1s})^2\}}\right]\cos(\psi_4 - \psi_1) - \left[\frac{g_{2r} a_2}{\{\omega_2^2 - (\Omega_s + \omega_2 + \gamma_{2s})^2\}}\right]\cos(\psi_4 + \psi_2)\Bigg] + o(\varepsilon^2),
\end{aligned}
\tag{2.67}
$$

where

$$
\psi_1 = \omega_1 t + \beta_1, \quad \psi_2 = \omega_2 t + \beta_2, \quad \psi_3 = \Omega_r t + \tau_{1r}, \quad \psi_4 = \Omega_s t + \gamma_{1n}. \tag{2.68}
$$

The steady state of Eq. 2.65 is given by

$$\begin{aligned}
&\mu_1 a_1 + \frac{3\alpha_2}{8\omega_1} a_1^2 a_2 \sin\theta_1 + \frac{1}{2\omega_1} f_{1r} a_1 \sin\theta_2 = 0,\\
&\frac{1}{2}\delta_2 a_1 - \frac{3\alpha_1}{8\omega_1} a_1^3 - \frac{3\alpha_3}{4\omega_1} a_1 a_2^2 - \frac{3\alpha_2}{8\omega_1} a_1^2 a_2 \cos\theta_1 - \frac{1}{2\omega_1} f_{1r} a_1 \cos\theta_2 = 0,\\
&-\mu_2 a_2 + \frac{3\alpha_2}{8\omega_1} a_1^2 a_2 \sin\theta_1 = 0,\\
&\left(\frac{3}{2}\delta_2 - \sigma_1\right) a_2 + \frac{3\alpha_3}{4\omega_2} a_1 a_2^2 + \frac{3\alpha_5}{8\omega_2} a_2^3 + \frac{3\alpha_2}{8\omega_1} a_1^2 a_2 \cos\theta_1 = 0.
\end{aligned} \tag{2.69}$$

There are two possibilities: first, $a_1 = a_2 = 0$ (this is the linear case); second, a_1 and $a_2 \neq 0$ and Eq. 2.69 yield the frequency response equations

$$\begin{aligned}
&\mu_1^2 + \frac{1}{4}\delta_2^2 + \left[\frac{9\alpha_2^2}{64\omega_1^2}\right] a_1^2 a_2^2 + \left[\frac{9\alpha_1^2}{64\omega_1^2}\right] a_1^4 - \frac{1}{4\omega_1^2} f_{1r}^2 - \left[\frac{3\alpha_3\sigma_2}{4\omega_1}\right] a_2^2\\
&- \left[\frac{3\alpha_2\sigma_2}{8\omega_1}\right] a_1 a_2 \cos\theta_1 + \left[\frac{9\alpha_1\alpha_2}{16\omega_1^2}\right] a_1^2 a_2^2 + \left[\frac{9\alpha_1\alpha_2}{32\omega_1^2}\right] a_1^3 a_2 \cos\theta_1 + \left[\frac{9\alpha_2\alpha_3}{16\omega_1^2}\right] a_1 a_2^3 \cos\theta_1 = 0,\\
&\left[\mu_2^2 + \left(\frac{3}{2}\delta_2 - \delta_1\right)^2\right] a_2^2 - \left[\frac{\alpha_2^2}{64\omega_2^2}\right] a_1^6 + \left[\frac{9\alpha_3^2}{16\omega_2^2}\right] a_1^4 a_2^2 + \left[\frac{9\alpha_5^2}{64\omega_2^2}\right] a_2^6\\
&- \left[\frac{3\alpha_3}{2\omega_2}\left(\frac{3}{2}\delta_2 - \delta_1\right)\right] a_1^2 a_2^2 - \left[\frac{3\alpha_5}{4\omega_2}\left(\frac{3}{2}\delta_2 - \delta_1\right)\right] a_2^4 + \left[\frac{3\alpha_3\alpha_5}{32\omega_2}\right] a_1^2 a_2^4 = 0.
\end{aligned} \tag{2.70}$$

Example 2.3

Introduction

In the present example, we consider the following two coupled Duffing–Van der Pol oscillators with a nonlinear coupling:

$$\ddot{x} - \varepsilon d_1 \dot{x}(1 - x^2) + \omega_1^2 x + \varepsilon\alpha_1 x^3 + \varepsilon\delta x y^2 = 0, \tag{2.71a}$$

$$\ddot{y} - \varepsilon d_2 \dot{y}(1 - y^2) + \omega_1^2 y + \varepsilon\alpha_2 y^3 + \varepsilon\delta x^2 y = 0. \tag{2.71b}$$

When $d_1 = d_2 = 0$ the system (Eq. 2.71a, b) consists of the two coupled anharmonic oscillators. If the nonlinear damping is replaced by a linear damping, the system (Eq. 2.71a,b) becomes two coupled Duffing oscillators. When the coupling parameter is set to zero, the system (Eq. 2.71a,b) becomes two uncoupled Duffing–Van der Pol oscillators. The Duffing oscillator corresponds to the choices $\delta = 0, d_1 = d_2 = 0$ and addition of a linear damping.

Analysis with the Method of Multiple Scales

We look for approximate asymptotic solutions of Eq. 2.71a,b by employing the multiple scales method. For small but finite x and y, we consider the solutions in the form of the power series

$$x(t, \varepsilon) = x_0(T_0, T_1) + \varepsilon x_1(T_0, T_1) + \cdots, \tag{2.72a}$$

$$y(t,\varepsilon) = y_0(T_0,T_1) + \varepsilon y_1(T_0,T_1) + \cdots, \tag{2.72b}$$

where $T_0 = t$ is a fast scale and $T_1 = \varepsilon t$ is a slow scale. The slow scale T_1 characterizes the modulation in the amplitude and phase caused by the nonlinearity, damping, and coupling. The fast scale T_0 is associated with the relatively fast changes in the response.

The first and second derivatives with respect to time t are given by

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \varepsilon D_1 + \cdots, \frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots, \tag{2.73}$$

where $D_n = \partial/\partial T_n$. Substituting Eq. 2.72a,b and 2.73 into Eq. 2.71a,b and equating coefficients of equal powers of ε, we obtain $O(\varepsilon^0)$:

$$D_0^2 x_0 + \omega_1^2 x_0 = 0, \tag{2.74a}$$

$$D_0^2 y_0 + \omega_2^2 y_0 = 0, \tag{2.74b}$$

$O(\varepsilon)$:

$$D_0^2 x_1 + \omega_1^2 x_1 = -2D_0 D_1 x_0 + d_1(1 - x_0^2) D_0 x_0 - \alpha_1 x_0^3 - \delta x_0 y_0^2, \tag{2.75a}$$

$$D_0^2 y_1 + \omega_2^2 y_1 = -2D_0 D_1 y_0 + d_2(1 - y_0^2) D_0 y_0 - \alpha_2 y_0^3 - \delta x_0^2 y_0. \tag{2.75b}$$

The general solutions of the linear Eq. 2.74a,b can be written in the form of the correlations

$$x_0 = A_1(T_1) e^{i\omega_1 T_0} + \text{c.c.}, \tag{2.76a}$$

$$y_0 = A_2(T_1) e^{i\omega_2 T_0} + \text{c.c.}, \tag{2.76b}$$

where $c.c.$ represents the complex conjugates of the preceding term. The quantities A_1 and A_2 are arbitrary to this order of approximation and are determined by imposing solvability conditions at the next order of approximation. We substitute Eq. 2.76a,b into Eq. 2.75a,b and obtain

$$\begin{aligned} D_0^2 x_1 + \omega_1^2 x_1 = {} & \left[-2i\omega_1 A_1' + id_1\omega_1 A_1(1 - A_1\overline{A}_1) - 3\alpha_1 A_1^2 \overline{A}_1 - 2\delta A_1 A_2 \overline{A}_2\right] e^{i\omega_1 T_0} \\ & - \left[\alpha_1 A_1^3 + id_1\omega_1 A_1^3\right] e^{3i\omega_1 T_0} - \delta A_1 A_2^2 e^{i(\omega_1 + 2\omega_2)T_0} - \delta \overline{A}_1 A_2^2 e^{i(2\omega_2 - \omega_1)T_0} + \text{ c.c.,} \end{aligned} \tag{2.77a}$$

$$\begin{aligned} D_0^2 y_1 + \omega_2^2 y_1 = {} & \left[-2i\omega_2 A_2' + id_2\omega_2 A_2(1 - A_2\overline{A}_2) - 3\alpha_2 A_2^2 \overline{A}_2 - 2\delta A_1 A_2 \overline{A}_1\right] e^{i\omega_2 T_0} \\ & - \left[\alpha_2 A_2^3 + id_2\omega_2 A_2^3\right] e^{3i\omega_2 T_0} - \delta A_2 A_1^2 e^{i(\omega_2 + 2\omega_1)T_0} - \delta A_1^2 \overline{A}_2 e^{i(2\omega_1 - \omega_2)T_0} + \text{ c.c.,} \end{aligned} \tag{2.77b}$$

where $\overline{A}_n$ denotes the complex conjugate of A_n, the prime denotes differentiation with respect to T_1, and now $c.c.$ represents complex conjugates of each preceding term.

Nonresonant Case

Consider the nonresonant case, $\omega_1 \neq \omega_2$. The arbitrary functions $A_1(T_1)$ and $A_2(T_1)$ are determined from Eq. 2.77a,b by satisfying the solvability conditions for boundedness of the solutions. Particular solutions of Eq. 2.77a,b contain secular terms generated by the first term on the right-hand sides of Eq. 2.77a,b. The conditions for the elimination of secular terms, in Eq. 2.77a,b are

$$2i\omega_1 A_1' - id_1\omega_1 A_1(1 - A_1\overline{A}_1) + 3\alpha_1 A_1^2\overline{A}_1 + 2\delta A_1 A_2 \overline{A}_2 = 0, \tag{2.78a}$$

$$2i\omega_2 A_2' - id_2\omega_2 A_2(1 - A_2\overline{A}_2) + 3\alpha_2 A_2^2\overline{A}_2 + 2\delta A_1 A_2 \overline{A}_1 = 0. \tag{2.78b}$$

At this step, we introduce the polar forms for the amplitudes A_1 and A_2 as

$$A_1(T_1) = \frac{1}{2}a_1(T_1)e^{i\theta_1(T_1)}, \tag{2.79a}$$

$$A_2(T_1) = \frac{1}{2}a_2(T_1)e^{i\theta_2(T_1)}. \tag{2.79b}$$

Substituting the above expressions for A_1 and A_2 in Eq. 2.78a,b and separating real and imaginary parts, we have

$$\frac{\mathrm{d}a_1}{\mathrm{d}T_1} = \frac{1}{2}d_1 a_1\left(1 - \frac{a_1^2}{4}\right), \tag{2.80a}$$

$$\frac{\mathrm{d}a_2}{\mathrm{d}T_1} = \frac{1}{2}d_2 a_2\left(1 - \frac{a_2^2}{4}\right), \tag{2.80b}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}T_1} = \frac{1}{8}a_1^2\left(\frac{2\delta}{\omega_2} - \frac{3\alpha_1}{\omega_1}\right) - \frac{1}{8}a_2^2\left(\frac{2\delta}{\omega_1} - \frac{3\alpha_2}{\omega_2}\right), \tag{2.80c}$$

where $\phi = \theta_2 - \theta_1$. Equations 2.80a, b, c are the first-order equations describing the variation of a_1, a_2, and ϕ. The Eqs. 2.80a and 2.80b are uncoupled, and the solutions are

$$a_1(T_1) = 2\left[1 - \left(1 - \frac{4}{a_{10}}\right)e^{-d_1 T_1}\right]^{-1/2}, \tag{2.81a}$$

$$a_2(T_1) = 2\left[1 - \left(1 - \frac{4}{a_{20}}\right)e^{-d_2 T_1}\right]^{-1/2}. \tag{2.81b}$$

The solutions x and y in the zeroth-order approximation are written as

$$x = x_0 = a_1\cos(\theta_1 + \omega_1 T_0), \tag{2.82a}$$

$$y = y_0 = a_2\cos(\theta_2 + \omega_2 T_0). \tag{2.82b}$$

Resonant Case

Next, we consider the resonant case. In order to describe the nearness of ω_2 to ω_1, we introduce a detuning parameter ε through the equation

$$\omega_2 = \omega_1 + \varepsilon\sigma. \tag{2.83}$$

Using the equations $(2\omega_2 - \omega_1)T_0 = \omega_1 T_0 + 2\sigma T_1$ and $(2\omega_1 - \omega_2)T_0 = \omega_2 T_0 - 2\sigma T_1$, the new solvability conditions are

$$2i\omega_1 A_1' - id_1\omega_1 A_1(1 - A_1\overline{A}_1) + 3\alpha_1 A_1^2\overline{A}_1 + 2\delta A_1 A_2\overline{A}_2 + \delta A_2^2\overline{A}_1 e^{i2\sigma T_1} = 0, \tag{2.84a}$$

$$2i\omega_2 A_2' - id_2\omega_2 A_2(1 - A_2\overline{A}_2) + 3\alpha_2 A_2^2\overline{A}_2 + 2\delta A_1 A_2\overline{A}_1 + \delta A_1^2\overline{A}_2 e^{-i2\sigma T_1} = 0. \tag{2.84b}$$

Substituting Eq. 2.80a,b,c into Eq. 2.84a,b and separating the real and imaginary parts of Eq. 2.84a,b, we obtain the following set of equations:

$$\frac{\mathrm{d}a_1}{\mathrm{d}T_1} = \frac{1}{2}d_1 a_1\left(1 - \frac{a_1^2}{4}\right) - \frac{\delta}{8\omega_1}a_1 a_2^2 \sin\phi, \tag{2.85a}$$

$$\frac{\mathrm{d}a_2}{\mathrm{d}T_1} = \frac{1}{2}d_2 a_2\left(1 - \frac{a_2^2}{4}\right) + \frac{\delta}{8\omega_2}a_1^2 a_2 \sin\phi, \tag{2.85b}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}T_1} = 2a_1^2\left(\frac{\delta}{4\omega_2} - \frac{3\alpha_1}{8\omega_1}\right) - 2a_2^2\left(\frac{\delta}{4\omega_1} - \frac{3\alpha_2}{8\omega_2}\right) + \frac{\delta}{4}\left(\frac{a_1^2}{\omega_2} - \frac{a_2^2}{\omega_1}\right)\cos\phi + 2\sigma, \tag{2.85c}$$

where $\phi = 2\theta_2 - 2\theta_1 + 2\sigma T_1$. For the resonant case, the solutions are given by Eq. 2.83, with the time evolution of the amplitudes and phases as described by Eq. 2.85a,b,c.

2.3 Parameterized Perturbation Method

2.3.1 Introduction

The parameterized perturbation method (PPM) was first proposed by He in 1999b and was further developed in He (2006a).

The method can be applied to nonlinear equations including differential-difference equations (Ding and Zhang 2009; Jalaal et al. 2011).

This approach is an explicit method with high validity for resolution of strong nonlinear systems, which can be used to derive the relationship between period and amplitude in a nonlinear oscillator. In addition, it is more convenient and more efficient, in comparison with traditional methods.

2.3.2 Application

In the following, we consider the method by applied examples.

Example 2.4

Consider the free response of the undamped and single-DOF system that is shown in Fig. 2.1. The restoring forces in the spring are given by

$$F_{sp} = -\left(k\,x(t) + \alpha\,x(t)^3\right), \tag{2.86}$$

with $\alpha > 0$. With this restoring force, the equation of motion of the system is

$$\ddot{x}\,(t) + k\,x(t) + \,\alpha\,x(t)^3 = 0, \tag{2.87}$$

where the equation of motion for this system with a cubic nonlinear stiffness is commonly known as Duffing's equation.

In order to use the parameterized perturbation method, it is necessary to introduce an artificial small parameter β:

$$x(t) = \beta\,v(t). \tag{2.88}$$

Substituting Eq. 2.88 in Eq. 2.84a,b, we obtain

$$\ddot{v}\,(t) + \;k.\,v(t) + \alpha\;\beta^2 v(t)^3 = 0, \qquad v(0) = A/\beta, \qquad \dot{v}\,(0) = 0. \tag{2.89}$$

Suppose that the solution of the Eq. 2.89 and the coefficient, k (or other coefficients), can be expressed in the forms

$$v(t) = v_0(t) + \beta^2 v_1(t) + \beta^4 v_2(t) + \cdots \tag{2.90}$$

$$k = \omega^2 + \beta^2\,\omega_1 + \beta^4\,\omega_2 + \cdots. \tag{2.91}$$

Substituting Eqs. 2.90 and 2.91 into Eq. 2.89 and equating the terms with the identical powers of β yields the following equations:

$$\ddot{v}_0\,(t) + \;\omega^2\,v_0(t) = 0, \qquad v_0(0) = A/\beta, \qquad \dot{v}_0(0) = 0. \tag{2.92}$$

$$\ddot{v}_1\,(t) + \;\omega^2 v_1(t) + \;\omega_1 v_0 + \alpha\;v_0^3 = 0, \qquad v_1(0) = 0, \qquad \dot{v}_1(0) = 0. \tag{2.93}$$

Considering the initial conditions $v_0 = A/\beta$ and $\dot{v}(0) = 0$, the solution of Eq. 2.92 is $v_0(t) = \frac{A}{\beta}\cos\omega t$.

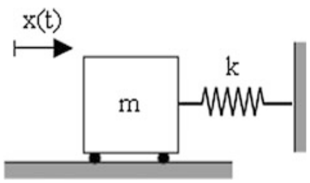

Fig. 2.1 Single-DOF system

Substituting the result into Eq. 2.93, it can be rewritten as

$$\ddot{v}_1(t) + \omega^2 v_1(t) + \frac{A}{\beta}(\omega_1 + \frac{3\,\alpha A^2}{4\beta^2})\cos(\omega t) + \frac{\alpha A^3}{4\beta^3}\cos(3\omega t) = 0. \tag{2.94}$$

Avoiding the presence of a secular term requires

$$\omega_1 = -\frac{3\,\alpha A^2}{4\beta^2}. \tag{2.95}$$

Solving Eq. 2.94, we obtain

$$v_1(t) = \frac{\alpha A^3}{32\omega^2\beta^3}(\cos(3\omega t) - \cos(\omega t)). \tag{2.96}$$

If, for example, its first-order approximation is sufficient, then we will have

$$x(t) = \beta v(t) = \beta\big(v_0(t) + \beta^2 v_1(t)\big) = A\,\cos\omega t + \left(\frac{A^3\alpha}{32\omega^2}\right)(\cos(3\omega t) - \cos(\omega t)). \tag{2.97}$$

Substituting Eq. 2.95 into Eq. 2.91, the first-order angular frequency can be written in the form

$$\omega = \frac{1}{2}\sqrt{4k + 3\,\alpha A^2}. \tag{2.98}$$

The period $T = 2\pi/\omega$ may then be written as

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{1 + \frac{3\,\alpha\,\varepsilon A^2}{4}}}. \tag{2.99}$$

2.4 Singular Perturbation Method

2.4.1 Introduction

The theory of singular perturbations has been with us, in one form or another, for a little over a century (although the term "singular perturbation" dates from the 1940s). The subject and the techniques associated with it have evolved over this period as a response to the need to find approximate solutions (in an analytical form) to complex problems.

Typically, such problems are expressed in terms of differential equations that contain at least one small parameter, and they can arise in many fields: fluid mechanics, particle physics, and combustion processes, to name but three. The essential hallmark of a singular perturbation problem is that a simple and straightforward approximation (based on the smallness of the parameter) does not give an accurate solution throughout the domain of that solution. Perforce, this leads to different approximations being valid in different parts of the domain (usually requiring a "scaling" of the variables with respect to the parameter). This, in turn, has led to the important concepts of breakdown, matching, and so on. The notion of a singular perturbation problem was first evident in the seminal work of L. Prandtl (1874–1953) on the viscous boundary layer.

The singular perturbation method concerns the study of problems featuring a parameter for which solutions of the problem at a limiting value of the parameter are different in character from the limit of solutions of the general problem—namely, the limit is singular. In contrast, for regular perturbation problems, solutions of the general problem converge to solutions of the limit-problem as the parameter approaches the limit-value.

Singular perturbation theory considers systems of the form $\dot{x} = f(x, \varepsilon)$ in which f behaves singularly in the limit $\varepsilon \to 0$. A simple example of such a system is the Van der Pol oscillator in the large damping limit.

2.4.2 Application

Example 2.5

The Van der Pol oscillator is a second-order system with nonlinear damping, of the form

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0. \tag{2.100}$$

The special form of the damping (which can be realized by an electric circuit) has the effect of decreasing the amplitude of large oscillations, while increasing the amplitude of small oscillations.

We are interested in the behavior of the large α. There are several ways to write Eq. 2.100 as a first-order system. For our purpose, a convenient representation is

$$\begin{aligned} \dot{x} &= \alpha\left(y + x - \frac{x^3}{3}\right), \\ \dot{y} &= -\frac{x}{\alpha}. \end{aligned} \tag{2.101}$$

One can easily check that this system is equivalent to Eq. 2.100 by computing $\ddot{x}$. If it is very large, x will move quickly, while y changes slowly. To analyze the limit $\alpha \to \infty$, we introduce a small parameter $\varepsilon = 1/\alpha^2$ and a slow time $t' = t/\alpha = \sqrt{\varepsilon}t$. This can then be rewritten as

$$\begin{aligned} \varepsilon\frac{\mathrm{d}x}{\mathrm{d}t'} &= y + x - \frac{x^3}{3}, \\ \frac{\mathrm{d}y}{\mathrm{d}t'} &= -x. \end{aligned} \tag{2.102}$$

In the limit $\varepsilon \to 0$, we obtain the system

$$\begin{aligned} 0 &= y + x - \frac{x^3}{3} \\ \frac{dy}{dt'} &= -x, \end{aligned} \tag{2.103}$$

which is no longer a system of differential equations. In fact, the solutions are constrained to move on the curve $C\colon y = \frac{1}{3}x^3 - x$, and eliminating y from the system, we have

$$-x = \frac{dy}{dt'} = (x^2 - 1)\frac{dx}{dt'} \Rightarrow \frac{dx}{dt'} = -\frac{x}{x^2 - 1}. \tag{2.104}$$

The dynamics is shown in Fig. 2.2a. Another possibility is to introduce the fast time $t'' = \alpha t = t/\sqrt{\varepsilon}$.

Then Eq. 2.100 becomes

$$\begin{aligned} \frac{dx}{dt''} &= y + x - \frac{x^3}{3}, \\ \frac{dy}{dt''} &= -\varepsilon x. \end{aligned} \tag{2.105}$$

In the limit $\varepsilon \to 0$, we get the system

$$\begin{aligned} \frac{dx}{dt''} &= y + x - \frac{x^3}{3}, \\ \frac{dy}{dt''} &= 0. \end{aligned} \tag{2.106}$$

In this case, y is a constant and acts as a parameter in the equation for x. Some orbits are shown in Fig. 2.2b. Of course, the systems (2.101, 2.102, and 2.105) are strictly equivalent for $\varepsilon > 0$. They only differ in the singular limit $\varepsilon \to 0$. The dynamics for small but positive ε can be understood by sketching the vector field. Let us note that $\dot{x}$ is positive if (x, y) lies above the curve C and negative when it

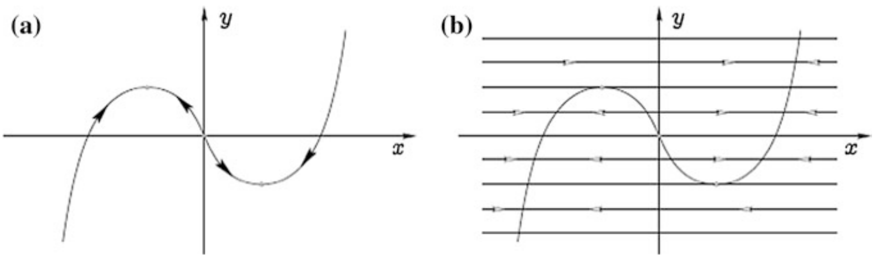

Fig. 2.2 Behavior of the Van der Pol equation in the singular limit $\varepsilon \to 0$, **a** on the slow time scale $t' = \sqrt{\varepsilon}t$, given by Eq. 2.103, and **b** on the fast time scale $t'' = t/\sqrt{\varepsilon}$ (see Eq. 2.105)

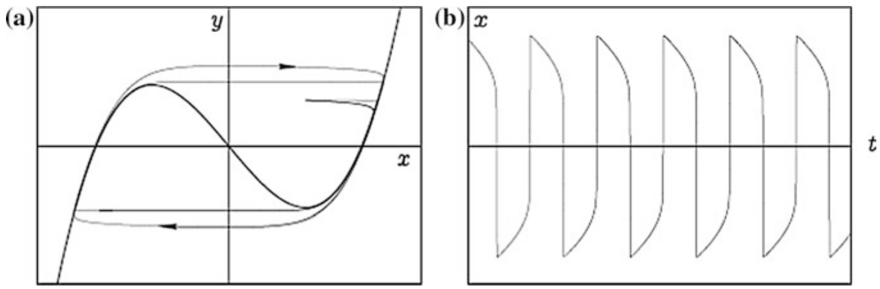

Fig. 2.3 **a** Two solutions of the Van der Pol equations (2.101) (*light curves*) for the same initial condition (1; 0:5), for $\alpha = 5$ and $\alpha = 20$. The heavy curve is the curve $C: y = \frac{1}{3}x^3 - x$. **b** The graph of $x(t)$ $(\alpha = 20)$ displays relaxation oscillations

lies below; this curve separates the plane into regions where x moves to the right or to the left and the orbit must cross the curve vertically; $\mathrm{d}y/\mathrm{d}x$: is very small unless the orbit is close to the curve C, so that the orbits will be almost horizontal except near this curve; Orbits move upward if $x < 0$ and downward if $x > 0$.

The resulting orbits are shown in Fig. 2.3a. An orbit starting somewhere in the plane will first approach the curve C on a nearly horizontal path, in a time t of order $1/\alpha$. Then it will track the curve at a small distance until it reaches a turning point, after a time t of order α. Since the equations forbid it to follow C beyond this point, the orbit will jump to another branch of C, where the behavior repeats. The graph of $x(t)$ thus contains some parts with a small slope and others with a large slope (Fig. 2.3b). This phenomenon is called a *relaxation oscillation*.

2.5 Homotopy Perturbation Method and Its Modifications

2.5.1 A Brief Introduction to the Homotopy Perturbation Method

The HPM was first proposed by He in 1999a (2000d). It has been worked out over a number of years by numerous authors and has matured into a relatively fledged theory thanks to the efforts of many researchers. For a relatively comprehensive survey on the concepts, theory, and applications of the HPM, the reader is referred to the review articles (He 2006a, d, 2008) and (Kachapi and Ganji 2013a, b; Kachapi et al. 2011; Ganji and Kachapi 2011; Hashemi et al. 2007; Tolou et al. 2009; Ganji and Hashemi 2007).

In contrast to the traditional perturbation methods, this technique does not require a small parameter in an equation. In this method, according to the homotopy technique, a homotopy with an imbedding parameter $p \in [0\ ,\ 1]$ is constructed, and the imbedding parameter is considered as a "small parameter," so the method is called the *homotopy perturbation method*, which can take full advantage of the traditional perturbation methods and homotopy techniques. In this method, the solution is considered as the sum of an infinite series, which converges rapidly to accurate solutions.

In this section, three cases of the HPM are applied for solving the governing equation. In the first case, standard HPM is applied. In the second case, time transformation $\theta = \omega t$ is used, and then a homotopy equation is constructed. The parameter expansion technique is used in both cases to expand the square of the unknown angular frequency. And the third case of an HPM is applied for fractional differential equations.

2.5.1.1 First Case of HPM

To explain the basic idea of the HPM for solving nonlinear differential equations, we consider the nonlinear differential equation

$$A(u) - f(r) = 0,\ r \in \Omega, \tag{2.145}$$

subject to boundary condition

$$B(u, \partial u/\partial n) = 0,\ r \in \Gamma, \tag{2.146}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of domain Ω, and $\partial u/\partial n$ denotes differentiation along the normal drawn outward from Ω.The operator A can, generally speaking, be divided into two parts: a linear part L and a nonlinear part N. Equation 2.145 therefore can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \tag{2.147}$$

In the case in which the nonlinear Eq. 2.145 has no "small parameter," we can construct the homotopy

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p(N(v) - f(r)) = 0, \tag{2.148}$$

where

$$v(r,p){:}\Omega \times [0,1] \to R, \tag{2.149}$$

In Eq. 2.146, $p \in [0\ ,\ 1]$ is an embedding parameter, and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. 2.146 can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots, \tag{2.150}$$

Also, the homotopy parameter p is used to expand the square of the unknown angular frequency ω as

$$\mu = \omega^2 - p\alpha_1 - p^2\alpha_2 - \cdots, \tag{2.151}$$

or

$$\omega^2 = \mu + p\alpha_1 + p^2\alpha_2 + \ldots, \tag{2.152}$$

where μ is the coefficient of $u(r)$ in Eq. 2.145, the right-hand side of Eq. 2.151 replaces it. Also $\boldsymbol{\alpha}$ ($i = 1, 2,\ldots$) are arbitrary parameters that are to be determined.

The best approximation for solution and angular frequency ω are

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots, \tag{2.153}$$

$$\omega^2 = 1 + \alpha_1 + \alpha_2 + \cdots, \tag{2.154}$$

when Eq. 2.148 corresponds to Eq. 2.145 and Eq. 2.153 becomes the approximate solution of Eq. 2.145.

2.5.1.2 Second Case of HPM

To explain the basic idea of this case for solving nonlinear differential equations, we consider the nonlinear differential equation

$$\ddot{u} = f(u, \dot{u}), \quad u(0) = A, \quad \dot{u}(0) = 0. \tag{2.155}$$

For the determination of the periodic solution of this equation, we first introduce a linear stiffness term with an unknown (constant) frequency ω into both sides of Eq. 2.155 as

$$\ddot{u} + \omega^2 u = f(u, \dot{u}) + \omega^2 u \equiv g(u, \dot{u}, \omega), \quad u(0) = A, \quad \dot{u}(0) = 0. \tag{2.156}$$

And then an artificial parameter p, is entered into Eq. 2.156, so we have

$$\ddot{u} + \omega^2 u = pg(u, \dot{u}, \omega), \quad u(0) = A, \quad \dot{u}(0) = 0. \tag{2.157}$$

It is obvious that Eq. 2.157 is equal to Eq. 2.156 for $p = 1$.

We now introduce a new independent variable $\theta = \omega t$ so that Eq. 2.157 can be written as

$$u'' + u = pg(u, u', \omega) = p\left(u + \frac{1}{\omega^2} f(u, u', \omega)\right), \quad u(0) = A, \quad \dot{u}(0) = 0. \tag{2.158}$$

The solution of Eq. 2.158 is

$$u(\theta) = u_0(\theta) + pu_1(\theta) + p^2 u_2(\theta) + \cdots, \tag{2.159}$$

with

$$\omega^2 = \omega_0^2(\theta) + p\omega_1^2(\theta) + p^2\omega_2^2(\theta) + \cdots, \tag{2.160}$$

Substituting Eqs. 2.159 and 2.160 into Eq. 2.158, expanding of f or G about (u_0, u_0, ω_0) and setting the coefficients of the monomials p^n, $n \geq 0$ in the resulting series to zero, yields the following sequential equations:

$$u_0'' + u_0 = 0\,, \qquad u_0(0) = A,\;\; u_0'(0) = 0, \tag{2.161}$$

$$u_1'' + u_1 = \left(u_0 + \frac{1}{\omega_0^2} f(u_0, u_0', \omega_0)\right), \qquad u_1(0) = 0,\;\; u_1'(0) = 0, \tag{2.162}$$

$$\begin{aligned} u_2'' + u_2 = \frac{\partial G}{\partial u}(u_0, u_0', \omega_0)u_1 + \frac{\partial G}{\partial u'}(u_0, u_0', \omega_0)u_1' + \frac{\partial G}{\partial \omega}(u_0, u_0', \omega_0)\omega_1\,,\;\; u_2(0) \\ = 0,\;\; u_2'(0) = 0. \end{aligned} \tag{2.163}$$

The solution of Eq. 2.161 is

$$u_0(\theta) = A\cos(\theta)\,, \tag{2.164}$$

which can be substituted into Eq. 2.162, and the condition that $u_1(\theta)$ is free from secular terms provides an algebraic equation for the determination of ω_0. Similar conditions applied to $u_k(\theta)$, $k \geq 2$, provide algebraic equations for ω_k, $k \geq 1$. Substitution of these values of $u_k(\theta)$ and ω_k, $k \geq 0$, into Eqs. 2.159 and 2.160, respectively, and setting $p = 1$ in those equations provide the solution and the frequency of oscillation, respectively.

2.5.1.3 Third Case of HPM

Recently, Shaher Momani applied the HPM to fractional differential equations in 2006 (Momani and Odibat 2006). To illustrate the basic ideas of the modification, we consider the following nonlinear differential equation of fractional order:

$$D_*^{\alpha} u(t) + L(u(t)) + N(u(t)) = f(t), \quad t > 0, \quad m-1 < \alpha < m, \tag{2.165}$$

where L is a linear operator that might include other fractional derivatives of order less than a, N is a nonlinear operator that also might include other fractional derivatives of order less than a, f is a known analytic function, and D_*^{α} is the Caputo fractional derivative of order a, subject to the initial conditions

$$u^k(0) = c_k, \qquad k = 0,\, 1,\, 2,\, \ldots, m-1 \tag{2.166}$$

In view of the homotopy technique, we can construct the homotopy

$$u^{(m)} - f(t) = p\left[u^{(m)} - L(u) - N(u) - D_*^{\alpha} u\right], \qquad p \in [0,\;1]. \tag{2.167}$$

2.5.2 Application

To illustrate its effectiveness and its convenience, several examples with a high order of nonlinearity are used; the result reveals that the first order of approximation obtained by the proposed method is valid uniformly even for a very large parameter and is more accurate than the perturbation solutions.

Example 2.6

Introduction

Consider the generalized Huxley equation

$$u_t - u_{xx} = \beta u(1 - u^\delta)(u^\delta - \gamma), \quad 0 \le x \le 1, \quad t \ge 0$$

with the initial condition

$$u(x,0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}},$$

which describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals. The exact solution of this equation was derived by Wang et al., using nonlinear transformations, and is given by

$$u(x,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(\sigma\gamma\left(x + \left\{\frac{(1+\delta-\gamma)\rho}{2(1+\delta)}\right\}t\right)\right)\right]^{\frac{1}{\delta}},$$

where $\sigma = \delta\rho/4(1+\delta)$ and $\rho = \sqrt{4\beta(1+\delta)}$.

Applications

After separating the linear and nonlinear parts of the equation, we apply homotopy-perturbation, which can be constructed as (Hashemi et al. 2007)

$$\begin{aligned}
&(1-p)\left(\left(\frac{\partial}{\partial t}v(x,t)\right) - \left(\frac{\partial}{\partial t}u_0(x,t)\right)\right) \\
&\quad + p\left(\begin{array}{l}\left(\frac{\partial}{\partial t}v(x,t)\right) - \left(\frac{\partial^2}{\partial x^2}v(x,t)\right) - \beta v(x,t) \\ (1 - v(x,t)^\delta)(v(x,t)^\delta - \gamma)\end{array}\right) \\
&\quad = 0.
\end{aligned}$$

Applying HPM into the previous equation and rearranging the resultant equation on the basis of powers of p-terms, we obtain

$$\left(\frac{\partial}{\partial t}v_0(x,t)\right) = 0,$$

$$\left(\begin{array}{l}\left(\frac{\partial}{\partial t}v_1(x,t)\right) - \beta v_0(x,t)v_0(x,t)^\delta - \left(\frac{\partial^2}{\partial x^2}v_0(x,t)\right) + \beta v_0(x,t)v_0(x,t)^{2\delta} - \\ \beta v_0(x,t)v_0(x,t)^\delta\gamma + \beta v_0(x,t)\gamma\end{array}\right) = 0.$$

Consider the following conditions:

$$u_0(x,0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}, \quad \frac{\mathrm{d}}{\mathrm{d}t}u_0(x,0) = 0,$$
$$u_i(x,0) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}u_i(x,0) = 0, \quad i = 1,2,\ldots$$

With the effective initial approximation for v_0 from the designated conditions and the solution of previous equations, we will have

$$v_0(x,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}},$$
$$v_1(x,t) = \frac{1}{\delta^2}\Bigg((1+\tanh(\sigma\gamma x))^{\frac{1}{\delta}}\Big(2^{\left(-\frac{1}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}\sigma^2\tanh(\sigma\gamma x)^2 + 2^{\left(-\frac{1}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}$$
$$\sigma^2\tanh(\sigma\gamma x)^2\delta - 2^{\left(\frac{-1+\delta}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}\sigma^2\tanh(\sigma\gamma x) + 2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}\sigma^2 + 2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1+\delta}{\delta}\right)}\beta$$
$$\left(2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}(1+\tanh(\sigma\gamma x))^{\frac{1}{\delta}}\right)^{\delta}\delta^2 - 2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1+\delta}{\delta}\right)}\beta\delta^2 - 2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}\sigma^2\delta + 2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}$$
$$\left(2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}(1+\tanh(\sigma\gamma x))^{\frac{1}{\delta}}\right)^{\delta}\beta\delta^2 - 2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}\left(2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}(1+\tanh(\sigma\gamma x))^{\frac{1}{\delta}}\right)^{2\delta}\delta^2\Big)t\Bigg)$$

In the same manner, the additional components were obtained using the Maple Package.

According to the HPM, we can conclude that

$$u(x,t) = \lim_{p\to 1} v(x,t) = v_0(x,t) + v_1(x,t) + \cdots$$

The numerical results of the exact solution and two-terms approximation of HPM, for $\beta = 1,\ \gamma = 0.001$, and $\delta = 1$, are given in Table 2.1.

Example 2.7

Introduction

Consider a one-dimensional finite beam excited at its center by a suddenly applied transverse force $Q(x,t)$ [variable coefficient fourth-order parabolic partial

Table 2.1 Numerical solutions for $\beta = 1,\ \gamma = 0.001$, and $\delta = 1$

X	t	Exact	HPM
0.1	0.05	5.00030171E-04	5.00005184E-04
	0.1	5.00042665E-04	4.99992690E-04
	1	5.00267553E-04	4.99767803E-04
0.5	0.05	5.00100882E-04	5.00075895E-04
	0.1	5.00113376E-04	5.00063401E-04
	1	5.00338263E-04	4.99838513E-04
0.9	0.05	5.00171593E-04	5.00146605E-04
	0.1	5.00184087E-04	5.00134111E-04
	1	5.00408974E-04	4.99909224E-04

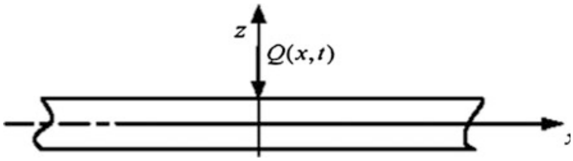

Fig. 2.4 One-dimensional finite beam subject to sudden shear load

differential equations, where $Q(x,t)=0$]. The applied point load at the origin on a finite beam is shown in Fig. 2.4 (Ganji and Hashemi 2007):

$$\frac{\partial^2}{\partial t^2}u(x,t)+\left(\frac{1}{x}+\frac{1}{120}x^4\right)\left(\frac{\partial^4}{\partial x^4}u(x,t)\right)=0,\quad \frac{1}{2}< x< 1,\quad t>0,$$

subject to the initial conditions

$$u(x,0)=0,\quad \frac{1}{2}< x< 1,$$
$$\frac{\partial u}{\partial t}(x,0)=1+\frac{x^5}{120},\quad \frac{1}{2}< x< 1$$

and the boundary conditions

$$u\left(\frac{1}{2},t\right)=\left(1+\frac{0.5^5}{120}\right)\sin t,\quad u(1,t)=\left(\frac{121}{120}\right)\sin t,\quad t>0,$$
$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2},t\right)=\frac{1}{6}\left(\frac{1}{2}\right)^3\sin t,\quad \frac{\partial^2 u}{\partial x^2}(1,t)=\frac{1}{6}\sin t,\quad t>0.$$

HPM Method

After separating the linear and nonlinear parts of the equation, we apply HPM as follows (Ganji and Hashemi 2007):

$$(1-p)\left(\frac{\partial^2 v(x,t)}{\partial t^2}-\frac{\partial^2 v_0(x,t)}{\partial t^2}\right)+p\left(\frac{\partial^2 v(x,t)}{\partial t^2}+\left(\frac{1}{x}+\frac{1}{120}x^4\right)\frac{\partial^4 v(x,t)}{4}\right)=0,$$
$$p\in[0,1].$$

Applying HPM and rearranging on the basis of powers of p-terms, we have

$$P^0: \frac{\partial^2 v_0(x,t)}{\partial t^2}=0,$$

$$p^1: \frac{120\left(\frac{\partial^2}{\partial t^2}v_1(x,t)\right)x+x^5\left(\frac{\partial^4}{\partial x^4}v_0(x,t)\right)+120\left(\frac{\partial^4}{\partial x^4}v_0(x,t)\right)}{120x}=0,$$

$$p^2: \frac{120\left(\frac{\partial^4}{\partial x^4}v_1(x,t)\right)+120x\left(\frac{\partial^2}{\partial t^2}v_2(x,t)\right)+x^5\left(\frac{\partial^4}{\partial x^4}v_1(x,t)\right)}{120x}=0,$$

Taking the following conditions into consideration,

$$v_0(x,0) = 0, \quad \frac{\partial v_0}{\partial t}(x,0) = 1 + \frac{x^5}{120},$$
$$v_i(x,0) = 0, \quad \frac{\partial v_i}{\partial t}(x,0)|_{t=0} = 0, \quad i = 1,2,\ldots$$

The solution of previous equations may be written as

$$v_0(x,t) = \left(1 + \frac{1}{120}x^5\right)t,$$
$$v_1(x,t) = \left(1 + \frac{1}{120}x^5\right)\left(-\frac{1}{6}t^3\right),$$
$$v_2(x,\ t) = \left(1 + \frac{1}{120}x^5\right)\left(\frac{1}{120}t^5\right),$$

In the same manner, the remaining components were obtained using the software Package.

According to the HPM, we can conclude that

$$u(x,t) = \lim_{p \to 1} v(x,t) = v_0(x,t) + v_1(x,t) + \cdots$$

Therefore,

$$u(x,t) = \left(1 + \frac{1}{120}x^5\right)t + \left(1 + \frac{1}{120}x^5\right)\left(-\frac{1}{6}t^3\right) + \left(1 + \frac{1}{120}x^5\right)\left(\frac{1}{120}t^5\right)$$
$$= \left(1 + \frac{1}{120}x^5\right)\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots\right)$$

The solution $u(x,t)$ in a closed form is

$$u(x,t) = \left(1 + \frac{x^5}{120}\right)\sin(t),$$

This is exactly the same as obtained by an exact solution.

Example 2.8

We next consider the parabolic equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + \left(\frac{x}{\sin(x)} - 1\right)\left(\frac{\partial^4}{\partial x^4}u(x,t)\right) = 0,$$
$$0 < x < 1, \quad t > 0$$

with the initial conditions

$$u(x,0) = x - \sin x, \quad 0 < x < 1,$$
$$\frac{\partial u}{\partial t}(x,0) = -(x - \sin x), \quad 0 < x < 1$$

and the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = e^{-t}(1 - \sin t), \quad t > 0,$$
$$\frac{\partial^2 u}{\partial x^2}(0,t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(1,t) = e^{-t}\sin t, \quad t > 0.$$

A homotopy can be constructed as follows (Ganji and Hashemi 2007):

$$(1-p)\left(\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\partial^2 u_0(x,t)}{\partial t^2}\right) + p\left(\frac{\partial^2 v(x,t)}{\partial t^2} + \left(\frac{x}{\sin(x)} - 1\right)\frac{\partial^4 v(x,t)}{\partial x^4}\right) = 0, \quad p \in [0,1].$$

Similar to previous example, after applying the HPM and rearranging based on powers of p-terms, we have

$$p^0: \left(\frac{\partial^2}{\partial t^2} v_0(x,t)\right) = 0,$$
$$p^1: \frac{\left(\frac{\partial^2}{\partial t^2} v_1(x,t)\right)\sin(x) + x\left(\frac{\partial^4}{\partial x^4} v_0(x,t)\right) - \sin(x)\left(\frac{\partial^4}{\partial x^4} v_0(x,t)\right)}{\sin(x)} = 0,$$
$$p^2: \frac{\left(\frac{\partial^2}{\partial t^2} v_2(x,t)\right)\sin(x) + x\left(\frac{\partial^4}{\partial x^4} v_1(x,t)\right) - \sin(x)\left(\frac{\partial^4}{\partial x^4} v_1(x,t)\right)}{\sin(x)} = 0,$$

with the conditions

$$v_0(x,0) = x - \sin(x), \quad \frac{\partial v_0}{\partial t}(x,0) = -(x - \sin(x)),$$
$$v_i(x,0) = 0, \quad \frac{\partial v_i}{\partial t}(x,0)|_{t=0} = 0, \quad i = 1, 2, \ldots$$

The solution of previous equations may be written as

$$v_0(x,t) = (x - \sin(x))(1 - t),$$
$$v_1(x,t) = (x - \sin(x))\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right),$$
$$\mathrm{v}_2(\mathrm{x,t}) = (x - \sin(x))\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right),$$

In the same manner, the remaining components were obtained using a software Package.

According to the HPM, we can conclude that

$$u(x,t) = \lim_{p \to 1} v(x,t) = v_0(x,t) + v_1(x,t) + \cdots$$

Therefore,

$$u(x,t) = (x - \sin(x))\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \cdots\right)$$

The solution of $u(x,t)$ in a closed form is

$$u(x,t) = (x - \sin(x))e^{-t},$$

This is exactly the same as that obtained by an exact solution.

Example 2.9

Introduction

In the following, we consider the fourth-order equation in two space variables

$$\frac{\partial^2}{\partial t^2}u(x,y,t) + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\left(\frac{\partial^4}{\partial x^4}u(x,y,t)\right) + 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\left(\frac{\partial^4}{\partial y^4}u(x,y,t)\right) = 0,$$
$$\frac{1}{2} < x,\ y < 1, \quad t > 0,$$

subject to the initial conditions

$$u(x,y,0) = 0, \quad \frac{1}{2} < x < 1,$$
$$\frac{\partial}{\partial t}u(x,y,0) = 2 + \frac{x^6}{6!} + \frac{y^4}{6!}, \quad \frac{1}{2} < x < 1,$$

and the boundary conditions

$$u\left(\frac{1}{2},y,t\right) = \left(2 + \frac{(1/2)^6}{6} + \frac{y^6}{6!}\right)\sin t, \quad u(1,y,t) = \left(2 + \frac{(1/2)^6}{6} + \frac{y^6}{6!}\right)\sin t,$$
$$\frac{\partial^2}{\partial x^2}u\left(\frac{1}{2},y,t\right) = \frac{(1/2)^4}{24}\sin t, \qquad \frac{\partial^2}{\partial x^2}u(1,y,t) = \frac{1}{24}\sin t,$$
$$\frac{\partial^2}{\partial y^2}u\left(x,\frac{1}{2},t\right) = \frac{(1/2)^4}{24}\sin t, \qquad \frac{\partial^2}{\partial y^2}u(x,1,t) = \frac{1}{24}\sin t, \quad t > 0.$$

HPM Method

After separating the linear and nonlinear parts of the equation, a homotopy can be constructed as follows (Ganji and Hashemi 2007):

$$(1-p)\left(\frac{\partial^2}{\partial t^2}v(x,y,t) - \frac{\partial^2}{\partial t^2}u(x,y,0)\right) + p\begin{pmatrix} \frac{\partial^2}{\partial t^2}v(x,y,t) + 2\left(\frac{1}{x^2} + \frac{1}{720}x^4\right)\frac{\partial^4}{\partial x^4}v(x,y,t) \\ + 2\left(\frac{1}{y^2} + \frac{1}{720}y^4\right)\frac{\partial^4}{\partial y^4}v(x,y,t) \end{pmatrix} = 0,$$

$p \in [0,1]$.

Applying HPM and rearranging on the basis of powers of p-terms, we have

$$p^0\colon\left(\frac{\partial^2}{\partial t^2}v_0(x,y,t)\right)=0,$$

$$p^1\colon\frac{1}{360x^2y^2}\left(\begin{array}{l}360\left(\frac{\partial^2}{\partial t^2}v_1(x,y,t)\right)x^2y^2+y^2x^6\left(\frac{\partial^4}{\partial x^4}v_0(x,y,t)\right)\\ +720y^2\left(\frac{\partial^4}{\partial x^4}v_0(x,y,t)\right)+x^2y^6\left(\frac{\partial^4}{\partial y^4}v_0(x,y,t)\right)\\ +720x^2\left(\frac{\partial^4}{\partial y^4}v_0(x,y,t)\right)\end{array}\right)=0,$$

$$p^2\colon\frac{1}{360x^2y^2}\left(\begin{array}{l}360\left(\frac{\partial^2}{\partial t^2}v_2(x,y,t)\right)x^2y^2+y^2x^6\left(\frac{\partial^4}{\partial x^4}v_1(x,y,t)\right)\\ +720y^2\left(\frac{\partial^4}{\partial x^4}v_1(x,y,t)\right)+x^2y^6\left(\frac{\partial^4}{\partial y^4}v_1(x,y,t)\right)\\ +720x^2\left(\frac{\partial^4}{\partial y^4}v_1(x,y,t)\right)\end{array}\right)=0,$$

with the conditions

$$\begin{aligned}&v_0(x,y,0)=0,\quad \frac{\partial}{\partial t}v_0(x,y,0)=2+\frac{x^6}{6!}+\frac{y^4}{6!},\\ &v_i(x,y,0)=0,\quad \frac{\partial}{\partial t}v_i(x,y,0)|_{t=0}=0,\quad i=1,2,\ldots\end{aligned}$$

With the effective initial approximation for v_0 from the designated conditions, the solutions of the equations may be written as

$$\begin{aligned}v_0(x,y,t)&=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)t,\\ v_1(x,y,t)&=-\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^3}{3!},\\ v_2(x,y,t)&=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^5}{5!},\\ &.\\ &.\end{aligned}$$

In the same manner, the remaining components were obtained using a software package.

According to the HPM, we can conclude that

$$u(x,y,t)=\lim_{p\to 1}v(x,y,t)=v_0(x,y,t)+v_1(x,y,t)+v_2(x,y,t)+\cdots$$

Therefore,

$$u(x,y,t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right),$$

The solution $u(x,t)$ in a closed form is

$$u(x,y,t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\ \sin t,$$

This is exactly the same as that obtained by an exact solution.

Example 2.10

This example considers the following nonlinear oscillator with discontinuity (Beléndez et al. 2008):

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \mathrm{sgn}(x) = 0, \tag{2.168}$$

with initial conditions

$$x(0) = A \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0, \tag{2.169}$$

and $\mathrm{sgn}(x)$ is defined by

$$\mathrm{sgn}(x) = \begin{cases} -1, & x < 0, \\ +1, & x \geq 0. \end{cases} \tag{2.170}$$

All the solutions to Eq. 2.168 are periodic. We denote the angular frequency of these oscillations by ω and note that one of our major tasks is to determine $\omega(A)$—that is, the functional behavior of ω as a function of the initial amplitude A.

Equation 2.168 can be rewritten in the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x = x - \mathrm{sgn}(x). \tag{2.171}$$

Now the homotopy parameter p is used to expand the solution $x(t)$ and the square of the unknown angular frequency ω as

$$x(t) = x_0(t) + px_1(t) + p^2x_2(t) + \cdots, \tag{2.172}$$

$$1 = \omega^2 - p\alpha_1 - p^2\alpha_2 - \cdots, \tag{2.173}$$

where $\alpha_i(i = 1, 2, \ldots)$ are to be determined. Substituting Eqs. 2.172 and 2.173 into Eq. 2.171,

$$\begin{aligned}&\left(x_0'' + px_1'' + p^2x_2'' + \cdots\right) + \left(\omega^2 - p\alpha_1 - p^2\alpha_2 - \cdots\right)\left(x_0 + px_1 + p^2x_2 + \cdots\right)\\ &\quad = p\left[\left(x_0 + px_1 + p^2x_2 + \cdots\right) - \mathrm{sgn}\left(x_0 + px_1 + p^2x_2 + \cdots\right)\right]\end{aligned} \tag{2.174}$$

and collecting the terms of the same power of p, we obtain a series of linear equations, of which we write only the first four:

$$x_0'' + \omega^2 x_0 = 0, \qquad x_0(0) = A, \quad x_0'(0) = 0, \tag{2.175}$$

$$x_1'' + \omega^2 x_1 = (1+\alpha_1)x_0 - \mathrm{sgn}(x_0), \qquad x_1(0) = x_1'(0) = 0, \tag{2.176}$$

$$x_2'' + \omega^2 x_2 = \alpha_2 x_0 + (1+\alpha_1)x_1, \qquad x_2(0) = x_2'(0) = 0, \tag{2.177}$$

$$x_3'' + \omega^2 x_3 = \alpha_3 x_0 + \alpha_2 x_1 + (1+\alpha_1)x_2, \qquad x_3(0) = x_3'(0) = 0. \tag{2.178}$$

In Eqs. 2.175–2.178, we have taken into account the expression

$$\begin{aligned} f(x) &= f\left(x_0 + px_1 + p^2 x_2 + p^3 x_3 + \cdots\right) \\ &= f(x_0) + p\left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)_{x=x_0} x_1 + p^2\left[\left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)_{x=x_0} x_2 + \frac{1}{2}\left(\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right)_{x=x_0} x_1^2\right] + O(p^3), \end{aligned} \tag{2.179}$$

where $f(x) = \mathrm{sgn}(x)$ and

$$\frac{\mathrm{dsgn}(x)}{\mathrm{d}x} = \frac{\mathrm{d}^2\mathrm{sgn}(x)}{\mathrm{d}x^2} = \cdots = 0 \quad \mathit{for} \quad x \neq 0,$$

and then

$$\mathrm{sgn}\left(x_0 + px_1 + p^2 x_2 + \cdots\right) = \mathrm{sgn}(x_0).$$

The solution of Eq. 2.175 is

$$x_0(t) = A\cos\omega t. \tag{2.180}$$

Substitution of this result into the right-hand side of Eq. 2.176 gives

$$x_1'' + \omega^2 x_1 = (1+\alpha_1)A\cos\omega t - \mathrm{sgn}(A\cos\omega t). \tag{2.181}$$

It is possible to do the following Fourier series expansion:

$$\mathrm{sgn}(\cos\omega t) = \sum_{n=0}^{\infty} a_{2n+1}\cos[(2n+1)\omega t] = a_1\cos\omega t + a_3\cos 3\omega t + \cdots, \tag{2.182}$$

where

$$a_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} \mathrm{sgn}(A\cos\tau)\cos[(2n+1)\tau]\mathrm{d}\tau = (-1)^n\frac{4}{(2n+1)\pi}. \tag{2.183}$$

The first term of the expansion in Eq. 2.183 is given by

$$a_1 = \frac{4}{\pi} \int_0^{\pi/2} \operatorname{sgn}(A \cos \tau) \cos \tau \mathrm{d}\tau = \frac{4}{\pi}. \tag{2.184}$$

Substituting Eq. 2.182 into Eq. 2.181, we have

$$x_1'' + \omega^2 x_1 = [(1 + \alpha_1)A - a_1] \cos \omega t - \sum_{n=1}^{\infty} a_{2n+1} \cos[(2n + 1)\omega t]. \tag{2.185}$$

No secular terms in $x_1(t)$ require eliminating contributions proportional to $\cos \omega t$ on the right-hand side of Eq. 2.185, and we obtain

$$\alpha_1 = -1 + \frac{a_1}{A} = -1 + \frac{4}{\pi A}. \tag{2.186}$$

From Eqs. 2.173, 2.184, and 2.186, writing $p = 1$, we can easily find that the first-order approximate frequency is

$$\omega_1(A) = \sqrt{\frac{a_1}{A}} = \frac{2}{\sqrt{\pi A}} = \frac{1.128379}{\sqrt{A}} \tag{2.187}$$

and the first-order approximation period can be obtained as

$$T_1(A) = \pi\sqrt{\pi A} = 5.568328\sqrt{A} \tag{2.188}$$

Now, in order to obtain the correction term x_1 for the periodic solution x_0, we consider the following procedure. Taking Eqs. 2.185 and 2.186 into account, we rewrite Eq. 2.185 in the form

$$x_1'' + \omega^2 x_1 = -\sum_{n=1}^{\infty} a_{2n+1} \cos[(2n + 1)\omega t] \tag{2.189}$$

with initial conditions $x_1(0) = 0$ and $x_1'(0) = 0$. The periodic solution of Eq. 2.189 can be written as

$$x_1(t) = \sum_{n=0}^{\infty} b_{2n+1} \cos[(2n + 1)\omega t]. \tag{2.190}$$

Substituting Eq. 2.190 into Eq. 2.189, we can write the following expression for the coefficients b_{2n+1}:

$$b_{2n+1} = \frac{a_{2n+1}}{4n(n+1)\omega^2} = \frac{(-1)^n}{n(n+1)(2n+1)\pi\omega^2}. \tag{2.191}$$

for $n \geq 1$. Taking into account that $x_1(0) = 0$, Eq. 2.186 gives

$$b_1 = -\sum_{n=1}^{\infty} b_{2n+1} = \frac{\pi - 3}{\pi\omega^2} = \frac{\sigma}{\omega^2} \tag{2.192}$$

where

$$\sigma = 1 - \frac{3}{\pi}. \tag{2.193}$$

Substituting Eqs. 2.180, 2.186, 2.190, 2.191, and 2.192 into Eq. 2.177 gives the following equation for $x_2(t)$:

$$x_2'' + \omega^2 x_2 = \alpha_2 A \cos\omega t + \frac{\sigma a_1}{A\omega^2}\cos\omega t + \sum_{n=1}^{\infty} \frac{a_{2n+1}}{4n(n+1)A\omega^2}\cos[(2n+1)\omega t]. \tag{2.194}$$

The secular term in the solution for $x_2(t)$ can be eliminated if

$$\alpha_2 = -\frac{\sigma a_1}{A^2\omega^2} = \frac{12 - 4\pi}{\pi^2 A^2 \omega^2}. \tag{2.195}$$

Similarly for ω_2, and taking $p = 1$, one can easily obtain the following expression for the second-order approximation frequency

$$\omega_2(A) = \frac{1}{\sqrt{2A}}\sqrt{a_1 + \sqrt{a_1^2 - 4\sigma a_1}} = \sqrt{\frac{2 + 2\sqrt{4 - \pi}}{\pi A}} = \frac{1.107452}{\sqrt{A}}, \tag{2.196}$$

and the second-order approximation period is given by

$$T_2(A) = 5.672551\sqrt{A}. \tag{2.197}$$

With the requirement of Eq. 2.198, we can rewrite Eq. 2.194 in the form

$$x_2'' + \omega^2 x_2 = \sum_{n=1}^{\infty} \frac{a_{2n+1}}{4n(n+1)A\omega^2}\cos[(2n+1)\omega t], \tag{2.198}$$

with initial conditions $x_2(0) = 0$ and $x_2'(0) = 0$. The solution of this equation is

$$x_2(t) = \sum_{n=0}^{\infty} c_{2n+1}\cos[(2n+1)\omega t]. \tag{2.199}$$

Substituting Eq. 2.199 into Eq. 2.198, we obtain the following expression for the coefficients c_{2n+1}:

$$c_{2n+1} = -\frac{a_1 a_{2n+1}}{16n^2(n+1)^2 A\omega^4} = \frac{(-1)^{n+1}}{n^2(n+1)^2(2n+1)\pi^2 A\omega^4}, \tag{2.200}$$

for $n \geq 1$. Taking into account that $x_2(0) = 0$, Eq. 2.199 gives

$$c_1 = -\sum_{n=1}^{\infty} c_{2n+1} = \frac{\pi^2 + 24\pi - 66}{6\pi^2 A\omega^4} = \frac{\lambda}{A\omega^4}. \tag{2.201}$$

where

$$\lambda = \frac{\pi^2 + 24\pi - 66}{6\pi^2}. \tag{2.202}$$

Substitution of Eqs. 2.180, 2.186, 2.190–2.192, 2.195, and 2.199 into Eq. 2.178 gives the following equation for $x_3(t)$:

$$\begin{aligned} x_3'' + \omega^2 x_3 = {} & \alpha_3 A \cos \omega t + \frac{\lambda a_1}{A^2\omega^4} \cos \omega t - \frac{\sigma^2 a_1}{A^2\omega^4} \cos \omega t \\ & - \sum_{n=1}^{\infty} \frac{\sigma^2 a_{2n+1}}{4n(n+1)A^2\omega^4} \cos[(2n+1)\omega t] - \sum_{n=1}^{\infty} \frac{a_1^2 a_{2n+1}}{16n^2(n+1)^2 A^2\omega^4} \cos[(2n+1)\omega t]. \end{aligned} \tag{2.203}$$

The secular term in the solution for $x_3(t)$ can be eliminated if

$$\alpha_3 = \frac{\sigma^2 a_1 - \lambda a_1}{A^3\omega^4} = \frac{240 - 120\pi + 14\pi^2}{3\pi^3 A^3 \omega^4}. \tag{2.204}$$

From Eqs. 2.173, 2.186, 2.195, and 2.204, and taking $p = 1$, one can easily obtain that the following expression for the third-order approximation frequency is

$$\omega_3(A) = \frac{1.111358}{\sqrt{A}}, \tag{2.205}$$

and the third-order approximate period is given by

$$T_3(A) = 5.633609\sqrt{A}. \tag{2.206}$$

Taking Eq. 2.204 into consideration, we can rewrite Eq. 2.203 in the form

$$\begin{aligned} x_3'' + \omega^2 x_3 = {} & - \sum_{n=1}^{\infty} \frac{\sigma^2 a_{2n+1}}{4n(n+1)A^2\omega^4} \cos[(2n+1)\omega t] \\ & - \sum_{n=1}^{\infty} \frac{a_1^2 a_{2n+1}}{16n^2(n+1)^2 A^2\omega^4} \cos[(2n+1)\omega t] \end{aligned} \tag{2.207}$$

with initial conditions $x_3(0) = 0$ and $x_3'(0) = 0$. The solution of this equation is

$$x_3(t) = \sum_{n=0}^{\infty} d_{2n+1} \cos[(2n+1)\omega t]. \tag{2.208}$$

Substituting Eq. 2.208 into Eq. 2.207, we obtain the following expression for the coefficients d_{2n+1}:

$$d_{2n+1} = \frac{(-1)^{n+1}[n(n+1)(\pi - 3) + 1]}{n^3(n+1)^3(2n+1)\pi^3 A^2 \omega^6} \tag{2.209}$$

for $n \geq 1$. Taking into account that $x_3(0) = 0$, Eq. 2.207 gives

$$d_1 = -\sum_{n=1}^{\infty} d_{2n+1} = \frac{\pi^3 - 32\pi^2 + 234\pi - 450}{n^3(n+1)^3(2n+1)\pi^3 A^2 \omega^6}. \tag{2.210}$$

For this nonlinear problem, the exact periodic solution and the exact period are given by the equations

$$x_e(t) = \begin{cases} -\dfrac{t^2}{2} + A, & 0 \le t \le \dfrac{T_e}{4}, \\ \dfrac{t^2}{2} - 2\sqrt{2At} + 3A, & \dfrac{T_e}{4} < t \le \dfrac{3T_e}{4}, \\ -\dfrac{t^2}{2} + 4\sqrt{2At} - 15A, & \dfrac{3T_e}{4} < t \le T_e, \end{cases} \tag{2.211}$$

$$T_e(A) = 4\sqrt{2A} = 5.656854\sqrt{A}. \tag{2.212}$$

An easy and direct calculation gives the following series representation for the exact solution $x_e(t)$ (Eq. 2.211):

$$x_e(t) = \frac{32A}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos[(2n+1)\omega_e t], \tag{2.213}$$

where

$$\omega_e(A) = \frac{2\pi}{T_e(A)} = \frac{\pi}{2\sqrt{2A}} = \frac{1.110721}{\sqrt{A}}. \tag{2.214}$$

2.6 Variational Iteration Method

2.6.1 Introduction

The VIM was proposed by the Chinese mathematician He in 1997a and is a modified general Lagrange's multiplier method (Inokuti et al. 1978).

VIM has been favorably applied to various kinds of nonlinear problems. The main property of the method lies in its flexibility and ability to solve nonlinear equations accurately and conveniently, using a linearization assumption as an initial approximation or trial function; then a more highly precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution. The confluence of modem mathematics and symbol computation has posed a challenge to developing technologies capable of handling strongly nonlinear equations, which cannot be successfully dealt with by classical methods. The VIM is uniquely qualified to address this challenge. The flexibility and adaptation provided by the method have made the method a strong candidate for approximate analytical solutions. A new iteration formulation is suggested for overcoming the shortcoming. A very useful formulation for determining approximately the period of a nonlinear oscillator is suggested. Examples are given to illustrate the solution procedure.

Consider the following general nonlinear differential equation of an oscillator:

$$u'' + f(u, u', u'') = 0, \tag{2.215}$$

subject to $u(0) = a$ and $u'(0) = b$, where t is time and u is the displacement. The prime denotes differentiation with respect to t.

We rewrite Eq. 2.215 in the form

$$u'' + \Omega^2 u = F(u), \qquad F(u) = \Omega^2 u - f(u). \tag{2.216}$$

We consider that the angular frequency of the oscillator is Ω, and we choose the trial function using an initial condition [such as, for the initial condition $u(0) = A$ and $u'(0) = 0$, the trial function is $u_0(t) = A\cos\Omega t$]. The angular frequency Ω is identified with the physical understanding that no secular terms should appear in $u_1(t)$, which leads to

$$\int_0^T \cos\Omega t\left[\Omega^2 u_0 - f(u_0)\right]\mathrm{d}t = 0, \qquad T = \frac{2\pi}{\Omega}. \tag{2.217}$$

From this equation, Ω can easily be found. It should be especially pointed out that the more accurate the identification of the multiplier, the faster the approximations converge to its exact solution, and for this reason, we identify the multiplier from Eq. 2.216 rather than Eq. 2.215.

According to the VIM, we can construct a correction functional as (He 1997a)

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left\{u_n''(\tau) + \Omega^2 u_n(\tau) - \tilde{F}_n\right\}\mathrm{d}\tau, \tag{2.218}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation, and $\tilde{F}_n$ is considered as a restricted variation—that is, $\delta\tilde{F}_n = 0$. Under this condition, its stationary conditions of the above correction functional can be written as

$$\begin{aligned} &\lambda''(\tau) + \Omega^2\lambda(\tau) = 0, \\ &\lambda(\tau)|_{\tau=t} = 0, \\ &1 - \lambda'(\tau)|_{\tau=t} = 0. \end{aligned} \tag{2.219}$$

The Lagrange multiplier can, therefore, be readily identified by

$$\lambda = \frac{1}{\Omega}\sin\Omega(\tau - t), \tag{2.220}$$

which leads to the iteration formula

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{\Omega} \sin \Omega(\tau - t) \left\{ u_n''(\tau) + f_n \right\} \mathrm{d}\tau. \tag{2.221}$$

As we will see in the forthcoming illustrative examples, we usually stop at the first-order approximation, and the obtained approximate and accurate solution is valid for the whole solution domain.

2.6.2 Application

In this section, several examples are considered for the comparison and usefulness of the method developed.

Example 2.11

Let us consider the following nonlinear oscillators with discontinuities:

$$u'' + u + \varepsilon u|u| = 0,$$

with initial conditions $u(0) = A$ and $u'(0) = 0$.

Here the discontinuous function is $f(u) = u + \varepsilon u|u|$. From Eq. 2.217, we can determine the angular frequency (Rafei et al. 2007a) as

$$\int_0^T \cos \Omega t \left[\Omega^2 A \cos \Omega t - (A \cos \Omega t + \varepsilon A \cos \Omega t |A \cos \Omega t|) \right] \mathrm{d}t = 0, \qquad T = \frac{2\pi}{\Omega}.$$

Noting that $|\cos \Omega t| = \cos \Omega t$ when $-\pi/2 \leq \Omega t \leq \pi/2$ and $|\cos \Omega t| = -\cos \Omega t$ when $\pi/2 \leq \Omega t \leq 3\pi/2$, we can write the previous equation in the form

$$\int_{-\pi/2\Omega}^{\pi/2\Omega} \left[(\Omega^2 - 1) A \cos^2 \Omega t - \varepsilon A^2 \cos^3 \Omega t \right] \mathrm{d}t$$
$$+ \int_{\pi/2\Omega}^{3\pi/2\Omega} \left[(\Omega^2 - 1) A \cos^2 \Omega t + \varepsilon A^2 \cos^3 \Omega t \right] \mathrm{d}t = 0.$$

From the above equation, one can easily conclude that

$$\Omega = \sqrt{1 + \frac{8}{3\pi} \varepsilon A}.$$

We rewrite Eq. 2.221 in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{\Omega} \sin \Omega(\tau - t) \left\{ u_n''(\tau) + u_n(\tau) + \varepsilon u_n(\tau) |u_n(\tau)| \right\} \mathrm{d}\tau.$$

By the above iteration formula, we can calculate the first-order approximation

$$u_1(t) = \begin{cases} A\cos\Omega t + \int\limits_0^t \frac{1}{\Omega}\sin\Omega(\tau - t)\left\{(1-\Omega^2)A\cos\Omega\tau + \varepsilon A^2\cos^2\Omega\tau\right\}\mathrm{d}\tau, \\ -\frac{\pi}{2} \le \Omega t \le \frac{\pi}{2}, \\ A\cos\Omega t + \int\limits_0^t \frac{1}{\Omega}\sin\Omega(\tau - t)\left\{(1-\Omega^2)A\cos\Omega\tau - \varepsilon A^2\cos^2\Omega\tau\right\}\mathrm{d}\tau, \\ \frac{\pi}{2} \le \Omega t \le \frac{3\pi}{2}, \end{cases}$$

which yields

$$u_1(t) = \begin{cases} A\cos\Omega t + \frac{1}{2\Omega}A(\Omega^2 - 1)t\sin\Omega t + \frac{\varepsilon A^2}{6\omega^2}(\cos 2\Omega t + 2\cos\Omega t) - \frac{\varepsilon A^2}{2\Omega^2}, \\ -\frac{\pi}{2} \le \Omega t \le \frac{\pi}{2}, \\ A\cos\Omega t + \frac{1}{2\Omega}A(\Omega^2 - 1)t\sin\Omega t - \frac{\varepsilon A^2}{6\omega^2}(\cos 2\Omega t + 2\cos\Omega t) + \frac{\varepsilon A^2}{2\Omega^2}, \\ \frac{\pi}{2} \le \Omega t \le \frac{3\pi}{2}, \end{cases}$$

where the angular frequency Ω is defined as $\Omega = \sqrt{1 + \frac{8}{3\pi}\varepsilon A}$.

In order to compare with a traditional perturbation solution, we write Ali Nayfeh's result:

$$u = A\cos\left(1 + \frac{4}{3\pi}\varepsilon A\right)t + \cdots,$$

Example 2.12

Introduction

Two-strand or Sirospun yarns are produced on a conventional ring frame by feeding two roving, drafted simultaneously, into the apron zone at a predetermined separation. Emerging from the nip point of the front rollers, the two strands are twisted together to form a two-ply structure (see Fig. 2.5).

Nonlinear Dynamical Model

Assume that the convergence point (equilibrium position) moves to an instantaneous position (see Fig. 2.6), and the distance x and y are measured from the equilibrium position. Then the motion equations in x and y directions can be expressed (Shou and He 2008) as

$$\begin{aligned} &m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + F_1\cos\alpha - F_2\cos\beta = 0, \\ &m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + F_1\sin\alpha + F_2\sin\beta - F = 0. \end{aligned} \tag{2.222}$$

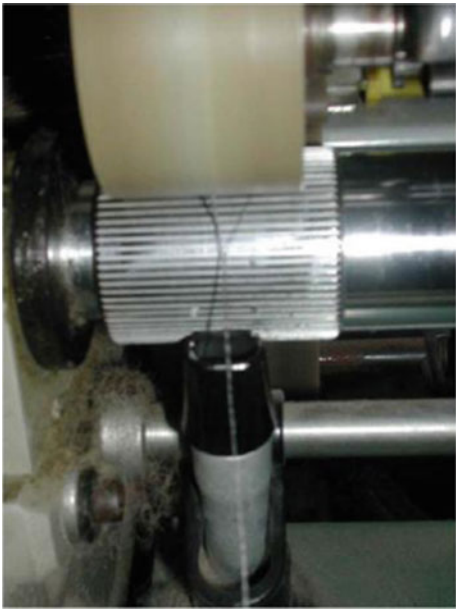

Fig. 2.5 Two-strand yarn spinning

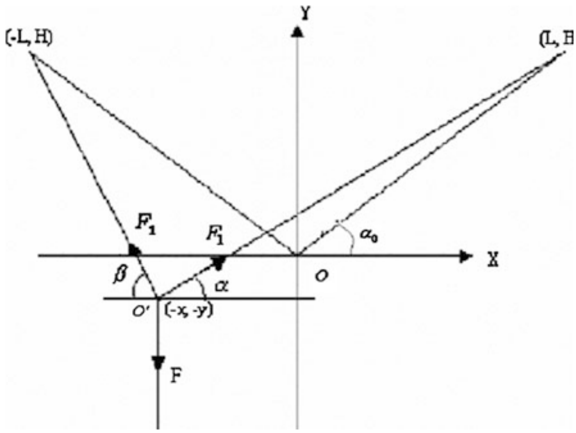

Fig. 2.6 Dynamical illustration of two-strand spun

Here, m is total mass of a fixed control volume, the control volume having been chosen in such a way that the mass center is located on the convergent point (O) of the two strands.

Expanding the trigonometric functions into series of x and y, we can obtain a coupled nonlinear oscillator. In this example, we consider the special

$$\begin{cases} \ddot{x} + \omega_1^2 x + \varepsilon_1 y^2 x = 0 \\ \ddot{y} + \omega_2^2 y + \varepsilon_2 x^2 y = 0 \end{cases}, \tag{2.223}$$

with the initial condition $x(0) = A, \dot{x}(0) = 0, y(0) = B, \dot{y}(0) = 0$.

In our study, ε_1 and ε_2 do not need to be small. We will apply the VIM to solve Eq. 2.222.

Applying the VIM, we can easily construct the following iteration formulations:

$$x_{n+1} = x_n + \frac{1}{\omega_1}\int_0^t \sin\omega_1(s-t)\left\{\frac{\mathrm{d}^2x}{\mathrm{d}s^2} + \omega_1^2 x + \varepsilon_1 y^2 x\right\}\mathrm{d}s, \tag{2.224}$$

$$y_{n+1} = y_n + \frac{1}{\omega_2}\int_0^t \sin\omega_2(s-t)\left\{\frac{\mathrm{d}^2y}{\mathrm{d}s^2} + \omega_1^2 y + \varepsilon_1 x^2 y\right\}\mathrm{d}s, \tag{2.225}$$

where $\lambda_x = \frac{1}{\omega_1}\sin\omega_1 t,\ \lambda_y = \frac{1}{\omega_2}\sin\omega_2 t.$

We begin with the initial solutions:

$$x_0 = A\cos\Omega_1 t, \tag{2.226}$$

$$y_0 = B\cos\Omega_2 t. \tag{2.227}$$

where Ω_1, Ω_2 are the frequencies in the x and y directions, respectively.

According to the iteration formulations 2.224 and 2.225, we obtain

$$\begin{aligned}
x_1 =& A\cos\Omega_1 t + \frac{1}{\omega_1}\int_0^t \sin\omega_1(s-t)\left\{A\left(\left(\omega_1^2-\Omega_1^2\right)\cos\Omega_1 s + \varepsilon_1 AB^2\cos\Omega_1 s\cos^2\Omega_2 s\right\}\mathrm{d}s\right. \\
=& A\cos\Omega_1 t + A\left(\omega_1^2-\Omega_1^2\right)\frac{\cos\omega_1 t-\cos\Omega_1 t}{\omega_1^2-\Omega_1^2} + \frac{\varepsilon_1 AB^2}{2}\frac{\cos\omega_1 t-\cos\Omega_1 t}{\omega_1^2-\Omega_1^2} \\
&+ \frac{\varepsilon_1 AB^2}{4}\frac{\cos\omega_1 t-\cos(2\Omega_2+\Omega_1)t}{\omega_1^2-(2\Omega_2+\Omega_1)^2} + \frac{\omega_2^2 AB^2}{4}\frac{\cos\omega_1 t-\cos(2\Omega_2-\Omega_1)t}{\omega_1^2-(2\Omega_2-\Omega_1)^2} \\
=& \left\{A + \frac{\varepsilon_1 AB^2}{2(\omega_1^2-\Omega_1^2)^2} + \frac{\varepsilon_1 AB^2}{4}\left[\frac{1}{\omega_1^2-(2\Omega_2+\Omega_1)^2} + \frac{1}{\omega_1^2-(2\Omega_2-\Omega_1)^2}\right]\right\}\cos\omega_1 t \\
&- \frac{\varepsilon_1 AB^2}{4}\left[\frac{2\cos\Omega_1 t}{\omega_1^2-\Omega_1^2} + \frac{\cos(2\Omega_2+\Omega_1)t}{\omega_1^2-(2\Omega_2+\Omega_1)^2} + \frac{\cos(2\Omega_2-\Omega_1)t}{\omega_1^2-(2\Omega_2-\Omega_1)^2}\right]
\end{aligned} \tag{2.228}$$

$$\begin{aligned}
y_1 &= B\cos\Omega_2 t + \frac{1}{\omega_2}\int_0^t \sin\omega_2(s-t)\left\{B(\omega_2^2-\Omega_2^2)\cos\Omega_2 s + \varepsilon_2 BA^2\cos\Omega_2 s\cos^2\Omega_1 s\right\}\mathrm{d}s \\
&= B\cos\Omega_2 t + B\left(\omega_2^2-\Omega_2^2\right)\frac{\cos\omega_2 t-\cos\Omega_2 t}{\omega_2^2-\Omega_2^2} + \frac{\varepsilon_2 BA^2}{2}\frac{\cos\omega_2 t-\cos\Omega_2 t}{\omega_2^2-\Omega_2^2} \\
&\quad + \frac{\varepsilon_2 BA^2}{4}\frac{\cos\omega_2 t-\cos(2\Omega_1+\Omega_2)t}{\omega_2^2-(2\Omega_1+\Omega_2)^2} + \frac{\omega_2^2 BA^2}{4}\frac{\cos\omega_2 t-\cos(2\Omega_1-\Omega_2)t}{\omega_2^2-(2\Omega_1-\Omega_2)^2} \\
&= \left\{B + \frac{\varepsilon_2 BA^2}{2\left(\omega_2^2-\Omega_2^2\right)^2} + \frac{\varepsilon_2 BA^2}{4}\left[\frac{1}{\omega_2^2-(2\Omega_1+\Omega_2)^2}+\frac{1}{\omega_2^2-(2\Omega_1-\Omega_2)^2}\right]\right\}\cos\omega_2 t \\
&\quad - \frac{\varepsilon_2 BA^2}{4}\left[\frac{2\cos\Omega_2 t}{\omega_2^2-\Omega_2^2}+\frac{\cos(2\Omega_1+\Omega_2)t}{\omega_2^2-(2\Omega_1+\Omega_2)^2}+\frac{\cos(2\Omega_1-\Omega_2)t}{\omega_2^2-(2\Omega_1-\Omega_2)^2}\right]
\end{aligned} \tag{2.229}$$

Eliminating the secular terms in x_2 and y_2, we require

$$A + \frac{\varepsilon_1 AB^2}{2(\omega_1^2-\Omega_1^2)} + \frac{\varepsilon_1 AB^2}{4}\left[\frac{1}{\omega_1^2-(2\Omega_2+\Omega_1)^2}+\frac{1}{\omega_1^2-(2\Omega_2-\Omega_1)^2}\right] = 0, \tag{2.230}$$

$$B + \frac{\varepsilon_2 BA^2}{2(\omega_2^2-\Omega_2^2)} + \frac{\varepsilon_2 BA^2}{4}\left[\frac{1}{\omega_2^2-(2\Omega_1+\Omega_2)^2}+\frac{1}{\omega_2^2-(2\Omega_1-\Omega_2)^2}\right] = 0, \tag{2.231}$$

from which the values of Ω_1 and Ω_2 can be determined.

From Eqs. 2.230 and 2.231, we can obtain the resonance condition of the coupled oscillator, which leads to

$$\omega_1^2 - (2\Omega_2-\Omega_1)^2 = 0, \tag{2.232}$$

$$\omega_2^2 - (2\Omega_1-\Omega_2)^2 = 0. \tag{2.233}$$

The condition for resonance can be obtained easily when the parameters are chosen. Resonance occurs when $\omega_1 = 2\Omega_2 - \Omega_1$ or $\omega_2 = 2\Omega_1 - \Omega_2$, where Ω_1 and Ω_2 can be determined from Eqs. 2.232 and 2.233.

2.7 He's Variational Approach

2.7.1 Basic Idea

He's variational approach was proposed by He in 2007. The main property of the method is to solve nonlinear equations accurately and conveniently with a linearization assumption used as an initial approximation or trial function, from which a more highly precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution. A very useful formulation for determining approximately the period of a nonlinear oscillator is suggested. Examples are given to illustrate the solution procedure.

Hereby, for a brief introduction of the method, we consider a general nonlinear oscillator in the form

$$\ddot{v}(t) + f(v(t)) = 0. \tag{2.234}$$

Its variational principle can be easily established using the semi-inverse method

$$J(v) = \int_0^{T/4} \left(-\frac{1}{2}\dot{v}^2 + F(v) \right) \mathrm{d}t, \tag{2.235}$$

where $T = 2\pi/\omega$ is the period of the nonlinear oscillator. Using Eq. 2.235 and $F(v) = \int (\alpha v + \beta v^3)\, \mathrm{d}v$, we obtain

$$J(v) = \int_0^{T/4} \left(-\frac{1}{2}\dot{v}^2 + \frac{1}{2}\alpha v^2 + \frac{1}{4}\beta v^4 \right) \mathrm{d}t. \tag{2.236}$$

Considering these initial conditions,

$$v(0) = A, \quad \dot{v}(0) = 0. \tag{2.237}$$

Assume that its solution can be expressed as

$$v(t) = A \cos \omega t. \tag{2.238}$$

Substituting Eq. 2.238 into Eq. 2.236 results in

$$\begin{aligned} J(A,\omega) &= \int_0^{T/4} \left(-\frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2}\alpha A^2 \cos^2 \omega t + \frac{1}{4}\beta A^4 \cos^4 \omega t \right) \mathrm{d}t \\ &= \frac{1}{\omega}\int_0^{\pi/2} \left(-\frac{1}{2}A^2\omega^2 \sin^2 t + \frac{1}{2}\alpha A^2 \cos^2 t + \frac{1}{4}\beta A^4 \cos^4 t \right) \mathrm{d}t \end{aligned} \tag{2.239}$$

Applying the Ritz method, we require

$$\partial J/\partial A = 0, \tag{2.240}$$

$$\partial J/\partial \omega = 0. \tag{2.241}$$

But by a careful inspection, for most cases, we find that

$$\partial J/\partial\omega < 0. \tag{2.242}$$

Thus, we modify the conditions 2.240 and 2.241 into the more simple form:

$$\partial J/\partial A = 0 \tag{2.243}$$

2.7.2 Application

Example 2.13

Introduction

The conservative autonomous system of a cubic Duffing equation is represented by the following second-order differential equation that one sees in Kachapi et al. (2010):

$$\ddot{v}(t) + \alpha v(t) + \beta v(t)^3 = 0, \tag{2.244}$$

with initial conditions

$$v(0) = A, \;\; \dot{v}(0) = 0. \tag{2.245}$$

where v and t are generalized dimensionless displacement and time variables, respectively, and α and β are any positive constant parameters.

Case 1

Consider a two-mass system connected with linear and nonlinear stiffnesses (the two-mass system model as shown in Fig. 2.7). The equation of motion is described as

$$m\ddot{x} + k_1(x - y) + k_2(x - y)^3 = 0, \tag{2.246a}$$

$$m\ddot{y} + k_1(y - x) + k_2(y - x)^3 = 0, \tag{2.246b}$$

with initial conditions

$$x(0) = X_0, \;\; \dot{x}(0) = 0. \tag{2.247a}$$

$$y(0) = Y_0, \;\; \dot{y}(0) = 0. \tag{2.247b}$$

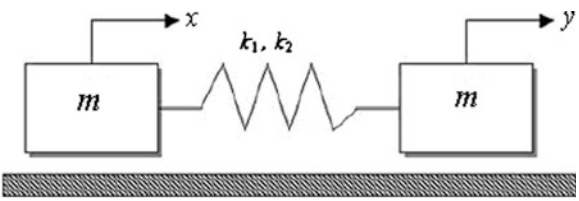

Fig. 2.7 Two masses connected by linear and nonlinear stiffnesses

whose double dots in Eqs. 2.246a and 2.246b denote double differentiation with respect to time t and k_1 and k_2 are linear and nonlinear coefficients of spring stiffness, respectively. Dividing Eqs. 2.246a and 2.246b by mass m yields

$$\ddot{x} + \frac{k_1}{m}(x - y) + \frac{k_2}{m}(x - y)^3 = 0, \tag{2.248a}$$

$$\ddot{y} + \frac{k_1}{m}(y - x) + \frac{k_2}{m}(y - x)^3 = 0. \tag{2.248b}$$

Introducing intermediate variables u and v as follows:

$$x := u, \tag{2.249}$$

$$y - x := v, \tag{2.250}$$

and transforming Eqs. 2.248a and 2.248b yield

$$\ddot{u} - \kappa\, v - \rho\, v^3 = 0, \tag{2.251a}$$

$$\ddot{v} + \ddot{u} + \kappa\, v + \rho\, v^3 = 0, \tag{2.251b}$$

where $\kappa = k_1/m$ and $\rho = k_2/m$. Equation 2.251a is rearranged as

$$\ddot{u} = \kappa\, v + \rho\, v^3 \tag{2.252}$$

Substituting Eq. 2.252 into Eq. 2.251b yields

$$\ddot{v} + 2\kappa\, v + 2\rho\, v^3 = 0, \tag{2.253}$$

with initial conditions

$$v(0) = y(0) - x(0) = Y_0 - X_0 = A, \quad \dot{v}(0) = 0. \tag{2.254}$$

Equation 2.253 is equivalent to Duffing's Eq. (a), with $\alpha = 2\kappa$ and $\beta = 2\rho$. For solving Eq. 2.253 using the variational approach, the approximate solutions of $v(\mathrm{t})$ can be back-substituted into Eq. 2.252 to obtain the intermediate variable $u(t)$ by double integration.

Its variational formulation can be readily obtained from Eq. 2.253 as

$$J(v) = \int_0^{T/4} \left(-\frac{1}{2}\dot{v}^2 + \kappa\, v^2 + \frac{1}{2}\rho\, v^4 \right) \mathrm{d}t. \tag{2.255}$$

Substituting Eq. 2.239 into Eq. 2.255, we obtain

$$J(A) = \int_0^{T/4} \left(-\frac{1}{2}A^2\omega^2 \sin^2 \omega\, t + \kappa A^2 \cos^2 \omega\, t + \frac{1}{2}\rho\, A^4 \cos^4 \omega\, t \right) \mathrm{d}t. \tag{2.256}$$

The stationary condition with respect to A leads to

$$\begin{aligned}\partial J/\partial A &= \int_0^{T/4} \left(-A\,\omega^2 \sin^2 \omega\, t + 2\,\kappa A\, \cos^2 \omega\, t + 2\,\rho\, A^3 \cos^4 \omega\, t\right) \mathrm{d}t \\ &= \int_0^{\pi/2} \left(-A\,\omega^2 \sin^2 t + 2\,\kappa A\, \cos^2 t + 2\,\rho\, A^3 \cos^4 t\right) \mathrm{d}t = 0.\end{aligned} \tag{2.257}$$

This leads to the result

$$\omega = \frac{1}{2}\sqrt{8\,\kappa + 6\,\rho\, A^2}. \tag{2.258}$$

According to Eqs. 2.239 and 2.258, we can obtain the approximate solution

$$v(t) = A\, \cos\left(\frac{1}{2} t\sqrt{8\,\kappa + 6\,\rho\, A^2}\right). \tag{2.259}$$

The first-order analytical approximation for $u(t)$ is

$$u(t) = \iint (\kappa\, v + \rho\, v^3)\, \mathrm{d}t\, \mathrm{d}t = \frac{1}{9\,\omega^2} A \cos(\omega\, t)\, \left(9\kappa + \rho A\, \cos^2(\omega\, t) + 6\rho A^2\right). \tag{2.260}$$

Therefore, the first-order analytically approximating displacements $x(t)$ and $y(t)$ are

$$x(t) = u(t), \tag{2.261}$$

$$y(t) = u(t) + A\cos(\omega t). \tag{2.262}$$

Case 2

Consider a two-mass system connected with linear and nonlinear springs and fixed to a body at two ends, as shown in Fig. 2.8. The equation of motion is described as

$$m\ddot{x} + k_1 x + k_2(x - y) + k_3(x - y)^3 = 0, \tag{2.263a}$$

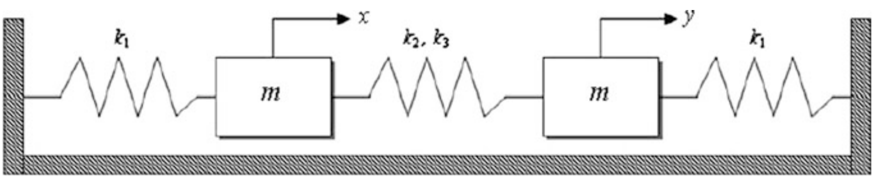

Fig. 2.8 Two-mass system connected with the fixed bodies

$$m\ddot{y} + k_1 y + k_2(y - x) + k_3(y - x)^3 = 0 \tag{2.263b}$$

with initial conditions

$$x(0) = X_0, \;\; \dot{x}(0) = 0. \tag{2.264}$$

$$y(0) = Y_0, \;\; \dot{y}(0) = 0. \tag{2.265}$$

whose double dots in Eqs. 2.263a and 2.263b denote double differentiation with respect to time t and where k_1 and k_2 are linear coefficients of spring stiffness and k_3 is the nonlinear coefficient of spring stiffness. Dividing Eqs. 2.263a and 2.263b by mass m yields

$$\ddot{x} + \frac{k_1}{m}x + \frac{k_2}{m}(x - y) + \frac{k_3}{m}(x - y)^3 = 0, \tag{2.266a}$$

$$\ddot{y} + \frac{k_1}{m}y + \frac{k_2}{m}(y - x) + \frac{k_3}{m}(y - x)^3 = 0. \tag{2.266b}$$

Similar to case 1, transforming the above equations, using intermediate variables, yields

$$\ddot{u} + \gamma\, u - \eta\, v - \lambda\, v^3 = 0, \tag{2.267a}$$

$$\ddot{u} + \ddot{v} + \gamma u + \gamma\, v + \eta v + \lambda\, v^3 = 0 \tag{2.267b}$$

in which $\gamma = k_1/m,\;\; \eta = k_2/m$, and $\lambda = k_3/m$. Rearranging Eq. 2.267a as

$$\ddot{u} = \eta v + \lambda\, v^3 - \gamma u \tag{2.268}$$

and back-substituting into Eq. 2.267b yields

$$\ddot{v} + (\gamma + 2\eta)\, v + 2\lambda\, v^3 = 0, \tag{2.269}$$

with initial conditions

$$v(0) = y(0) - x(0) = Y_0 - X_0 = A, \quad \dot{v}(0) = 0 \tag{2.270}$$

Equation 2.269 is again equivalent to Duffing's Eq. (a) with $\alpha = \gamma + 2\eta$ and $\beta = 2\lambda$. For solving Eq. 2.270 using a coupled variational approach, the approximate solutions of $v(t)$ can be back-substituted into Eq. 2.268 to yield

$$\ddot{u} + \gamma u = \eta v + \lambda\, v^3, \tag{2.271}$$

with initial conditions

$$u(0) = x(0) = X_0, \quad \dot{u}(0) = 0. \tag{2.272}$$

Equation 2.271 is a linear nonhomogeneous second-order ordinary differential equation, and it can be solved readily using a standard method such as a Laplace transformation.

Its variational formulation Eq. 2.273 can be readily obtained as

$$J(v) = \int_0^{T/4} \left(-\frac{1}{2}\dot{v}^2 + \frac{1}{2}(\gamma + 2\eta)\, v^2 + \frac{1}{2}\lambda\, v^4 \right) \mathrm{d}t. \tag{2.273}$$

Substituting Eq. 2.239 into Eq. 2.273, we obtain

$$J(A) = \int_0^{T/4} \left(-\frac{1}{2}\, A^2\omega^2 \sin^2 \omega\, t + \frac{1}{2}(\gamma + 2\eta) A^2 \cos^2 \omega\, t + \frac{1}{2}\lambda A^4 \cos^4 \omega\, t \right) \mathrm{d}t. \tag{2.274}$$

The stationary condition with respect to A reads

$$\begin{aligned} \partial J/\partial A &= \int_0^{T/4} \left(-A\,\omega^2 \sin^2 \omega\, t + (\gamma + 2\eta)\, A\, \cos^2 \omega\, t + 2\,\lambda A^3 \cos^4 \omega\, t \right) \mathrm{d}t \\ &= \int_0^{\pi/2} \left(-A\,\omega^2 \sin^2 t + (\gamma + 2\eta)\, A\, \cos^2 t + 2\,\lambda A^3 \cos^4 t \right) \mathrm{d}t = 0 \end{aligned} \tag{2.275}$$

This leads to the result

$$\omega = \frac{1}{2}\sqrt{8\,\eta + 4\gamma + 6\,\lambda\, A^2}. \tag{2.276}$$

According to Eqs. 2.239 and 2.276, we can obtain the approximate solution

$$v(t) = A\, \cos\left(\frac{1}{2} t \sqrt{8\,\eta + 4\gamma + 6\,\lambda A^2} \right). \tag{2.277}$$

By Eq. 2.275, the first-order analytical approximation for $u(t)$ is

$$\begin{aligned} u(t) = {} & \frac{\cos\left(\sqrt{\gamma}\, t\right) \left(\lambda A^3\, \gamma - 7\lambda A^3\, \omega^2 - \eta\, \gamma A + 9\,\eta\, \omega^2 A + 5\gamma^2 - 50\,\gamma\,\omega^2 + 45\,\omega^4 \right)}{\gamma^2 - 10\,\gamma\,\omega^2 + 9\omega^4} \\ & - \frac{36\left(-\frac{1}{36}(\omega^2 - \gamma)\lambda A^2 \cos(3\,\omega t) + \cos(\omega t)\left(\eta - \frac{3}{4}\,\lambda A^2\right)\left(\omega^2 - \frac{1}{9}\gamma\right)\right) A}{4\gamma^2 - 40\,\gamma\,\omega^2 + 36\omega^4} \end{aligned} \tag{2.278}$$

Therefore, the first-order analytically approximates displacements $x(t)$ and $y(t)$ are

$$x(t) = u(t), \tag{2.279}$$

$$y(t) = u(t) + A\cos(\omega t). \tag{2.280}$$

Discussion of Examples

The exact solution of the dynamical system can be obtained by integrating the governing equation (2.244) and imposing the initial conditions (2.245) as follows:

$$T(A) = \frac{4}{\sqrt{\alpha + \beta A^2}} \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{1 - m \sin^2 t}}, \tag{2.281}$$

for which

$$m = \frac{\beta A^2}{2(\alpha + \beta A^2)}. \tag{2.282}$$

The exact frequency ω_e is also a function of A and can be obtained from the period of the motion as

$$\omega_e(A) = \frac{\pi\sqrt{\alpha + \beta A^2}}{2} \left(\int_0^{\pi/2} \frac{\mathrm{d}t}{1 - m \sin^2 t} \right)^{-1}. \tag{2.283}$$

It should be noted that ω_e contains an integral, which could only be solved numerically in general.

Plotting the exact solution and variational solution, it is clear that the results are in excellent agreement (Figs. 2.9, 2.10).

Example 2.14

As a last example, we consider the following nonlinear Duffing-harmonic oscillation:

$$u'' + u^3/(1 + u^2) = 0, \qquad u(0) = A, \quad u'(0) = 0.$$

in which $f(u) = u^3/(1 + u^2)$.

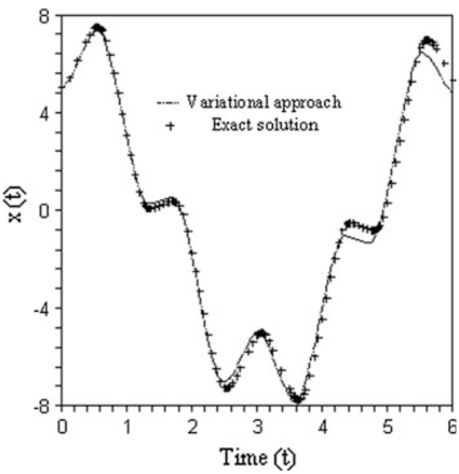

Fig. 2.9 Comparison of the analytical approximates Example 2.13 with the exact solution for $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, with x(0) = 5

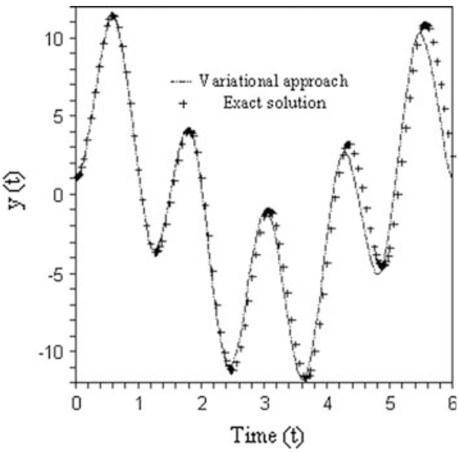

Fig. 2.10 Comparison of the analytical approximates Example 2.13 with the exact solution for $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, with $y(0) = 1$

Its variational formulation (Naghipour et al. 2008) is

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2}u'^2 + \frac{1}{2}u^2 - \frac{1}{2}\ln(1+u^2) \right) dt.$$

For similar previous examples, we have

$$J(A) = \int_0^{T/4} \left(-\frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2}A^2 \cos^2 \omega t - \frac{1}{2}\ln\left(1 + A^2 \cos^2 \omega t\right) \right) dt,$$

$$\frac{\partial J}{\partial A} = \int_0^{T/4} \left(-A\,\omega^2 \sin^2 \omega t + A\,\cos^2 \omega t - \left(A\,\cos^2 \omega t\right)/\left(1 + A^2\,\cos^2 \omega t\right)\right) dt$$

$$= \int_0^{\pi/2} \left(-A\,\omega^2 \sin^2 t + A\,\cos^2 t - \left(A\,\cos^2 t\right)/\left(1 + A^2\,\cos^2 t\right)\right) dt = 0.$$

From the previous equation, we have

$$\omega = \left((A^2+1)^{1/2}(2\,\mathrm{csgn}\,((A^2+1)^{1/2}) + A^2\,(A^2+1)^{1/2} - 2\,(A^2+1)^{1/2}\right)^{1/2} \Big/ \left(A\,(A^2+1)^{1/2}\right).$$

The csgn is defined in Maple Package Software.

2.8 Couple Variational Method

2.8.1 Introduction

The couple variational method (CVM) is a procedure for studying periodic solutions of strongly nonlinear systems (Kachapi et al. 2009b). The method consists of a combination of variational approaches to determine frequency and amplitude of the system and VIMs to obtain the time response of the system. Some examples are given to illustrate the effectiveness and convenience of the method.

2.8.2 Application

Example 2.15

As a first example, let us consider a family of nonlinear differential equations

$$u'' + \alpha u + \gamma u^{2m+1} = 0\,, \qquad \alpha \geq 0, \quad \gamma > 0, \quad m = 1,\ 2,\ 3,\ \ldots, \tag{2.284}$$

where $\alpha,\ \gamma$, and $m$are constant values. With the initial conditions,

$$u(0) = A, \quad u'(0) = 0. \tag{2.285}$$

For this problem,

$$f(u) = \alpha u + \gamma u^{2m+1} \text{ and } F(u) = \frac{1}{2}\alpha u^2 + \frac{\gamma u^{2m+2}}{2m+2}.$$

Its variational formulation can be readily obtained as

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2}u'^2 + \frac{1}{2}\alpha u^2 + \frac{\gamma u^{2m+2}}{2m+2} \right) \mathrm{d}t \tag{2.286}$$

Substituting $u_0(t) = A\cos\omega t$ into Eq. 2.286, we obtain

$$J(A) = \int_0^{T/4} \left(-\frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2}\alpha A^2 \cos^2 \omega t + \frac{\gamma (A \cos \omega t)^{2m+2}}{2m+2} \right) \mathrm{d}t \tag{2.287}$$

The stationary condition with respect to A leads to

$$\begin{aligned}\frac{\partial J}{\partial A} &= \int\limits_0^{T/4} \left(-A\,\omega^2 \sin^2 \omega\, t + \alpha A\, \cos^2 \omega\, t + \frac{\gamma\, (A\, \cos\, \omega\, t)^{2m+2}}{A} \right) \mathrm{d}t \\ &= \int\limits_0^{\pi/2} \left(-A\,\omega^2\, \sin^2 t + \alpha A\, \cos^2 t + \frac{\gamma\, (A\, \cos\, t)^{2m+2}}{A} \right) \mathrm{d}t = 0\end{aligned} \tag{2.288}$$

This leads to the result

$$\omega = \frac{\sqrt{A\,\pi\, \Gamma(m+2)\, \left(\alpha A\, \pi\, \Gamma(m+2)\, + 2\,\gamma A^{2m+1}\, \Gamma\left(\frac{3}{2} + m\right)\, \sqrt{\pi}\right)}}{A\,\pi\, \Gamma(m+2)}, \tag{2.289}$$

Function Gamma (Γ) is defined in the Mathematical package.
with $T = \frac{2\pi}{\omega}$, yield

$$T = \frac{2A\,\pi^2\,\Gamma(m+2)}{\sqrt{A\,\pi\, \Gamma(m+2)\, \left(\alpha A\, \pi\, \Gamma(m+2)\, + 2\,\gamma A^{2m+1}\, \Gamma\left(\frac{3}{2} + m\right)\, \sqrt{\pi}\right)}} \tag{2.290}$$

Thus, we apply VIM and rewrite in the form

$$u_{n+1}(t) = u_n(t) + \int\limits_0^t \frac{1}{\omega} \sin \omega\, (\tau - t) \left(u_n''(\tau) + \alpha\, u_n(\tau) + \gamma\, u_n^{2m+1}(\tau) \right) \mathrm{d}\tau. \tag{2.291}$$

By the above iteration formula, we can calculate the first-order approximation

$$\begin{aligned}u_1(t) &= A \cos \omega\, \mathrm{t} \\ &\quad + \frac{\int_0^t \sin(\tau - t) \left(-\omega^2 A \cos \omega\tau + \alpha A \cos \omega\tau \,+\, \gamma (A \cos \omega\tau)^{2m+1} \right) \mathrm{d}\tau}{\omega}.\end{aligned} \tag{2.292}$$

The angular frequency ω is defined as in Eq. 2.289. For example, for $\alpha = \gamma = A = m = 1$, it yields

$$u_1(t) = \cos (1.3229\, \mathrm{t}) - 0.012813 \cos \mathrm{t} + 0.012813 \cos (3.9687\, \mathrm{t})\,. \tag{2.293}$$

The above results are in good agreement with the results obtained by the exact solutions.

Example 2.16

In dimensionless form, a mass attached to the center of a stretched elastic wire has the equation of motion (Kachapi et al. 2009b)

$$u'' + u - \frac{\lambda\, u}{\sqrt{1+u^2}} = 0, \tag{2.294}$$

This is an example of a conservative nonlinear oscillatory system having an irrational elastic item. All the motions corresponding to Eq. 2.294 are periodic, the system will oscillate between symmetric bounds $[-A,\ A]$, and its angular frequency and corresponding periodic solution are dependent on the amplitude A.

Its variational formulation can be readily obtained as

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2}\, u'^2 + \frac{1}{2}\, u^2 - \lambda \sqrt{1+u^2} \right) \mathrm{d}t \tag{2.295}$$

By a similar manipulation, as illustrated in the previous example, we have

$$J(A) = \int_0^{T/4} \left(-\frac{1}{2}\, A^2\omega^2 \sin^2 \omega\, t + \frac{1}{2} A^2 \cos^2 \omega\, t - \lambda\sqrt{1 + A^2\, \cos^2 \omega\, t} \right) \mathrm{d}t. \tag{2.296}$$

The stationary condition with respect to A reads

$$\begin{aligned} \frac{\partial J}{\partial A} &= \int_0^{T/4} \left(-A\, \omega^2 \sin^2 \omega\, t + A \cos^2 \omega\, t - \left(\lambda A\, \cos^2 \omega\, t \Big/ \sqrt{1 + A^2 \cos^2\, \omega\, t} \right) \right) \mathrm{d}t \\ &= \int_0^{\pi/2} \left(-A\, \omega^2 \sin^2 t + A\, \cos^2 t - \left(\lambda A\, \cos^2 t \Big/ \sqrt{1 + A^2 \cos^2 t} \right) \right) \mathrm{d}t = 0 \end{aligned} \tag{2.297}$$

This leads to the result

$$\omega = \frac{\sqrt{\pi \left(4\lambda \left(\mathrm{Elliptic}K\left(\sqrt{-A^2} \right) \right) - 4\lambda \left(\mathrm{Elliptic}E\left(\sqrt{-A^2} \right) \right) + A^2\pi \right)}}{A\, \pi}, \tag{2.298}$$

where the incomplete elliptic integral Elliptic$E$ and Elliptic$K$ are defined in the Mathematical package. Hence, the approximate period is

$$\mathrm{T} = \frac{2\, \pi}{\omega} = \frac{2\, A\, \pi^2}{\sqrt{\pi \left(4\lambda \left(\mathrm{Elliptic}K\left(\sqrt{-A^2} \right) \right) - \left(4\lambda\, \mathrm{Elliptic}E\left(\sqrt{-A^2} \right) \right) + A^2\pi \right)}} \tag{2.299}$$

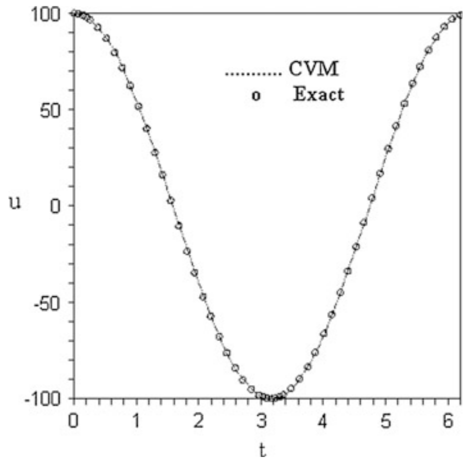

Fig. 2.11 The comparison of the approximate solution (CVM) with the exact solution for $\lambda = 0.1,\ A = 100$

We rewrite Eq. 2.294 in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{\omega} \sin \omega\,(\tau - t) \left(u_n''(\tau) + u_n(\tau) - \frac{\lambda\, u_n(\tau)}{\sqrt{1 + u_n^2(\tau)}} \right) \mathrm{d}\tau. \quad (2.300)$$

By the above iteration formula, we can calculate the first-order approximation

$$u_1(t) = A \cos \omega \mathrm{t} + \frac{\int_0^t \sin(\tau - t) \left(-\omega^2 A \cos \omega\tau + A \cos \omega\tau - \frac{\lambda A \cos \omega\tau}{\sqrt{1 + A^2 \cos^2 \omega\tau}} \right) \mathrm{d}\tau}{\omega}. \quad (2.301)$$

in which the angular frequency ω is defined as Eq. 2.298. The above results are in good agreement with the results obtained by the exact solution, as illustrated in Fig. 2.11.

2.9 Energy Balance Method

2.9.1 Introduction

In this section, we will introduce a heuristic approach, called the He's energy balance method (EBM), to nonlinear oscillators that were proposed by He in 2002a. In this method, a variational principle for the nonlinear oscillation is established; then a Hamiltonian is constructed, from which the angular frequency can be readily obtained by the collocation method. The results are valid not only for weakly nonlinear systems, but also for strongly nonlinear ones.

Some examples reveal that even the lowest order approximations are of high accuracy. They illustrate that the energy balance methodology is very effective and convenient and does not require linearization or small perturbation. It is predicted that the energy balance method will find wide application in engineering problems, as indicated in the following examples.

In order to represent the EBM, we consider a general nonlinear oscillator in the form (He 2002a)

$$u'' + f(u(t)) = 0 \tag{2.302}$$

in which u and t are generalized dimensionless displacement and time variables, respectively.

Its variational principle can be easily obtained as

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + F(u)\right)\mathrm{d}t \tag{2.303}$$

Its Hamiltonian, therefore, can be written as

$$H = \frac{1}{2}u'^2 + F(u) = F(A) \tag{2.304}$$

or

$$R(t) = \frac{1}{2}u'^2 + F(u) - F(A) = 0 \tag{2.305}$$

Oscillatory systems contain two important physical parameters—that is, the frequency ω and the amplitude of oscillation, A. So let us consider initial conditions such as

$$u(0) = A, \qquad u'(0) = 0 \tag{2.306}$$

Assume that its initial approximation can be expressed as

$$u(t) = A\cos(\omega t) \tag{2.307}$$

Substituting Eq. 2.307 as the u term of Eq. 2.305 yields

$$R(t) = \frac{1}{2}\omega^2 A^2 \sin^2 \omega t + F(A\ \cos\ \omega t) - F(A) = 0 \tag{2.308}$$

If, by any chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Eq. 2.306 is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega t = {}^{\pi}/_{4}$ gives

$$\omega = \sqrt{\frac{2(F(A) - F(A\ \cos\ \omega t))}{A^2 \sin^2\ \omega t}} \tag{2.309}$$

Its period can be written in the form

$$T = \frac{2\pi}{\sqrt{\frac{2(F(A)-F(A\ \cos\ \omega t))}{A^2 \sin^2\ \omega t}}} \tag{2.310}$$

2.9.2 Application

Example 2.17

We consider the nonlinear oscillator

$$u'' + u^3 + u^{\frac{1}{3}} = 0$$

with the boundary conditions

$$u(0) = A, u'(0) = 0.$$

Its Hamiltonian, therefore, can be written in the form (Ganji et al. 2009b)

$$\Delta H = \frac{1}{2}u'^2 + \frac{1}{4}u^4 + \frac{3}{4}u^{\frac{4}{3}} - \frac{1}{4}A^4 - \frac{3}{4}A^{\frac{4}{3}} = 0.$$

Choosing the trial function $u = A\cos(\omega t)$, we obtain the residual equation

$$R(t) = \frac{1}{2}A^2\omega^2\sin^2(\omega t) + \frac{1}{4}A^4\cos^4(\omega t) + \frac{3}{4}(A\cos(\omega t))^{\frac{4}{3}} - \frac{1}{4}A^4 - \frac{3}{4}A^{\frac{4}{3}} = 0.$$

If we collocate at $\omega t = \pi/4$, we obtain

$$\omega = \sqrt{\frac{3}{4}A^2 + 1.1101184A^{-\frac{2}{3}}},\ T = \frac{\omega}{2\pi}$$

We can obtain the approximate solution

$$u = A\cos\sqrt{\frac{3}{4}A^2 + 1.1101184A^{-\frac{2}{3}}}t$$

Example 2.18

The governing equation of motion and initial conditions of a particle on a rotating parabola can be expressed (Ganji et al. 2009b) as

$$\left(1 + 4q^2u^2\right)\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + 4q^2u\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + \Delta u = 0,\quad u(0) = A,\ \frac{\mathrm{d}u}{\mathrm{d}t}(0) = 0.$$

where $q > 0$ and $\Delta > 0$ are known positive constants. For this problem,

$$f(u) = 4q^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + 4q^2u\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + \Delta u \text{ and } F(u) = -2q^2u^2u'^2 + \frac{1}{2}\Delta u^2.$$

Its variational and Hamiltonian formulations can be readily obtained as

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 - 2q^2u^2u'^2 + \frac{1}{2}\Delta u^2\right)\mathrm{d}t,$$

$$H = \frac{1}{2}u'^2 + 2q^2u^2u'^2 + \frac{1}{2}\Delta u^2 = \frac{1}{2}\Delta A^2,$$

$$R(t) = \frac{1}{2}u'^2 + 2q^2u^2u'^2 + \frac{1}{2}\Delta u^2 - \frac{1}{2}\Delta A^2 = 0,$$

Substituting $u = A\cos(\omega t)$ into $R(t)$, we obtain

$$R(t) = \frac{1}{2}A^2\omega^2\sin^2(\omega t) + 2q^2\omega^2A^4\cos^2(\omega t)\,\sin^2(\omega t) + \frac{1}{2}\Delta A^2\cos^2(\omega t) - \frac{1}{2}\Delta A^2 = 0,$$

which gives us the result

$$\omega = \sqrt{\frac{\Delta}{(4A^2q^2\cos^2(\omega t) + 1)}},$$

with $T = \frac{2\pi}{\omega}$, which yields

$$T = \frac{2\pi}{\sqrt{\frac{\Delta}{(4A^2q^2\cos^2(\omega t)+1)}}},$$

If we collocate at $\omega t = \pi/4$, we obtain

$$\omega_{\mathrm{EBM}} = \sqrt{\frac{\Delta}{(2A^2q^2 + 1)}},$$

with $T = \frac{2\pi}{\omega}$, yielding

$$T_{\mathrm{EBM}} = \frac{2\pi}{\sqrt{\frac{\Delta}{(2A^2q^2+1)}}},$$

The exact period is

$$T_{\mathrm{ex}} = 4\Delta^{-1/2}\int_0^{\pi/2} \left(1 + 4q^2\beta^2\cos^2\varphi\right)^{1/2}\mathrm{d}\varphi.$$

Table 2.2 Comparison between analytical EBM and exact solutions when $\Delta = 1$ and $q = 1$

A	T_{EBM}	T_{ex}	Error percentage
0.1	6.34570	6.34555	0.0024
1.0	10.8827	10.5407	3.2451
10	89.0795	80.4880	10.674
100	888.598	800.071	11.064
1000	8885.76	8000.00	11.071

For comparison, the exact period $T_{ex}$and the approximate period T_{EBM} are listed in Table 2.2; they can give a good approximate period for values of oscillation amplitude.

Example 2.19

Consider the motion of a rigid rod rocking back and forth on a circular surface without slipping. The governing equation of motion can be expressed as

$$\left(\frac{1}{12}+\frac{1}{16}u^2\right)\frac{\mathrm{d}^2u}{\mathrm{d}t}+\frac{1}{16}u\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2+\frac{g}{4l}u\cos u=0,\quad u(0)=A,\quad \frac{\mathrm{d}u}{\mathrm{d}t}(0)=0,$$

where $g > 0$ and $l > 0$ are known positive constants.

For the problem, its variational formulation (Ganji et al. 2008b) can be obtained as

$$J(u)=\int_0^t\left(-\frac{1}{2}u'^2-\frac{3}{8}u^2u'^2+\frac{3g(\cos u+\sin u)}{l}\right)\mathrm{d}t,$$

By a similar manipulation, as illustrated in the previous example by using Eqs. (2.307) and (2.308) and with $T=\frac{2\pi}{\omega}$, we obtain the result

$$\begin{aligned}R(t)=&\frac{1}{2}A^2\omega^2\sin^2(\omega t)+\frac{3}{8}A^4\omega^2\cos^2(\omega t)\ \sin^2(\omega t)\\&+\frac{3g(\cos(A\cos(\omega t))+A\cos(\omega t)\sin(A\cos(\omega t))-\cos(A)-A\ \sin(A))}{l}=0,\end{aligned}$$

$$\omega=\frac{\sqrt{2\left(-6lg(3A^2\cos^2(\omega t)+4)\left(\begin{array}{r}\cos(A\ \cos(\omega t))+A\ \cos(\omega t)\ \sin(A\ \cos(\omega t))\\-\cos(A)-A\ \sin(A)\end{array}\right)\right)}}{(lA(3A^2\cos^2(\omega t)+4)\sin(\omega t))},$$

$$T=\frac{2\,\pi(lA(3A^2\cos^2(\omega t)+4)\sin(\omega t))}{\sqrt{2\left(-6lg(3A^2\cos^2(\omega t)+4)\left(\begin{array}{r}\cos(A\ \cos(\omega t))+A\ \cos(\omega t)\ \sin(A\ \cos(\omega t))\\-\cos(A)-A\ \sin(A)\end{array}\right)\right)}},$$

Substituting $\omega t = {}^{\pi}\!/_{4}$ in the previous equations for ω and T, we have

$$\omega_{\mathrm{EBM}} = \frac{4\sqrt{-3\,l g(8 + 3A^2)(\eta)}}{l A(3A^2 + 8)},$$
$$T_{\mathrm{EBM}} = \frac{2\,\pi\, l A(3A^2 + 8)}{4\sqrt{-3\,l g(8 + 3A^2)(\eta)}}.$$

where η is

$$\eta = 2\cos\left(\frac{A\sqrt{2}}{2}\right) + A\sqrt{2}\sin\left(\frac{A\sqrt{2}}{2}\right) - 2\cos(A) - 2A\sin(A)$$

The exact period of the equation of motion is

$$T_{\mathrm{ex}} = 4\Delta^{-1/2}\int\limits_{0}^{\pi/2}\left(\frac{(4 + 3\beta^2\sin^2\varphi)\beta^2\cos^2\varphi}{8[\beta\sin\beta + \cos\beta - \beta\sin\varphi\ \sin(\beta\sin\varphi) - \cos](\beta\sin\varphi)}\right)^{1/2}\mathrm{d}\varphi.$$

The exact period T_{ex} and the approximate period T_{EBM} are shown in Table 2.3. Note that for the problem, the maximum amplitude of oscillation should satisfy $\mathrm{A} < \pi/2$.

Example 2.20

Consider a system that undergoes rotary motion with linear and nonlinear stiffness and is fixed to a body at two ends, as shown in Fig. 2.12. Suppose that the two discs of moment of inertia (second moment of mass) are J about their center, the torsional stiffness between the two discs is $f = k_2\theta + k_3\theta^3$, and torsional stiffness at the two ends is $f = k_1\theta$ [].

The equation of motion is given as

$$J\ddot{\theta}_1 + k_1\theta_1 + k_2(\theta_1 - \theta_2) + k_3(\theta_1 - \theta_2)^3 = 0 \qquad (2.311)$$

$$J\ddot{\theta}_2 + k_1\theta_2 + k_2(\theta_2 - \theta_3) + k_3(\theta_2 - \theta_3)^3 = 0, \qquad (2.312)$$

with initial conditions

$$\theta_1(0) = \Theta_{10}\,,\ \dot{\theta}_1(0) = 0$$
$$\theta_2(0) = \Theta_{20}\,,\ \dot{\theta}_2(0) = 0.$$

Table 2.3 Comparison of approximate periods with exact solution

A	T_{EBM}	T_{ex}	Error percentage
0.10π	3.76397	3.76397	0.0008
0.15π	3.94064	3.94086	0.0056
0.20π	4.20181	4.20292	0.02642
0.30π	5.05831	5.07728	0.37348
0.40π	6.70586	6.89564	2.7521
0.45π	8.60226	8.94333	3.8136

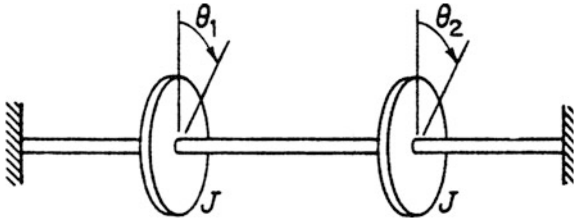

Fig. 2.12 Rotary motion with linear and nonlinear stiffness

Transforming the above equations using intermediate variables in Eqs. 2.311 and 2.312 yields

$$\ddot{u} + \gamma u - \eta v - \lambda v^3 = 0 \tag{2.313}$$

$$\ddot{v} + \ddot{u} + \gamma u + \gamma v + \eta v + \lambda v^3 = 0, \tag{2.314}$$

where $\gamma = \frac{k_1}{J}$, $\eta = \frac{k_2}{J}$ and $\lambda = \frac{k_3}{J}$. Rearranging Eq. 2.313 as follows,

$$\ddot{u} = -\gamma u + \eta v + \lambda v^3 = 0 \tag{2.315}$$

and substituting back into Eq. 2.314 yields

$$\ddot{v} + (\gamma + 2\eta)v + 2\lambda v^3 = 0, \tag{2.316}$$

with initial conditions

$$\begin{aligned} &v(0) = \theta_2(0) - \theta_1(0) = \Theta_{20} - \Theta_{10} = A \\ &\dot{v}(0) = 0 \end{aligned}$$

We choose two trial functions $v_1 = A\cos t$ and $v_2 = A\cos\omega t$.
Substituting v_1 and v_2 into Eq. 2.316, we obtain, respectively, the residuals

$$R_1 = -A\cos(t) + (\gamma + 2\eta)A\cos(t) + 2\lambda A^3\cos^3(t) \tag{2.317}$$

and

$$R_2 = -A\omega^2\cos(\omega t) + (\gamma + 2\eta)A\cos(\omega t) + 2\lambda A^3\cos^3(\omega t) \tag{2.318}$$

Also,

$$R_{11} = \frac{2\left(-\frac{1}{4}A\pi + \frac{1}{4}\gamma A\pi + \frac{1}{2}\eta A\pi + \frac{3}{8}\lambda A^3\pi\right)}{\pi} \tag{2.319}$$

and

$$R_{22} = \frac{1}{4}\frac{A(-2\omega^2\pi + 3\lambda A^2\pi + 2\gamma\pi + 4\eta\pi)}{\pi}. \tag{2.320}$$

We therefore obtain

$$\omega = \sqrt{\gamma + 2\eta + \frac{3}{2}A^2\lambda}. \tag{2.321}$$

According to Eq. 2.307, we can obtain the approximate solution

$$v(t) = A\cos\left(\sqrt{\gamma + 2\eta + \frac{3}{2}A^2\lambda}\, t\right). \tag{2.322}$$

By Eq. 2.315, the first-order analytical approximation for $u(t)$ is

$$x(t) = u(t) = \frac{\cos(\sqrt{\gamma}t)(\lambda A^3\gamma - 7\lambda A^3\omega^2 - \eta\gamma A + 9\eta\omega^2 A + 5\gamma^2 - 50\gamma\omega^2 + 45\omega^4)}{\gamma^2 - 10\gamma\omega^2 + 9\omega^4} - \frac{36\left(-\frac{1}{36}(\omega^2-\gamma)\lambda A^2\cos(3\omega t) + \cos(\omega t)\left(\eta - \frac{3}{4}\lambda A^2\right)\left(\omega^2 - \frac{1}{9}\gamma\right)\right)A}{4\gamma^2 - 40\gamma\omega^2 + 36\omega^4} \tag{2.323}$$

2.10 Coupled Method of Homotopy Perturbation and Variational Method

2.10.1 Introduction

The coupled method of homotopy perturbation and variational method has been given much attention recently; it has been proved that this method is very effective in determining the natural frequencies of strongly nonlinear oscillators with high accuracy (He 2006c).

In the coupled method of homotopy perturbation and variational method, the following homotopy is constructed, and a variational formulation for the nonlinear oscillation is established, from which the natural frequency and approximate solution can be readily obtained.

To illustrate the basic ideas of this method, we consider the following equation (see Sect. 2.5):

$$A(u) - f(r) = 0 \quad r \in \Omega, \tag{2.324}$$

with the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0 \quad r \in \Gamma, \tag{2.325}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function, and Γ is the boundary of the domain Ω.

A can be divided into two parts, which are L and N, where L is linear and N is nonlinear. Equation 2.324 therefore can be rewritten as

$$L(u) + N(u) - f(r) = 0 \quad r \in \Omega. \tag{2.326}$$

Homotopy perturbation structure is shown as

$$H(v,p) = (1-p)\ [L(v) - L(u_0)] + p[A(v) - f(r)\ ] = 0, \tag{2.327}$$

where

$$v(r,p) : \Omega \times [0,1] \to R \tag{2.328}$$

In Eq. 2.328, $p \in [0\ ,\ 1]$ is an embedding parameter, and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. 2.328 can be written as a power series in p,

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{2.329}$$

And the best approximation for the solution is

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{2.330}$$

Consider the following generalized nonlinear oscillations without forced terms:

$$u'' + \omega_0^2 u + \varepsilon f(u) = 0, \tag{2.331}$$

where f is a nonlinear function of u'', u', u.Its variational functional can be easily obtained as (see Sect. 2.6)

$$J(u) = \int_0^t \left\{ -\frac{1}{2}u'^2 + \frac{1}{2}\omega_0^2 u_2 + \varepsilon F(u) \right\} \mathrm{d}t, \tag{2.332}$$

where F is the potential,

$$\frac{\mathrm{d}F}{\mathrm{d}u} = f \tag{2.333}$$

2.10.2 Application

As an example of the method, a nonlinear oscillator with discontinuities of a conservative autonomous system can be described by the second-order differential equation

$$u'' + \alpha u' + \beta u^2 \mathrm{sgn}(u) + \gamma u^3 = 0 \text{ or } u'' + \alpha u' + \beta u|\,u| + \gamma u^3 = 0 \tag{2.334}$$

with initial conditions

$$u(0) = A\,,\ u'(0) = 0 \tag{2.335}$$

where $\operatorname{sgn}(u)$ is the sign function, equal to +1 if $u > 0$, 0 if $u = 0$ and -1 if $u < 0$.

As we will see in the forthcoming illustrative examples, we always stop at the first-order approximation, and the obtained approximate and accurate solution is valid for the whole solution domain.

In order to assess the advantages and the accuracy of the coupled method of homotopy perturbation and variational, we will consider the following two examples (Akbarzade et al. 2011).

Example 2.21

Let us consider the following nonlinear oscillators with discontinuities:

$$u'' + u + \varepsilon u^2 \operatorname{sgn}(u) = 0 \tag{2.336}$$

with initial conditions

$$u(0) = A\,,\ u'(0) = 0.$$

If $u > 0$,

$$u'' + u + \varepsilon u^2 = 0 \tag{2.337}$$

Suppose that the frequency of Eq. 2.337 is ω. We construct the following homotopy by the same manipulation as the basic idea:

$$u'' + \omega^2 u + p[\varepsilon u^2 + (1 - \omega^2)u] = 0\,,\ p \in [0\ ,\ 1]. \tag{2.338}$$

We assume that the periodic solution to Eq. 2.338 may be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots \tag{2.339}$$

Substituting Eq. 2.339 into Eq. 2.338 and collecting terms of the same power of p, gives

$$u_0'' + \omega^2 u_0 = 0\,,\ u_0(0) = A\,,\ u_0'(0) = 0 \tag{2.340}$$

and

$$u_1'' + \omega^2 u_1 + \varepsilon u_0^2 + (1 - \omega^2) u_0 = 0\,,\ u_1(0) = 0\,,\ u_1'(0) = 0. \tag{2.341}$$

The solution of Eq. 2.340 is $u_0 = A\cos\omega t$, where ω will be identified from the variational formulation for u_1, which leads to

$$J(u_1) = \int_0^T \left\{ -\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1^2 + (1 - \omega^2)\, u_0 u_1\ + \varepsilon u_0^2 u_1 \right\} \mathrm{d}t,\ \ \mathrm{T} = \frac{2\pi}{\omega} \tag{2.342}$$

To better illustrate the procedure, we choose the simplest trail function,

$$u_1 = B\left(\cos\omega t - \frac{5}{3}\cos 2\omega t\right) \tag{2.343}$$

Substituting u_1 into the functional Eq. 2.342 results in

$$J(B,\omega) = \left\{\frac{1}{18}\frac{(18A\pi - 75\omega^2 B\pi - 18A\omega^2\pi + 15A^2\pi)B}{\omega}\right\} \tag{2.344}$$

Setting

$$\frac{\partial J}{\partial B} = 0, \quad \text{and} \quad \frac{\partial J}{\partial \omega} = 0, \tag{2.345}$$

we obtain

$$-0.833\varepsilon A - 1 + \omega^2 = 0, \quad \text{and,} \quad \mathrm{B} = 0. \tag{2.346}$$

A first-order approximate solution is obtained, which leads to

$$\omega_1 = \sqrt{1 + 0.833\varepsilon A}. \tag{2.347}$$

The approximate period is

$$T_1 = \frac{2\pi}{\sqrt{1 + 0.833\varepsilon A}}. \tag{2.348}$$

In order to compare with harmonic balance, we write Hu's result:

$$T_1 = \frac{2\pi}{\sqrt{1 + \frac{8}{3\pi}\varepsilon A}}. \tag{2.349}$$

If $u < 0$,

$$u'' + u - \varepsilon u^2 = 0. \tag{2.350}$$

Suppose that the frequency of Eq. 2.350 is ω.

We construct the following homotopy by the same manipulation as the basic idea:

$$u'' + \omega^2 u + p[-\varepsilon u^2 + (1 - \omega^2)u] = 0\,, p \in [0\,,\ 1]. \tag{2.351}$$

We assume that the periodic solution to Eq. 2.351 may be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots \tag{2.352}$$

Substituting Eq. 2.352 into Eq. 2.351, collecting terms of the same power of p, gives

$$u_0'' + \omega^2 u_0 = 0\,,\; u_0(0) = A\,,\; u_0'(0) = 0 \tag{2.353}$$

and

$$u_1'' + \omega^2 u_1 - \varepsilon u_0^2 + (1 - \omega^2)u_0 = 0\,,\; u_1(0) = 0\,,\; u_1'(0) = 0. \tag{2.354}$$

The solution of Eq. 2.353 is $u_0 = A\cos\omega t$, where ω will be identified from the variational formulation for u_1, which reads

$$J(u_1) = \int_0^T \left\{ -\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1^2 + (1 - \omega^2)\, u_0 u_1 - \varepsilon u_0^2 u_1 \right\} \mathrm{d}t,\;\; \mathrm{T} = \frac{2\pi}{\omega}. \tag{2.355}$$

To better illustrate the procedure, we choose the simplest trail function,

$$u_1 = B\left(\cos\omega t - \frac{5}{3}\cos 2\omega t\right) \tag{2.356}$$

Substituting u_1 into the functional Eq. 2.355 results in

$$J(B,\omega) = \left\{ \frac{1}{18} \frac{(18A\pi - 75\omega^2 B\pi - 18A\omega^2\pi - 15A^2\pi)B}{\omega} \right\} \tag{2.357}$$

Setting

$$\frac{\partial J}{\partial B} = 0, \quad \text{and} \quad \frac{\partial J}{\partial \omega} = 0, \tag{2.358}$$

we obtain

$$+0.833\varepsilon A - 1 + \omega^2 = 0,\; \text{and, B} = 0 \tag{2.359}$$

The first-order approximate solution is obtained, which reads

$$\omega_2 = \sqrt{1 - 0.833\varepsilon A}. \tag{2.360}$$

The approximate period is

$$T_2 = \frac{2\pi}{\sqrt{1 - 0.833\varepsilon A}}. \tag{2.361}$$

In order to compare with harmonic balance, we write He's result

$$T_2 = \frac{2\pi}{\sqrt{1 - \frac{8}{3\pi}\varepsilon A}}. \tag{2.362}$$

We obtain the first approximate period T:

$$T = \frac{T_1 + T_2}{2} \tag{2.363}$$

Example 2.22

Consider the following nonlinear oscillator with discontinuities:

$$u'' + u + \varepsilon u^2 \operatorname{sgn}(u) + u^3 = 0, \tag{2.364}$$

with the initial conditions

$$u(0) = A\,,\ u'(0) = 0.$$

If $u > 0$,

$$u'' + u + \varepsilon u^2 + u^3 = 0 \tag{2.365}$$

Suppose that the frequency of Eq. 2.365 is ω. We construct the following homotopy by the same manipulation as the basic idea:

$$u'' + \omega^2 u + p[u^3 + \varepsilon u^2 + (1 - \omega^2)u] = 0\,,\ p \in [0\ ,\ 1]. \tag{2.366}$$

We assume that the periodic solution to equation Eq. 2.364 may be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots \tag{2.367}$$

Substituting Eq. 2.367 into Eq. 2.366, collecting terms of the same power of p gives

$$u_0'' + \omega^2 u_0 = 0\,,\ u_0(0) = A\,,\ u_0'(0) = 0 \tag{2.368}$$

and

$$u_1'' + \omega^2 u_1 + u_0^3 + \varepsilon u_0^2 + (1 - \omega^2)u_0 = 0\,,\ u_1(0) = 0\,,\ u_1'(0) = 0. \tag{2.369}$$

The solution of Eq. 2.368 is $u_0 = A\cos\omega t$, where ω will be identified from the variational formulation for u_1, which reads

$$J(u_1) = \int_0^T \left\{ -\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1^2 + (1 - \omega^2)\, u_0 u_1 + u_0^3 u_1 + \varepsilon u_0^2 u_1 \right\} \mathrm{d}t,\ \mathrm{T} = \frac{2\pi}{\omega} \tag{2.370}$$

To better illustrate the procedure, we choose the simplest trail function,

$$u_1 = B\left(\cos\omega t - \frac{5}{3}\cos 2\omega t\right) \tag{2.371}$$

Substituting u_1 into the functional Eq. 2.370 results in

$$J(B,\omega) = \left\{ -\frac{1}{36}\frac{(-30\varepsilon A^2\pi - 36A\pi + 36\omega^2 A\pi + 150B\omega^2\pi - 27A^3\pi)B}{\omega} \right\}. \tag{2.372}$$

Setting

$$\frac{\partial J}{\partial B} = 0, \text{ and } \frac{\partial J}{\partial \omega} = 0 \tag{2.373}$$

we obtain

$$-0.833\varepsilon A - 075A^2 - 1 + \omega^2 = 0, \text{and, B = 0.} \tag{2.374}$$

The first-order approximate solution is obtained, which reads

$$\omega_1 = \sqrt{1 + 0.75A^2 + 0.833\varepsilon A}. \tag{2.375}$$

The approximate period is

$$T_1 = \frac{2\pi}{\sqrt{1 + 0.75A^2 + 0.833\varepsilon A}}. \tag{2.376}$$

In order to compare with the harmonic balance solution, we write He's result,

$$T_1 = \frac{2\pi}{\sqrt{1 + \frac{3}{4}A^2 + \frac{8}{3\pi}\varepsilon A}}. \tag{2.377}$$

If $u < 0$,

$$u'' + u - \varepsilon u^2 + u^3 = 0. \tag{2.378}$$

Suppose that the frequency of Eq. 2.378 is ω. We construct the following homotopy by the same manipulation as the basic idea:

$$u'' + \omega^2 u + p[u^3 - \varepsilon u^2 + (1 - \omega^2)u] = 0,\ p \in [0, 1]. \tag{2.379}$$

We assume that the periodic solution to Eq. 2.379 may be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots \tag{2.380}$$

Substituting Eq. 2.380 into Eq. 2.379, collecting terms of the same power of p gives

$$u_0'' + \omega^2 u_0 = 0, u_0(0) = A, u_0'(0) = 0 \tag{2.381}$$

and

$$u_1'' + \omega^2 u_1 + u_0^3 + \varepsilon u_0^2 + (1 - \omega^2)u_0 = 0\,,\ u_1(0) = 0\,,\ u_1'(0) = 0. \tag{2.382}$$

The solution of Eq. 2.381 is $u_0 = A\cos\omega t$, where ω will be identified from the variational formulation for u_1, which reads

$$J(u_1) = \int_0^T \left\{ -\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1^2 + (1-\omega^2)\,u_0 u_1 \,+ u_0^3 u_1 - \varepsilon u_0^2 u_1 \right\} \mathrm{d}t,\ \ \mathrm{T} = \frac{2\pi}{\omega}. \tag{2.383}$$

To better illustrate the procedure, we choose the simplest trail function,

$$u_1 = B\left(\cos\omega t - \frac{5}{3}\cos 2\omega t\right) \tag{2.384}$$

Substituting u_1 into the functional Eq. 2.383 results in

$$J(B,\omega) = \left\{ -\frac{1}{36}\frac{(+30\varepsilon A^2\pi - 36A\pi + 36\omega^2 A\pi + 150B\omega^2\pi - 27A^3\pi)B}{\omega} \right\}. \tag{2.385}$$

Setting

$$\frac{\partial J}{\partial B} = 0, \text{ and } \frac{\partial J}{\partial \omega} = 0, \tag{2.386}$$

we obtain

$$+0.833\varepsilon A - 075A^2 - 1 + \omega^2 = 0,\ \text{ and, B = 0} \tag{2.387}$$

The first-order approximate solution is obtained, which reads

$$\omega_2 = \sqrt{1 + 0.75A^2 - 0.833\varepsilon A}. \tag{2.388}$$

The approximate period is

$$T_2 = \frac{2\pi}{\sqrt{1 + 0.75A^2 - 0.833\varepsilon A}}. \tag{2.389}$$

In order to compare with a harmonic balance solution, we write He's result as

$$T_2 = \frac{2\pi}{\sqrt{1 + \frac{3}{4}A^2 - \frac{8}{3\pi}\varepsilon A}}. \tag{2.390}$$

We obtain the first approximate period:

$$T = \frac{T_1 + T_2}{2}. \tag{2.391}$$

In order to compare with the exact solution,

$$T_e = \int_0^A \frac{2\mathrm{d}x}{\sqrt{A^2 - x^2 + \frac{2}{3}\varepsilon(A^3 - x^3) + \frac{1}{2}(A^4 - x^4)}} + \int_0^A \frac{2\mathrm{d}x}{\sqrt{A^2 - x^2 - \frac{2}{3}\varepsilon(A^3 - x^3) + \frac{1}{2}(A^4 - x^4)}} \tag{2.392}$$

For a relatively comprehensive survey on the concepts, theory, and applications of the methods mentioned in this chapter, see more examples in Hashemi et al. (2009), Kachapi et al. (2009a), Shou and He (2008), Ganji et al. (2007a, b, 2008a, c, d, e, 2009a, c, d, e, f), Varedi et al. (2007), Kimiaeifar et al. (2009), Pashaei et al. (2008), Barari et al. (2010), Ghotbi and Barari (2008), Jamshidi and Ganji (2009), Mehdipour et al. (2009), Ganji and Esmaeilpour (2010), Rafei et al. (2007b, c, d), Babazadeh et al. (2008).

Problems

Solve the following problems using methods presented in this chapter.

2.1 Consider the free response of the undamped, single-DOF system with $\alpha > 0$ so that the equation of motion of the system is

$$\ddot{x}(t) + k\,x(t) + \alpha\,x(t)^3 = 0,$$

where the initial condition is zero.

2.2 We consider the motion of a ring of mass m sliding freely on the wire described by the parabola $y = qu^2$, which rotates with a constant angular velocity λ about the y-axis. The equation describing the motion of the ring is

$$\ddot{u} + \omega^2 u = -4qu(u\ddot{u} + \dot{u}^2),$$

where $\omega^2 = 2gq - \lambda^2$ and the initial conditions are $u(0) = A, \quad \dot{u}(0) = 0$

2.3 The Van der Pol oscillator is a second-order system with nonlinear damping, of the form

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0.$$

2.4 Here, a system consisting of a block of mass m that hangs from a viscous damper with coefficient c and a nonlinear spring of stiffness k_1 and k_3 is considered. Equation of motion is given by the following nonlinear differential equation:

$$\frac{d^2x(t)}{\mathrm{d}t^2} + \frac{k_1}{m}\,x(t) + \frac{k_3}{m}\,x^3(t) + \frac{c}{m}\frac{\mathrm{d}x(t)}{\mathrm{d}t} = 0,$$

with the following initial conditions:

$$x_0(0) = A, \quad \frac{\mathrm{d}x_0}{\mathrm{d}t}(0) = 0.$$

2.5 In this problem, we shall consider a system of (1 + 1)-dimensional long-wave equations:

$$u_t + uu_x + v_x = 0,$$

$$v_t + (vu)_x + \frac{1}{3}u_{xxx} = 0,$$

with the initial conditions $u(x,0) = f(x)$ and $v(x,0) = g(x)$, where v is the elevation of the water wave and u is the surface velocity of water along the x-direction.

2.6 We consider a uniform cantilever beam in which μ is the constant mass density, EI is bending stiffness, and l is the length of the beam, when its base is given a motion $w_b(t)$ normal to the beam axis. The corresponding fourth-order differential equation of this case is

$$\mu\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} = -\mu\,\ddot{w}_b(t),$$

where we can assume

$$w_b(t) = W\sin(\omega t)$$

Thus,

$$\ddot{w}_b(t) = -W\omega^2\sin(\omega t)$$

The boundary conditions are

$$w(0,t) = 0, \quad \frac{\partial w}{\partial x}(0,t) = 0,$$

$$\frac{\partial}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right](l,t) = 0, \quad \frac{\partial^2 w}{\partial x^2}(l,t) = 0$$

and the initial displacement and velocity of the beam is assumed to be zero; thus,

$$w(x,0) = g(x) = 0,$$

$$\frac{\partial w}{\partial t}(x,0) = h(x) = 0.$$

2.7 We consider a generalized (scalar) Boussinesq equation of the form

$$u_{tt} - [f(u)]_{xx} - u_{xxtt} = h(x,t) \quad -\infty < x < +\infty, t \geq 0,$$

subject to the initial conditions

$$u(x,0) = a(x), \quad u_t(x,0) = b(x) \quad -\infty < x < +\infty,$$

2.8 We consider the nonlinear oscillator

$$u'' + u^3 = 0$$

with the boundary conditions

$$u(0) = A, u'(0) = 0$$

2.9 In this problem, we consider periodic solutions for sub-harmonic resonances of nonlinear oscillations with parametric excitation. The governing equation is

$$X'' + (1 + \varepsilon \cos(\lambda t))[\alpha X + \beta X^3] = 0,$$

and the boundary conditions for this equation are

$$X(0) = X_0,\ X'(0) = 0.$$

2.10 We consider the coupled Whitham–Broer–Kaup (WBK) equations, which have been studied by Whitham, Broer, and Kaup. The equations describe the propagation of shallow water waves, with different dispersion relations. The WBK equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \beta\frac{\partial^2 u}{\partial x^2} = 0,$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \alpha\frac{\partial^3 u}{\partial x^3} - \beta\frac{\partial^2 v}{\partial x^2} = 0,$$

with the initial conditions

$$u(x,0) = \lambda - 2k(\alpha + \beta^2)^{0.5}\coth[k(x + x_0)],$$

$$v(x,0) = -2k^2(\alpha + \beta^2 + \beta(\alpha + \beta^2)^{0.5})\mathrm{csch}^2[k(x + x_0)].$$

2.11 We consider the linear Schrödinger equation, which occurs in various areas of physics, including nonlinear optics, plasma physics, superconductivity, and quantum mechanics:

$$u_t + iu_{xx} = 0, \quad u(x,0) = f(x), \quad i^2 = -1$$

and the nonlinear Schrödinger equation

$$iu_t + u_{xx} + \gamma|u|^2 u = 0,\ u(x,0) = f(x), \quad i^2 = -1.$$

2.12 In this problem, we shall consider the Schrödinger equation in the form

$$i\frac{\partial\psi(x,t)}{\partial t} = -{}^1\!/_2\nabla^2\psi + V_d(x)\psi + \beta_d|\psi|^2\psi,\ x \in R^d, \quad t \geq 0$$

where $V_d(x)$ is the trapping potential and β_d is a real constant. With initial data

$$\psi(x,0) = \psi_0(x),\ x \in R^d$$

2.13 We consider nonlinear oscillation systems with parametric excitations, governed by

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + (1-\varepsilon\cos(\varphi t))(\lambda x(t) + \beta x(t)^3) = 0 \quad x(0) = A, \dot{x}(0) = 0.$$

where $\varepsilon, \varphi, \beta, \lambda$ are known as physical parameters.

2.14 Consider the Van der Pol equation

$$y''(t) + y(t) = \varepsilon\left[1 - y^2(t)\right]y'(t), \qquad y(0) = 0, \quad y'(0) = 2,$$

2.15 We consider a conservative system with a single degree of freedom for which the equation of motion is

$$(1 + 4r^2x^2(t))x'^2(t) + Ax(t) + 4r^2rx'^2(t)x(t) = 0,$$

where A is:

$$A = 2gr - \Omega^2$$

2.16 The rigid frame is forced to rotate at the fixed rate Ω while the frame rotates and the simple pendulum oscillates. The governing equation can be obtained as

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + (1 - A\cos(x(t)))\sin(x(t)) = 0.$$

Here, by using the Taylor's series expansion for $\cos(x(t))$ and $\sin(x(t))$, the above equation reduces to

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + (1-A)x(t) - \left(\frac{1}{6}x^3(t) + \frac{2}{3}Ax^3(t) - \frac{1}{2}Ax^5(t)\right) = 0,$$

with the boundary conditions

$$x(0) = \lambda,\ x'(0) = 0,$$

2.17 Consider the following nonlinear oscillator governed by

$$u'' + u = \varepsilon u'^2 u$$

with initial conditions of

$$u(0) = A,\ u'(0) = 0$$

2.18 We consider the motion of a particle on a rotating parabola. The governing equation of motion can be expressed as

$$u''(1+4q^2u^2)+\alpha^2u+4q^2uu'^2=0$$

with initial conditions

$$u(0)=A,\ u'(0)=0$$

2.19 Consider the following nonlinear oscillator governed by

$$u''+\Omega^2u+4\varepsilon u^2u''+4\varepsilon uu'^2=0$$

with initial conditions

$$u(0)=A\,,\ u'(0)=0.$$

2.20 In this problem, we have a rigid frame that is forced to rotate at the fixed rate Ω. The governing equation can be simply derived as

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+(1-\Lambda\cos\theta)\sin\theta=0,\ \theta(0)=A,\ \ \frac{\mathrm{d}\theta}{\mathrm{d}t}(0)=0.$$

2.21 Consider free vibration of a conservative system with a single degree of freedom containing a mass attached to linear and nonlinear springs in series. After transformation, the motion is governed by a nonlinear differential equation of motions:

$$(1+3\varepsilon z\,v^2)v''+6\varepsilon z\,v\,v'^2+\omega_e^2v+\varepsilon\,\omega_e^2v^3=0,$$

with the initial conditions

$$v(0)=A,\quad v'(0)=0$$

2.22 We consider the RLW equation

$$u_t+u_x+uu_x+u_{xxt}=0.$$

For the special case of this equation, the solitary wave solution is given in the form

$$u(x,t)=3B\sec h^2[k(x-(1+B)t)],$$

where

$$k=\frac{\sqrt{B}}{2\sqrt{1+B}},$$

and the exact solution is

$$u(x,t)=3\alpha\sec h^2(\beta(x-(1+\alpha)t)).$$

2.23 Generalize the KDV equation of two space variables and formulate the well-known Kadomtsev–Petviashvili (KP) equation to provide an explanation of the general weakly dispersive waves. The (2 + 1) KP equation is given in the form

$$(u_t + 6\mu u u_x + u_{xxx})_x + 3u_{yy} = 0,$$

or equivalently,

$$u_{xt} + 6\mu u_x^2 + 6\mu u u_{xx} + u_{xxxx} + 3u_{yy} = 0, \quad \mu = (\pm 1)$$

2.24 We consider the Burger equation

$$u_t + uu_x - u_{xx} = 0,\ x \in R$$

with the exact solution being

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{4}\left(x - \frac{1}{2}t\right)\right).$$

2.25 Consider a nonlinear equation of fourth order:

$$u_{tt} + 3u_{xt}u_{xx} + u_{xxxt} = 0,$$

2.26 The Whitham–Broer–Kaup equation is

$$u_t + uu_x + v_x + \beta\, u_{xx} = 0,$$
$$v_t + (uv)_x + \alpha\, u_{xxx} - \beta\, v_{xx} = 0.$$

2.27 The motivation of this part is to extend the analysis of the sine–cosine method to solve different types of nonlinear equations—namely, sSK and LsKdV and water wave equations, which can be shown in the form

$$u_t + \left(63u^4 + 63\left(2u^2u_{xx} + uu_x^2\right) + 21\left(uu_{xxx} + u_{xx}^2 + u_xu_{xxx}\right) + u_{xxxxxx}\right)_x = 0,$$
$$u_t + \left(35u^4 + 70\left(u^2u_{xx} + uu_x^2\right) + 7\left(2uu_{xxx} + 3u_{xx}^2 + 4u_xu_{xxx}\right) + u_{xxxxxx}\right)_x = 0,$$
$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} - u_{xxxxx} = 0,$$

Respectively, the first equation is known as the seventh-order Sawada–Kotera equation, the second equation is known as Lax's seventh-order KdV equation, and the third equation is known as the water wave equation.

2.28 In this problem, we consider the generalized Zakharov equations, which can be shown in the form

$$i\psi_t + \psi_{xx} - 2\,\alpha\,|\psi|^2\,\psi + 2\,\psi\,v = 0,$$

$$v_{tt} - v_{xx} + \left(|\psi|^2\right)_{xx} = 0.$$

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