Chapter 2
Vector Bundles

Vector bundles constitute a special class of manifolds, which is of great importance in physics. In particular, all sorts of tensor fields occurring in physical models may be viewed in a coordinate-free manner as sections of certain vector bundles. We start with the observation that the tangent spaces of a manifold combine in a natural way into a bundle, which is called tangent bundle. Next, by taking its typical properties as axioms, we arrive at the general notion of vector bundle. In Sect. 2.2, we discuss elementary aspects of this notion, including the proof that—up to isomorphy—vector bundles are completely determined by families of transition functions. In Sect. 2.3 we discuss sections and frames,[1] and in Sect. 2.4 we present the tool kit for vector bundle operations. We will see that, given some vector bundles over the same base manifold, by applying fibrewise the standard algebraic operations of taking the dual vector space, of building the direct sum and of taking the tensor product, we obtain a universal construction recipe for building new vector bundles. In Sect. 2.5, by applying these operations to the tangent bundle of a manifold, we get the whole variety of tensor bundles over this manifold. The remaining two sections contain further operations, which will be frequently used in this book. In Sect. 2.6, we discuss the notion of induced bundle and Sect. 2.7 is devoted to subbundles and quotient bundles. There is a variety of special cases occurring in applications: regular distributions, kernel and image bundles, annihilators, normal and conormal bundles.

2.1 The Tangent Bundle

Let M be a C^k-manifold, let $I \subset \mathbb{R}$ be an open interval and let $\gamma : I \to M$ be a C^k-curve. According to Example 1.5.6, for every $t \in I$, the tangent vector $\dot{\gamma}(t)$ of γ at t is an element of the tangent space $\mathrm{T}_{\gamma(t)}M$. Hence, while t runs through I, $\dot{\gamma}(t)$ runs through the tangent spaces along γ, see Fig. 2.1.

[1] Here, as well as in Sect. 2.5, in order to keep in touch with the physics literature, the local description is presented in some detail. In particular, we discuss transformation properties. This way, we make contact with classical tensor analysis.

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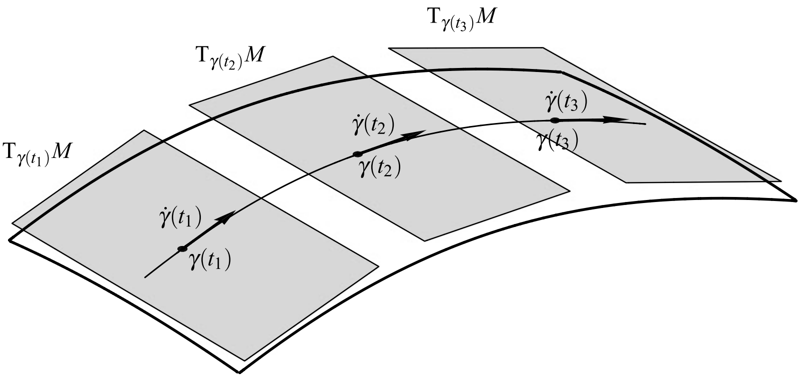

Fig. 2.1 Tangent vectors along a curve γ in M

To follow the tangent vectors along γ it is convenient to consider the totality of all tangent spaces of M. This leads to the notion of tangent bundle of a manifold M, denoted by $\mathrm{T}M$. As a set, $\mathrm{T}M$ is given by the disjoint union of the tangent spaces at all points of M, that is,

$$\mathrm{T}M := \bigsqcup_{m \in M} \mathrm{T}_m M. \tag{2.1.1}$$

Let $\pi : \mathrm{T}M \to M$ be the canonical projection which assigns to an element of $\mathrm{T}_m M$ the point m for every $m \in M$. $\mathrm{T}M$ can be equipped with a manifold structure as follows. Denote $n = \dim M$. Choose a countable atlas $\{(U_\alpha, \kappa_\alpha) : \alpha \in A\}$ on M and define the mappings

$$\kappa_\alpha^{\mathrm{T}} : \pi^{-1}(U_\alpha) \to \mathbb{R}^n \times \mathbb{R}^n, \qquad \kappa_\alpha^{\mathrm{T}}(X_m) := \big(\kappa_\alpha(m), \mathbf{X}_m^{\kappa_\alpha}\big). \tag{2.1.2}$$

The image of $\kappa_\alpha^{\mathrm{T}}$ is given by $\kappa_\alpha(U_\alpha) \times \mathbb{R}^n$ and is hence open in $\mathbb{R}^n \times \mathbb{R}^n$. Using (1.4.9), for the transition mappings we obtain

$$\kappa_\beta^{\mathrm{T}} \circ \big(\kappa_\alpha^{\mathrm{T}}\big)^{-1}(\mathbf{x}, \mathbf{X}) = \big(\kappa_\beta \circ \kappa_\alpha^{-1}(\mathbf{x}), \big(\kappa_\beta \circ \kappa_\alpha^{-1}\big)'(\mathbf{x}) \cdot \mathbf{X}\big), \tag{2.1.3}$$

where $(\mathbf{x}, \mathbf{X}) \in \kappa_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n$. Since $\kappa_\alpha \circ \kappa_\beta^{-1}$ is of class C^k, the transition mappings are of class C^{k-1}. Finally, it is obvious that the subsets $\pi^{-1}(U_\alpha)$ cover $\mathrm{T}M$. Thus, according to Remark 1.1.10, the family of bijections $\{(\pi^{-1}(U_\alpha), \kappa_\alpha^{\mathrm{T}}) : \alpha \in A\}$ defines a differentiable structure of class C^{k-1} and dimension $2n$ on $\mathrm{T}M$, which has the following properties. First, due to (2.1.3), it is independent of the choice of an atlas on M used to construct it. Second, the local representative of the projection $\pi : \mathrm{T}M \to M$ with respect to the charts $\kappa_\alpha^{\mathrm{T}}$ and κ_α is given by the natural projection pr_1 to the first factor in $\kappa_\alpha(U_\alpha) \times \mathbb{R}^n$. Hence, π is a submersion of class C^{k-1}. Third, the charts $\kappa_\alpha^{\mathrm{T}}$ identify the open submanifolds $\pi^{-1}(U_\alpha)$ of $\mathrm{T}M$ with direct products of an open subset of M with a copy of $\mathbb{R}^n$. Under this identification, both the natural projection and the vector space structure on every tangent space

$\mathrm{T}_m M$, $m \in U_\alpha$, is preserved. To formalize this, for every $\alpha \in A$, define a mapping

$$\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n, \qquad \chi_\alpha(X_m) := \big(m, \mathbf{X}_m^{\kappa_\alpha}\big). \tag{2.1.4}$$

Then $\kappa_\alpha^{\mathrm{T}} = (\kappa_\alpha \times \mathrm{id}_{\mathbb{R}^n}) \circ \chi_\alpha$. In particular, the local representative of χ_α with respect to the global charts $\kappa_\alpha^{\mathrm{T}}$ on $\pi^{-1}(U_\alpha)$ and $\kappa_\alpha \times \mathrm{id}_{\mathbb{R}^n}$ on $U_\alpha \times \mathbb{R}^n$ is given by the identical mapping of $\mathbb{R}^n \times \mathbb{R}^n$, restricted to the open subset $\kappa_\alpha(U_\alpha) \times \mathbb{R}^n$. Hence, χ_α is a C^{k-1}-diffeomorphism. Moreover, $\mathrm{pr}_1 \circ \chi_\alpha = \pi_{\restriction \pi^{-1}(U_\alpha)}$ and the restrictions $\chi_{\alpha \restriction \mathrm{T}_m M}$ are vector space isomorphisms for all $m \in U_\alpha$. Let us summarize.

Proposition 2.1.1 *Let M be a C^k-manifold of dimension n and let $\mathrm{T}M$ be defined by (2.1.1). There exists a unique C^{k-1}-structure on $\mathrm{T}M$ such that for every local chart (U, κ) on M, the mapping $\kappa^{\mathrm{T}} : \pi^{-1}(U) \to \mathbb{R}^n \times \mathbb{R}^n$, defined by (2.1.2), is a local chart on $\mathrm{T}M$. With respect to this structure, $\mathrm{T}M$ has dimension $2n$ and the following holds.*

1. *The natural projection $\pi : \mathrm{T}M \to M$ is a surjective submersion.*
2. *There exists an open covering $\{U_\alpha\}$ of M and an associated family of diffeomorphisms $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ such that*
   (a) *the following diagram commutes,*

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\chi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^n \arrow[dl, "\mathrm{pr}_1"] \\ & U_\alpha & \end{tikzcd}$$

   (b) *for every $m \in U_\alpha$, the induced mapping $\mathrm{pr}_2 \circ \chi_{\alpha \restriction \mathrm{T}_m M} : \mathrm{T}_m M \to \mathbb{R}^n$ is a vector space isomorphism.*

Definition 2.1.2 The triple $(\mathrm{T}M, M, \pi)$ is called the tangent bundle of M. $\mathrm{T}M$ is called the total space or the bundle manifold, M the base manifold and π the natural projection. For $m \in M$, $\pi^{-1}(m) \equiv \mathrm{T}_m M$ is called the fibre over m. The vector space $\mathbb{R}^n$ is called the typical fibre and the pairs (U_α, χ_α) are called local trivializations of $\mathrm{T}M$ over U_α.

By an abuse of notation, the tangent bundle will usually be denoted by $\mathrm{T}M$.

Example 2.1.3 Let $M = \mathrm{S}^1$ be realized as the unit circle in $\mathbb{R}^2$. For every $\mathbf{x} \in \mathrm{S}^1$, the tangent space $\mathrm{T}_{\mathbf{x}}\mathrm{S}^1$ can be identified with the subspace of vectors orthogonal to $\mathbf{x}$. This yields a bijection Φ from TS^1 onto the subset

$$T = \big\{(\mathbf{x}, \mathbf{X}) \in \mathrm{S}^1 \times \mathbb{R}^2 : \mathbf{x} \perp \mathbf{X}\big\}$$

of $\mathbb{R}^4$. This is the level set of the smooth mapping

$$F : \mathbb{R}^4 \to \mathbb{R}^2, \qquad F(\mathbf{x}, \mathbf{X}) := \big(\|\mathbf{x}\|^2, \mathbf{x} \cdot \mathbf{X}\big)$$

at the regular value $\mathbf{c} = (1, 0)$. Hence, it carries a smooth structure. One can check that Φ is a diffeomorphism with respect to this structure. (To see this, let $\mathrm{pr}_k : \mathbb{R}^2 \to \mathbb{R}$ denote the natural projection to the k-th component and choose the charts on S^1 and T to be restrictions of pr_k and $\mathrm{pr}_k \times \mathrm{pr}_k$, respectively, $k = 1, 2$.) Thus, TS^1 can naturally be identified with T. The construction carries over to higher-dimensional spheres: as a manifold, the tangent bundle TS^n can be identified with the subset $\{(\mathbf{x}, \mathbf{X}) \in \mathrm{S}^n \times \mathbb{R}^{n+1} : \mathbf{x} \perp \mathbf{X}\}$ of $\mathbb{R}^{2(n+1)}$ which is the level set of a function similar to F at the regular value $\mathbf{c} = (1, 0)$, see also Remark 2.1.4/2 below.

Remark 2.1.4

1. Let V be a finite-dimensional real vector space and let M be an open subset of V. The natural identifications of the tangent spaces $\mathrm{T}_v M$ with V for all $v \in M$, cf. Example 1.4.3/1, combine to a smooth diffeomorphism $\chi : \mathrm{T}M \to M \times V$ which is fibrewise linear. We will refer to χ as the natural identification of $\mathrm{T}M$ with $M \times V$. After choosing a basis in V, this bijection coincides with the (global) trivialization induced via (2.1.4) by the corresponding global chart on M.
2. The construction of Example 2.1.3 generalizes to arbitrary level sets. Let V, W be finite-dimensional real vector spaces and let M be the level set of a C^k-mapping $f : V \to W$ at a regular value $c \in W$. Identifying the tangent space $\mathrm{T}_v M$ with $\ker f'(v)$ for all $v \in M$, see Remark 1.2.2/1, we obtain a bijection Φ from $\mathrm{T}M$ onto the subset

$$T = \big\{(v, X) \in M \times V : f'(v)X = 0\big\}$$

of $V \times V$. This is the level set of the C^{k-1}-mapping

$$F : V \times V \to W \times W, \qquad F(v, X) := \big(f(v), f'(v)X\big)$$

at the value $(c, 0)$, whose regularity follows from that of c with respect to f. It follows that T is an embedded C^{k-1}-submanifold of $V \times V$ and that Φ is a C^{k-1}-diffeomorphism (Exercise 2.1.1). Thus, the tangent bundle of a level set in V can be naturally identified with a level set in $V \times V$.

Just as the tangent spaces of a manifold combine to the tangent bundle, the tangent mappings of a differentiable mapping combine to a mapping of the tangent bundles.

Definition 2.1.5 (Tangent mapping) Let M, N be C^k-manifolds and let $\Phi : M \to N$ be a C^k-mapping. The tangent mapping of Φ is defined by

$$\Phi' : \mathrm{T}M \to \mathrm{T}N, \qquad \Phi'(X_m) := \Phi'_m(X_m).$$

The tangent mapping is of class C^{k-1} (Exercise 2.1.6). The basic properties of the tangent mapping are stated in the next section (Proposition 2.2.9).

Exercises

2.1.1 Prove that the mapping Φ of Remark 2.1.4/2 is a diffeomorphism.
Hint. As local charts on M, use those constructed in the proof of the Level Set Theorem 1.2.1.

2.1.2 Determine the tangent bundle in the form of the level set T of Remark 2.1.4/2 for
(a) the spheres S^n, see Example 1.2.3,
(b) the hyperboloid of Example 1.2.4,
(c) the paraboloid, the ellipsoid and the rotational torus of Exercise 1.2.5,
(d) the classical groups, see Example 1.2.6.
Compare your result for the spheres S^n with Example 2.1.3.

2.1.3 Let M be the level set of a differentiable mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ at a regular value $\mathbf{c} \in \mathbb{R}^m$. Identify $\mathrm{T}M$ with the level set T of Remark 2.1.4/2. The bundle of unit tangent vectors of M is defined to be $\mathrm{E}M := \{(\mathbf{x}, \mathbf{X}) \in \mathrm{T}M : \|\mathbf{X}\| = 1\}$. Show that $\mathrm{E}M$ is an embedded submanifold of $\mathrm{T}M$. What does one get for ES^1 and ES^2?

2.1.4 Let (U, κ) be a local chart on M and let κ^{T} be the local chart induced by κ on the tangent bundle $\mathrm{T}M$ via (2.1.2). Determine the local trivialization (2.1.4) of the tangent bundle $\mathrm{T}(\mathrm{T}M)$ of $\mathrm{T}M$ induced by κ^{T}.

2.1.5 Iterate the construction of Remark 2.1.4/2 by determining the level set T for the tangent bundle $\mathrm{T}(\mathrm{T}M)$ of the tangent bundle $\mathrm{T}M$ of a level set M. Write down the defining equations explicitly for $M = \mathrm{S}^n$.

2.1.6 Let $\Phi : M \to N$ be of class C^k. Show that Φ' is of class C^{k-1}.

2.2 Vector Bundles

The notion of vector bundle arises from the notion of tangent bundle of a manifold by allowing the fibres to be arbitrary finite-dimensional vector spaces, rather than the tangent spaces of that manifold.

Definition 2.2.1 (Vector bundle) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $k \geq 0$. A $\mathbb{K}$-vector bundle of class C^k is a triple (E, M, π), where E and M are C^k-manifolds and $\pi : E \to M$ is a surjective C^k-mapping satisfying the following conditions.

1. For every $m \in M$, $E_m := \pi^{-1}(m)$ carries the structure of a vector space over $\mathbb{K}$.
2. There exists a finite-dimensional vector space F over $\mathbb{K}$, an open covering $\{U_\alpha\}$ of M and an associated family of C^k-diffeomorphisms $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ such that, for all α,
   (a) the following diagram commutes,

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\chi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U_\alpha & \end{tikzcd} \tag{2.2.1}$$

(b) for every $m \in U_\alpha$, the induced mapping $\chi_{\alpha,m} := \mathrm{pr}_2 \circ \chi_{\alpha \restriction E_m} : E_m \to F$ is linear.

Like in the case of the tangent bundle, by an abuse of notation, a vector bundle (E, M, π) will usually be denoted by E alone. Like for the tangent bundle, E is called the total space or the bundle manifold, M the base manifold, π the bundle projection and F the typical fibre. For $m \in M$, E_m is called the fibre over m and m is called the base point. The pairs (U_α, χ_α) are called local trivializations. A local trivialization (U, χ) with $U = M$ is called a global trivialization. If a global trivialization exists, the vector bundle is called (globally) trivial.

Remark 2.2.2

1. By condition 2a, since the χ_α are diffeomorphisms, the bundle projection π is a submersion (because so is the projection to a factor of a direct product) and the fibres E_m are embedded submanifolds (because by χ_α they are mapped onto the subsets $\{m\} \times F$ of $U_\alpha \times F$). Being bijective and linear, the mappings $\chi_{\alpha,m}$ are vector space isomorphisms. Hence, all fibres have the same dimension as F; this number is called the dimension or the rank of the vector bundle. Thus, $\dim E = \dim M + \dim F$ for $\mathbb{K} = \mathbb{R}$, and $\dim E = \dim M + 2 \dim F$ for $\mathbb{K} = \mathbb{C}$. For a $\mathbb{K}$-vector bundle of dimension n, one can always choose $F = \mathbb{K}^n$.
2. Let A denote the index set of a family of local trivializations $\{(U_\alpha, \chi_\alpha)\}$. The mappings

$$\chi_\beta \circ \chi_\alpha^{-1} : U_\alpha \cap U_\beta \times F \to U_\alpha \cap U_\beta \times F, \quad (\alpha, \beta) \in A \times A, \tag{2.2.2}$$

which are of class C^k, are called the transition mappings of the system of local trivializations $\{(U_\alpha, \chi_\alpha) : \alpha \in A\}$. Since for every $(\alpha, \beta) \in A \times A$, $\chi_\beta \circ \chi_\alpha^{-1}$ maps the subsets $\{m\} \times F$, where $m \in U_\alpha \cap U_\beta$, linearly and bijectively onto themselves, there exists a mapping $\rho_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(F)$ such that

$$\chi_\beta \circ \chi_\alpha^{-1}(m, u) = \big(m, \rho_{\beta\alpha}(m)u\big) \tag{2.2.3}$$

for all $m \in U_\alpha \cap U_\beta$ and $u \in F$. The mappings $\rho_{\beta\alpha}$ are called the transition functions of the system of local trivializations $\{(U_\alpha, \chi_\alpha) : \alpha \in A\}$. To see that they are of class C^k it suffices to check that for every $(\alpha, \beta) \in A \times A$ and $u \in F$, the mapping $U_\alpha \cap U_\beta \to F$ defined by $m \mapsto (m, \rho_{\beta\alpha}(m)u)$ is of class C^k. This follows at once from the differentiability of the transition mappings $\chi_\beta \circ \chi_\alpha^{-1}$. One can check that the transition functions satisfy

$$\rho_{\gamma\beta}(m)\rho_{\beta\alpha}(m) = \rho_{\gamma\alpha}(m) \tag{2.2.4}$$

for all $\alpha, \beta, \gamma \in A$ and $m \in U_\alpha \cap U_\beta \cap U_\gamma$.
3. A vector bundle is said to be orientable if there exists a family of local trivializations whose transition mappings have positive determinant.

Example 2.2.3

1. Let M be a C^k-manifold, let F be a vector space of dimension r over $\mathbb{K}$ and let $\mathrm{pr}_M : M \times F \to M$ denote the natural projection to the first component. Then, $(M \times F, M, \mathrm{pr}_M)$ is a $\mathbb{K}$-vector bundle of class C^k and dimension r. It is called the product vector bundle of M and F. A product vector bundle is obviously trivial.
2. According to Proposition 2.1.1, the tangent bundle of an n-dimensional C^k-manifold is an n-dimensional real vector bundle of class C^{k-1}.
3. Let (E, M, π) be a vector bundle of class C^k and let $U \subset M$ be open. Define $E_U := \pi^{-1}(U)$. This is an open subset of E and hence a C^k-manifold. By restriction, π induces a surjective C^k-mapping $\pi_U : E_U \to U$, and a system of local trivializations $\{(U_\alpha, \chi_\alpha)\}$ of E induces the system of local trivializations $\{(U_\alpha \cap U, \chi_{\alpha \restriction U_\alpha \cap U})\}$ of E_U. Thus, (E_U, U, π_U) is a $\mathbb{K}$-vector bundle of class C^k. It has the same dimension as E.

Example 2.2.4 (Möbius strip) Let $M = \mathrm{S}^1$ be realized as the unit circle in $\mathbb{C}$ and let E be the Möbius strip of Example 1.1.12 with the open interval $(-1, 1)$ replaced by the whole of $\mathbb{R}$. That is, $E := \mathbb{R}^2/\sim$, where $(s_1, t_1) \sim (s_2, t_2)$ iff there exists $k \in \mathbb{Z}$ such that $s_2 = s_1 + 2\pi k$ and $t_2 = (-1)^k t_1$. Define the projection by

$$\pi : E \to \mathrm{S}^1, \qquad \pi\big(\big[(s,t)\big]\big) := \mathrm{e}^{\mathrm{i}s}.$$

Using the local charts on E constructed in Example 1.1.12 one can easily check that π is smooth. The fibres are $E_{\mathrm{e}^{\mathrm{i}s}} = \pi^{-1}(\mathrm{e}^{\mathrm{i}s}) = \{[(s,t)] : t \in \mathbb{R}\}$. For every $s \in \mathbb{R}$, define

$$\lambda\big[(s,t_1)\big] + \big[(s,t_2)\big] := \big[(s, \lambda t_1 + t_2)\big], \quad \lambda, t_1, t_2 \in \mathbb{R}.$$

This way, the fibres become real vector spaces of dimension one. To construct local trivializations, we choose $U_\pm := \mathrm{S}^1 \setminus \{\pm 1\}$ and define mappings

$$\chi_\pm : \pi^{-1}(U_\pm) \to U_\pm \times \mathbb{R}, \qquad \chi_\pm\big(\big[(s,t)\big]\big) := \big(\mathrm{e}^{\mathrm{i}s}, t\big),$$

where in case of χ_+ and χ_- the representative (s,t) of $[(s,t)]$ used to compute the right hand side is chosen from $]0, 2\pi[\times \mathbb{R}$ and from $]-\pi, \pi[\times \mathbb{R}$, respectively. We leave it to the reader to check that the $\chi_\pm$ are diffeomorphisms and satisfy conditions 2a and 2b of Definition 2.2.1. Thus, (E, M, π) is a smooth real vector bundle of dimension 1. Figure 2.2 shows E together with the product vector bundle $\mathrm{S}^1 \times \mathbb{R}$. It is quite obvious that E is not trivial. We will be able to give a precise argument for that in the next section.

Remark 2.2.5 Let M be a C^k-manifold, let E be a set and let $\pi : E \to M$ be a surjective mapping such that conditions 1 and 2 of Definition 2.2.1 are satisfied, however, with the following difference. Instead of assuming the χ_α to be C^k-diffeomorphisms, assume that they are bijective and that their transition mappings

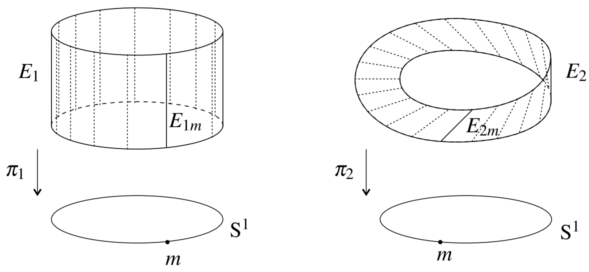

Fig. 2.2 The product vector bundle $S^1 \times \mathbb{R}$ and the Möbius strip as a vector bundle over S^1

(2.2.2) are of class C^k. Since M is second countable, the open covering $\{U_\alpha\}$ contains a countable subcovering. According to Remark 1.1.10, the corresponding subfamily of the family $\{\chi_\alpha\}$ defines a C^k-structure on E. With respect to this structure, (E, M, π) is a $\mathbb{K}$-vector bundle of class C^k and the (U_α, χ_α) are local trivializations. Conversely, if (E, M, π) is a vector bundle of class C^k, then the C^k-structure on E induced in this way coincides with the original one (Exercise 2.2.1).

Next, we consider mappings of vector bundles.

Definition 2.2.6 (Vector bundle morphism) Let (E_1, M_1, π_1) and (E_2, M_2, π_2) be $\mathbb{K}$-vector bundles of class C^k. A C^k-mapping $\Phi : E_1 \to E_2$ is called a morphism if for every $m_1 \in M_1$ there exists $m_2 \in M_2$ such that

1. $\Phi(E_{1,m_1}) \subset E_{2,m_2}$,
2. the induced mapping $\Phi_{m_1} := \Phi_{\restriction E_{1,m_1}} : E_{1,m_1} \to E_{2,m_2}$ is linear.

The rank of Φ is defined to be the integer-valued function which assigns to $m_1 \in M_1$ the rank of the linear mapping Φ_{m_1}. In case $M_1 = M_2 = M$, Φ is called a vertical morphism or a morphism over M if conditions 1 and 2 hold with $m_1 = m_2 = m$.

As usual, together with the notion of morphism there comes the notion of isomorphism (a bijective morphism whose inverse is also a morphism), endomorphism (a morphism of a vector bundle to itself), automorphism (an isomorphism of a vector bundle onto itself). For a vector bundle morphism Φ to be an isomorphism it is obviously sufficient for Φ to be a diffeomorphism. If Φ is vertical, it is sufficient for Φ to be bijective, because then the tangent mapping Φ' is bijective at any point and Theorem 1.5.7 yields that the inverse mapping is of class C^k.

Remark 2.2.7

1. Since Φ is a mapping, condition 1 implies that the point m_2 is uniquely determined by m_1. Thus, every morphism Φ induces a mapping $\varphi : M_1 \to M_2$, defined by

$$\varphi \circ \pi_1 = \pi_2 \circ \Phi.$$

One says that Φ covers φ and calls φ the projection of Φ. If Φ is of class C^k, so is φ. Indeed, if (U_1, χ_1) is a local trivialization of E_1, $\varphi_{\restriction U_1}$ coincides with

the composition of the embedding $U_1 \to U_1 \times \{0\} \subset U_1 \times F_1$ with the mapping $\pi_2 \circ \Phi \circ \chi_1^{-1}$. In case $M_1 = M_2 = M$, Φ is a vertical morphism iff $\varphi = \mathrm{id}_M$.
2. Let (U_i, χ_i) be local trivializations of E_i, $i = 1, 2$. The mapping

$$\chi_2 \circ \Phi \circ \chi_1^{-1} : \big(U_1 \cap \varphi^{-1}(U_2)\big) \times F_1 \to U_2 \times F_2 \tag{2.2.5}$$

is called the local representative of Φ with respect to (U_1, χ_1) and (U_2, χ_2). A fibre-preserving and fibrewise linear mapping $\Phi : E_1 \to E_2$ is a morphism iff all of its local representatives are of class C^k.
3. Let E_1, E_2 be $\mathbb{K}$-vector bundles over M of class C^k. For $\lambda \in \mathbb{K}$ and vertical morphisms $\Phi, \Psi : E_1 \to E_2$ we can define

$$(\lambda\Phi + \Psi)(x) := \lambda\Phi(x) + \Psi(x), \quad x \in E_1,$$

because for all $x \in E_1$, $\Phi(x)$ and $\Psi(x)$ belong to the same fibre of E_2. This provides a $\mathbb{K}$-vector space structure on the set of vertical morphisms from E_1 to E_2.

Example 2.2.8 A local trivialization (U, χ) of a $\mathbb{K}$-vector bundle (E, M, π) with typical fibre F is a vertical isomorphism from the vector bundle E_U, see Example 2.2.3/3, onto the product vector bundle $U \times F$. Accordingly, a global trivialization is a vertical isomorphism from E onto $M \times F$. Thus, a vector bundle is trivial iff it is isomorphic to a product vector bundle.

Probably the most important example of a vector bundle morphism is the tangent mapping. The reader may convince himself that Proposition 1.5.2 implies the following (Exercise 2.2.4).

Proposition 2.2.9 (Properties of the tangent mapping) *Let M and N be C^k-manifolds and let $\varphi \in C^k(M, N)$. The tangent mapping $\varphi' : \mathrm{T}M \to \mathrm{T}N$ has the following properties.*

1. *φ' is a vector bundle morphism of class C^{k-1} with projection φ.*
2. $(\mathrm{id}_M)' = \mathrm{id}_{\mathrm{T}M}$.
3. *If P is another C^k-manifold and $\psi \in C^k(N, P)$, then $(\psi \circ \varphi)' = \psi' \circ \varphi'$.*
4. *If φ is a diffeomorphism, then φ' is an isomorphism and $(\varphi')^{-1} = (\varphi^{-1})'$.*

Remark 2.2.10 (Partial derivatives and product rule) Let M_1, M_2, N be C^k-manifolds and let $\varphi \in C^k(M_1 \times M_2, N)$. We discuss the properties of the tangent mapping φ' which are related to the direct product structure of its domain. Proofs are left to the reader (Exercise 2.2.5). The induced partial mappings

$$\begin{aligned} &\varphi_{m_2} : M_1 \to N, \qquad \varphi_{m_2}(m_1) := \varphi(m_1, m_2), \quad m_2 \in M_2, \\ &\varphi_{m_1} : M_2 \to N, \qquad \varphi_{m_1}(m_2) := \varphi(m_1, m_2), \quad m_1 \in M_1, \end{aligned}$$

are of class C^k. Their tangent mappings combine to C^{k-1}-mappings

$$\begin{aligned} \mathrm{T}M_1 \times M_2 &\to \mathrm{T}N, & (X_1, m_2) &\mapsto (\varphi_{m_2})'(X_1), \\ M_1 \times \mathrm{T}M_2 &\to \mathrm{T}N, & (m_1, X_2) &\mapsto (\varphi_{m_1})'(X_2), \end{aligned}$$

called the partial derivatives of φ. They fulfil the product rule,

$$\varphi'_{(m_1,m_2)}(X_1, X_2) = (\varphi_{m_2})'(X_1) + (\varphi_{m_1})'(X_2), \quad m_i \in M_i,\ X_i \in \mathrm{T}_{m_i} M_i. \tag{2.2.6}$$

If $M_1 = M_2 = M$ and if φ is composed with the diagonal mapping $\Delta : M \to M \times M$, then

$$(\varphi \circ \Delta)'_m(X) = (\varphi_{1,m})'(X) + (\varphi_{2,m})'(X), \quad m \in M,\ X \in \mathrm{T}_m M. \tag{2.2.7}$$

In particular, if $M = I$ is some open interval, then $\varphi \circ \Delta$, φ_{t_1} and φ_{t_2} are C^k-curves in N. For the corresponding tangent vectors at $t \in I$ there holds

$$\frac{\mathrm{d}}{\mathrm{d}s}_{\restriction_t} \varphi(s,s) = \frac{\mathrm{d}}{\mathrm{d}s}_{\restriction_t} \varphi(s,t) + \frac{\mathrm{d}}{\mathrm{d}s}_{\restriction_t} \varphi(t,s), \quad t \in I. \tag{2.2.8}$$

To conclude this section, we show that—up to isomorphy—vector bundles are completely determined by the family of transition functions associated with a system of local trivializations.

Theorem 2.2.11 (Reconstruction theorem) *Let M be a C^k-manifold. Assume that the following data are given*:

1. *a finite-dimensional vector space F over $\mathbb{K}$*,
2. *an open covering $\{U_\alpha : \alpha \in A\}$ of M*,
3. *a family of C^k-mappings $\rho_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(F)$, $(\alpha, \beta) \in A \times A$, satisfying* (2.2.4).

Then, there exists a $\mathbb{K}$-vector bundle E over M of class C^k and a family of local trivializations $\{(U_\alpha, \chi_\alpha) : \alpha \in A\}$ of E whose transition functions are given by the functions $\rho_{\beta\alpha}$. E is uniquely determined up to vertical isomorphisms.

In particular, the last assertion implies that if the $\rho_{\alpha\beta}$ are the transition functions of a vector bundle, then the vector bundle provided by Theorem 2.2.11 is isomorphic over M to the original one.

Proof First, we prove existence. Since M is second countable, the covering $\{U_\alpha : \alpha \in A\}$ contains a countable subcovering. Hence, for the following construction we may assume that A is countable. Moreover, we notice that (2.2.4) implies that $\rho_{\alpha\alpha} = \mathbb{1}$ for all $\alpha \in A$. Take the topological direct sum

$$\mathscr{X} := \bigsqcup_{\alpha \in A} U_\alpha \times F,$$

denote its elements by (α, m, u), where $\alpha \in A$, $m \in U_\alpha$ and $u \in F$, and define a relation on $\mathscr{X}$ by $(\alpha_1, m_1, u_1) \sim (\alpha_2, m_2, u_2)$ iff $m_1 = m_2$ and $u_2 = \rho_{\alpha_2\alpha_1}(m_1)u_1$. Due to $\rho_{\alpha\alpha} = \mathbb{1}$ and (2.2.4), this is an equivalence relation. Let E denote the set of equivalence classes. The mapping $\pi : E \to M$, given by $\pi[(\alpha, m, u)] := m$, is well-defined and surjective. To construct a vector space structure on $\pi^{-1}(m)$ for every $m \in M$, choose α such that $m \in U_\alpha$. Every class in $\pi^{-1}(m)$ has a unique representative of the form (α, m, u) with $u \in F$. Using this, we transport the linear structure from F to $\pi^{-1}(m)$,

$$\lambda\big[(\alpha, m, u_1)\big] + \big[(\alpha, m, u_2)\big] := \big[(\alpha, m, \lambda u_1 + u_2)\big], \quad u_1, u_2 \in F,\ \lambda \in \mathbb{K}.$$

By linearity of the mappings $\rho_{\beta\alpha}(m) : F \to F$, this definition does not depend on the choice of α. The natural injections $U_\alpha \times F \to \mathscr{X}$ induce mappings $U_\alpha \times F \to E$. Due to $\rho_{\alpha\alpha} = \mathbb{1}$, these mappings are injective and hence induce bijective mappings $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$. A brief computation shows that the transition mappings of the family of bijections $\{\chi_\alpha : \alpha \in A\}$ are given by (2.2.3). Therefore, they are of class C^k and hence define a C^k-structure on E with respect to which (E, M, π) is a $\mathbb{K}$-vector bundle of class C^k and the χ_α are local trivializations, see Remark 2.2.5. To prove uniqueness up to vertical isomorphisms, let $\tilde{E}$ be a $\mathbb{K}$-vector bundle over M of class C^k with projection $\tilde{\pi}$ and let $\tilde{\chi}_\alpha : \tilde{\pi}^{-1}(U_\alpha) \to U_\alpha \times F$ be local trivializations whose transition functions coincide with the $\rho_{\beta\alpha}$. Then, on $\pi^{-1}(U_\alpha \cap U_\beta) \subset E$ we have $\tilde{\chi}_\alpha^{-1} \circ \chi_\alpha = \tilde{\chi}_\beta^{-1} \circ \chi_\beta$ and on $\tilde{\pi}^{-1}(U_\alpha \cap U_\beta) \subset \tilde{E}$ there holds $\chi_\alpha^{-1} \circ \tilde{\chi}_\alpha = \chi_\beta^{-1} \circ \tilde{\chi}_\beta$. Hence, the mappings $\tilde{\chi}_\alpha^{-1} \circ \chi_\alpha$ and $\chi_\alpha^{-1} \circ \tilde{\chi}_\alpha$, $\alpha \in A$, combine to mappings $E \to \tilde{E}$ and $\tilde{E} \to E$, respectively, which are morphisms and inverse to one another. □

Remark 2.2.12

1. Given two finite-dimensional vector spaces F_1, F_2 and two open coverings $\{U_{i,\alpha_i} : \alpha_i \in A_i\}$, $i = 1, 2$, of M with associated systems of C^k-mappings

$$\rho_{i,\beta_i\alpha_i} : U_{i,\alpha_i} \cap U_{i,\beta_i} \to \mathrm{GL}(F), \quad (\alpha_i, \beta_i) \in A_i \times A_i,$$

there arises the question under which conditions the vector bundles E_1 and E_2, defined by these data according to Theorem 2.2.11, are isomorphic over M. The answer is as follows. First, F_1 and F_2 have to be isomorphic so that they can be replaced by $\mathbb{K}^r$ for some $r \in \mathbb{N}$. Second, there exists a common refinement $\{U_\alpha : \alpha \in A\}$ of the open coverings $\{U_{i,\alpha_i} : \alpha_i \in A_i\}$, $i = 1, 2$. By restriction, the $\rho_{i,\beta_i\alpha_i}$ induce mappings

$$\rho_{i,\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{K}), \quad (\alpha, \beta) \in A \times A,\ i = 1, 2.$$

Now, E_1 and E_2 are isomorphic iff there exists a system of C^k-mappings $\rho_\alpha : U_\alpha \to \mathrm{GL}(r, \mathbb{K})$, $\alpha \in A$, such that

$$\rho_{2,\beta\alpha}(m) = \rho_\beta^{-1}(m) \cdot \rho_{1,\beta\alpha}(m) \cdot \rho_\alpha(m), \quad m \in U_\alpha \cap U_\beta. \tag{2.2.9}$$

The proof is left to the reader (Exercise 2.2.6).

2. An open covering $\{U_\alpha : \alpha \in A\}$ together with an associated family of C^k-mappings $\rho_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{K})$, $(\alpha, \beta) \in A \times A$, with the property (2.2.4) is called a 1-cocycle on M with values in the structure group $\mathrm{GL}(r, \mathbb{K})$. Two 1-cocycles are called cohomologous if there exists a system of C^k-mappings $\rho_\alpha : U_\alpha \to \mathrm{GL}(r, \mathbb{K})$, $\alpha \in A$, such that (2.2.9) holds. To be cohomologous is an equivalence relation in the set of 1-cocycles. Passage to equivalence classes, that is, cohomology classes of 1-cocycles, yields a cohomology theory on M which is called the first Čech cohomology of M with values in the structure group $\mathrm{GL}(r, \mathbb{K})$. According to point 1, the cohomology classes correspond bijectively to the isomorphism classes of $\mathbb{K}$-vector bundles over M of class C^k and dimension r.
3. One can show that the first Čech cohomology of M and, correspondingly, the set of isomorphism classes of vector bundles over M do not depend on the degree of differentiability k, see [130, Ch. 4, Thm. 3.5].

Exercises

2.2.1 Let (E, M, π) be a C^k-vector bundle. Consider the C^k-structure on E induced by a system of local trivializations via the method of Remark 2.2.5. Show that this structure coincides with the original C^k-structure.

2.2.2 Let M be a C^k-manifold. Use the system of bijections (2.1.4) associated with an atlas on M to construct a C^k-structure on $\mathrm{T}M$ via the method of Remark 2.2.5.

2.2.3 Construct a smooth structure on the Möbius strip by means of the method of Remark 2.2.5, using the local trivializations $(U_\pm, \chi_\pm)$ of Example 2.2.4. Show that this structure coincides with the one constructed in Example 1.1.12.

2.2.4 Prove Proposition 2.2.9.

2.2.5 Prove the assertions about partial derivatives stated in Remark 2.2.10.

2.2.6 Prove the criterion for the isomorphy of two vector bundles over M stated in Remark 2.2.12/1.

2.2.7 Let (E_1, π_1, M_1) and (E_2, π_2, M_2) be $\mathbb{K}$-vector bundles of class C^k and of dimensions r_1 and r_2. Define $E := E_1 \times E_2$, $M := M_1 \times M_2$ and $\pi := \pi_1 \times \pi_2 : E_1 \times E_2 \to M_1 \times M_2$. For $(m_1, m_2) \in M_1 \times M_2$, equip $E_{(m_1,m_2)} := \pi^{-1}(m) \equiv E_{1,m_1} \times E_{2,m_2}$ with the linear structure of the direct sum $E_{1,m_1} \oplus E_{2,m_2}$. Show that (E, M, π) is a $\mathbb{K}$-vector bundle of class C^k and dimension $r_1 + r_2$. It is called the direct product of (E_1, π_1, M_1) and (E_2, π_2, M_2).

2.2.8 Let M_1 and M_2 be C^k-manifolds. Let $\mathrm{pr}_i : M_1 \times M_2 \to M_i$ denote the natural projections to the factors. Show that the following mapping is a vertical vector bundle isomorphism:

$$\Phi : \mathrm{T}(M_1 \times M_2) \to \mathrm{T}M_1 \times \mathrm{T}M_2, \qquad \Phi(X) := \big(\mathrm{pr}_1'(X), \mathrm{pr}_2'(X)\big).$$

2.3 Sections and Frames

The notion of section generalizes the concept of a function on a manifold with values in a finite-dimensional vector space.

Definition 2.3.1 (Section) Let (E, M, π) be a $\mathbb{K}$-vector bundle of class C^k. A section (or cross section) of (E, M, π) is a C^k-mapping $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$.

A local section of (E, M, π) over an open subset $U \subset M$ is a section of the vector bundle (E_U, U, π_U).

Remark 2.3.2

1. Let s be a section and let (U, χ) be a local trivialization of the $\mathbb{K}$-vector bundle (E, M, π) of class C^k. The mapping
$$\mathrm{pr}_F \circ \chi \circ s_{\restriction U} : U \to F \tag{2.3.1}$$
is called the local representative of s with respect to (U, χ). Since local trivializations are diffeomorphisms, a mapping $s : M \to E$ satisfying $\pi \circ s = \mathrm{id}_M$ is of class C^k (and hence a section) iff so are all local representatives of s with respect to a system of local trivializations.
2. Every vector bundle admits a distinguished section $m \mapsto 0_m$, called the zero section.
3. The set of all sections of a $\mathbb{K}$-vector bundle (E, M, π) of class C^k is denoted by $\Gamma(E)$. It carries the structure of a real vector space and of a bimodule over the algebra $C^\infty(M)$, with all operations defined pointwise (Exercise 2.3.1).
4. A local section need not be extendable to a global section, as is shown by the example of $M = \mathbb{R}$, $E = M \times \mathbb{R}$, $U = \mathbb{R}_+$ and $s(x) = (x, \frac{1}{x})$. There holds, however, the following weaker extension property. For every $m \in U$, there exists an open neighbourhood V of m in U and a section $\tilde{s}$ of E such that $s_{\restriction V} = \tilde{s}_{\restriction V}$ (Exercise 2.3.2).

Example 2.3.3

1. (Local) sections of the tangent bundle $\mathrm{T}M$ of a C^k-manifold M are called (local) vector fields on M. They will usually be denoted by $X, Y, \dots$ and the vector space $\Gamma(\mathrm{T}M)$ will be denoted by $\mathfrak{X}(M)$. To be consistent with the previous notation X_m for a tangent vector at $m \in M$, for the value of the vector field X at the point m we will often write X_m instead of $X(m)$. Note that since $\mathrm{T}M$ is of class C^{k-1}, so are vector fields. If (U, κ) is a local chart on M, for every $i = 1, \dots, \dim M$, the mapping
$$\partial_i^\kappa : U \to (\mathrm{T}M)_U, \qquad \partial_i^\kappa(m) := \partial_{i,m}^\kappa,$$
is a section of $(\mathrm{T}M)_U = \mathrm{T}U$. Since the representative of this section with respect to the global trivialization of $\mathrm{T}U$ induced by κ is given by the constant mapping

whose value is the i-th standard basis vector in $\mathbb{R}^{\dim M}$, ∂_i^κ is of class C^{k-1}. Hence, ∂_i^κ is a local vector field on M.

2. If the vector bundle E over M is given in terms of a $\mathbb{K}$-vector space F, an open covering $\{U_\alpha\}$ of M and a family of C^k-mappings $\rho_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(F)$ satisfying (2.2.4), then its sections of class C^k correspond bijectively to families $\{s_\alpha\}$ of C^k-mappings $s_\alpha : U_\alpha \to F$ satisfying $s_\alpha(m) = \rho_{\alpha\beta}(m)s_\beta(m)$ for all $\alpha, \beta \in A$ and $m \in U_\alpha \cap U_\beta$.

Remark 2.3.4

1. Let V be a finite-dimensional real vector space and let $M \subset V$ be an open subset. Via the natural identification of $\mathrm{T}M$ with $M \times V$ of Remark 2.1.4/1, vector fields X on M correspond bijectively to smooth mappings[2] $X : M \to V$. By construction, for all $v \in M$ and $f \in C^\infty(M)$, we have

$$X_v(f) = \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} f\big(v + tX(v)\big). \tag{2.3.2}$$

2. Let V and W be finite-dimensional real vector spaces and let $M \subset V$ be the level set of a C^k-mapping $f : V \to W$ at a regular value. Via the natural identification of $\mathrm{T}M$ with the embedded C^{k-1}-submanifold $\{(v, X) \in M \times V : X \in \ker(f'(v))\}$ of $V \times V$, see Remark 2.1.4/2, vector fields X on M correspond bijectively to C^{k-1}-mappings $X : M \to V$ satisfying $X(v) \in \ker f'(v)$.

By means of a local trivialization, sections are identified locally with the graphs of their local representatives. This implies

Proposition 2.3.5 *Let (E, M, π) be a $\mathbb{K}$-vector bundle of class C^k and let $s \in \Gamma(E)$. Then, (M, s) is an embedded C^k-submanifold of E.*

Proof Let $m \in M$. According to Remark 1.6.13/3, we have to show that there exists an open neighbourhood V of $s(m)$ in E such that $(s^{-1}(V), s_{\restriction s^{-1}(V)})$ is an embedded C^k-submanifold of E. Choose a local trivialization (U, χ) of E at m and let $V = \pi^{-1}(U)$. Then, $s^{-1}(V) = U$ and hence we have to show that $(U, s_{\restriction U})$ is an embedded C^k-submanifold of $\pi^{-1}(U)$. Since χ is a diffeomorphism and $(U, \chi \circ s_{\restriction U})$ is the graph of the local representative of s with respect to the local trivialization (U, χ), the latter follows from Example 1.6.12/2. $\square$

Now let (E_1, M_1, π_1) and (E_2, M_2, π_2) be $\mathbb{K}$-vector bundles of class C^k, let $\Phi : E_1 \to E_2$ be a morphism and let $\varphi : M_1 \to M_2$ be the projection of Φ.

[2] Denoted by the same symbol.

Definition 2.3.6 (Φ-relation and transport operator)

1. Sections $s_1 \in \Gamma(E_1)$ and $s_2 \in \Gamma(E_2)$ are said to be Φ-related if they satisfy

$$\Phi \circ s_1 = s_2 \circ \varphi.$$

2. If φ is a C^k-diffeomorphism, the following mapping is called the transport operator of Φ:

$$\Phi_* : \Gamma(E_1) \to \Gamma(E_2), \qquad \Phi_* s := \Phi \circ s \circ \varphi^{-1}. \tag{2.3.3}$$

The following proposition lists the properties of the transport operator (Exercise 2.3.3).

Proposition 2.3.7 *Let (E_1, M_1, π_1) and (E_2, M_2, π_2) be $\mathbb{K}$-vector bundles of class C^k and let $\Phi : E_1 \to E_2$ be a morphism whose projection $\varphi : M_1 \to M_2$ is a diffeomorphism. The transport operator Φ_* has the following properties.*

1. *Φ_* is linear. If Φ is an isomorphism of vector bundles, Φ_* is an isomorphism of vector spaces and there holds $(\Phi^{-1})_* = (\Phi_*)^{-1}$.*
2. *For every $s \in \Gamma(E_1)$, s is Φ-related to $\Phi_* s$.*
3. *For every $s \in \Gamma(E_1)$ and $f \in C^k(M_1)$, there holds $\Phi_*(fs) = ((\varphi^{-1})^* f)\Phi_* s$.*
4. *If $\Psi : E_2 \to E_3$ is another morphism whose projection is a diffeomorphism, then $(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*$.*

Remark 2.3.8 In the case of vector fields, it is common to speak of φ-relation rather than φ'-relation. Thus, $X_i \in \mathfrak{X}(M_i)$, $i = 1, 2$, are φ-related iff

$$\varphi' \circ X_1 = X_2 \circ \varphi. \tag{2.3.4}$$

Next, we turn to the discussion of (local) frames.

Definition 2.3.9 (Local frame) Let (E, M, π) be a $\mathbb{K}$-vector bundle of class C^k and dimension l, let $U \subset M$ be open and let $\mathscr{B} = \{s_1, \dots, s_r\}$ be a system of local sections of E over U. $\mathscr{B}$ is said to be pointwise linearly independent if the system $\{s_1(m), \dots, s_r(m)\}$ is linearly independent in E_m for all $m \in U$. In this case, $\mathscr{B}$ is called a local r-frame (frame if $r = l$) in E over U. If $U = M$, $\mathscr{B}$ is called a global r-frame (global frame if $r = l$).

Local frames provide bases in the fibres over their domain and hence allow for the expansion of local sections.

Proposition 2.3.10 *Let (E, M, π) be a $\mathbb{K}$-vector bundle of class C^k, let $U \subset M$ be open and let $\{s_1, \dots, s_l\}$ be a local frame in E over U. The assignment of $f^i s_i$ (summation convention) to an l-tuple $(f^1, \dots, f^l)$ of $\mathbb{K}$-valued C^k-functions on U defines a bijection from $\prod_{i=1}^l C^k(U, \mathbb{K})$ onto $\Gamma(E_U)$.*

Proof Obviously, for every $(f^1, \ldots, f^l) \in \prod_{i=1}^l C^k(U, \mathbb{K})$, the sum $f^i s_i$ is a C^k-section of E_U. Conversely, let $s \in \Gamma(E_U)$. By expanding $s(m)$ with respect to the basis $\{s_1(m), \ldots, s_l(m)\}$ of E_m for all $m \in U$, we obtain functions $f^i : U \to \mathbb{K}$ satisfying $s_{\restriction U} = f^i s_i$. For every $m \in U$, $(f^1(m), \ldots, f^l(m))$ is the unique solution of a system of linear equations whose coefficients depend differentiably of class C^k on m. Hence, $f^i \in C^k(U)$. □

Example 2.3.11

1. Let M be a C^k-manifold of dimension n and let (U, κ) be a local chart on M. Since $\{\partial^\kappa_{1,m}, \ldots, \partial^\kappa_{n,m}\}$ is a basis in $\mathrm{T}_m M$ for all $m \in U$, the system $\{\partial^\kappa_1, \ldots, \partial^\kappa_n\}$ is a local frame in $\mathrm{T}M$ over U. Thus, over U, vector fields $X \in \mathfrak{X}(M)$ can be represented as $X_{\restriction U} = X^i \partial^\kappa_i$ with $X^i \in C^{k-1}(U)$. According to (1.4.15) and (1.4.16), the coefficient functions X^i are given by $X^i(m) = X_m(\kappa^i)$, where $i = 1, \ldots, n$.
2. Let (E, M, π) be a $\mathbb{K}$-vector bundle of class C^k with typical fibre F, let (U, χ) be a local trivialization and let $\{e_1, \ldots, e_r\}$ be a linearly independent system in F. Define local sections s_i of E over U by

$$s_i(m) := \chi^{-1}(m, e_i), \quad i = 1, \ldots, r. \tag{2.3.5}$$

   These sections are of class C^k, because their local representatives with respect to (U, χ) are the constant mappings $m \mapsto e_i$. Hence, the system $\{s_1, \ldots, s_r\}$ is a local r-frame in E over U.

As the second example suggests, local frames are closely related to local trivializations.

Proposition 2.3.12 *Let (E, M, π) be a $\mathbb{K}$-vector bundle of class C^k with typical fibre F and let $U \subset M$ be open. By virtue of* (2.3.5), *every basis of F defines a bijection between local trivializations $\chi : \pi^{-1}(U) \to U \times F$ and local frames in E over U. In particular,*

1. *there exists a local trivialization of E over U iff there exists a local frame in E over U.*
2. *E is trivial iff there exists a global frame.*

Proof Let $\{e_1, \ldots, e_l\}$ be a basis of F. That a local trivialization over U defines a local frame over U has been shown in Example 2.3.11/2. Conversely, for a given local frame $\{s_1, \ldots, s_l\}$ in E over U, expand $x \in E_U$ as $x = x^i s_i(\pi(x))$ and define a mapping $\chi : \pi^{-1}(U) \to U \times F$ by $\chi(x) := (\pi(x), x^i e_i)$, $x \in E_U$. The mapping χ is a bijection and satisfies conditions 2a and 2b of Definition 2.2.1. Thus, to show that χ is a local trivialization, it remains to check that χ and χ^{-1} are of class C^k (Exercise 2.3.4). Finally, assertions 1 and 2 are obvious. □

Example 2.3.13

1. Let M be a C^k-manifold of dimension n, let (U, κ) be a local chart on M and let (U, χ) be the local trivialization of $\mathrm{T}M$ induced by this chart via (2.1.4). The bijection between local frames over U and local trivializations over U, defined by the standard basis of $\mathbb{R}^n$ via (2.3.5), assigns to (U, χ) the local frame $\{\partial_1^\kappa, \dots, \partial_n^\kappa\}$.
2. Consider the smooth real vector bundle E given by the Möbius strip, cf. Example 2.2.4. Since E has dimension 1, a global frame in E is just a nowhere vanishing section. Since the base manifold is S^1, sections of E correspond to closed smooth curves in E winding around exactly once.[3] Since any such curve must cross the zero section, E does not admit a global frame and is hence not globally trivial, cf. Proposition 2.3.12.

Remark 2.3.14 Using the description of vector bundles in terms of coverings and transition functions as explained in Remark 2.2.12, one can show that, up to isomorphy over S^1, the Möbius strip and the product vector bundle $\mathrm{S}^1 \times \mathbb{R}$ are the only real vector bundles of dimension 1 over S^1.

The following proposition collects useful extension results. The proof is left to the reader (Exercise 2.3.5).

Proposition 2.3.15 *Let (E, M, π) be a $\mathbb{K}$-vector bundle of class C^k and dimension l and let $m \in M$.*

1. *Let $\{e_1, \dots, e_l\}$ be a basis of E_m. There exists an open neighbourhood U of m and a local frame $\{s_1, \dots, s_l\}$ over U such that $s_i(m) = e_i$, $i = 1, \dots, l$.*
2. *Let $s_1, \dots, s_r$ be local sections over neighbourhoods $U_1, \dots U_r$ of m such that the system $\{s_1(m), \dots, s_r(m)\}$ is linearly independent in E_m. Then, there exists an open neighbourhood $U \subset U_1 \cap \cdots \cap U_r$ of m such that the system $\{s_{1\restriction U}, \dots, s_{r\restriction U}\}$ is a local r-frame in E over U.*
3. *Let $\{s_1, \dots, s_r\}$ be a local r-frame over a neighbourhood U of m. Then, there exist local sections $s_{r+1}, \dots, s_l$ over $V \subset U$ such that the system $\{s_{1\restriction V}, \dots, s_{r\restriction V}, s_{r+1}, \dots, s_l\}$ is a local frame over V.*

As an application, we briefly discuss manifolds whose tangent bundle is trivial.

Definition 2.3.16 A C^k-manifold is called parallelizable if its tangent bundle is trivial.

According to Proposition 2.3.12, a differentiable manifold M of dimension n is parallelizable iff there exist n pointwise linearly independent vector fields on M.

[3] And with tangent vectors being nowhere parallel to the fibres, but this is not relevant for the argument.

Proposition 2.3.17 *The spheres* S^1, S^3 *and* S^7 *are parallelizable.*

Proof Since S^n is a level set of the smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f(\mathbf{x}) = \|\mathbf{x}\|^2$, we can use the natural representation of smooth vector fields on S^n by smooth mappings $X : S^n \to \mathbb{R}^{n+1}$ satisfying $\mathbf{x} \cdot X(\mathbf{x}) = 0$, cf. Example 2.1.3 and Remark 2.3.4/2. In the case of S^1 we identify $\mathbb{R}^2$ with $\mathbb{C}$ via $\mathbf{x} = (x_1, x_2) \mapsto \hat{\mathbf{x}} := x_1 + \mathrm{i}x_2$. Then, $\mathbf{x} \cdot \mathbf{y} = \mathrm{Re}(\bar{\hat{\mathbf{x}}}\hat{\mathbf{y}})$ and vector fields on S^1 are represented by mappings $X : S^1 \subset \mathbb{C} \to \mathbb{C}$ satisfying $\mathrm{Re}(\bar{z}X(z)) = 0$. This condition holds for example for $X(z) := z\mathrm{i}$. Since this function is nowhere vanishing, the corresponding vector field is nowhere vanishing and hence forms a frame in TS^1. In the case of S^3, we identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ via $\mathbf{x} \mapsto \hat{\mathbf{x}} := x_1\mathbf{1} + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$. Then, $\mathbf{x} \cdot \mathbf{y} = \mathrm{Re}(\bar{\hat{\mathbf{x}}}\hat{\mathbf{y}})$, where $\bar{\hat{\mathbf{x}}}$ now denotes quaternionic conjugation, and vector fields on S^3 are represented by mappings $X : S^3 \subset \mathbb{H} \to \mathbb{H}$ satisfying $\mathrm{Re}(\bar{\mathbf{q}}X(\mathbf{q})) = 0$. For $l = 1, 2, 3$, define $X_l : \mathbb{H} \to \mathbb{H}$ by

$$X_1(\mathbf{q}) = \mathbf{qi}, \qquad X_2(\mathbf{q}) = \mathbf{qj}, \qquad X_3(\mathbf{q}) = \mathbf{qk}.$$

Then, $\mathrm{Re}(\bar{\mathbf{q}}X_l(\mathbf{q})) = 0$ and $\mathrm{Re}(\overline{X_l(\mathbf{q})}X_j(\mathbf{q})) = \delta_{lj}$. Hence, the X_l restrict to vector fields on S^3 and these vector fields are pointwise linearly independent. In the case of S^7, the proof is analogous, with quaternions replaced by octonions.[4] □

Remark 2.3.18

1. Since TS^1 is isomorphic to the product vector bundle $S^1 \times \mathbb{R}$, one can rephrase Remark 2.3.14 as follows. Up to isomorphy over S^1, the tangent bundle of S^1 and the Möbius strip are the only real vector bundles of dimension 1 over S^1.
2. The construction of pointwise linearly independent vector fields on the spheres S^1, S^3 and S^7 presented in the proof of Proposition 2.3.17 carries over to the unit spheres of $\mathbb{C}^k$, $\mathbb{H}^k$ and $\mathbb{O}^k$, where $\mathbb{O}$ denotes the octonions. Thus, for $r = 2, 4, 8$ and $k = 1, 2, \ldots$ there exist $r - 1$ pointwise linearly independent vector fields on the sphere S^{rk-1}. In case $k = 1$, these vector fields constitute a global frame, whereas in the other cases they constitute just a global $(r - 1)$-frame. While there may exist more than $r - 1$ pointwise linearly independent vector fields, there does not exist a global frame for any odd-dimensional sphere except for S^1, S^3 and S^7. More precisely, Adams showed that the maximum number of pointwise linearly independent vector fields on an odd-dimensional sphere is given by the corresponding Radon-Hurwitz number [4]. On the other hand, on an even-dimensional sphere, every vector field has a zero. This is known as the Hairy Ball Theorem. For a proof, see for example [6]. As a consequence, S^1, S^3 and S^7 are the only spheres which are parallelizable.

Exercises

2.3.1 Show that $\Gamma(E)$ carries the structure of a real vector space and of a bimodule over the algebra $C^\infty(M)$, cf. Remark 2.3.2/3.

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[4]For a guide to octonions, see [29].

2.3.2 Prove the statement of Remark 2.3.2/4.
2.3.3 Prove Proposition 2.3.7.
2.3.4 Complete the proof of Proposition 2.3.12 by showing that the mapping χ defined there as well as its inverse are of class C^k.
2.3.5 Prove Proposition 2.3.15.

2.4 Vector Bundle Operations

Every operation with vector spaces defines an operation with vector bundles by fibrewise application. Below, we will discuss the most important of these operations in the form of examples. The construction uses the method of Remark 2.2.5. It will be explained in some detail for the dual vector bundle and the direct sum of vector bundles. The other operations are then given without further explanations.

Throughout this section, let E, E_1 and E_2 be $\mathbb{K}$-vector bundles over M of class C^k. Let, respectively, π, π_1 and π_2 be their projections and l, l_1 and l_2 their dimensions. Choose, respectively, typical fibres F, F_1 and F_2 and local trivializations (U_α, χ_α), $(U_\alpha, \chi_{1\alpha})$ and $(U_\alpha, \chi_{2\alpha})$ over an appropriate open covering $\{U_\alpha : \alpha \in A\}$ of M.

Example 2.4.1 (Dual vector bundle) Take the dual vector space E^*_m of each fibre E_m of E and define the set E^* as the disjoint union

$$E^* = \bigsqcup_{m\in M} E^*_m.$$

Let $\pi^{E^*} : E^* \to M$ be the natural projection to the index set. Define mappings

$$\chi^{E^*}_\alpha : \big(\pi^{E^*}\big)^{-1}(U_\alpha) \to U_\alpha \times F^*, \qquad \chi^{E^*}_\alpha(\xi) = \big(m, \big(\chi^{\mathrm{T}}_{\alpha,m}\big)^{-1}(\xi)\big), \tag{2.4.1}$$

where $m = \pi^{E^*}(\xi)$ and $\chi^{\mathrm{T}}_{\alpha,m} : F^* \to E^*_m$ denotes the dual linear mapping of $\chi_{\alpha,m} : E_m \to F$. The corresponding transition mappings are given by

$$\chi^{E^*}_\beta \circ \big(\chi^{E^*}_\alpha\big)^{-1}(m, \mu) = \big(m, \big(\big(\chi_\alpha \circ \chi_\beta^{-1}\big)_{\restriction\{m\}\times F}\big)^{\mathrm{T}}\mu\big)$$

with $m \in U_\alpha \cap U_\beta$ and $\mu \in F^*$. They are of class C^k, because so are the transition mappings $\chi_\alpha \circ \chi_\beta^{-1}$ of E. Thus, according to Remark 2.2.5, if we equip E^* with the C^k-structure induced by the family of mappings $\{\chi^{E^*}_\alpha, \alpha \in A\}$, then (E^*, M, π^{E^*}) is a $\mathbb{K}$-vector bundle of class C^k, called the dual vector bundle of E. It has the same dimension as E, typical fibre F^*, and $\{(U_\alpha, \chi^{E^*}_\alpha) : \alpha \in A\}$ is a system of local trivializations.

Let $\{s_1, \dots, s_l\}$ be a local frame in E over $U \subset M$. For $m \in U$, let $s(m)^{*1}, \dots, s(m)^{*l}$ denote the elements of the basis of E^*_m which is dual to the basis $\{s_1(m), \dots, s_l(m)\}$ of E_m. Define local sections s^{*i} in E^* by

$$s^{*i}(m) := s(m)^{*i}, \quad i = 1, \dots, l. \tag{2.4.2}$$

Using Proposition 2.3.12 it is easy to see that these local sections are of class C^k and form a local frame of E^*, called the dual local frame or coframe.

The pointwise evaluation mappings $E^*_m \times E_m \to \mathbb{K}$ combine to a natural pairing

$$\Gamma\big(E^*_U\big) \times \Gamma(E_U) \to C^k(U, \mathbb{K}), \qquad (\sigma, s) \mapsto \langle \sigma, s \rangle,$$

also denoted by $\sigma(s)$ or $s(\sigma)$. In terms of this pairing, sections of E^* can be expanded over U as

$$\sigma = \sigma(s_i)s^{*i} \equiv \langle \sigma, s_i \rangle s^{*i}. \tag{2.4.3}$$

Let E_a and E_b be $\mathbb{K}$-vector bundles of class C^k over M_a and M_b, respectively, and let $\Phi : E_a \to E_b$ be a morphism projecting to a diffeomorphism $\varphi : M_a \to M_b$. For every $m \in M_b$, the linear mappings $\Phi_{\varphi^{-1}(m)} : E_{a,\varphi^{-1}(m)} \to E_{b,m}$ induce dual linear mappings which combine to a fibre-preserving and fibrewise linear mapping

$$\Phi^{\mathrm{T}} : E^*_b \to E^*_a, \qquad \big\langle \Phi^{\mathrm{T}}(\xi), x \big\rangle := \big\langle \xi, \Phi_{\varphi^{-1}(m)}(x) \big\rangle, \tag{2.4.4}$$

where $m \in M_b$, $\xi \in E^*_{b,m}$ and $x \in E_{a,\varphi^{-1}(m)}$, which is a morphism projecting to the C^k-diffeomorphism φ^{-1} (Exercise 2.4.1). It is called the dual morphism of Φ. Via (2.3.3), the dual morphism induces a transport operator Φ^{T}_* of sections. More generally, by duality, every morphism $\Phi : E_a \to E_b$ induces the following operation on sections, called the pull-back,

$$\Phi^* : \Gamma\big(E^*_b\big) \to \Gamma\big(E^*_a\big), \qquad \big\langle \big(\Phi^*\sigma\big)(m), x \big\rangle := \big\langle \sigma \circ \varphi(m), \Phi(x) \big\rangle, \tag{2.4.5}$$

where $m \in M_a$ and $x \in E_{a,m}$. Indeed, if Φ projects to a diffeomorphism, then the pull-back is given by

$$\Phi^*\sigma = \Phi^{\mathrm{T}} \circ \sigma \circ \varphi = \Phi^{\mathrm{T}}_* \sigma, \tag{2.4.6}$$

that is, it coincides with the transport operator of the dual morphism.[5]

Example 2.4.2 (Direct sum) Take the direct sum $E_{1,m} \oplus E_{2,m}$ of the fibres over each point $m \in M$ and define

$$E_1 \oplus E_2 = \bigsqcup_{m \in M} E_{1,m} \oplus E_{2,m}.$$

Let $\pi^{\oplus} : E_1 \oplus E_2 \to M$ be the natural projection. Define mappings

$$\chi^{\oplus}_\alpha : \big(\pi^{\oplus}\big)^{-1}(U_\alpha) \to U_\alpha \times (F_1 \oplus F_2)$$

by

$$\chi^{\oplus}_\alpha(x_1, x_2) := \big(m, \big(\chi_{1\alpha,m}(x_1), \chi_{2\alpha,m}(x_2)\big)\big),$$

[5] Taking into account that Φ^{T} projects to φ^{-1}.

where $m = \pi^{\oplus}((x_1, x_2))$. By similar arguments as for the dual vector bundle, one can check that $(E_1 \oplus E_2, M, \pi^{\oplus})$ is a $\mathbb{K}$-vector bundle of class C^k and that the mappings $\chi_\alpha^{\oplus}$ provide a system of local trivializations. $E_1 \oplus E_2$ is called the direct sum of E_1 and E_2. It has dimension $l_1 + l_2$ and typical fibre $F_1 \oplus F_2$. Next, note that every local section s_i of E_i over U can be viewed as a local section of $E_1 \oplus E_2$ in an obvious way. Thus, given local frames $\{s_{i,1}, \dots, s_{i,l_i}\}$ in E_i, the collection

$$\{s_{1,1}, \dots, s_{1,l_1}, s_{2,1}, \dots, s_{2,l_2}\}$$

constitutes a local frame in $E_1 \oplus E_2$. Finally, let E_{a1}, E_{a2} and E_{b1}, E_{b2} be $\mathbb{K}$-vector bundles of class C^k over M_a and M_b, respectively, and let $\Phi_i : E_{ai} \to E_{bi}$, $i = 1, 2$, be morphisms projecting to the same mapping $\varphi : M_a \to M_b$. The linear mappings $\Phi_{i,m} : E_{ai,m} \to E_{bi,\varphi(m)}$ induce linear mappings

$$\Phi_{1,m} \oplus \Phi_{2,m} : E_{a1,m} \oplus E_{a2,m} \to E_{b1,\varphi(m)} \oplus E_{b2,\varphi(m)},$$

which combine to a morphism projecting to φ,

$$\Phi_1 \oplus \Phi_2 : E_{a1} \oplus E_{a2} \to E_{b1} \oplus E_{b2}, \qquad (\Phi_1 \oplus \Phi_2)_m := \Phi_{1,m} \oplus \Phi_{2,m}, \quad (2.4.7)$$

where $m \in M_a$. It is called the direct sum of Φ_1 and Φ_2.

Example 2.4.3 (Tensor product) Define

$$E_1 \otimes E_2 = \bigsqcup_{m \in M} E_{1,m} \otimes E_{2,m},$$

denote the canonical projection by $\pi^{\otimes} : E_1 \otimes E_2 \to M$ and take the system of induced local trivializations $\chi_\alpha^{\otimes} : (\pi^{\otimes})^{-1}(U_\alpha) \to U_\alpha \times (F_1 \otimes F_2)$ defined by

$$\chi_\alpha^{\otimes}(x_1 \otimes x_2) = \big(m, \chi_{1\alpha,m}(x_1) \otimes \chi_{2\alpha,m}(x_2)\big),$$

where $m = \pi^{\otimes}(x_1 \otimes x_2)$. Then, $(E_1 \otimes E_2, M, \pi^{\otimes})$ is a C^k-vector bundle, called the tensor product of E_1 and E_2. Its typical fibre is $F_1 \otimes F_2$ and its dimension is $l_1 l_2$. Every pair of local sections s_i of E_i over U, $i = 1, 2$, defines a local section $s_1 \otimes s_2$ of $E_1 \otimes E_2$ by

$$(s_1 \otimes s_2)(m) := s_1(m) \otimes s_2(m), \quad m \in U, \qquad (2.4.8)$$

which is called the tensor product of s_1 and s_2. If $\{s_{i,1}, \dots, s_{i,l_i}\}$ are local frames in E_i, $i = 1, 2$, then

$$\{s_{1,i} \otimes s_{2,j} : i = 1, \dots, l_1, j = 1, \dots, l_2\}$$

is a local frame in $E_1 \otimes E_2$. For $\mathbb{K}$-vector bundle morphisms $\Phi_j : E_{aj} \to E_{bj}$, $j = 1, 2$, projecting to the same mapping $\varphi : M_a \to M_b$, the tensor product is the morphism $\Phi_1 \otimes \Phi_2 : E_{a1} \otimes E_{a2} \to E_{b1} \otimes E_{b2}$ defined by

$$(\Phi_1 \otimes \Phi_2)_m(x_1 \otimes x_2) := \Phi_{1,m}(x_1) \otimes \Phi_{2,m}(x_2). \qquad (2.4.9)$$

It projects to φ as well.

Example 2.4.4 (Tensor bundles) The tensor bundle of E of type (p,q) is defined to be

$$\mathbb{T}^q_p E := E^* \otimes \overset{p}{\cdots} \otimes E^* \otimes E \otimes \overset{q}{\cdots} \otimes E.$$

Its fibres are the p-fold covariant and q-fold contravariant tensor products $\mathbb{T}^q_p E_m$. Hence, the dimension is l^{p+q} and the elements of $\mathbb{T}^q_p E_m$ are linear combinations of $\xi_1 \otimes \cdots \otimes \xi_p \otimes x_1 \otimes \cdots \otimes x_q$, where $x_i \in E_m$ and $\xi_i \in E^*_m$. The projection is denoted by $\pi^{\otimes} : \mathbb{T}^q_p E \to M$ and the typical fibre is $\mathbb{T}^q_p F$. We will view elements of $\mathbb{T}^q_p E_m$ as $(p+q)$-linear mappings

$$u : E_m \times \overset{p}{\cdots} \times E_m \times E^*_m \times \overset{q}{\cdots} \times E^*_m \to \mathbb{R},$$

thus using the natural isomorphism which assigns to $\xi_1 \otimes \cdots \otimes \xi_p \otimes x_1 \otimes \cdots \otimes x_q$ the mapping

$$u(y_1, \ldots, y_p, \eta_1, \ldots, \eta_q) = \xi_1(y_1) \cdots \xi_p(y_p) \eta_1(x_1) \cdots \eta_q(x_q).$$

Then, the tensor product of $u_i \in \mathbb{T}^{q_i}_{p_i} E_m$, $i = 1, 2$, is given by

$$\begin{aligned} &u_1 \otimes u_2(x_1, \ldots, x_{p_1+p_2}, \xi_1, \ldots, \xi_{q_1+q_2}) \\ &\quad := u_1(x_1, \ldots, x_{p_1}, \xi_1, \ldots, \xi_{q_1}) u_2(x_{p_1+1}, \ldots, x_{p_1+p_2}, \xi_{q_1+1}, \ldots, \xi_{q_1+q_2}) \end{aligned} \tag{2.4.10}$$

for all $x_j \in E_m$ and $\xi_j \in E^*_m$. Accordingly, local sections τ of $\mathbb{T}^q_p E$ over U can be viewed as mappings

$$\tau : \Gamma(E_U) \times \overset{p}{\cdots} \times \Gamma(E_U) \times \Gamma\big(E^*_U\big) \times \overset{q}{\cdots} \times \Gamma\big(E^*_U\big) \to C^k(U) \tag{2.4.11}$$

which are $C^k(U)$-linear in every argument. Every pair of local sections τ_i of $\mathbb{T}^{q_i}_{p_i} E$, $i = 1, 2$, defines a local section $\tau_1 \otimes \tau_2$ in $\mathbb{T}^{q_1+q_2}_{p_1+p_2} E$ by

$$(\tau_1 \otimes \tau_2)(m) := \tau_1(m) \otimes \tau_2(m).$$

On the level of the mappings (2.4.11), $\tau_1 \otimes \tau_2$ is given by (2.4.10), with u_i replaced by τ_i and x_j and ξ_j replaced by local sections in E and E^*, respectively. In particular, if $\{s_1, \ldots, s_l\}$ is a local frame in E over U, then

$$\big\{s^{*i_1} \otimes \cdots \otimes s^{*i_p} \otimes s_{j_1} \otimes \cdots \otimes s_{j_q} : i_1, \ldots, i_p, j_1, \ldots, j_q = 1, \ldots, l\big\}$$

is a local frame in $\mathbb{T}^q_p E$. Every $\tau \in \Gamma(\mathbb{T}^q_p E)$ can be decomposed over U as

$$\tau_{\restriction U} = \tau^{j_1 \ldots j_q}_{i_1 \ldots i_p} s^{*i_1} \otimes \cdots \otimes s^{*i_p} \otimes s_{j_1} \otimes \cdots \otimes s_{j_q}$$

with

$$\tau^{j_1\ldots j_q}_{i_1\ldots i_p}(m) = \tau(m)\big(s_{i_1}(m), \ldots, s_{i_p}(m), s^{*j_1}(m), \ldots, s^{*j_q}(m)\big) \tag{2.4.12}$$

(Exercise 2.4.2). Finally, according to (2.4.9), every isomorphism $\Phi : E_a \to E_b$ of $\mathbb{K}$-vector bundles of class C^k induces isomorphisms

$$\Phi^{\otimes} : \mathbb{T}^q_p E_a \to \mathbb{T}^q_p E_b, \qquad \Phi^{\otimes} := \big(\Phi^{-1}\big)^{\mathrm{T}} \otimes \overset{p}{\cdots} \otimes \big(\Phi^{-1}\big)^{\mathrm{T}} \otimes \Phi \otimes \overset{q}{\cdots} \otimes \Phi, \tag{2.4.13}$$

with the same projection. On the level of $(p+q)$-linear mappings, $\Phi^{\otimes}$ takes the form

$$\begin{aligned}&\big(\Phi^{\otimes} u\big)(x_1, \ldots, x_p, \xi_1, \ldots, \xi_q)\\&\quad = u\big(\Phi^{-1}(x_1), \ldots, \Phi^{-1}(x_p), \Phi^{\mathrm{T}}(\xi_1), \ldots, \Phi^{\mathrm{T}}(\xi_q)\big)\end{aligned} \tag{2.4.14}$$

for all $u \in \mathbb{T}^q_p E_{a,m}$, $x_i \in E_{b,\varphi(m)}$ and $\xi_i \in E^*_{b,\varphi(m)}$. The corresponding transport operators $\Phi^{\otimes}_*$ satisfy

$$\Phi^{\otimes}_*(\tau_1 \otimes \tau_2) = \big(\Phi^{\otimes}_* \tau_1\big) \otimes \big(\Phi^{\otimes}_* \tau_2\big) \tag{2.4.15}$$

for all $\tau_i \in \Gamma(\mathbb{T}^{q_i}_{p_i} E_a)$, and

$$\begin{aligned}&\big(\Phi^{\otimes}_* \tau\big)(s_1, \ldots, s_p, \sigma_1, \ldots, \sigma_q)\\&\quad = \tau\big(\Phi^{-1}_* s_1, \ldots, \Phi^{-1}_* s_p, \Phi^* \sigma_1, \ldots, \Phi^* \sigma_q\big) \circ \varphi^{-1}\end{aligned} \tag{2.4.16}$$

for all $\tau \in \Gamma(\mathbb{T}^q_p E_a)$, $s_i \in \Gamma(E_b)$ and $\sigma_i \in \Gamma(E^*_b)$. Here, $\varphi : M_a \to M_b$ is the projection of Φ.

Example 2.4.5 (Exterior powers) The r-fold exterior power $\bigwedge^r E^*$ has the vector spaces $\bigwedge^r E^*_m$ of antisymmetric r-linear forms on E_m as its fibres. Hence, the dimension is $\binom{l}{r}$. In particular, $\bigwedge^0 E^* = M \times \mathbb{K}$, $\bigwedge^1 E^* = E^*$ and $\bigwedge^r E^* = M \times \{0\}$ (the zero-dimensional vector bundle over M) for $r > l$. The projection is denoted by $\pi^{\wedge} : \bigwedge^r E^* \to M$ and the typical fibre is $\bigwedge^r F^*$. The exterior product of $\eta_i \in \bigwedge^{r_i} E^*_m$ is defined to be the $(r_1 + r_2)$-linear form on E_m given by[6]

$$\begin{aligned}&(\eta_1 \wedge \eta_2)(x_1, \ldots, x_{r_1+r_2})\\&\quad := \frac{1}{r_1! r_2!} \sum_{\pi \in \mathrm{S}_{r_1+r_2}} \operatorname{sign}(\pi)\eta_1(x_{\pi(1)}, \ldots, x_{\pi(r_1)})\eta_2(x_{\pi(r_1+1)}, \ldots, x_{\pi(r_1+r_2)}),\end{aligned} \tag{2.4.17}$$

[6] Beware that there exist different conventions concerning the choice of the factor in Formula (2.4.17).

for all $x_i \in E_m$. A local section σ in $\bigwedge^r E^*$ over U can be viewed as an antisymmetric mapping

$$\sigma : \Gamma(E_U) \times \overset{r}{\cdots} \times \Gamma(E_U) \to C^k(U) \tag{2.4.18}$$

which is $C^k(U)$-linear in every argument. Every pair of local sections σ_i of $\bigwedge^{r_i} E^*$, $i = 1, 2$, defines a local section $\sigma_1 \wedge \sigma_2$ of $\bigwedge^{r_1+r_2} E^*$ by

$$(\sigma_1 \wedge \sigma_2)(m) := \sigma_1(m) \wedge \sigma_2(m), \quad m \in U. \tag{2.4.19}$$

If we view $\sigma_1 \wedge \sigma_2$ as a mapping (2.4.18), it is given by (2.4.17) with ξ_i replaced by σ_i and x_i replaced by local sections in E. If $\{s_1, \ldots, s_l\}$ is a local frame in E, then

$$\left\{s^{*i_1} \wedge \cdots \wedge s^{*i_r} : 1 \le i_1 < \cdots < i_r \le l\right\} \tag{2.4.20}$$

is a local frame in $\bigwedge^r E^*$. Every $\sigma \in \Gamma(\bigwedge^r E^*)$ can be decomposed over U as

$$\sigma_{\restriction U} = \sum_{i_1 < \cdots < i_r} \sigma_{i_1 \ldots i_r} s^{*i_1} \wedge \cdots \wedge s^{*i_r} \tag{2.4.21}$$

with $\sigma_{i_1 \ldots i_r}(m) = \sigma(m)(s_{i_1}(m), \ldots, s_{i_r}(m))$ (Exercise 2.4.2). Next, every $\mathbb{K}$-vector bundle morphism $\Phi : E_a \to E_b$ projecting to a diffeomorphism $\varphi : M_a \to M_b$ induces a morphism $\Phi^{\mathrm{T}\wedge} : \bigwedge^r E_b^* \to \bigwedge^r E_a^*$ projecting to φ^{-1}, defined by

$$\big(\Phi_m^{\mathrm{T}\wedge}(\eta)\big)(x_1, \ldots, x_r) := \eta\big(\Phi_{\varphi^{-1}(m)}(x_1), \ldots, \Phi_{\varphi^{-1}(m)}(x_r)\big). \tag{2.4.22}$$

This generalizes Formula (2.4.4). Via (2.3.3), $\Phi^{\mathrm{T}\wedge}$ induces a transport operator $(\Phi^{\mathrm{T}\wedge})_*$ of sections. Moreover, the pull-back operation (2.4.5) generalizes in an obvious way to a mapping $\Phi^* : \Gamma(\bigwedge^r E_b^*) \to \Gamma(\bigwedge^r E_a^*)$, given by

$$\big(\big(\Phi^*\sigma\big)(m)\big)(x_1, \ldots, x_r) := \big(\sigma \circ \varphi(m)\big)\big(\Phi(x_1), \ldots, \Phi(x_r)\big). \tag{2.4.23}$$

Again, if Φ projects to a diffeomorphism, then $\Phi^* = (\Phi^{\mathrm{T}\wedge})_*$.

Example 2.4.6 (Exterior algebra bundle) By composing the operations of exterior power and direct sum one obtains the exterior algebra bundle $\bigwedge E^* = \bigoplus_{i=0}^{l} \bigwedge^i E^*$, which has dimension 2^l. We retain the notations $\pi^\wedge : \bigwedge E \to M$ for the projection and $\Phi^{\mathrm{T}\wedge} : \bigwedge E_b^* \to \bigwedge E_a^*$ for the morphism induced by a morphism $\Phi : E_a \to E_b$. The local frame in $\bigwedge E^*$ associated with a local frame $\{s_1, \ldots, s_l\}$ in E consists of the constant mapping $U \to \mathbb{K}$ given by $m \mapsto 1$ and the local sections (2.4.20) with $r = 1, \ldots, l$. In addition to being a vector bundle, $\bigwedge E^*$ is an associative $\mathbb{K}$-algebra bundle[7] of class C^k over M. The exterior product of local sections (2.4.19) induces

[7] In the definition of vector bundle, replace "$\mathbb{K}$-vector space" by "$\mathbb{K}$-algebra" and "linear mapping" by "algebra homomorphism".

a bilinear mapping

$$\Gamma\Big(\bigwedge^{r_1} E^*\Big) \times \Gamma\Big(\bigwedge^{r_2} E^*\Big) \to \Gamma\Big(\bigwedge^{r_1+r_2} E^*\Big)$$

and hence defines on $\Gamma(\bigwedge E^*)$ the structure of an associative $\mathbb{K}$-algebra. By (2.4.17) and (2.4.23), the pull-back is a homomorphism with respect to this algebra structure,

$$\Phi^*(\sigma_1 \wedge \sigma_2) = \big(\Phi^*\sigma_1\big) \wedge \big(\Phi^*\sigma_2\big), \quad \sigma_1, \sigma_2 \in \Gamma\Big(\bigwedge E_b^*\Big). \tag{2.4.24}$$

Remark 2.4.7 (Homomorphism and endomorphism bundles) Analogously, one can construct the homomorphism bundle $\mathrm{Hom}(E_1, E_2)$ of E_1 and E_2, which has the fibres $\mathrm{Hom}(E_{1,m}, E_{2,m})$, and the endomorphism bundle $\mathrm{End}(E)$ of E, which has the fibres $\mathrm{End}(E_m)$. For every m, the vector space $\mathrm{Hom}(E_{1,m}, E_{2,m})$ is naturally isomorphic to the vector space $E_{1,m}^* \otimes E_{2,m}$ and all these isomorphisms combine to a natural isomorphism of $\mathrm{Hom}(E_1, E_2)$ with the tensor product $E_1^* \otimes E_2$ (Exercise 2.4.4). Therefore, we may always identify $\mathrm{Hom}(E_1, E_2)$ with $E_1^* \otimes E_2$. Accordingly, we may identify $\mathrm{End}(E)$ with the tensor bundle $E^* \otimes E \equiv \mathbb{T}^1_1 E$ of E. Then, since vertical C^k-morphisms $E_1 \to E_2$ correspond to C^k-sections of $\mathrm{Hom}(E_1, E_2)$, the vector space of these morphisms is naturally isomorphic to $\Gamma(E_1^* \otimes E_2)$. Accordingly, since vertical endomorphisms of E correspond to sections of $\mathrm{End}(E)$, the vector space of these endomorphisms is naturally isomorphic to $\Gamma(\mathbb{T}^1_1 E)$. The proof is left to the reader (Exercise 2.4.5).

Exercises

2.4.1 Show that the mapping Φ^{T} defined by (2.4.4) is a morphism of vector bundles.

2.4.2 Verify Formulae (2.4.12) and (2.4.21).

2.4.3 Let (E, M, π) be a smooth $\mathbb{K}$-vector bundle. Consider the tangent mapping $\pi' : \mathrm{T}E \to \mathrm{T}M$.
- (a) Show that $(\mathrm{T}E, \mathrm{T}M, \pi')$ is a $\mathbb{K}$-vector bundle by determining the linear structure of the fibres and constructing a system of local trivializations.
- (b) Show that in the cases $E = \mathrm{T}M$ and $E = \mathrm{T}^*M$, local charts on M induce local trivializations of $(\mathrm{T}E, \mathrm{T}M, \pi')$.
- (c) If $E = \mathrm{T}M$, then $(\mathrm{T}E, \mathrm{T}M, \pi')$ has the same base manifold as the tangent bundle of E. Are these two vector bundles isomorphic?

2.4.4 Let E, E_1 and E_2 be $\mathbb{K}$-vector bundles over M of class C^k. Construct the homomorphism bundle $\mathrm{Hom}(E_1, E_2)$ and the endomorphism bundle $\mathrm{End}(E)$ as explained in Remark 2.4.7. Show that $\mathrm{Hom}(E_1, E_2)$ and $\mathrm{End}(E)$ are naturally isomorphic to $E_1^* \otimes E_2$ and $\mathbb{T}^1_1 E$, respectively.

2.4.5 Show that the natural isomorphisms of Exercise 2.4.4 induce natural isomorphisms between the vector space of vertical C^k-morphisms $E_1 \to E_2$ and $\Gamma(E_1^* \otimes E_2)$, as well as between the vector space of vertical C^k-endomorphisms of E and $\Gamma(\mathbb{T}^1_1 E)$, cf. Remark 2.4.7.

2.4.6 The image of the identical mapping id_E under the isomorphism from the vector space of C^k-endomorphisms of E to $\Gamma(\mathbb{T}^1_1 E)$ of Exercise 2.4.5 is called the Kronecker tensor field of E and is denoted by δ. Determine the coefficient functions δ^i_j of δ with respect to the local frame in $\mathbb{T}^1_1 E$ induced by a local frame in E.

2.4.7 Show that if E is one-dimensional, the tensor bundles $\mathbb{T}^2_0 E$, $\mathbb{T}^0_2 E$ and $\mathbb{T}^1_1 E$ are trivial.

2.5 Tensor Bundles and Tensor Fields

Let M be a C^k-manifold of dimension n. By tensor bundles over M one means the various vector bundles arising from the tangent bundle $\mathrm{T}M$ by applying the vector bundle operations of Sect. 2.4. These are

(a) the cotangent bundle $\mathrm{T}^*M := (\mathrm{T}M)^*$. Its fibres are the cotangent spaces[8] $\mathrm{T}^*_m M$ introduced in Sect. 1.4. (Local) sections of T^*M are called (local) covector fields or (local) differential 1-forms.

(b) The bundle of alternating r-vectors $\bigwedge^r \mathrm{T}M$, the bundle of alternating r-forms $\bigwedge^r \mathrm{T}^*M$ and the bundles of exterior algebras

$$\bigwedge \mathrm{T}M = \bigoplus_{r=0}^{n} \bigwedge^r \mathrm{T}M, \qquad \bigwedge \mathrm{T}^*M = \bigoplus_{r=0}^{n} \bigwedge^r \mathrm{T}^*M.$$

Their (local) sections are called (local) multivector fields and (local) differential forms, respectively. The number r is called the degree. We denote

$$\mathfrak{X}^r(M) := \Gamma\Big(\bigwedge^r \mathrm{T}M\Big),\ \Omega^r(M) := \Gamma\Big(\bigwedge^r \mathrm{T}^*M\Big),\ \Omega^*(M) := \Gamma\Big(\bigwedge \mathrm{T}^*M\Big)$$

and, as before, $\mathfrak{X}(M) \equiv \mathfrak{X}^1(M)$. One has $\mathfrak{X}^0(M) = \Omega^0(M) = C^k(M)$.

(c) The tensor bundles $\mathbb{T}^q_p M := \mathbb{T}^q_p(\mathrm{T}M)$, $p, q = 0, 1, 2, \dots$. Their (local) sections are called (local) tensor fields of type (p, q). The algebraic operations of symmetrization, antisymmetrization and contraction of tensors over a vector space carry over to tensor fields in an obvious way.

Since $\mathrm{T}M$ is of class C^{k-1}, so are all the tensor bundles over M. Recall from Example 2.4.1 that pointwise evaluation $\mathrm{T}^*_m M \times \mathrm{T}_m M \to \mathbb{R}$ defines a natural pairing

$$\Omega^1(M) \times \mathfrak{X}(M) \to C^{k-1}(M), \qquad (\alpha, X) \mapsto \langle \alpha, X \rangle, \tag{2.5.1}$$

which depending on the context can also be written as $\alpha(X)$ or $X(\alpha)$.

[8] Like for the tangent bundle we will stick to this notation (instead of writing $(\mathrm{T}^*M)_m$).

Example 2.5.1 Let $f : U \to \mathbb{R}$, with $U \subset M$ open, be a real-valued local C^k-function. Then, the differentials $(\mathrm{d}f)_m$ of f at $m \in U$, defined by (1.4.20), combine to a local C^{k-1}-covector field $\mathrm{d}f$ on U, called the differential of f.

Now, let (U, κ) be a local chart on M. The differentials of the coordinate functions κ^i form a local frame $\{\mathrm{d}\kappa^1, \dots, \mathrm{d}\kappa^n\}$ in T^*M which is dual to $\{\partial_1, \dots, \partial_n\}$, cf. Examples 2.3.11/1 and 2.4.1 and Formula (1.4.21). The induced local frame in the tensor bundle $\mathbb{T}^q_p M$ consists of the local sections

$$\mathrm{d}\kappa^{i_1} \otimes \cdots \otimes \mathrm{d}\kappa^{i_p} \otimes \partial_{j_1} \otimes \cdots \otimes \partial_{j_q} : \quad i_1, \dots, i_p, j_1, \dots, j_q = 1, \dots, n,$$

see Example 2.4.4. Using these local frames, a tensor field T of type (p, q) can be represented locally as follows:

$$T_{\restriction U} = \left(T^\kappa\right)^{j_1 \dots j_q}_{i_1 \dots i_p} \mathrm{d}\kappa^{i_1} \otimes \cdots \otimes \mathrm{d}\kappa^{i_p} \otimes \partial_{j_1} \otimes \cdots \otimes \partial_{j_q}, \tag{2.5.2}$$

where, according to (2.4.12), pointwise we have

$$\left(T^\kappa\right)^{j_1 \dots j_q}_{i_1 \dots i_p} = T\left(\partial_{i_1}, \dots, \partial_{i_p}, \mathrm{d}\kappa^{j_1}, \dots, \mathrm{d}\kappa^{j_q}\right). \tag{2.5.3}$$

Remark 2.5.2 We determine the transformation laws for the local frames and for the corresponding coefficient functions of tensor fields under a change of local chart. Thus, let (V, ρ) be another local chart on M. The following formulae hold over $U \cap V$. From (1.4.17) and (1.4.23) we read off

$$\partial^\rho_i = \tilde{A}^j_i \partial^\kappa_j, \qquad \mathrm{d}\rho^i = A^i_j \mathrm{d}\kappa^j$$

where

$$A^i_j := \left[\left(\rho \circ \kappa^{-1}\right)' \circ \kappa\right]^i_j, \qquad \tilde{A}^i_j := \left[\left(\kappa \circ \rho^{-1}\right)' \circ \rho\right]^i_j,$$

and an according formula for the induced local frames in $\mathbb{T}^q_p M$. Then, (2.5.3) implies

$$\left(T^\rho\right)^{j_1 \dots j_q}_{i_1 \dots i_p} = \tilde{A}^{k_1}_{i_1} \cdots \tilde{A}^{k_p}_{i_p} A^{j_1}_{l_1} \cdots A^{j_q}_{l_q} \left(T^\kappa\right)^{l_1 \dots l_q}_{k_1 \dots k_p}. \tag{2.5.4}$$

To pass to coefficient functions which depend on the coordinates, denote the elements of $\kappa(U \cap V)$ by $\mathbf{x}$ and the elements of $\rho(U \cap V)$ by $\mathbf{y}$ and write

$$y^i(\mathbf{x}) := \rho^i \circ \kappa^{-1}(\mathbf{x}), \qquad x^i(\mathbf{y}) := \kappa \circ \rho^{-1}(\mathbf{y}).$$

Then, from (2.5.4) we read off

$$\left(T^\rho\right)^{j_1 \dots j_q}_{i_1 \dots i_p} \circ \rho^{-1} = \frac{\partial x^{k_1}}{\partial y^{i_1}} \cdots \frac{\partial x^{k_p}}{\partial y^{i_p}} \frac{\partial y^{j_1}}{\partial x^{l_1}} \cdots \frac{\partial y^{j_q}}{\partial x^{l_q}} \left(T^\kappa\right)^{l_1 \dots l_q}_{k_1 \dots k_p} \circ \kappa^{-1}. \tag{2.5.5}$$

This formula is well-known from classical tensor analysis. The argument in (2.5.5) can be either $\mathbf{x}$, in which case $\frac{\partial x^{i_a}}{\partial y^{k_a}}$ and $(T^\rho)^{j_1\dots j_q}_{i_1\dots i_p} \circ \rho^{-1}$ have to be evaluated at $\mathbf{y}(\mathbf{x})$, or $\mathbf{y}$, in which case $\frac{\partial y^{l_a}}{\partial x^{j_a}}$ and $(T^\kappa)^{l_1\dots l_q}_{k_1\dots k_p} \circ \kappa^{-1}$ have to be evaluated at $\mathbf{x}(\mathbf{y})$.

Next, let M and N be C^k-manifolds and let $\varphi : M \to N$ be a C^k-mapping. According to Proposition 2.2.9, $\varphi' : \mathrm{T}M \to \mathrm{T}N$ is a vector bundle morphism of class C^{k-1} projecting to φ. The corresponding pull-back operation (2.4.23) applies to differential r-forms of class C^{k-1}. It will be denoted by $\varphi^* : \Omega^r(N) \to \Omega^r(M)$. According to Examples 2.4.1–2.4.5, if φ is a diffeomorphism, φ' induces isomorphisms of tensor bundles. The corresponding transport operator (2.3.3) will be denoted by φ_* in case the induced isomorphism projects to φ and by φ^* in case it projects to φ^{-1}. Then, for $T \in \Gamma(\mathbb{T}^q_p M)$, we have

$$\varphi_* T \equiv \big((\varphi')^{\otimes}\big)_* T = (\varphi')^{\otimes} \circ T \circ \varphi^{-1}, \tag{2.5.6}$$

with $(\varphi')^{\otimes}$ given by (2.4.13), and Formula (2.4.16) takes the form

$$\begin{aligned}(\varphi_* T)&(X_1, \dots, X_p, \alpha_1, \dots, \alpha_q) \\ &= T\big(\varphi_*^{-1} X_1, \dots, \varphi_*^{-1} X_p, \varphi^* \alpha_1, \dots, \varphi^* \alpha_q\big) \circ \varphi^{-1}\end{aligned} \tag{2.5.7}$$

with $X_i \in \mathfrak{X}(M)$ and $\alpha_i \in \Omega^1(M)$. Moreover, Eq. (2.4.15) reads

$$\varphi_*(T_1 \otimes T_2) = (\varphi_* T_1) \otimes (\varphi_* T_2). \tag{2.5.8}$$

Recall from Example 2.4.5 that for differential forms, the transport operation φ^* coincides with the pull-back under φ.

Remark 2.5.3

1. Let $T \in \Gamma(\mathbb{T}^q_p M)$ and let $\varphi : M \to N$ be a diffeomorphism. Given local charts (U, κ) and (V, ρ) on M and N, respectively, the local formula for the transport (2.5.6) of T is given by (2.5.4), with T replaced by $\varphi_* T$ on the left hand side and by $T \circ \varphi^{-1}$ on the right hand side, and with A and $\tilde{A}$ given by

$$A^i_j = \big[\big(\rho \circ \varphi \circ \kappa^{-1}\big)' \circ \kappa \circ \varphi^{-1}\big]^i_k, \qquad \tilde{A}^i_j = \big[\big(\kappa \circ \varphi^{-1} \circ \rho^{-1}\big)' \circ \rho\big]^i_j.$$

   The proof of this fact is left to the reader (Exercise 2.5.3).
2. Let (U, κ) be a local chart on M. We compare the corresponding local representative $\kappa_*(T_{\restriction U})$ of a tensor field $T \in \Gamma(\mathbb{T}^q_p M)$ with the local representative of the mapping $T : M \to \mathbb{T}^q_p M$ with respect to the induced chart $((\pi^{\otimes})^{-1}(U), \kappa^{\otimes})$ on $\mathbb{T}^q_p M$, given by

$$\kappa^{\otimes} \circ T_{\restriction U} \circ \kappa^{-1} : \kappa(U) \to \kappa(U) \times \mathbb{T}^q_p \mathbb{R}^n.$$

Proofs are left to the reader (Exercise 2.5.4). Since

$$\kappa_* \partial_i = \frac{\partial}{\partial x^i}, \qquad \kappa^* \mathrm{d}x^i = \mathrm{d}\kappa^i, \tag{2.5.9}$$

we have

$$\kappa_*(T_{\restriction U}) = \left(\left(T^\kappa\right)^{j_1 \dots j_q}_{i_1 \dots i_p} \circ \kappa^{-1}\right) \mathrm{d}x^{i_1} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_q}}. \tag{2.5.10}$$

On the other hand,

$$\left(\kappa^{\otimes} \circ T_{\restriction U} \circ \kappa^{-1}\right)(\mathbf{x}) = \left(\mathbf{x}, \left(\left(T^\kappa\right)^{j_1 \dots j_q}_{i_1 \dots i_p} \circ \kappa^{-1}(\mathbf{x})\right) \mathbf{e}^{*i_1} \otimes \cdots \otimes \mathbf{e}_{j_q}\right),$$

where, as before, $\mathbf{e}_i$ denote the elements of the standard basis of $\mathbb{R}^n$ and $\mathbf{e}^{*i}$ the elements of the dual basis. The relation to $\kappa_*(T_{\restriction U})$ is as follows. The natural identifications of the tangent spaces $\mathrm{T}_{\mathbf{x}}(\kappa(U))$ with $\mathbb{R}^n$ and of the cotangent spaces $\mathrm{T}^*_{\mathbf{x}}(\kappa(U))$ with $\mathbb{R}^{n*}$ induce a natural identification of tensor fields on $\kappa(U)$ with C^{k-1}-mappings $\kappa(U) \to \mathbb{T}^q_p \mathbb{R}^n$. Since the latter identifies the elements of the global frames $\{\frac{\partial}{\partial x^i}\}$ in $\mathrm{T}(\kappa(U))$ and $\{\mathrm{d}x^i\}$ in $\mathrm{T}^*(\kappa(U))$ with the constant mappings $\mathbf{x} \mapsto \mathbf{e}_i$ and $\mathbf{x} \mapsto \mathbf{e}^{*i}$, respectively, it identifies $\kappa_*(T_{\restriction U})$ with $\kappa^{\otimes} \circ T_{\restriction U} \circ \kappa^{-1}$. Note that for $M = \mathbb{R}^n$ and $\kappa = \mathrm{id}$, (2.5.9) yields $\partial_i = \frac{\partial}{\partial x^i}$.

Exercises

2.5.1 Let $M_1 = \mathbb{R}_+ \times \mathrm{S}^1$, with S^1 realized as the unit sphere in $\mathbb{R}^2$, and $M_2 = \mathbb{R}^2 \setminus \{0\}$. Consider the mapping $\varphi : M_1 \to M_2$, $\varphi(r, (a, b)) := (ra, rb)$. Let r denote the standard coordinate on $\mathbb{R}_+$ and let ϕ denote the angle coordinate of S^1. Determine the coefficient functions of $\varphi_* \frac{\partial}{\partial r}$ and $\varphi_* \frac{\partial}{\partial \phi}$ with respect to the global frame $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ in $\mathrm{T}(\mathbb{R}^2 \setminus \{0\})$.

2.5.2 Let $M_1 = \mathbb{R}_+ \times \mathrm{S}^2$, with S^2 realized as the unit sphere in $\mathbb{R}^3$, and $M_2 = \mathbb{R}^3 \setminus \{0\}$. Consider the mapping $\varphi : M_1 \to M_2$, $\varphi(r, (a, b, c)) = (ra, rb, rc)$. Let r denote the natural coordinate on $\mathbb{R}_+$ and let the angle coordinates ϑ, ϕ on S^2 be defined by $a = \cos\phi \sin\vartheta$, $b = \sin\phi \sin\vartheta$, $c = \cos\vartheta$. Determine the coefficient functions of $\varphi_* \frac{\partial}{\partial r}$, $\varphi_* \frac{\partial}{\partial \phi}$ and $\varphi_* \frac{\partial}{\partial \theta}$ with respect to the global frame $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ in $\mathrm{T}(\mathbb{R}^3 \setminus \{0\})$.

2.5.3 Prove the transformation formula for the transport of tensor fields under diffeomorphisms given in Remark 2.5.3/1.

2.5.4 Prove the assertions of Remark 2.5.3/2.

2.6 Induced Bundles

Let (E, N, π) be a $\mathbb{K}$-vector bundle of class C^k, let M be a C^k-manifold and let $\varphi \in C^k(M, N)$. Using φ, one can construct from E a vector bundle $\varphi^* E$ over M by attaching to $m \in M$ the fibre $E_{\varphi(m)}$ as follows. Define

$$\varphi^* E := \{(m, x) \in M \times E : \varphi(m) = \pi(x)\}$$

and consider the surjective mapping $\pi^{\varphi^*} : \varphi^* E \to M$ defined by $\pi^{\varphi^*}(m, x) := m$. The fibres are

$$(\varphi^* E)_m \equiv (\pi^{\varphi^*})^{-1}(m) = \{m\} \times E_{\varphi(m)}.$$

They inherit a natural $\mathbb{K}$-vector space structure from E.

Proposition 2.6.1 *Under the above assumptions, $\varphi^* E$ admits a C^k-structure such that it is an embedded submanifold of $M \times E$. Then,*

1. *$(\varphi^* E, M, \pi^{\varphi^*})$ is a $\mathbb{K}$-vector bundle of class C^k,*
2. *the natural projection $M \times E \to E$ restricts to a C^k-morphism $\varphi^* E \to E$ covering φ,*
3. *every local section s of E induces a local section of $\varphi^* E$ defined by*

$$(\varphi^* s)(m) := (m, s \circ \varphi(m)).$$

Proof We apply Proposition 1.7.3 in the formulation of Remark 1.7.4. Choose a typical fibre F and a system of local trivializations $\{(U_\alpha, \chi_\alpha) : \alpha \in A\}$ for E. For every $\alpha \in A$, consider the open subset $V_\alpha := \varphi^{-1}(U_\alpha)$ of M and the mapping

$$\psi_\alpha : V_\alpha \times F \to M \times E, \qquad \psi_\alpha(m, u) := (m, \chi_\alpha^{-1}(\varphi(m), u)).$$

Since ψ_α is obtained by composing the diffeomorphism χ_α^{-1} with the natural inclusion mapping of the graph of $\varphi_{\restriction V_\alpha} : V_\alpha \to U_\alpha$, by Example 1.6.12/2, it is a C^k-embedding. Hence, the image $\psi_\alpha(V_\alpha \times F)$ inherits a C^k-structure from $V_\alpha \times F$ and with respect to this structure it is an embedded C^k-submanifold of $M \times E$. Since the image is $\varphi^* E \cap (V_\alpha \times \pi^{-1}(U_\alpha))$ and since the $V_\alpha \times \pi^{-1}(U_\alpha)$ are open subsets of $M \times E$ covering $\varphi^* E$, we conclude that $\varphi^* E$ is an embedded submanifold. It remains to prove assertion 1; assertions 2 and 3 are then obvious. Since π^{φ^*} is the restriction of the natural projection $M \times E \to M$ to the C^k-submanifold $\varphi^* E$, it is of class C^k. Since, by construction, the ψ_α restrict to C^k-diffeomorphisms from $V_\alpha \times F$ to $\varphi^* E \cap (V_\alpha \times \pi^{-1}(U_\alpha)) = (\pi^{\varphi^*})^{-1}(V_\alpha)$, by inverting them we obtain C^k-diffeomorphisms

$$\chi_\alpha^{\varphi^*} : (\pi^{\varphi^*})^{-1}(V_\alpha) \to V_\alpha \times F, \qquad \chi_\alpha^{\varphi^*}(m, x) = (m, \chi_{\alpha, \varphi(m)}(x)). \tag{2.6.1}$$

The latter satisfy conditions 2a and 2b of Definition 2.2.1. Thus, $(\varphi^* E, M, \pi^{\varphi^*})$ is a $\mathbb{K}$-vector bundle of class C^k. □

Definition 2.6.2 (Induced vector bundle) The $\mathbb{K}$-vector bundle $(\varphi^* E, M, \pi^{\varphi^*})$ is called the vector bundle induced from E by φ or the pull-back of E by φ. For a local section s of E, the local section $\varphi^* s$ of $\varphi^* E$ is said to be induced from s by φ or to be the pull-back of s by φ.

Another common notation for the induced vector bundle is $\varphi^* E \equiv M \times_N E$.

Remark 2.6.3

1. From the proof of Proposition 2.6.1 we note that via (2.6.1), every local trivialization (U, χ) of E induces a local trivialization $(\varphi^{-1}(U), \chi^{\varphi^*})$ of $\varphi^* E$. In particular, the pull-back of a trivial vector bundle is trivial. Moreover, if $\rho_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(F)$ are the transition functions of a system of local trivializations of E, then $\varphi^* \rho_{\alpha\beta} : \varphi^{-1}(U_\alpha) \cap \varphi^{-1}(U_\beta) \to \mathrm{GL}(F)$ are the transition functions of the induced system of local trivializations of $\varphi^* E$.
2. Let (E_i, M_i, π_i), $i = 1, 2$, be $\mathbb{K}$-vector bundles of class C^k and let $\Phi : E_1 \to E_2$ be a morphism with projection $\varphi : M_1 \to M_2$. Φ naturally decomposes as
$$E_1 \xrightarrow{\Phi_{\mathrm{ver}}} \varphi^* E_2 \xrightarrow{\Phi_{\mathrm{hor}}} E_2, \tag{2.6.2}$$
where Φ_{ver} is given by $\Phi_{\mathrm{ver}}(x) = (\pi_1(x), \Phi(x))$, $x \in E_1$, and Φ_{hor} denotes the induced vector bundle morphism of Proposition 2.6.1/2. One can check that Φ_{ver} is a vertical morphism, with differentiability of class C^k following from Proposition 1.6.10 and the fact that $\varphi^* E$ is an embedded submanifold of $M \times E$. Using this decomposition, one can derive the following characterization of isomorphisms in terms of their projections and fibre mappings (Exercise 2.6.1): a morphism is an isomorphism iff its projection is a diffeomorphism and its fibre mappings are bijective.

Example 2.6.4

1. If $\varphi : M \to N$ is constant with $\varphi(m) = p$, then $\varphi^* E$ coincides with the product vector bundle $M \times E_p$.
2. If $M \subset N$ is an open subset and $j : M \to N$ is the natural inclusion mapping, $j^* E$ can be identified with the restriction $E_{\restriction M}$, see Example 2.2.3/3.
3. Let $M = N = \mathrm{S}^1$ and let E be the Möbius strip of Example 2.2.4. Realize S^1 as the unit circle in $\mathbb{C}$ and consider the n-fold covering $\varphi_n : \mathrm{S}^1 \to \mathrm{S}^1$, $\varphi_n(z) = z^n$. Since $\varphi_n^* E$ is a differentiable real vector bundle over S^1 of dimension 1, according to Remark 2.3.14, it must be isomorphic to either E or the product vector bundle $\mathrm{S}^1 \times \mathbb{R}$. Indeed, one finds (Exercise 2.6.2)
$$\varphi_n^* E \cong \begin{cases} E \mid n \text{ odd}, \\ \mathrm{S}^1 \times \mathbb{R} \mid n \text{ even}. \end{cases}$$
4. Let E_1 and E_2 be $\mathbb{K}$-vector bundles of class C^k over M, let $E_1 \times E_2$ denote the product vector bundle over $M \times M$, see Exercise 2.2.7, and let $\Delta : M \to M \times M$ denote the diagonal mapping, $\Delta(m) = (m, m)$. The pull-back $\Delta^*(E_1 \times E_2)$ is naturally isomorphic to the direct sum $E_1 \oplus E_2$ (Exercise 2.6.3).
5. If E is a $\mathbb{K}$-vector bundle of class C^k over N and (M, φ) is a C^k-submanifold of N, the induced vector bundle $\varphi^* E$ is referred to as the restriction of E to M and is usually denoted by $E_{\restriction M}$. This applies in particular to $E = \mathrm{T}N$, where $\varphi^* \mathrm{T}N$ is a real vector bundle over M of class C^{k-1} and dimension $\dim N$.

Exercises

2.6.1 Use the natural decomposition (2.6.2) of vector bundle morphisms to show that a morphism is an isomorphism iff its projection is a diffeomorphism and the fibre mappings are bijective, cf. Remark 2.6.3/2.

2.6.2 Prove the statement of Example 2.6.4/3 about the pull-back of the Möbius strip by means of a covering of S^1.

2.6.3 Show that $\Delta^*(E_1 \times E_2) \cong E_1 \oplus E_2$, see Example 2.6.4/4.

2.7 Subbundles and Quotient Bundles

Definition 2.7.1 (Vector subbundle) Let (E_i, M_i, π_i), $i = 1, 2$, be $\mathbb{K}$-vector bundles of class C^k and let $\Phi : E_1 \to E_2$ be a morphism. The pair (E_1, Φ) is called a subbundle, an initial subbundle or an embedded subbundle of E_2 if it is, respectively, a submanifold, an initial submanifold or an embedded submanifold. If $M_1 = M_2 = M$ and Φ is vertical, (E_1, Φ) is called a vertical subbundle or a subbundle over M.

At the very beginning, we observe that Propositions 1.6.10 and 1.6.14 remain true if the term submanifold is replaced by subbundle and C^k-mapping by morphism. The following two specific types of subbundles are the building blocks for arbitrary subbundles.

Example 2.7.2 (Vertical subbundle) If E_1 and E_2 are $\mathbb{K}$-vector bundles of class C^k over M and $\Phi : E_1 \to E_2$ is an injective vertical morphism, then (E_1, Φ) is a vertical subbundle of E_2. Vertical subbundles are embedded. To see this, it suffices to show that (E_1, Φ) is an embedded submanifold of E_2. Let l_i denote the dimensions of E_i. Necessarily, $l_1 \leq l_2$. Let $x \in E_1$ and $m := \pi_1(x)$. Choose a local frame in E_1 at m. By injectivity, the image under Φ is a local l_1-frame in E_2. According to Proposition 2.3.15/3, the latter can be complemented, over a possibly smaller domain U, to a local frame in E_2 at m. The local representative of Φ with respect to the local trivializations associated with these local frames in E_1 and E_2 is given by

$$U \times \mathbb{K}^{l_1} \to U \times \mathbb{K}^{l_2}, \qquad (m, \mathbf{x}) \mapsto \big(m, (\mathbf{x}, 0)\big).$$

Hence, it is an embedding. Since $\pi_1^{-1}(U) = \Phi^{-1}(\pi_2^{-1}(U))$, this implies that the restriction $\Phi_{\restriction \Phi^{-1}(\pi_2^{-1}(U))}$ is an embedding. Since $\pi_2^{-1}(U)$ is an open neighbourhood of $\Phi(x)$ and x was arbitrary, Remark 1.6.13/3 yields the assertion.

Example 2.7.3 (Restriction of the base manifold) Let (E, N, π) be a $\mathbb{K}$-vector bundle of class C^k and let (M, φ) be a C^k-submanifold of N. Let $\Phi : \varphi^* E \to E$ denote the induced vector bundle morphism of Proposition 2.6.1/2. Recall from Example 2.6.4/5 that $\varphi^* E$ is referred to as the restriction of E to M and is alternatively denoted by $E_{\restriction M}$. We show that $(\varphi^* E, \Phi)$ is a C^k-subbundle of E. If (M, φ) is initial or embedded, so is $(\varphi^* E, \Phi)$. Indeed, the local representatives of Φ with respect to a system of local trivializations $\{(U_\alpha, \chi_\alpha) : \alpha \in A\}$ of E and the induced system of local trivializations of $\varphi^* E$ are given by

$$\chi_\alpha \circ \Phi_{\restriction (\pi^{\varphi^*})^{-1}(\varphi^{-1}(U_\alpha))} \circ \big(\chi_\alpha^{\varphi^*}\big)^{-1} : \varphi^{-1}(U_\alpha) \times F \to U_\alpha \times F,$$
$$(m, u) \mapsto \big(\varphi(m), u\big).$$

First, this implies that Φ is an immersion. Second, since χ_α and $\chi_\alpha^{\varphi^*}$ are diffeomorphisms and since

$$\big(\pi^{\varphi^*}\big)^{-1}\big(\varphi^{-1}(U_\alpha)\big) = \Phi^{-1}\big(\pi^{-1}(U_\alpha)\big),$$

this implies that the submanifolds $(\Phi^{-1}(\pi^{-1}(U_\alpha)), \Phi_{\restriction \Phi^{-1}(\pi^{-1}(U_\alpha))})$ inherit the property of being initial or embedded from (M, φ). Then, Remark 1.6.13/3 yields the assertion.

The following proposition states criteria for a morphism to define a subbundle.

Proposition 2.7.4 *Let (E_i, M_i, π_i), $i = 1, 2$, be $\mathbb{K}$-vector bundles of class C^k, let $\Phi : E_1 \to E_2$ be a morphism and let $\varphi : M_1 \to M_2$ be the projection. The following statements are equivalent.*

1. *(E_1, Φ) is, respectively, a subbundle, initial subbundle or embedded subbundle of E_2.*
2. *(M_1, φ) is, respectively, a submanifold, initial submanifold or embedded submanifold of M_2 and the fibre mappings $\Phi_m : E_{1,m} \to E_{2,\varphi(m)}$ are injective for all $m \in M_1$.*
3. *In the decomposition (2.6.2), (E_1, Φ_{ver}) is a vertical subbundle of $\varphi^* E_2$ and $(\varphi^* E_2, \Phi_{\text{hor}})$ is, respectively, a subbundle, initial subbundle or embedded subbundle of E_2.*

Item 3 gives a precise meaning to the statement made above that vertical subbundles (Example 2.7.2) and restrictions of the base manifold (Example 2.7.3) provide the building blocks for arbitrary subbundles.

Proof $1 \Rightarrow 2$: The fibre mappings Φ_m are obviously injective. Since they are linear, one has the commutative diagram

$$\begin{tikzcd}
M_1 \arrow[r, "\Phi \circ s_{0,1}"] \arrow[dr, "\varphi"'] & E_2 \\
 & M_2 \arrow[u, "s_{0,2}"']
\end{tikzcd}$$

where $s_{0,i}$ denotes the zero sections of E_i. According to Proposition 2.3.5, $s_{0,1}$ and $s_{0,2}$ are embeddings. Hence, the assertion follows by applying Proposition 1.6.14. (We encourage the reader to work out the argument for each case.)

2 $\Rightarrow$ 3: Since the mappings Φ_m are injective, Φ_{ver} is injective, hence the assertion on $(E_1, \Phi_{\mathrm{ver}})$ holds due to Example 2.7.2. The assertion on $(\varphi^* E_2, \Phi_{\mathrm{hor}})$ was proved in Example 2.7.3.

3 $\Rightarrow$ 1: Since vertical subbundles are embedded, this follows from Proposition 1.6.14/1. $\square$

In the following proposition we give criteria for a family of fibre subspaces of a vector bundle to define a vertical subbundle. The proof is left to the reader (Exercise 2.7.1).

Proposition 2.7.5 (Families of fibre subspaces) *Let (E_2, M, π) be a $\mathbb{K}$-vector bundle of class C^k. For every $m \in M$, let $E_{1,m} \subset E_{2,m}$ be a linear subspace. Define $E_1 := \bigcup_{m\in M} E_{1,m}$. The following statements are equivalent.*

1. *E_1 admits a C^k-structure such that it is a vertical subbundle of E_2 of dimension r.*
2. *There exists a covering of M by local r-frames in E_2 which span E_1.*
3. *There exists a covering of M by local frames in E_2 whose first r elements span E_1.*
4. *There exists a system of local trivializations $\{(U_\alpha, \chi_\alpha) : \alpha \in A\}$ of E_2 and a subspace F_1 of dimension r of the typical fibre F_2 such that the restrictions of the χ_α to E_1 take values in $U_\alpha \times F_1$.*

Example 2.7.6 (Regular distribution) Let M be a C^k-manifold. A vertical subbundle (D, Φ) of $\mathrm{T}M$ is called a regular distribution (in the geometrical sense) on M. According to Proposition 2.7.5, a family of r-dimensional subspaces $D_m \subset \mathrm{T}_m M$, $m \in M$, defines a distribution iff for every $m_0 \in M$ there exists an open neighbourhood U and pointwise linearly independent local vector fields $X_1, \dots, X_r$ on U such that D_m is spanned by $X_{1,m}, \dots, X_{r,m}$ for all $m \in U$. There is a more general notion of distribution on M which will be defined and studied in Sect. 3.5.

Example 2.7.7 (Kernel and image) Let E_i be $\mathbb{K}$-vector bundles over M of class C^k and dimension l_i, $i = 1, 2$, and let $\Phi : E_1 \to E_2$ be a vertical morphism of constant rank r. Define the image and the kernel of Φ to be

$$\operatorname{im}\Phi := \bigcup_{m\in M} \operatorname{im}\Phi_m, \qquad \ker\Phi := \bigcup_{m\in M} \ker\Phi_m,$$

respectively. We show that $\operatorname{im}\Phi$ is a vertical subbundle of E_2 of dimension r and that $\ker\Phi$ is a vertical subbundle of E_1 of dimension $l_1 - r$.

Let $m_0 \in M$. Choose a basis $\{e_1, \dots, e_{l_1}\}$ of E_{1,m_0} such that $e_{r+1}, \dots, e_{l_1}$ span $\ker\Phi_{m_0}$. Extend this basis to a local frame $\{s_1, \dots, s_{l_1}\}$ in E_1, cf. Proposition 2.3.15. By construction, the vectors $\Phi_{m_0}(e_1), \dots, \Phi_{m_0}(e_r)$ form a basis of the subspace

$\operatorname{im} \Phi_{m_0} \subset E_{2,m_0}$. In particular, the local sections $\Phi \circ s_1, \ldots, \Phi \circ s_r$ of E_2 are linearly independent at m_0 so that, by possibly shrinking the domain of definition of the s_i, we may assume that they form a local r-frame in E_2. Since Φ has rank r, this local r-frame spans $\operatorname{im} \Phi_m$ for all m belonging to the domain of definition. First, in view of Proposition 2.7.5, this yields the assertion for $\operatorname{im}\Phi$. Second, this implies that there exist local C^k-functions a_{ij}, $i = r+1, \ldots, l_1$, $j = 1, \ldots, r$, on M such that

$$\Phi \circ s_i = \sum_{j=1}^{r} a_{ij} \Phi \circ s_j, \quad r+1 \leq i \leq l_1.$$

Then the local sections $\tilde{s}_{r+1}, \ldots, \tilde{s}_{l_1}$ given by

$$\tilde{s}_i := s_i - \sum_{j=1}^{r} a_{ij} s_j, \quad r+1 \leq i \leq l_1,$$

form a local $(l_1 - r)$-frame in E_1 spanning $\ker \Phi_m$. Applying Proposition 2.7.5 once again, we obtain the assertion for $\ker \Phi$.

Example 2.7.8 (Annihilator) Let V be a vector space. The annihilator of a subspace $W \subset V$ is the subspace

$$W^0 := \left\{ \nu \in V^* : \nu_{\restriction W} = 0 \right\}$$

of the dual vector space V^*. Let E_2 be a $\mathbb{K}$-vector bundle over M of class C^k and dimension l_2 and let (E_1, Φ) be a vertical subbundle of dimension l_1. Then,

$$E_1^0 := \bigcup_{m \in M} \left(\Phi(E_{1,m}) \right)^0$$

is a vertical subbundle of dimension $l_2 - l_1$ of the dual vector bundle E_2^*, called the annihilator of E_1 in E_2. In view of Proposition 2.7.5/3, this follows from the obvious fact that for every local frame in E_2 whose first l_1 elements span (E_1, Φ), the last $l_2 - l_1$ elements of the corresponding dual local frame in E_2^* span E_1^0. The annihilator of a general vector subbundle (E_1, Φ) is defined to be $(E_1^0, (\Phi_{\mathrm{hor}})_{\restriction E_1^0})$, where E_1^0 is the annihilator of the vertical subbundle $(E_1, \Phi_{\mathrm{ver}})$ of $\varphi^* E_2$ and φ is the projection of Φ. It has the same base manifold as E_1.

Remark 2.7.9

1. For every vertical subbundle (E_1, Φ) of E_2 there exists a complement in E_2, that is, a vertical subbundle $(\tilde{E}_1, \tilde{\Phi})$ of E_2 such that $E_2 = E_1 \oplus \tilde{E}_1$. The proof is in two steps.
   (a) Show that for every $\mathbb{K}$-vector bundle (E, M, π) of class C^k there exists a C^k-function $\mathsf{h} : E \otimes E \to \mathbb{K}$ such that $\mathsf{h}_m := \mathsf{h}_{\restriction E_m \otimes E_m}$ is a scalar product on E_m for all $m \in M$ (Exercise 2.7.2).[9]

[9] (E, h) is called a Euclidean vector bundle if $\mathbb{K} = \mathbb{R}$ and a Hermitian vector bundle if $\mathbb{K} = \mathbb{C}$.

(b) Show that the family of h_m-orthogonal complements of the subspaces $E_{1,m} \subset E_{2,m}$ defines a vertical subbundle of E_2 (Exercise 2.7.3).

2. Let M be a compact smooth manifold. The statement of 1 provides part of the proof that for every smooth vector bundle E over M there exists a smooth vector bundle $\tilde{E}$ over M such that $E \oplus \tilde{E}$ is trivial. For the remaining part, see for example [125, Prop. 1.4].[10] This is known as the cancellation property and is an important ingredient in what is called the K-theory of M. Let us have a glimpse at the reduced version of the latter. Two smooth $\mathbb{K}$-vector bundles E and $\tilde{E}$ over M are said to be stably equivalent if $E \oplus (M \times \mathbb{K}^r)$ is isomorphic to $\tilde{E} \oplus (M \times \mathbb{K}^s)$ for some r, s. The set of stable equivalence classes is an Abelian semigroup with respect to the operation of direct sum, where the unit element is given by the class of trivial bundles. Now, the cancellation property yields that every element of this semigroup has an inverse, hence the semigroup is in fact a group, called the reduced real (for $\mathbb{K} = \mathbb{R}$) or complex (for $\mathbb{K} = \mathbb{C}$) K-group of M. Together with the operation of tensor product, it is an Abelian ring.

Next, we discuss quotient vector bundles. Let (E_2, M, π_2) be a $\mathbb{K}$-vector bundle of class C^k and let (E_1, Φ) be a vertical subbundle of E_2 of rank r with projection π_1. Since vertical subbundles are embedded, we may assume that $E_1 \subset E_2$ and Φ is the natural inclusion mapping. $E_{1,m}$ is a vector subspace of $E_{2,m}$ for all $m \in M$, and we can form the quotient spaces $E_{2,m}/E_{1,m}$. Let

$$E_2/E_1 := \bigsqcup_{m \in M} E_{2,m}/E_{1,m}$$

and let $\pi : E_2/E_1 \to M$ denote the natural projection to the index set. By construction, the fibres $\pi^{-1}(m)$ are vector spaces. According to Proposition 2.7.5, there exists a family of local trivializations $\{(U_\alpha, \chi_{2\alpha}) : \alpha \in A\}$ of E_2 and an r-dimensional subspace F_1 of the typical fibre F_2 of E_2 such that the restrictions of $\chi_{2\alpha}$ to E_1 take values in $U_\alpha \times F_1$. For any such $\chi_{2\alpha}$, we define a mapping

$$\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F_2/F_1, \qquad \chi_\alpha\big([x]\big) := \big(m, \big[\chi_{2\alpha,m}(x)\big]\big),$$

where $m = \pi_2(x)$. To check differentiability of the corresponding transition mappings, choose a complement $\tilde{F}_1$ of F_1 in F_2 and let $\lambda : F_2/F_1 \to F_2$ denote the linear mapping which assigns to each class its unique representative in $\tilde{F}_1$. Moreover, let $\mathrm{pr} : F_2 \to F_2/F_1$ be the natural projection. Since $\chi_\beta \circ \chi_\alpha^{-1}$ decomposes as

$$\chi_\beta \circ \chi_\alpha^{-1} = (\mathrm{id} \times \mathrm{pr}) \circ \big(\chi_{2\beta} \circ \chi_{2\alpha}^{-1}\big) \circ (\mathrm{id} \times \lambda),$$

it is of class C^k. Then, Remark 2.2.5 yields that the family $\{(U_\alpha, \chi_\alpha) : \alpha \in A\}$ defines a C^k-structure on E_2/E_1 such that $(E_2/E_1, M, \pi)$ is a $\mathbb{K}$-vector bundle of class C^k over M. This C^k-structure obviously does not depend on the choice of the subspace F_1.

[10]Compactness of M is necessary here, see Example 3.6 in [125].

Definition 2.7.10 (Quotient vector bundle) The vector bundle $(E_2/E_1, M, \pi)$ constructed above is called the quotient vector bundle of E_2 by E_1.

Remark 2.7.11

1. The fibrewise natural projections $E_{2,m} \to E_{2,m}/E_{1,m}$ to classes combine to a natural projection $E_2 \to E_2/E_1$. The latter is a vertical morphism, because its local representative with respect to a local trivialization $(U_\alpha, \chi_{2\alpha})$ of E_2 whose restriction to E_1 takes values in $U_\alpha \times F_1$, and the induced local trivialization of E_2/E_1 is given by the natural projection $F_2 \to F_2/F_1$. By composing a local section s of E_2 with the natural projection $E_2 \to E_2/E_1$ one obtains a local section of E_2/E_1, denoted by $[s]$.
2. Let l_i denote the dimension of E_i, $i = 1, 2$. For any local frame $\{s_1, \dots, s_{l_2}\}$ in E_2 with the property[11] that $s_1, \dots, s_{l_1}$ span E_1, $\{[s_{l_1+1}], \dots, [s_{l_2}]\}$ is a local frame in E_2/E_1.
3. According to Remark 2.7.9/1, E_1 admits a complement $\tilde{E}_1$ in E_2. For any such complement, the natural projection $E_2 \to E_2/E_1$ restricts to a vertical isomorphism $\tilde{E}_1 \to E_2/E_1$. This follows at once by observing that the induced mapping is a bijective vertical morphism. Thus, every complement defines a vector bundle isomorphism
$$E_2 \cong E_1 \oplus (E_2/E_1).$$
4. By a coorientation, or transversal orientation, of E_1 in E_2 one means an orientation of the quotient vector bundle E_2/E_1. Accordingly, E_1 is said to be coorientable, or transversally orientable, in E_2 if E_2/E_1 is orientable.

Example 2.7.12 (Homomorphism theorem) Let E_1 and E_2 be $\mathbb{K}$-vector bundles of class C^k over M and let $\Phi : E_1 \to E_2$ be a vertical morphism of constant rank. Then, the induced mapping

$$\tilde{\Phi} : E_1 / \ker \Phi \to \operatorname{im} \Phi$$

is an isomorphism. Indeed, $\tilde{\Phi}$ is obviously bijective and fibrewise linear. To see that it is of class C^k, one may choose a complement E_0 of $\ker \Phi$ in E_1 and write $\tilde{\Phi}$ as the composition of the isomorphism $E_1/\ker \Phi \to E_0$ and the restriction of $\tilde{\Phi}$ in domain to E_0. Thus, $\tilde{\Phi}$ is a bijective vertical morphism and hence an isomorphism.

Example 2.7.13 (Dual quotient vector bundle) Let E_2 be a $\mathbb{K}$-vector bundle of class C^k over M and let E_1 be a vertical subbundle. The dual vector bundle $(E_2/E_1)^*$ is called the dual quotient vector bundle. It is naturally isomorphic over M to the annihilator E_1^0. Indeed, the mapping

$$\Phi : E_1^0 \to (E_2/E_1)^*, \qquad \big(\Phi_m(\xi)\big)\big([x]\big) := \xi(x), \quad \xi \in E_{2,m}^*, \ x \in E_{2,m},$$

[11] Such local frames exist by Proposition 2.7.5.

is well-defined, bijective and fibrewise linear. Hence, it remains to show that Φ is of class C^k. To see this, choose a local frame $\{s_1, \ldots, s_{l_2}\}$, whose first l_1 elements span E_1 over U. Then, the elements s^{*i}, $i = l_1 + 1, \ldots, l_2$, of the dual local frame span E_1^0 over U and, according to Remark 2.7.11/2, the dual local frame $\{[s_{l_1+1}]^*, \ldots, [s_{l_2}]^*\}$ spans $(E_2/E_1)^*$ over U. By construction, the local representative of Φ with respect to the local trivializations defined by these local frames does not depend on m and is hence of class C^k, as asserted.

To conclude this section, we discuss vector bundle structures induced by submanifolds. Thus, let N be a C^k-manifold and let (M, φ) be a C^k-submanifold of N.

Proposition 2.7.14 $(\mathrm{T}M, \varphi')$ *is a vector subbundle of* $\mathrm{T}N$. *It is initial or embedded iff so is* (M, φ).

Proof Recall from Example 2.6.4/5 that the restriction of $\mathrm{T}N$ to the submanifold (M, φ) is defined to be the induced vector bundle $(\mathrm{T}N)_{\restriction M} := \varphi^*\mathrm{T}N$. Since φ is an immersion, the vertical morphism $(\varphi')_{\mathrm{ver}} : \mathrm{T}M \to (\mathrm{T}N)_{\restriction M}$ in the natural decomposition (2.6.2) of φ' is injective. Hence, $(\mathrm{T}M, (\varphi')_{\mathrm{ver}})$ is a vertical subbundle of $(\mathrm{T}N)_{\restriction M}$. Then, Proposition 2.7.4/3 yields that $(\mathrm{T}M, \varphi')$ is a subbundle of $\mathrm{T}N$ and that it is initial or embedded if so is (M, φ). The converse direction follows from Proposition 1.6.14 and the fact that the zero sections of the tangent bundles of M and N are embeddings. The details are left to the reader (Exercise 2.7.4). □

Remark 2.7.15 Let V be a finite-dimensional real vector space and let $M \subset V$ be an embedded C^k-submanifold. For every $v \in M$, the natural identification of $\mathrm{T}_v V$ with V of Example 1.4.3/2 identifies $\mathrm{T}_v M$ with a subspace of V, which we denote by the same symbol. In particular, in case M is open in V, one has $\mathrm{T}_v M = V$; and in case M is a level set of a C^k-mapping f, one has $\mathrm{T}_v M = \ker f'(v)$. In the general case, $\mathrm{T}_v M$ is just the tangent plane of M at v, shifted by $-v$ to the origin. Thus, together with the induced natural identification of $\mathrm{T}V$ with $V \times V$, Proposition 2.7.14 yields a natural identification of $\mathrm{T}M$ with the embedded C^{k-1}-submanifold

$$\big\{(v, X) \in M \times V : X \in \mathrm{T}_v M\big\}$$

of $M \times V$ and a natural representation of vector fields on M by C^{k-1}-mappings $X : M \to V$ satisfying $X(v) \in \mathrm{T}_v M$ for all $v \in M$. This generalizes Remarks 2.1.4/2 and 2.3.4/2.

A further consequence of the observation that $(\mathrm{T}M, (\varphi')_{\mathrm{ver}})$ is a vertical subbundle of $(\mathrm{T}N)_{\restriction M}$ is the following. A vector field X on N is said to be tangent to the submanifold (M, φ) if $X_{\varphi(m)} \in \varphi'(\mathrm{T}_m M)$ for all $m \in M$.

Proposition 2.7.16 *Let N be a manifold and let (M, φ) be a submanifold of N. For every vector field X on N which is tangent to (M, φ), there exists a unique vector field $\tilde{X}$ on M such that $\varphi' \circ \tilde{X} = X \circ \varphi$, that is, $\tilde{X}$ and X are φ-related.*

We will say that $\tilde{X}$ is induced from X and call it the restriction of X to (M, φ).

Proof Due to the assumption, the equation $\varphi' \circ \tilde{X} = X \circ \varphi$ defines a mapping $\tilde{X} : M \to \mathrm{T}M$. $\tilde{X}$ is the restriction in range to $\mathrm{T}M$ of the section of $(\mathrm{T}N)_{\restriction M} = \varphi^*\mathrm{T}N$ induced from X by φ. Since vertical subbundles are embedded, Proposition 1.6.10 yields that $\tilde{X}$ is differentiable,[12] that is, of class C^{k-1}. □

Finally, we introduce

Definition 2.7.17 (Normal and conormal bundle) Let N be a manifold and let (M, φ) be a submanifold of N.

1. The quotient vector bundle $\mathrm{N}M := (\mathrm{T}N)_{\restriction M}/\mathrm{T}M$ is called the normal bundle of (M, φ). Its fibres are called the normal spaces of M at $m \in M$. They are denoted by $\mathrm{N}_m M$.
2. The dual vector bundle $\mathrm{N}^*M := (\mathrm{N}M)^*$ is called the conormal bundle of (M, φ). Its fibres are called the conormal spaces of M at $m \in M$. They are denoted by $\mathrm{N}^*_m M$.

Remark 2.7.18

1. The normal and the conormal bundle of (M, φ) are real vector bundles over M of class C^{k-1} and dimension $\dim N - \dim M$. According to Remark 2.7.11/3, $\mathrm{N}M$ is isomorphic to an arbitrary complement of $\mathrm{T}M$ in $(\mathrm{T}N)_{\restriction M}$, and it is often realized in this way. For an example, see Exercise 2.7.6. According to Example 2.7.13, N^*M is naturally isomorphic to the annihilator $(\mathrm{T}M)^0$ of $\mathrm{T}M$ in $(\mathrm{T}N)_{\restriction M}$.
2. By a coorientation, or a transversal orientation, of (M, φ) one means an orientation of $\mathrm{N}M$. Accordingly, (M, φ) is said to be coorientable, or transversally orientable, if the normal bundle $\mathrm{N}M$ of (M, φ) is orientable. This is consistent with the terminology for vector subbundles introduced in Remark 2.7.11/4: a coorientation of (M, φ) is the same as a coorientation of $\mathrm{T}M$ in $(\mathrm{T}N)_{\restriction M}$.
3. We discuss local frames in $\mathrm{N}M$ and N^*M induced by local charts on N adapted to M. Denote $r := \dim M$ and $s := \dim N$. For simplicity, we consider the case of M being a subset of N. We leave it to the reader to write down the respective local frames for the general situation. According to Proposition 1.6.7, for every $m \in M$, there exists an open neighbourhood U of m in M and a local chart (V, ρ) on N at m such that $U \subset V$ and $(U, \rho_{\restriction U})$ is a local chart on M, taking values in the subspace $\mathbb{R}^r \times \{0\} \subset \mathbb{R}^s$. Then, $\{\partial_{i \restriction U} : i = 1, \ldots, s\}$ is a local frame in $(\mathrm{T}N)_{\restriction M}$ whose first r elements span $\mathrm{T}M$ over U. According to Remark 2.7.11/2, then $\{[\partial_{i \restriction U}] : i = r + 1, \ldots, s\}$ is a local frame in $\mathrm{N}M$. This, in turn, induces a dual local frame $\{[\partial_{i \restriction U}]^* : i = r + 1, \ldots, s\}$ in N^*M, see Example 2.4.1. According

[12]By construction, $\tilde{X}$ is also the restriction of X in domain to the submanifold (M, φ) and in range to the subbundle $(\mathrm{T}M, \varphi')$. This does not help for the argument though, because the latter need not be embedded, so that Proposition 1.6.10 does not apply here.

to Example 2.7.13, the natural isomorphism $\mathrm{N}^*M \to (\mathrm{T}M)^0$ maps the latter to the local frame in $(\mathrm{T}M)^0$ consisting of the last $s-r$ elements of the local frame $\{(\mathrm{d}\rho^i)_{\restriction U} : i = 1, \ldots, s\}$ in $((\mathrm{T}N)_{\restriction M})^* \equiv (\mathrm{T}^*N)_{\restriction M}$.

4. Assume that (M, φ) is embedded. The subset

$$C^k_M(N) = \big\{f \in C^k(N) : \varphi^* f = 0\big\} \tag{2.7.1}$$

is an ideal of the associative algebra $C^k(N)$, called the vanishing ideal of M. By means of this ideal, for $m \in M$, the subspaces $\mathrm{T}_m M$ of $\mathrm{T}_{\varphi(m)}N$ and $\mathrm{N}^*_m M \cong (\mathrm{T}_m M)^0$ of $\mathrm{T}^*_{\varphi(m)}(N)$ can be characterized as follows:

$$\varphi'(\mathrm{T}_m M) = \big\{X \in \mathrm{T}_{\varphi(m)}N : X(f) = 0 \text{ for all } f \in C^k_M(N)\big\}, \tag{2.7.2}$$

$$\mathrm{N}^*_m M = \big\{\xi \in \mathrm{T}^*_{\varphi(m)}N : \xi = \mathrm{d}f\big(\varphi(m)\big) \text{ for some } f \in C^k_M(N)\big\}. \tag{2.7.3}$$

The proof is left to the reader (Exercise 2.7.5). Beware that (2.7.2) or (2.7.3) need not hold if M is not embedded. A counterexample is provided by the figure eight submanifold $(\mathbb{R}, \gamma_\pm)$ of Example 1.6.6/2. At the crossing point, the derivative of any element of $C^\infty_M(N)$ vanishes. Hence, for the right hand side of (2.7.2) one obtains $\mathrm{T}_{\varphi(m)}N$.

Exercises

2.7.1 Prove Proposition 2.7.5 by means of Proposition 2.3.15.

2.7.2 Let (E, M, π) be a $\mathbb{K}$-vector bundle of class C^k. Use a system of local trivializations and a subordinate partition of unity of M to construct a C^k-function $\mathsf{h} : E \otimes E \to \mathbb{K}$ such that $\mathsf{h}_m := \mathsf{h}_{\restriction E_m \otimes E_m}$ is a scalar product on E_m for all $m \in M$.

2.7.3 Show that every vertical subbundle admits a complement.

2.7.4 Complete the proof of Proposition 2.7.14.

2.7.5 Prove Eqs. (2.7.2) and (2.7.3) of Remark 2.7.18, characterizing the tangent and the conormal spaces of an embedded submanifold.

2.7.6 Using the Euclidean metric, construct the normal bundle of the submanifold S^n of $\mathbb{R}^{n+1}$ as a complement of TS^n in $(\mathrm{T}\mathbb{R}^{n+1})_{\restriction \mathrm{S}^n}$. Is this bundle trivial?

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