Preface

The main goal of this book is to give a presentation of various types of coherent states introduced and studied in the physics and mathematics literature during almost a century. We describe their mathematical properties together with application to quantum physics problems. It is intended to serve as a compendium on coherent states and their applications for physicists and mathematicians, stretching from the basic mathematical structures of generalized coherent states in the sense of Gilmore and Perelomov[1] via the semiclassical evolution of coherent states to various specific examples of coherent states (hydrogen atom, torus quantization, quantum oscillator).

We have tried to show that the field of applications of coherent states is wide, diversified and still alive. Because of our own ability limitations we have not covered the whole field. Besides this would be impossible in one book. We have chosen some parts of the subject which are significant for us. Other colleagues may have different opinions.

There exist several definitions of coherent states which are not equivalent. Nowadays the most well known is the Gilmore–Perelomov [84, 85, 155] definition: a coherent state system is an orbit for an irreducible group action in an Hilbert space. From a mathematical point of view coherent states appear like a part of group representation theory.

In particular canonical coherent states are obtained with the Weyl–Heisenberg group action in $L^2(\mathbb{R})$ and the standard Gaussian $\varphi_0(x) = \pi^{1/4}\mathrm{e}^{-x^2/2}$. Modulo multiplication by a complex number, the orbit of φ_0 is described by two parameters $(q, p) \in \mathbb{R}^2$ and the L^2-normalized canonical coherent states are

$$\varphi_{q,p}(x) = \pi^{-1/4}\mathrm{e}^{-(x-q)^2/2}\mathrm{e}^{i((x-q)p+qp/2)}.$$

Wavelets are included in the group definition of coherent states: they are obtained from the action of the affine group of $\mathbb{R}$ $(x \mapsto ax + b)$ on a "mother function" $\psi \in L^2(\mathbb{R})$. The wavelet system has two parameters: $\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi(\frac{x-b}{a})$.

[1] They have discovered independently the relationship with group theory in 1972.

One of the most useful property of coherent system ψ_z is that they are an "over-complete" system in the Hilbert space in the sense that we can analyze any $\eta \in \mathcal{H}$ with its coefficient $\langle \psi_z, \eta \rangle$ and we have a reconstruction formula of η like

$$\eta = \int dz\, \tilde{\eta}(z) \psi_z,$$

where $\tilde{\eta}$ is a complex valued function depending on $\langle \psi_z, \eta \rangle$.

Coherent states (being given no name) were discovered by Schrödinger (1926) when he searched solutions of the quantum harmonic oscillator being the closest possible to the classical state or minimizing the uncertainty principle. He found that the solutions are exactly the canonical coherent states φ_z.

Glauber (1963) has extended the Schrödinger approach to quantum electrodynamic and he called these states *coherent states* because he succeeded to explain coherence phenomena in light propagation using them. After the works of Glauber, coherent states became a very popular subject of research in physics and in mathematics.

There exist several books discussing coherent states. Perelomov's book [156] played an important role in the development of the group aspect of the subject and in its applications in mathematical physics. Several other books brought contributions to the theory of coherent states and worked out their applications in several fields of physics; among them we have [3, 80, 126] but many others could be quoted as well. There is a huge number of original papers and review papers on the subject; we have quoted some of them in the bibliography. We apologize the authors for forgotten references.

In this book we put emphasis on applications of coherent states to semi-classical analysis of Schrödinger type equation (time dependent or time independent). Semi-classical analysis means that we try to understand how solutions of the Schrödinger equation behave as the Planck constant $\hbar$ is negligible and how classical mechanics is a limit of quantum mechanics. It is not surprising that semi-classical analysis and coherent states are closely related because coherent states (which are particular quantum states) will be chosen localized close to classical states. Nevertheless we think that in this book we have given more mathematical details concerning these connections than in the other monographs on that subjects.

Let us give now a quick overview of the content of the book.

The first half of the book (Chap. 1 to Chap. 5) is concerned with the canonical (standard) Gaussian Coherent States and their applications in semi-classical analysis of the time dependent and the time independent Schrödinger equation.

The basic ingredient here is the Weyl–Heisenberg algebra and its irreducible representations. The relationship between coherent states and Weyl quantization is explained in Chaps. 2 and 3. In Chap. 4 we compute the quantum time evolution of coherent states in the semi-classical régime: the result is a squeezed coherent states whose shape is deformed, depending on the classical evolution of the system. The main outcome is a proof of the Gutzwiller trace formula given in Chap. 5.

The second half of the book (Chap. 6 to Chap. 12) is concerned with extensions of coherent states systems to other geometry settings. In Chap. 6 we consider quantization of the 2-torus with application to the cat map and an example of "quantum chaos".

Chapters 7 and 8 explain the first examples of non canonical coherent states where the Weyl–Heisenberg group is replaced successively by the compact group $SU(2)$ and the non-compact group $SU(1,1)$. We shall see that some representations of $SU(1,1)$ are related with squeezed canonical coherent states, with quantum dynamics for singular potentials and with wavelets.

We show in Chap. 9 how it is possible to study the hydrogen atom with coherent states related with the group $SO(4)$.

In Chap. 10 we consider infinite systems of bosons for which it is possible to extend the definition of canonical coherent states. This is used to prove mean-field limit result for two-body interactions: the linear field equation can be approximated by a non linear Schrödinger equation in $\mathbb{R}^3$ in the semi-classical limit (large number of particles or small Planck constant are mathematically equivalent problems).

Chapters 11 and 12 are concerned with extension of coherent states for fermions with applications to supersymmetric systems.

Finally in the appendices we have a technical section A around the stationary phase theorem, and in section B we recall some basic facts concerning Lie algebras, Lie groups and their representations. We explain how this is used to build generalized coherent systems in the sense of Gilmore–Perelomov.

The material covered in these book is designed for an advanced graduate student, or researcher, who wishes to acquaint himself with applications of coherent states in mathematics or in theoretical physics. We have assumed that the reader has a good founding in linear algebra and classical analysis and some familiarity with functional analysis, group theory, linear partial differential equations and quantum mechanics.

We would like to thank our colleagues of Lyon, Nantes and elsewhere, for discussions concerning coherent states. In particular we thank our collaborator Jim Ralston with whom we have given a new proof of the trace formula, Stephan Debièvre, Alain Joye and André Martinez for stimulating meetings.

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