Preface

I don’t see any problem with the math, but this is not a dissertation in economics. We can’t give you a Ph.D. in economics for a dissertation that isn’t about economics. It’s not economics. It’s not mathematics. It’s not even business administration.

Milton Friedman, about H. Markowitz’s manuscript

Balance laws appear in many areas of application, ranging from fluid mechanics modeling, or semi-classical WKB approximations of linear quantum models, to discrete-ordinate reduction of multi-dimensional kinetic equations. These are partial differential equations describing the evolution in time of intensive (or bulk) quantities which are submitted to a physical process involving both convection and another mechanism (reaction, relaxation, or even diffusion). In many situations, such a system of equations stabilizes onto a large-time behavior which is characterized by an accurate balancing between the transport terms and the other ones. Another interesting configuration is the one in which the system contains an independent parameter which variation deeply affects the qualitative behavior of the solutions. We shall therefore speak about qualitatively correct numerical approximations when either (or both) aforementioned distinguished behaviors can be reproduced algorithmically without salient restrictions on the computational grid. Such accurate computations usually result from the use of sophisticate numerical flux functions, which display consistency not only with the convection terms, but with other parts of the equation. Perceiving simultaneously several (if not all) the terms appearing in the partial differential equation helps in preserving at the numerical level desirable qualitative properties, like dissipation of certain norms, respect of positively invariant domains, entropy inequalities or Lyapunov functionals in a robust manner. The objective of the present book is to raise the reader’s awareness of how such elaborate flux functions can be built, mainly in a one-dimensional context for hyperbolic systems admitting shock-type solutions and for kinetic equations in the discrete-ordinate approximation as well. An effort will be dedicated to rigorous mathematical derivations and to the analysis of the net gain retrieved from this approach.

In particular, one should often keep in mind that an equilibrium has to be sought between the three edges of the golden triangle\(^1\): observations, modeling and analysis, numerical simulation. While observations are imposed by our surrounding world, modeling can be instead achieved at several levels of complexity. A more intricate

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\(^1\) I learnt this nice expression from Prof. Vincent Courtillot.
model can lead to bigger difficulties in terms of mathematical analysis, even if the
development of powerful tools in the field of non-linear analysis allowed to suc-
cessfully resolve delicate problems in terms of existence, uniqueness and stability of
appropriate weak solutions (arousing some reflexions\textsuperscript{2} about what is called solving).
Impressive achievements in theoretical analysis don’t yield automatically powerful
algorithms to simulate efficiently these weak solutions on a computer: concerning
balance laws, only Tai-Ping Liu’s extension of James Glimm’s theorem was actu-
ally based on an astute numerical algorithm. One insight in that work was a seem-
ingly simple finite-difference scheme which building block contains a complete time-
asymptotic wave pattern, including both convection and source terms. Slightly later,
Gary Sod developed a similar processing for convection-diffusion systems, involving
again a solver consistent with all the terms. There is an unpleasant fact about increasing
the complexity of a physical model: even if mathematical issues can be over-
come by means of an elegant theory, usually the level of noise produced by standard
approximation algorithms increases too. Second-order accurate numerical schemes
which behave nicely on smooth classical solutions can display spurious oscillations
when asked to compute discontinuous waves emanating from models endowed with
degenerate or vanishing viscosity: the case of the Lax-Wendroff scheme is quite
revealing of this type of drawback. Shock solutions are a visual expression of the
mathematical fact that no strong dissipation has been kept at the Sobolev level: how-
ever dissipation helps when designing algorithms because it smears off part of the
numerical truncation errors. The gain in accuracy when reproducing real-life obser-
vations that one obtains by increasing the complexity of a mathematical model must
always be vastly superior to the increase of numerical noise resulting from dissipation
processes being removed. There’s little doubt that homogeneous systems of conserv-
aton laws are somewhat limited when it comes to rendering certain situations: when
thinking about large-scale gas dynamics, gravity is an external force which can hardly
be bypassed thus leading to the inclusion of source terms on the right-hand side of
both momentum and total energy equations. Such terms make the system “less dis-
sipative”, therefore more sensitive to truncation errors as new mechanisms appear
likely to amplify them. Solvers involving a whole non-interacting, time-asymptotic
wave pattern sometimes can help.

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Laurent Gosse

\textsuperscript{2} Clément Mouhot, Que signifie résoudre les équations de la physique pour un mathématicien?
Computing Qualitatively Correct Approximations of
Balance Laws
Exponential-Fit, Well-Balanced and
Asymptotic-Preserving
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