# Contents

1 Preliminaries ............................................ 1
   1.1 Operator Algebras and Hilbert Modules ....................... 1
      1.1.1 $C^*$-Algebras .................................. 1
      1.1.2 Von Neumann Algebras .......................... 4
      1.1.3 Free Product and Tensor Product ..................... 5
      1.1.4 Hilbert Modules . ................................ 6
   1.2 Quantum Groups ...................................... 8
      1.2.1 Hopf Algebras ....... 8
      1.2.2 Compact Quantum Groups: Basic Definitions and Examples ................................ 10
      1.2.3 The CQG $U_{\mu}(2)$ ................................ 17
      1.2.4 The CQG $SU_{\mu}(2)$ .................................. 18
      1.2.5 The Hopf $*$-algebras $O(SU_{\mu}(2))$ and $U_{\mu}(su(2))$ ...................... 19
      1.2.6 The CQG $SO_{\mu}(3)$ .................................. 20
   1.3 Coaction of Compact Quantum Groups on a $C^*$-Algebra ..... 21
      1.3.1 Coactions on Finite Quantum Spaces ................. 22
      1.3.2 Free and Half-Liberated Quantum Groups .............. 25
      1.3.3 The Coaction of $SO_{\mu}(3)$ on the Podles’ Spheres .... 27
   1.4 Dual of a Compact Quantum Group .......................... 29
   1.5 Coaction on von Neumann Algebras by Conjugation of Unitary Corepresentation ...................... 30
   References ............................................... 33

2 Classical and Noncommutative Geometry ............................ 37
   2.1 Classical Riemannian Geometry .......................... 37
      2.1.1 Forms and Connections ................................ 37
      2.1.2 The Hodge Laplacian of a Riemannian Manifold .......... 38
      2.1.3 Spin Groups and Spin Manifolds ..................... 39
2.1.4 Dirac Operators .................................. 40
2.1.5 Isometry Groups of Classical Manifolds .......... 41
2.2 Noncommutative Geometry .......................... 50
  2.2.1 Spectral Triples: Definition and Examples ...... 50
  2.2.2 The Noncommutative Space of Forms .......... 53
  2.2.3 Laplacian in Noncommutative Geometry ....... 56
2.3 Quantum Group Equivariance in Noncommutative
  Geometry ........................................ 58
  2.3.1 The Example of $SU(2)$ ......................... 58
  2.3.2 The Example of the Podles’ Spheres .......... 58
  2.3.3 Constructions from Coactions by Quantum
       Isometries .................................... 60
  2.3.4 $R$-twisted Volume Form Coming
       from the Modularity of a Quantum Group ....... 63
References ............................................. 66

3 Definition and Existence of Quantum Isometry Groups .. 69
  3.1 The Approach Based on Laplacian ................. 69
    3.1.1 The Definition and Existence of the Quantum
        Isometry Group ............................. 70
    3.1.2 Discussions on the Admissibility Conditions .. 75
  3.2 Definition and Existence of the Quantum Group
    of Orientation Preserving Isometries ............... 77
    3.2.1 Motivation .................................. 77
    3.2.2 Quantum Group of Orientation-Preserving
        Isometries of an $R$-twisted Spectral Triple .... 78
    3.2.3 Stability and $C^*$ Coaction ................ 83
    3.2.4 Comparison with the Approach Based
        on Laplacian .................................. 86
  3.3 The Case of $J$ Preserving Quantum Isometries ..... 90
  3.4 A Sufficient Condition for Existence of Quantum Isometry
    Groups Without Fixing the Volume Form .............. 92
References ............................................. 95

4 Quantum Isometry Groups of Classical and Quantum Spheres .. 97
  4.1 Classical Spheres: No Quantum Isometries .......... 97
  4.2 Quantum Isometry Group of a Spectral Triple
    on Podles’ Sphere ................................ 99
  4.3 Descriptions of the Podles’ Spheres ............... 100
    4.3.1 The Description as in [3] ..................... 100
    4.3.2 ‘Volume Form’ on the Podles’ Spheres ....... 101
  4.4 Computation of the Quantum Isometry Groups ....... 102
    4.4.1 Affineness of the Coaction .................. 104
    4.4.2 Homomorphism Conditions .................... 108
4.4.3 Relations Coming from the Antipode .............. 109
4.4.4 Identification of $SO_3(3)$ as the Quantum Isometry
Group ............................................. 111
4.5 Another Spectral Triple on the Podles’ Sphere:
A Counterexample ................................... 114
4.5.1 The Spectral Triple ............................ 115
4.5.2 Computation of the Quantum Isometry Group .... 117
 References ........................................... 127

5 Quantum Isometry Groups of Discrete Quantum Spaces .... 129
5.1 Quantum Isometry Groups of Finite Metric Spaces
and Finite Graphs .................................... 130
5.1.2 Noncommutative Geometry on Finite Metric
Spaces ............................................. 131
5.1.3 Quantum Symmetry Groups of Banica and Bichon
as Quantum Isometry Groups .................... 133
5.2 Quantum Isometry Groups for Inductive Limits ......... 136
5.2.1 Examples Coming from AF Algebras .......... 138
5.2.2 The Example of the Middle-Third Cantor Set .... 143
 References .......................................... 146

6 Nonexistence of Genuine Smooth CQG Coactions
on Classical Connected Manifolds .......................... 149
6.1 Smooth Coaction of a Compact Quantum Group
and the No-Go Conjecture ............................ 149
6.1.1 Definition of Smooth Coaction .................. 149
6.1.2 Statement of the Conjecture and Some Positive
Evidence ........................................... 150
6.1.3 Defining the ‘Differential’ of the Coaction ...... 153
6.2 Brief Sketch of Proof of Nonexistence of Genuine
Quantum Isometries .................................. 155
6.3 An Example of No-Go Result Without Quadratic
Independence ....................................... 158
 References ......................................... 161

7 Deformation of Spectral Triples and Their Quantum
Isometry Groups ........................................ 163
7.1 Cocycle Twisting ................................... 163
7.1.1 Cocycle Twist of a Compact Quantum Group .... 164
7.1.2 Unitary Corepresentations of a Twisted Compact
Quantum Group ................................... 166
7.1.3 Deformation of a von Neumann Algebra by Dual
Unitary 2-Cocycles ................................. 167
7.2 Deformation of Spectral Triples by Unitary Dual Cocycles 168
7.3 Quantum Isometry Groups of Deformed Spectral Triples 169
7.4 Examples and Computations 172
References 177
8 Spectral Triples and Quantum Isometry Groups on Group $C^*$-Algebras 179
8.1 Connes’ Spectral Triple on Group $C^*$-Algebras and Their Quantum Isometry Groups 180
8.1.1 Quantum Isometry Groups of $(\mathbb{C}[\Gamma], l^2(\Gamma), D_F)$ 181
8.2 The Case of Finitely Generated Abelian Groups 183
8.2.1 Computation for the Groups $\mathbb{Z}_n$ and $\mathbb{Z}$ 183
8.2.2 Results for the General Case 188
8.3 The Case of Free Products of Groups 190
8.3.1 Some Quantum Groups 190
8.3.2 Results for the Free Groups $\mathbb{F}_n$ 192
8.3.3 Quantum Isometry Groups of Free Product of Finite Cyclic Groups 192
8.4 Quantum Isometry Groups as Doublings 193
8.4.1 Result for a Generating Set of Transpositions 194
8.4.2 The Case When $S$ Has a Cycle 197
References 197
9 An Example of Physical Interest 199
9.1 Notations and Preliminaries 201
9.1.1 Generalities on Real $C^*$-Algebras 201
9.1.2 Quantum Isometries 202
9.2 The Finite Noncommutative Space $F$ 203
9.2.1 The Elementary Particles and the Hilbert Space of Fermions 203
9.2.2 The Spectral Triple 204
9.2.3 A Hypothesis on the $\mathcal{Y}$ Matrices 205
9.3 Quantum Isometries of $F$ 206
9.3.1 $\text{QISO}_F^+$ in two special cases 209
9.3.2 Quantum Isometries for the Real $C^*$ Algebra $A_F$ 210
9.4 Quantum Isometries of $M \times F$ 211
9.5 Physical Significance of the Results 211
9.5.1 Analysis of the Result for the Minimal Standard Model 213
9.6 Invariance of the Spectral Action 214
References 217