Chapter 2
Classical and Noncommutative Geometry

Abstract We discuss classical Riemannian geometry and its noncommutative geometric counterparts. At first the definition and properties of the Hodge Laplacian and the Dirac operator are given. We also derive the characterizations of isometries (resp. orientation preserving isometries) in terms of the Laplacian (resp. Dirac operator). This is followed by discussion on noncommutative manifolds given by spectral triples, including the definitions of noncommutative space of forms and the Laplacian in this set up. The last section of this chapter deals with the quantum group equivariance in noncommutative geometry where we discuss some natural examples of equivariant spectral triples on the Podles' spheres.

2.1 Classical Riemannian Geometry

In this section, we recall some classical facts regarding classical differential geometry manifolds that will be useful for us.

2.1.1 Forms and Connections

Let M be an n-dimensional compact Riemannian manifold. Let $\chi(M)$ denote the $C^\infty(M)$-module of smooth vector fields on the manifold M. A linear or affine connection ∇ on M is given by an assignment $\chi(M) \ni X \mapsto \nabla_X$, where ∇_X is an $\mathbb{R}$-linear map from $\chi(M)$ to $\chi(M)$ such that $\chi(M) \ni X \mapsto \nabla_X$ is $C^\infty(M)$-linear and $\nabla_X(fY) = f\nabla_X(Y) + X(f)Y$, for all $Y \in \chi(M)$, $f \in C^\infty(M)$. Given a local chart in M and coordinates x_i, the Christoffel symbols of the connection ∇ are the functions Γ^k_{ij} defined by: $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x_k}$. A linear connection is called symmetric or torsionless if $\nabla_X(Y) - \nabla_Y(X) = [X, Y]$ for all $X, Y \in \chi(M)$. It is said to be compatible with the Riemannian metric if $\langle \nabla_X(Y), Z\rangle + \langle Y, \nabla_X(Z)\rangle = X\langle Y, Z\rangle$ for all $X, Y, Z \in \chi(M)$, where $\langle \cdot, \cdot \rangle$ denotes the Riemannian inner product on the tangent bundle. There is a unique linear connection on M [1], which is torsionless and compatible with the metric, called the Levi-Civita connection on M.

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Let $\Omega^k(M)$ $(k = 0, 1, 2, ...n)$ be the space of smooth k-forms. Set $\Omega^k(M) = \{0\}$ for $k > n$. The de-Rham differential d maps $\Omega^k(M)$ to $\Omega^{k+1}(M)$. Let $\Omega \equiv \Omega(M) = \oplus_k \Omega^k(M)$. We will denote the Riemannian volume element by dvol. We recall that the Hilbert space $L^2(M)$ is obtained by completing the space of compactly supported smooth functions on M with respect to the pre-inner product given by $\langle f_1, f_2 \rangle = \int_M \overline{f_1} f_2 dvol$.

In an analogous way, one can construct a canonical Hilbert space of forms. The Riemannian metric $\langle . , . \rangle_m$ (for m in M) on $T_m M$ induces an inner product on the vector space $T_m^* M$ and hence also $\Lambda^k T_m^* M$, which will be again denoted by $\langle . , . \rangle_m$. This gives a natural pre-inner product on the space of compactly supported k-forms by integrating the compactly supported smooth function $m \mapsto \langle \omega(m), \eta(m) \rangle_m$ over M. We will denote the completion of this space by $\mathcal{H}^k(M)$. Let $\mathcal{H} = \oplus_k \mathcal{H}^k(M)$.

Then, one can view $d : \Omega \to \Omega$ as an unbounded, densely defined operator (again denoted by d) on the Hilbert space $\mathcal{H}$ with the domain Ω. It can be verified that it is closable.

2.1.2 The Hodge Laplacian of a Riemannian Manifold

We recall that the Laplacian $\mathcal{L}$ on M is an unbounded densely defined self-adjoint operator $-d^* d$ on the space of zero forms $\mathcal{H}^0(D) = L^2(M, \text{dvol})$ which has the local expression

$$\mathcal{L}(f) = \frac{1}{\sqrt{det(g)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} (g^{ij} \sqrt{\det(g)} \frac{\partial}{\partial x_i} f)$$

for f in $C^\infty(M)$ and where $g = ((g_{ij}))$ is the Riemannian metric and $g^{-1} = ((g^{ij}))$.

We begin with a well-known characterization of the isometry group of a (classical) compact Riemannian manifold. Let (M, g) be a compact Riemannian manifold and let $\Omega^1 = \Omega^1(M)$ be the space of smooth one forms, which has a right Hilbert-$C^\infty(M)$-module structure given by the $C^\infty(M)$-valued inner product $<< \cdot, \cdot >>$ defined by

$$\langle \omega, \eta \rangle (m) = \langle \omega(m), \eta(m) \rangle |_m,$$

where $< \cdot, \cdot > |_m$ is the Riemannian metric on the cotangent space $T_m^* M$ at the point $m \in M$. The Riemannian volume form allows us to make Ω^1 a pre-Hilbert space, and we denote its completion by $\mathcal{H}_1$. Let $\mathcal{H}_0 = L^2(M, \text{dvol})$ and consider the de-Rham differential d as an unbounded linear map from $\mathcal{H}_0$ to $\mathcal{H}_1$, with the natural domain $C^\infty(M) \subset \mathcal{H}_0$, and also denote its closure by d. Let $\mathcal{L} := -d^* d$. The following identity can be verified by direct and easy computation using the local coordinates:

$$(\partial \mathcal{L})(\phi, \psi) \equiv \mathcal{L}(\bar{\phi}\psi) - \mathcal{L}(\bar{\phi})\psi - \bar{\phi}\mathcal{L}(\psi) = 2 << d\phi, d\psi >> \quad \text{for } \phi, \psi \in C^\infty(M). \tag{2.1.1}$$

Let us recall a few well-known facts about the Laplacian $\mathcal{L}$, viewed as a negative self-adjoint operator on the Hilbert space $L^2(M, \text{dvol})$. It is known (see [2] and references therein) that $\mathcal{L}$ has compact resolvents and all its eigenvectors belong to $C^{\infty}(M)$. Moreover, it follows from the Sobolev Embedding Theorem that

$$\bigcap_{n\geq 1} \mathrm{Dom}(\mathcal{L}^n) = C^{\infty}(M).$$

Let $\{e_{ij}, j = 1, ..., d_i; i = 0, 1, 2, ...\}$ be the set of (normalized) eigenvectors of $\mathcal{L}$, where $e_{ij} \in C^{\infty}(M)$ is an eigenvector corresponding to the eigenvalue λ_i, $0 = |\lambda_0| < |\lambda_1| < |\lambda_2| < ....$ We have the following:

Lemma 2.1.1 *The complex linear span of* $\{e_{ij}\}$ *is norm-dense in* $C(M)$.

Proof This is a consequence of the asymptotic estimates of eigenvalues λ_i, as well as the uniform bound of the eigenfunctions e_{ij}. For example, it is known ([3], Theorem 1.2) that there exist constants C, C' such that $\|e_{ij}\|_{\infty} \leq C|\lambda_i|^{\frac{n-1}{4}}$, $d_i \leq C'|\lambda_i|^{\frac{n-1}{2}}$, where n is the dimension of the manifold M. Now, for $f \in C^{\infty}(M) \subseteq \bigcap_{k\geq 1} \mathrm{Dom}(\mathcal{L}^k)$, we write f as an a priori L^2-convergent series $\sum_{ij} f_{ij}e_{ij}$ ($f_{ij} \in \mathbb{C}$), and observe that $\sum |f_{ij}|^2|\lambda_i|^{2k} < \infty$ for every $k \geq 1$. Choose and fix sufficiently large k such that $\sum_{i\geq 0} |\lambda_i|^{n-1-2k} < \infty$, which is possible due to the well-known Weyl asymptotics of eigenvalues of $\mathcal{L}$. Now, by the Cauchy–Schwarz inequality and the estimate for d_i, we have

$$\sum_{ij} |f_{ij}|\|e_{ij}\|_{\infty} \leq C(C')^{\frac{1}{2}} \left(\sum_{ij} |f_{ij}|^2|\lambda_i|^{2k}\right)^{\frac{1}{2}} \left(\sum_{i\geq 0} |\lambda_i|^{n-1-2k}\right)^{\frac{1}{2}} < \infty.$$

Thus, the series $\sum_{ij} f_{ij}e_{ij}$ converges to f in sup-norm, so $\mathrm{Sp}\{e_{ij}, j = 1, 2, ..., d_i; i = 0, 1, 2, ...\}$ is dense in sup-norm in $C^{\infty}(M)$, hence in $C(M)$ as well. □

2.1.3 Spin Groups and Spin Manifolds

We begin with the Clifford algebras. Let Q be a quadratic form on an n-dimensional vector space V. Then $Cl(V, Q)$ will denote the universal associative algebra $\mathcal{C}$ equipped with a linear map $i : V \to \mathcal{C}$, such that $i(V)$ generates $\mathcal{C}$ as a unital algebra satisfying $i(V)^2 = Q(V).1$

Let $\beta : V \to Cl(V, Q)$ be defined by $\beta(x) = -i(x)$. Then, $Cl(V, Q) = Cl^0(V, Q) \oplus Cl^1(V, Q)$ where $Cl^0(V, Q) = \{x \in Cl(V, Q) : \beta(x) = x\}$, $Cl^1(V, Q) = \{x \in Cl(V, Q) : \beta(x) = -x\}$.

We will denote by $\mathcal{C}_n$ and $\mathcal{C}_n^{\mathbb{C}}$ the Clifford algebras $Cl(\mathbb{R}^n, -x_1^2 - ... - x_n^2)$ and $Cl(\mathbb{C}^n, z_1^2 + ... + z_n^2)$, respectively.

We will denote the vector space $\mathbb{C}^{2^{[\frac{n}{2}]}}$ by the symbol Δ_n. It follows that $\mathcal{C}_n^{\mathbb{C}} = \mathrm{End}(\Delta_n)$ if n is even and equals $\mathrm{End}(\Delta_n) \oplus \mathrm{End}(\Delta_n)$ is n is odd. There is a representation $\mathcal{C}_n^{\mathbb{C}} \to \mathrm{End}(\Delta_n)$ that is the isomorphism with $\mathrm{End}(\Delta_n)$ when n is even and in the odd case, it is the isomorphism with $\mathrm{End}(\Delta_n) \oplus \mathrm{End}(\Delta_n)$ followed by the projection onto the first component. This representation restricts to $\mathcal{C}_n$, to be denoted by κ_n and called the spin representation. This representation is irreducible when n is odd and for even n, it decomposes into two irreducible representations, which decomposes Δ_n into a direct sum of two vector spaces Δ_n^+ and Δ_n^-.

$\mathrm{Pin}(n)$ is defined to be the subgroup of $\mathcal{C}_n$ generated by elements of the form $\{x : \|x\| = 1, x \in \mathbb{R}^n\}$. Spin (n) is the group given by $\mathrm{Pin}(n) \cap \mathcal{C}_n^0$. There exists a continuous group homomorphism from Pin(n) to $O(n)$, which restricts to a 2-covering map $\lambda : \mathrm{Spin(n)} \to SO(n)$.

Let M be an n-dimensional orientable Riemannian manifold. Then we have the oriented orthonormal bundle of frames over M (which is a principal $SO(n)$ bundle) which we will denote by F.

Such a manifold M is said to be a **spin manifold** if there exists a pair (P, Λ) (called a spin structure) where

(1) P is a Spin(n) principal bundle over M.

(2) Λ is a map from P to F such that it is a 2-covering as well as a bundle map over M.

(3) $\Lambda(p.\widehat{g}) = \Lambda(p).g$ where $\lambda(\widehat{g}) = g,\ \widehat{g} \in \mathrm{Spin}(n)$.

Given such a spin structure, we consider the associated bundle $S = P \times_{\mathrm{Spin}(n)} \Delta_n$ called the **'bundle of spinors'**.

2.1.4 *Dirac Operators*

We follow the notations of the previous subsection. On the space of smooth sections of the bundle of spinors S on a compact Riemannian spin manifold M, one can define an inner product by

$$\langle s_1, s_2 \rangle_S = \int_M \langle s_1(x), s_2(x) \rangle \, dvol(x).$$

The Hilbert space obtained by completing the space of smooth sections with respect to this inner product is denoted by $L^2(S)$ and its members are called the square integrable spinors. The Levi-Civita connection on M induces a canonical connection on S which we will denote by ∇^S.

Definition 2.1.2 The **Dirac operator** on M is the self-adjoint extension of the following operator D defined on the space of smooth sections of S:

$$(Ds)(m) = \sum_{i=1}^{n} \kappa_n(X_i(m))(\nabla^S_{X_i} s)(m),$$

where $(X_1, ...X_n)$ are local orthonormal (with respect to the Riemannian metric) vector fields defined in a neighborhood of m. In this definition, we have viewed

$X_i(m)$ belonging to $T_m(M)$ as an element of the Clifford algebra $Cl_{\mathbb{C}}(T_m M)$, hence $\kappa_n(X_i(m))$ is a map on the fiber of S at m, which is isomorphic with Δ_n. The self-adjoint extension of D is again denoted by the same symbol.

We recall three important facts about the Dirac operator:

Proposition 2.1.3 *(1) $C^\infty(M)$ acts on S by multiplication and this action extends to a representation, say π, of the C^* algebra $C(M)$ on the Hilbert space $L^2(S)$.*

(2) For f in $C^\infty(M)$, $[D, \pi(f)]$ has a bounded extension.

(3) Furthermore, the Dirac operator on a compact manifold has compact resolvents.

As the action of an element f in $C^\infty(M)$ on $L^2(S)$ is by multiplication operator, we will use the symbol M_f in place of $\pi(f)$.

The Dirac operator carries a lot of geometric and topological information. We give two examples.

(a) The Riemannian metric of the manifold is recovered by

$$d(p,q) = \sup_{\phi \in C^\infty(M),\ \|[D, M_\phi]\| \leq 1} |\phi(P) - \phi(q)|\,. \tag{2.1.2}$$

(b) For a compact manifold, the operator e^{-tD^2} is trace class for all $t > 0$. Then the volume form of the manifold can be recovered by the formula

$$\int_M f dvol = c(n) \lim_{t \to 0} \frac{\mathrm{Tr}(M_f e^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$$

where $\dim M = n$, $c(n)$ *is a constant depending on the dimension.*

2.1.5 Isometry Groups of Classical Manifolds

Let M be a Riemannian manifold of dimension n. Then the collection of all isometries of M has a natural group structure and is denoted by ISO(M). The aim of this subsection is to prepare the necessary background for defining the notion of "quantum isometry"of a noncommutative manifold. Therefore, for a classical Riemannian (resp, spin) manifold, we give characterizations of an isometry (resp, orientation preserving isometry) in terms of the Hodge Laplacian (resp, Dirac operator). moreover, motivated by the work of Woronowicz and Soltan on "quantum families", we give characterizations of classical families of isometries (resp, orientation preserving isometries). We should mention that Proposition 2.1.4 and Theorem 2.1.12 are well known [4, 5], but for the sake of completeness, we give detailed proofs.

The topology on ISO(M) is defined in the following way. Let C and U be, respectively, a compact and open subset of M and let $W(C, U) = \{h \in \mathrm{ISO}(M) : h.C \subseteq U\}$. The compact open topology on ISO(M) is the smallest topology on ISO(M) for which the sets $W(C, U)$ are open. It follows (see [4]) that under this topology,

$\mathrm{ISO}(M)$ is a closed locally compact topological group. Moreover, if M is compact, $\mathrm{ISO}(M)$ is also compact.

Characterization of $\mathrm{ISO}(M)$ **for a Riemannian manifold**

We start with the characterization of a single isometry.

Proposition 2.1.4 *A smooth map $\gamma : M \to M$ is a Riemannian isometry if and only if γ commutes with $\mathcal{L}$ in the sense that $\mathcal{L}(f \circ \gamma) = (\mathcal{L}(f)) \circ \gamma$ for all $f \in C^\infty(M)$.*

Proof If γ commutes with $\mathcal{L}$ then from the identity (2.1.1), we get for $m \in M$ and $\phi, \psi \in C^\infty(M)$:

$$
\begin{aligned}
&< d\phi|_{\gamma(m)}, d\psi|_{\gamma(m)} > |_{\gamma(m)} \\
&=<< d\phi, d\psi >> (\gamma(m)) \\
&= \frac{1}{2}(\partial\mathcal{L}(\phi, \psi) \circ \gamma)(m) \\
&= \frac{1}{2}\partial\mathcal{L}(\phi \circ \gamma, \psi \circ \gamma)(m) \\
&=<< d(\phi \circ \gamma), d(\psi \circ \gamma) >> (m) \\
&=< d(\phi \circ \gamma)|_m, d(\psi \circ \gamma)|_m > |_m \\
&=< (d\gamma|_m)^*(d\phi|_{\gamma(m)}), (d\gamma|_m)^*(d\psi|_{\gamma(m)}) > |_m,
\end{aligned}
$$

which proves that $(d\gamma|_m)^* : T^*_{\gamma(m)}M \to T^*_m M$ is an isometry. Thus, γ is a Riemannian isometry.

Conversely, if γ is an isometry, both the maps induced by γ on $\mathcal{H}_0$ and $\mathcal{H}_1$, i.e., $U^0_\gamma : \mathcal{H}_0 \to \mathcal{H}_0$ given by $U^0_\gamma(f) = f \circ \gamma$ and $U^1_\gamma : \mathcal{H}^1 \to \mathcal{H}^1$ given by $U^1_\gamma(f d\phi) = (f \circ \gamma)d(\phi \circ \gamma)$ are unitaries. Moreover, $d \circ U^0_\gamma = U^1_\gamma \circ d$ on $C^\infty(M) \subset \mathcal{H}_0$. From this, it follows that $\mathcal{L} = -d^*d$ commutes with U^0_γ. □

Next, we move on to the characterization of a family of isometries, which will need the following lemma.

Lemma 2.1.5 *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and for $i = 1, 2$, let $\mathcal{L}_i$ be (possibly unbounded) self-adjoint operator on $\mathcal{H}_i$ with compact resolvents, and let $\mathcal{V}_i$ be the linear span of eigenvectors of $\mathcal{L}_i$. Moreover, assume that there is an eigenvalue of $\mathcal{L}_i$ for which the eigenspace is one-dimensional, say spanned by a unit vector ξ_i. Let Ψ be a linear map from $\mathcal{V}_1$ to $\mathcal{V}_2$ such that $\mathcal{L}_2\Psi = \Psi\mathcal{L}_1$ and $\Psi(\xi_1) = \xi_2$. Then we have*

$$\langle \xi_2, \Psi(x)\rangle = \langle \xi_1, x\rangle \ \ \forall x \in \mathcal{V}_1. \tag{2.1.3}$$

Proof By hypothesis on Ψ, it is clear that there is a common eigenvalue, say λ_0, of $\mathcal{L}_1$ and $\mathcal{L}_2$, with the eigenvectors ξ_1 and ξ_2, respectively. Let us write the set of eigenvalues of $\mathcal{L}_i$ as a disjoint union $\{\lambda_0\} \bigcup \Lambda_i$ $(i = 1, 2)$, and let the corresponding orthogonal decomposition of $\mathcal{V}_i$ be given by $\mathcal{V}_i = \mathbb{C}\xi_i \bigoplus_{\lambda \in \Lambda_i} \mathcal{V}_i^\lambda \equiv \mathbb{C}\xi_i \oplus \mathcal{V}_i'$, say, where $\mathcal{V}_i^\lambda$ denotes the eigenspace of $\mathcal{L}_i$ corresponding to the eigenvalue λ. By assumption, Ψ maps $\mathcal{V}_1^\lambda$ to $\mathcal{V}_2^\lambda$ whenever λ is an eigenvalue of $\mathcal{L}_2$, i.e., $\mathcal{V}_2^\lambda \neq \{0\}$, and otherwise it maps $\mathcal{V}_1^\lambda$ into $\{0\}$. Thus, $\Psi(\mathcal{V}_1') \subseteq \mathcal{V}_2'$. Now, (2.1.3) is obviously satisfied for $x = \xi_1$, so it is enough to prove (2.1.3) for all $x \in \mathcal{V}_1'$. But we have $\langle \xi, x \rangle = 0$ for $x \in \mathcal{V}_1'$, and since $\Psi(x) \in \mathcal{V}_2' = \mathcal{V}_2 \bigcap \{\xi_2\}^\perp$, it follows that $\langle \xi_2, \Psi(x) \rangle = 0 = \langle \xi_1, x \rangle$. □

Now let us consider a compact metrizable (i.e., second countable) space Y with a continuous map $\theta : M \times Y \to M$. We abbreviate $\theta(m, y)$ as my and denote by ξ_y the map $M \ni m \mapsto my$. Let $\alpha : C(M) \to C(M) \otimes C(Y) \cong C(M \times Y)$ be the map given by $\alpha(f)(m, y) := f(my)$ for $y \in Y$, $m \in M$ and $f \in C(M)$. For a state ϕ on $C(Y)$, denote by α_ϕ the map $(\mathrm{id} \otimes \phi) \circ \alpha : C(M) \to C(M)$. We shall also denote by $\mathcal{C}$ the subspace of $C(M) \otimes C(Y)$ generated by elements of the form $\alpha(f)(1 \otimes \psi)$, $f \in C(M)$, $\psi \in C(Y)$. Since $C(M)$ and $C(Y)$ are commutative algebras, it is easy to see that $\mathcal{C}$ is a $*$-subalgebra of $C(M) \otimes C(Y)$. Then we have the following

Theorem 2.1.6 *(i) $\mathcal{C}$ is norm-dense in $C(M) \otimes C(Y)$ if and only if for every $y \in Y$, ξ_y is one-to-one.*
(ii) The map ξ_y is C^∞ for every $y \in Y$ if and only if $\alpha_\phi(C^\infty(M)) \subseteq C^\infty(M)$ for all ϕ.
(iii) Under the hypothesis of (ii), each ξ_y is also an isometry if and only if α_ϕ commutes with $(\mathcal{L} - \lambda)^{-1}$ for all state ϕ and all λ in the resolvent of $\mathcal{L}$ (equivalently, α_ϕ commutes with the Laplacian $\mathcal{L}$ on $C^\infty(M)$).

Proof (i) First, assume that ξ_y is one-to-one for all y. By Stone-Weierstrass Theorem, it is enough to show that $\mathcal{C}$ separates points. Take $(m_1, y_1) \neq (m_2, y_2)$ in $M \times Y$. If $y_1 \neq y_2$, we can choose $\psi \in C(Y)$ that separates y_1 and y_2, hence $(1 \otimes \psi) \in \mathcal{C}$ separates (m_1, y_1) and (m_2, y_2). So, we can consider the case when $y_1 = y_2 = y$ (say), but $m_1 \neq m_2$. By injectivity of ξ_y, we have $m_1 y \neq m_2 y$, so there exists $f \in C(M)$ such that $f(m_1 y) \neq f(m_2 y)$, i.e., $\alpha(f)(m_1, y) \neq \alpha(f)(m_2, y)$. This proves the density of $\mathcal{C}$.

For the converse, we argue as in the proof of Proposition 3.3 of [6]. Assume that $\mathcal{C}$ is dense in $C(M) \otimes C(Y)$, and let $y \in Y$, $m_1, m_2 \in M$ such that $m_1 y = m_2 y$. That is, $\alpha(f)(1 \otimes \psi)(m_1, y) = \alpha(f)(1 \otimes \psi)(m_2, y)$ for all $f \in C(M)$, $\psi \in C(Y)$. By the density of $\mathcal{C}$, we get $\chi(m_1, y) = \chi(m_2, y)$ for all $\chi \in C(M \times Y)$, so $(m_1, y) = (m_2, y)$, i.e., $m_1 = m_2$.

(ii) The 'if part' of (ii) follows by considering the states corresponding to point evaluation, i.e., $C(Y) \ni \psi \mapsto \psi(y)$, $y \in Y$. For the converse, we note that an arbitrary state ϕ corresponds to a regular Borel measure μ on Y so that $\phi(h) = \int h d\mu$, and thus, $\alpha_\phi(f)(m) = \int f(my) d\mu(y)$ for $f \in C(M)$. From this, by interchanging differentiation and integration (which is allowed by the Dominated Convergence Theorem, since μ is a finite measure), we can prove that $\alpha_\phi(f)$ is C^∞ whenever f is so.

The assertion (iii) follows from Proposition 2.1.4 in a straightforward way. □

Lemma 2.1.7 *Let Y and α be as in Theorem 2.1.6 and let $\mathcal{A}_0^\infty$ denote the complex linear span of the eigenvectors of $\mathcal{L}$, where $\mathcal{A}^\infty = C^\infty(M)$. Then the following are equivalent.*
(a) For every $y \in Y$, ξ_y is smooth isometric.
(b) For every state ϕ on $C(Y)$, we have $\alpha_\phi(\mathcal{A}_0^\infty) \subseteq \mathcal{A}_0^\infty$, and $\alpha_\phi \mathcal{L} = \mathcal{L}\alpha_\phi$ on $\mathcal{A}_0^\infty$.

Proof We prove only the nontrivial implication $(b) \Rightarrow (a)$. Assume that α_ϕ leaves $\mathcal{A}_0^\infty$ invariant and commutes with $\mathcal{L}$ on it, for every state ϕ. To prove that α is smooth and isometric, it is enough (see the proof of Theorem 2.1.6) to prove that $\alpha_y(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$ for all $y \in Y$, where $\alpha_y(f) := (\mathrm{id} \otimes \mathrm{ev}_y)(f) = f \circ \xi_y$, ev_y being the evaluation at the point y. Let $M_1, ..., M_k$ be the connected components of the compact manifold M. Thus, the Hilbert space $L^2(M, \mathrm{dvol})$ admits an orthogonal decomposition $\oplus_{i=1}^k L^2(M_i, \mathrm{dvol})$, and the Laplacian $\mathcal{L}$ is of the form $\oplus_i \mathcal{L}_i$, where $\mathcal{L}_i$ denotes the Laplacian on M_i. Since each M_i is connected, we have $\mathrm{Ker}(\mathcal{L}_i) = \mathbb{C}\chi_i$, where χ_i is the constant function on M_i equal to 1. Now, we note that for fixed y and i, the image of M_i under the continuous function ξ_y must be mapped into a component, say M_j. Thus, by applying Lemma 2.1.5 with $\mathcal{H}_1 = L^2(M_i), \mathcal{H}_2 = L^2(M_j)$, $\Psi = \xi_y$ and the L^2-continuity of the map $f \mapsto \alpha_y(f) = f \circ \xi_y$, we have

$$\int_{M_j} \alpha_y(f)(x)\mathrm{dvol}(x) = \int_{M_i} f(x)\mathrm{dvol}(x)$$

for all f in the linear span of eigenvectors of $\mathcal{L}_i$, hence (by density) for all f in $L^2(M_i)$. It follows that $\int_M \alpha_y(f)\mathrm{dvol} = \int_M f\mathrm{dvol}$ for all $f \in L^2(M)$, in particular for all $f \in C(M)$. Since α_y is a $*$-homomorphism on $C(M)$, we have

$$\langle \alpha_y(f), \alpha_y(g) \rangle = \int_M \alpha_y(\overline{f}g)\mathrm{dvol} = \int_M \overline{f}g\mathrm{dvol} = \langle f, g \rangle,$$

for all $f, g \in C(M)$. Thus, α_y extends to an isometry on $L^2(M)$, to be denoted by the same notation, which by our assumption commutes with the self-adjoint operator $\mathcal{L}$ on the core $\mathcal{A}_0^\infty$, and hence α_y commutes with $\mathcal{L}^n$ for all n. In particular, it leaves invariant the domains of each $\mathcal{L}^n$, which implies $\alpha_y(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$. □

Consider the category with objects being the pairs (G, α), where G is a compact metrizable group acting on M by the smooth and isometric action α. If (G_1, α) and (G_2, β) are two objects in this category, $\mathrm{Mor}((G_1, \alpha), (G_2, \beta))$ consists of group homomorphisms π from G_1 to G_2 such that $\beta \circ \pi = \alpha$. Then the isometry group of M is the universal object in this category.

More generally, the isometry group of a classical compact Riemannian manifold, viewed as a compact metrizable space (forgetting the group structure), can be seen to be the universal object of a category whose object class consists of subsets (not generally subgroups) of the set of smooth isometries of the manifold. Then it can be proved that this universal compact set has a canonical group structure. Thus, motivated by the ideas of Woronowicz and Soltan [7, 8], one can consider a bigger category with objects as the pair (S, f) where S is a compact metrizable space and

$f : S \times M \to M$ such that the map from M to itself defined by $m \mapsto f(s, m)$ is a smooth isometry for all s in S. The morphism set is defined as above (replacing group homomorphisms by continuous set maps). Thus, summarizing the above discussion and recalling that the span of eigenvectors of the Laplacian is norm-dense in $C(M)$, we have the following result.

Theorem 2.1.8 *Let M be a smooth Riemannian compact manifold and $(C^\infty(M))_0$ denote the span of eigenvectors of the Laplacian. Then* $\mathrm{ISO}(M)$ *is the universal object of the category with objects as pairs $(C(Y), \alpha)$ where Y is a compact metrizable space and α is a unital C^*-homomorphism from $C(M)$ to $C(M) \otimes C(Y)$ satisfying the following:*

a. $\overline{\mathrm{Sp}}(\alpha(C(M))(1 \otimes C(Y)) = C(M) \otimes C(Y)$,

b. $\alpha_\phi = (\mathrm{id} \otimes \phi)\alpha$ *maps* $(C^\infty(M))_0$ *into itself and commutes with* $\mathcal{L}$ *on* $(C^\infty(M))_0$, *for every state ϕ on $C(Y)$.*

Example 2.1.9 1. The isometry group of the n-sphere S^n is $O(n+1)$ where the action is given by the usual action of $O(n+1)$ on $\mathbb{R}^{n+1}$. The subgroup of $O(n+1)$ consisting of all orientation preserving isometries on S^n is $SO(n+1)$.

2. The isometry group of the circle S^1 is $S^1 \rtimes Z_2$. Here the $\mathbf{Z}_2(=\{0, 1\})$ action on S^1 is given by $1.z = \overline{z}$, where z is in S^1 while the action of S^1 is its action on itself.

3. $\mathrm{ISO}(\mathbb{T}^n) \cong \mathbb{T}^n \rtimes (\mathbf{Z}_2^n \rtimes S_n)$ where S_n is the permutation group on n symbols. Here an element of S_n acts on an element $(z_1, z_2, ..., z_n) \in \mathbb{T}^n$ by permutation. If the generator of i-th copy of $\mathbf{Z}_2^n$ is denoted by 1_i, then the action of 1_i is given by $1_i(z_1, z_2, ..., z_n) = (z_1, ..., z_{i-1}, \overline{z_i}, z_{i+1}, ..., z_n)$ where $(z_1, z_2, ..., z_n) \in \mathbb{T}^n$. Lastly, the action of $\mathbb{T}^n$ on itself is its usual action.

Characterization of orientation preserving isometries of a spin manifold

This characterization is in the terms of the Dirac operator [9]. For the characterization of isometries of a Riemannian manifold in terms of the Hodge Dirac operator, we refer to [10].

We begin with a few basic facts about topologizing the space $C^\infty(M, N)$ where M, N are smooth manifolds. Let Ω be an open set of $\mathbb{R}^n$. We endow $C^\infty(\Omega)$ with the usual Fre'chet topology coming from uniform convergence (over compact subsets) of partial derivatives of all orders. The space $C^\infty(\Omega)$ is complete with respect to this topology, so is a Polish space in particular. Moreover, by the Sobolev imbedding Theorem (Corollary 1.21, [2]), $\cap_{k \geq 0} H_k(\Omega) = C^\infty(\Omega)$ as a set, where $H_k(\Omega)$ denotes the k-th Sobolev space. Thus, $C^\infty(\Omega)$ has also the Hilbertian seminorms coming from the Sobolev spaces, hence the corresponding Frechet topology. We claim that these two topologies on $C^\infty(\Omega)$ coincide. Indeed, the inclusion map from $C^\infty(\Omega)$ into $\cap_k H_k(\Omega)$, is continuous and surjective, so by the open mapping theorem for Frechet space, the inverse is also continuous, proving our claim.

Given two second countable smooth manifolds M, N, we shall equip $C^\infty(M, N)$ with the weakest locally convex topology making $C^\infty(M, N) \ni \phi \mapsto f \circ \phi \in C^\infty(M)$ Frechet continuous for every f in $C^\infty(N)$.

For topological or smooth fiber or principal bundles E, F over a second countable smooth manifold M, we shall denote by $\mathrm{Hom}(E, F)$ the set of bundle morphisms from E to F. We remark that the total space of a locally trivial topological bundle such that the base and the fiber spaces are locally compact Hausdorff second countable must itself be so, hence in particular Polish (that is, a complete separable metric space).

In particular, if E, F are locally trivial principal G-bundles over a common base, such that the (common) base as well as the structure group G are locally compact Hausdorff and second countable, then $\mathrm{Hom}(E, F)$ is a Polish space.

We need a standard fact, stated below as Lemma 2.1.11, about the measurable lift of Polish space valued functions.

Before that, we introduce some notions.

A multifunction $G : X \to Y$ is a map with domain X and whose values are non-empty subsets of Y. For $A \subseteq Y$, we put $G^{-1}(A) = \{x \in X : G(x) \cap A \neq \phi\}$.

A selection of a multifunction $G : X \to Y$ is a point map $s : X \to Y$ such that $s(x)$ belongs to $G(x)$ for all x in X. Now let Y be a Polish space and σ_X a σ-algebra on X. A multifunction $G : X \to Y$ is called σ_X measurable if $G^{-1}(U)$ belongs to σ_X for every open set U in Y.

The following well-known selection theorem is Theorem 5.2.1 of [11] and was proved by Kuratowski and Ryll-Nardzewski.

Proposition 2.1.10 *Let σ_X be a σ algebra on X and Y a Polish space. Then, every σ_X measurable, closed valued multifunction $F : X \to Y$ admits a σ_X measurable selection.*

A trivial consequence of this result is the following:

Lemma 2.1.11 *Let M be a compact metrizable space, B, $\tilde{B}$ Polish spaces such that there is an n-covering map $\Lambda : \tilde{B} \to B$. Then any continuous map $\xi : M \to B$ admits a lifting $\tilde{\xi} : M \to \tilde{B}$, which is Borel measurable and $\Lambda \circ \tilde{\xi} = \xi$. In particular, if $\tilde{B}$ and B are topological bundles over M, with Λ being a bundle map, any continuous section of B admits a lifting which is a measurable section of $\tilde{B}$.*

We shall now give an operator-theoretic characterization of the classical group of orientation preserving Riemannian isometries, which will be the motivation of our definition of its quantum counterpart. Let M be a compact Riemannian n-dimensional spin manifold, with a fixed choice of orientation. We recall the notations as in Sect. 2.1.3. In particular, the spinor bundle S is the associated bundle of a principal $Spin(n)$-bundle P on M which has a canonical 2-covering bundle-map Λ from P to the frame-bundle F (which is an $SO(n)$-principal bundle), such that Λ is locally of the form $(\mathrm{id}_M \otimes \lambda)$ where λ is the two covering map from $Spin(n)$ to $SO(n)$. Moreover, the spinor space will be denoted by Δ_n. Let f be a smooth orientation preserving Riemannian isometry of M, and consider the bundles $E = \mathrm{Hom}(F, f^*(F))$

and $\tilde{E} = \mathrm{Hom}(P, f^*(P))$ (where Hom denotes the set of bundle maps). We view df as a section of the bundle E in the natural way. By the Lemma 2.1.11 we obtain a measurable lift $\tilde{df} : M \to \tilde{E}$, which is a measurable section of $\tilde{E}$. Using this, we define a map on the space of measurable section of $S = P \times_{Spin(n)} \Delta_n$ as follows: given a (measurable) section ξ of S, say of the form $\xi(m) = [p(m), v]$, with $p(m)$ in P_m, v in Δ_n, we define $U\xi$ by $(U\xi)(m) = [\tilde{df}(f^{-1}(m))(p(f^{-1}(m))), v]$. Note that sections of the above form constitute a total subset in $L^2(S)$, and the map $\xi \mapsto U\xi$ is clearly a densely defined linear map on $L^2(S)$, whose fiber-wise action is unitary since the $Spin(n)$ action is so on Δ_n. Thus it extends to a unitary U on $\mathcal{H} = L^2(S)$. Any such U, induced by the map f, will be denoted by U_f. It is not unique since the choice of the lifting used in its construction is not unique.

Theorem 2.1.12 *Let M be a compact Riemannian spin manifold (hence orientable, and fix a choice of orientation) with the usual Dirac operator D acting as an unbounded self-adjoint operator on the Hilbert space $\mathcal{H}$ of the square integrable spinors, and let S denote the spinor bundle, with $\Gamma(S)$ being the $C^\infty(M)$ module of smooth sections of S. Let $f : M \to M$ be a smooth one-to-one map which is a Riemannian orientation preserving isometry. Then the unitary U_f on $\mathcal{H}$ commutes with D and $U_f M_\phi U_f^* = M_{\phi \circ f}$, for any ϕ in $C(M)$, where M_ϕ denotes the operator of multiplication by ϕ on $L^2(S)$. Moreover, when the dimension of M is even, U_f commutes with the canonical grading γ on $L^2(S)$.*

Conversely, suppose that U is a unitary on $\mathcal{H}$ such that $UD = DU$ and the map $\alpha_U(X) = UXU^{-1}$ for X in $\mathcal{B}(\mathcal{H})$ maps $\mathcal{A} = C(M)$ into $L^\infty(M) = \mathcal{A}''$. If the dimension of M is even, assume furthermore that U commutes with the grading operator γ. Then there is a smooth one-to-one orientation preserving Riemannian isometry f on M such that $U = U_f$.

Proof From the construction of U_f, it is clear that $U_f M_\phi U_f^{-1} = M_{\phi \circ f}$. Moreover, since the Dirac operator D commutes with the $Spin(n)$-action on S, we have $U_f D = DU_f$ on each fiber, hence on $L^2(S)$. In the even dimensional case, it is easy to see that the $Spin(n)$ action commutes with γ (the grading operator), hence U_f does so.

For the converse, first note that α_U is a unital $*$-homomorphism on $L^\infty(M, dvol)$ and thus must be of the form $\psi \mapsto \psi \circ f$ for some measurable f. We claim that f must be smooth. Fix any smooth g on M and consider $\phi = g \circ f$. We have to argue that ϕ is smooth. Let δ_D denote the generator of the strongly continuous one-parameter group of automorphism $\beta_t(X) = e^{itD} X e^{-itD}$ on $\mathcal{B}(\mathcal{H})$ (with respect to the weak operator topology, say). From the assumption that D and U commute it is clear that α_U maps $\mathcal{D} := \bigcap_{n \geq 1} \mathrm{Dom}(\delta_D^n)$ into itself and since $C^\infty(M) \subset \mathcal{D}$, we conclude that $\alpha_U(M_\phi) = M_{\phi \circ g}$ belongs to $\mathcal{D}$. We claim that this implies the smoothness of ϕ. Let m be a point of M and choose a local chart (V, ψ) at m, with the coordinates $(x_1, ..., x_n)$, such that $\Omega = \psi(V) \subseteq \mathbb{R}^n$ has compact closure, $S|_V$ is trivial and D has the local expression $D = i \sum_{j=1}^n \mu(e_j) \nabla_j$, where $\nabla_j = \nabla_{\frac{\partial}{\partial x_j}}$ denotes the covariant derivative (with respect to the canonical Levi-Civita connection) operator along the vector field $\frac{\partial}{\partial x_j}$ on $L^2(\Omega)$ and $\mu(v)$ denotes the Clifford multiplication by a vector v. Now, $\phi \circ \psi^{-1} \in L^\infty(\Omega) \subseteq L^2(\Omega)$ and it is easy to observe from the above local structure of D that $[D, M_\phi]$ has the local expression $\sum_j iM_{\frac{\partial}{\partial x_j}\phi} \otimes \mu(e_j)$. Thus, the fact $M_\phi \in \bigcap_{n\geq 1} \mathrm{Dom}(\delta_D^n)$ implies $\phi \circ \psi^{-1}$ is in $\mathrm{Dom}(d_{j_1}...d_{j_k})$ for every integer tuple $(j_1, ..., j_k)$, $j_i \in \{1, ..., n\}$, where $d_j := \frac{\partial}{\partial x_j}$. In other words, $\phi \circ \psi^{-1}$ is in $H^k(\Omega)$ for all $k \geq 1$, where $H^k(\Omega)$ denotes the k-th Sobolev space on Ω (see [2]). By Sobolev's theorem (see, for example. [2], Corollary 1.21, page 24) it follows that $\phi \circ \psi^{-1}$ is in $C^\infty(\Omega)$.

We note that f is one-to-one as $\phi \to \phi \circ f$ is an automorphism of L^∞. Now, we shall show that f is an isometry of the metric space (M, d), where d is the metric coming from the Riemannian structure, and we have the explicit formula (2.1.2)

$$d(p, q) = \sup_{\phi\in C^\infty(M), \|[D,M_\phi]\|\leq 1} |\phi(p) - \phi(q)|.$$

Since U commutes with D, we have $\|[D, M_{\phi\circ f}]\| = \|[D, UM_\phi U^*]\| = \|U[D, M_\phi]U^*\| = \|[D, M_\phi]\|$ for every ϕ, from which it follows that $d(f(p), f(q)) = d(p, q)$. Finally, f is orientation preserving if and only if the volume form (say ω), which defines the choice of orientation, is preserved by the natural action of df on the space of n-forms. This will follow from the explicit description of ω in terms of D, given by (see [12] page 26, also see [13])

$$\omega(\phi_0 d\phi_1...d\phi_n) = \tau(\epsilon M_{\phi_0}[D, M_{\phi_1}]...[D, M_{\phi_n}]),$$

where $\phi_0, ..., \phi_n$ belong to $C^\infty(M)$, $\epsilon = 1$ in the odd case and $\epsilon = \gamma$ (the grading operator) in the even case and τ denotes the volume integral. In fact, $\tau(X) = \mathrm{Lim}_{t\to 0+} \frac{\mathrm{Tr}(e^{-tD^2}X)}{\mathrm{Tr}(e^{-tD^2})}$ (where Lim is as in Sect. 2.2.2), which implies $\tau(UXU^*) = \tau(X)$ for all X in $\mathcal{B}(\mathcal{H})$ (using the fact that D and U commute). Thus,

$$\omega(\phi_0 \circ f\ d(\phi_1 \circ f) \dots d(\phi_n \circ f))$$

$$= \tau(\epsilon UM_{\phi_0}U^*U[D, M_{\phi_1}]U^*...U[D, M_{\phi_n}]U^*)$$

$$= \tau(U\epsilon M_{\phi_0}[D, M_{\phi_1}]...[D, M_{\phi_n}]U^*)$$

$$= \tau(\epsilon M_{\phi_0}[D, M_{\phi_1}]...[D, M_{\phi_n}])$$

$$= \omega(\phi_0 d\phi_1...d\phi_n).$$

□

Now we turn to the case of a family of maps. We are ready to state and prove the operator-theoretic characterization of a 'family of orientation preserving isometries'.

Theorem 2.1.13 *Let X be a compact metrizable space and $\psi : X \times M \to M$ is a map such that ψ_x defined by $\psi_x(m) = \psi(x, m)$ is a smooth orientation preserving Riemannian isometry and $x \mapsto \psi_x \in C^\infty(M, M)$ is continuous with respect to the locally convex topology of $C^\infty(M, M)$ mentioned before.*

Then there exists a ($C(X)$-linear) unitary U_ψ on the Hilbert $C(X)$-module $\mathcal{H} \otimes C(X)$ (where $\mathcal{H} = L^2(S)$ as in Theorem 2.1.12) such that for all x belonging to X, $U_x := (id \otimes ev_x)U_\psi$ is a unitary of the form U_{ψ_x} on the Hilbert space $\mathcal{H}$ commuting with D and $U_x M_\phi U_x^{-1} = M_{\phi \circ \psi_x^{-1}}$. If in addition, the manifold is even dimensional, then U_{ψ_x} commutes with the grading operator γ.

Conversely, if there exists a $C(X)$-linear unitary U on $\mathcal{H} \otimes C(X)$ such that $U_x := (\mathrm{id} \otimes \mathrm{ev}_x)(U)$ is a unitary commuting with D for all x, (and U_x commutes with the grading operator γ if the manifold is even dimensional) and $(\mathrm{id} \otimes \mathrm{ev}_x)\alpha_U(L^\infty(M)) \subseteq L^\infty(M)$ *for all x in X, then there exists a map $\psi : X \times M \to M$ satisfying the conditions mentioned above such that $U = U_\psi$.*

Proof Consider the bundles $\hat{F} = X \times F$ and $\hat{P} = X \times P$ over $X \times M$, with fibers at (x, m) isomorphic with F_m and P_m, respectively, and where F and P are, respectively, the bundles of orthonormal frames and the Spin(n) bundle discussed before. Moreover, denote by Ψ the map from $X \times M$ to itself given by $(x, m) \mapsto (x, \psi(x, m))$. Let $\pi_X : \mathrm{Hom}(\hat{F}, \Psi^*(\hat{F})) \to X$ be the obvious map obtained by composing the projection map of the $X \times M$ bundle with the projection from $X \times M$ to X and let us denote by B the closed subset of the Polish space $C(X, \mathrm{Hom}(\hat{F}, \Psi^*(\hat{F})))$ consisting of those f such that for all x, $\pi_X(f(x)) = x$. Define $\tilde{B}$ in a similar way replacing $\hat{F}$ by $\hat{P}$. The covering map from P to F induces a covering map from $\tilde{B}$ to B as well. Let $d_\psi^{'} : M \to B$ be the map given by $d'_\psi(m)(x) \equiv d'_\psi(x, m) = d\psi_x|_m$. Then by Lemma 2.1.11 there exists a measurable lift of $d_\psi^{'}$, say $\widetilde{d_\psi^{'}}$ from M into $\tilde{B}$. Since $d'_\psi(x, m) \in \mathrm{Hom}(F_m, F_{\psi(x,m)})$, it is clear that the lift $\widetilde{d'_\psi}(x, m)$ will be an element of $\mathrm{Hom}(P_m, P_{\psi(x,m)})$.

We can identify $\mathcal{H} \otimes C(X)$ with $C(X \to \mathcal{H})$, and since $\mathcal{H}$ has a total set $\mathcal{F}$ (say) consisting of sections of the form $[p(\cdot), v]$, where $p : M \to P$ is a measurable section of P and v belongs to Δ_n, we have a total set $\tilde{\mathcal{F}}$ of $\mathcal{H} \otimes C(X)$ consisting of $\mathcal{F}$ valued continuous functions from X. Any such function can be written as $[\Xi, v]$ with $\Xi : X \times M \to P$, $v \in \Delta_n$, and $\Xi(x, m) \in P_m$, and we define U on $\tilde{\mathcal{F}}$ by $U[\Xi, v] = [\Theta, v]$, where

$$\Theta(x, m) = \widetilde{d_\psi^{'}}(x, \psi_x^{-1}(m))(\Xi(x, \psi_x^{-1}(m))).$$

It is clear from the construction of the lift that U is indeed a $C(X)$-linear isometry that maps the total set $\tilde{\mathcal{F}}$ onto itself, so extends to a unitary on the whole of $\mathcal{H} \otimes C(X)$ with the desired properties.

Conversely, given U as in the statement of the converse part of the theorem, we observe that for each x in X, by Theorem 2.1.12, $(\mathrm{id} \otimes ev_x)U = U_{\psi_x}$ for some ψ_x such that ψ_x is a smooth orientation preserving Riemannian isometry. This defines the map ψ by setting $\psi(x, m) = \psi_x(m)$. The proof will be complete if we can show that $x \mapsto \psi_x \in C^\infty(M, M)$ is continuous, which is equivalent to showing that whenever $x_n \to x$ in the topology of X, we must have $\phi \circ \psi_{x_n} \to \phi \circ \psi_x$ in the Fre'chet topology of $C^\infty(M)$, for any $\phi \in C^\infty(M)$. However, by Lemma 1.1.10, we have $(\mathrm{id} \otimes ev_{x_n})\alpha_U([D,\ M_\phi]) \to (\mathrm{id} \otimes ev_x)\alpha_U([D,\ M_\phi])$ in the strong operator topology where $\alpha_U(X) = UXU^{-1}$. Since U commutes with D, this implies

$$(\mathrm{id} \otimes ev_{x_n})[D \otimes \mathrm{id},\ \alpha_U(M_\phi)] \to (\mathrm{id} \otimes ev_x)[D \otimes \mathrm{id},\ \alpha_U(M_\phi)],$$

that is, for all ξ in $L^2(S)$,

$$[D,\, M_{\phi\circ\psi_{x_n}}]\xi \xrightarrow{L^2} [D,\ M_{\phi\circ\psi_x}]\xi.$$

By choosing ϕ with support in a local trivializing coordinate neighborhood for S, and then using the local expression of D used in the proof of Theorem 2.1.12, we conclude that $d_k(\phi \circ \psi_{x_n}) \xrightarrow{L^2} d_k(\phi \circ \psi_x)$ (where d_k is as in the proof of Theorem 2.1.12). Similarly, by taking repeated commutators with D, we can show the L^2 convergence with d_k replaced by $d_{k_1}...d_{k_m}$ for any finite tuple $(k_1, ..., k_m)$. In other words, $\phi \circ \psi_{x_n} \to \phi \circ \psi_x$ in the topology of $C^\infty(M)$ described before. □

2.2 Noncommutative Geometry

In this section, we recall those basic concepts of noncommutative geometry, which we are going to need. We refer to [14–19] for more details.

2.2.1 Spectral Triples: Definition and Examples

Motivated by the facts in Proposition 2.1.3, Alain Connes defined a noncommutative manifold based on the idea of a spectral triple:

Definition 2.2.1 A **spectral triple** or **spectral data** is a triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ where $\mathcal{H}$ is a separable Hilbert space, $\mathcal{A}^\infty$ is a $*$ subalgebra of $\mathcal{B}(\mathcal{H})$, (not necessarily norm closed) and D is a self-adjoint (typically unbounded) operator such that for all a in $\mathcal{A}^\infty$, the operator $[D, a]$ has a bounded extension. Such a spectral triple is also called an odd spectral triple. If in addition, we have γ in $\mathcal{B}(\mathcal{H})$ satisfying $\gamma = \gamma^* = \gamma^{-1},\ D\gamma = -\gamma D$ and $[a, \gamma] = 0$ for all a in $\mathcal{A}^\infty$, then we say that the quadruplet $(\mathcal{A}^\infty, \mathcal{H}, D, \gamma)$ is an even spectral triple. The operator D is called the Dirac operator corresponding to the spectral triple.

Furthermore, given an abstract $*$-algebra $\mathcal{B}$, an odd (even) spectral triple on $\mathcal{B}$ is an odd (even) spectral triple $(\pi(\mathcal{B}), \mathcal{H}, D)$ (respectively, $(\pi(\mathcal{B}), \mathcal{H}, D, \gamma)$) where $\pi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism.

Since in the classical case, the Dirac operator has compact resolvent if the manifold is compact, we say that the spectral triple is of **compact type** if $\mathcal{A}^\infty$ is unital and D has compact resolvent. For nonunital C^* algebras, interesting spectral triples are not of compact type. Examples of such spectral triples include semifinite spectral triples for which we refer to [20, 21], and the references therein. Since, our final goal is to study quantum isometry groups of spectral triples of compact type, all the spectral triples under discussion will be assumed to be of compact type.

Definition 2.2.2 We say that two spectral triples $(\pi_1(\mathcal{A}), \mathcal{H}_1, D_1)$ and $(\pi_2(\mathcal{A}), \mathcal{H}_2, D_2)$ are said to be unitarily equivalent if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $D_2 = UD_1U^*$ and $\pi_2(.) = U\pi_1(.)U^*$ where π_j, $j = 1, 2$ are the representations of $\mathcal{A}$ in $\mathcal{H}_j$,, respectively.

Real structure on a spectral triple

We now give a definition of the real structure along the lines of [22, 23], which is a suitable modification of Connes' original definition (see [14, 24]) to accommodate the examples coming from quantum groups and quantum homogeneous spaces.

Definition 2.2.3 An odd spectral triple with a real structure is given by a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ along with a (possibly unbounded, invertible) closed antilinear operator $\widetilde{J}$ on $\mathcal{H}$ such that $\mathcal{D} := \mathrm{Dom}(D) \subseteq \mathrm{Dom}(\widetilde{J})$, $\widetilde{J}\mathcal{D} \subseteq \mathcal{D}$, $\widetilde{J}$ commutes with D on $\mathcal{D}$, and the antilinear isometry J obtained from the polar decomposition of $\widetilde{J}$ satisfies the usual conditions for a real structure in the sense of [23], for a suitable sign-convention given by $(\epsilon, \epsilon') \in \{\pm 1\} \times \{\pm 1\}$ as described in [12], page 30, i.e., $J^2 = \epsilon I$, $JD = \epsilon' DJ$, and for all $x, y \in \mathcal{A}^\infty$, the commutators $[x, JyJ^{-1}]$ and $[JxJ^{-1}, [D, y]]$ are compact operators.

If the spectral triple is even, a real structure with the sign-convention given by a triplet $(\epsilon, \epsilon', \epsilon'')$ as in [12], page 30, is similar to a real structure in the odd case (with the sign-convention (ϵ, ϵ')), but with the additional requirement that $J\gamma = \epsilon''\gamma J$.

Next, we give a few examples of spectral triples in classical and noncommutative geometry. We will give more examples in the later chapters of the book.

Example 2.2.4 Let M be a smooth spin manifold. Then from Proposition 2.1.3, we see that $(C^\infty(M), \mathcal{H}, D)$ is a spectral triple over $C^\infty(M)$ and it is of compact type if M is compact.

We recall that when the dimension of the manifold is even, $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$. An L^2 section s has a decomposition $s = s_1 + s_2$ where $s_1(m), s_2(m)$ belongs to $\Delta_n^+(m)$ and $\Delta_n^-(m)$ (for all m), respectively, where $\Delta_n^\pm(m)$ denotes the subspace of the fiber over m. This decomposition of $L^2(S)$ induces a grading operator γ on $L^2(S)$. It can be seen that D anticommutes with γ.

Example 2.2.5 This example comes from the classical Hilbert space of forms discussed in Sect. 2.2.2. One considers the self-adjoint extension of the operator $d + d^*$ on $\mathcal{H} = \oplus_k \mathcal{H}^k(M)$, which is again denoted by $d + d^*$. $C^\infty(M)$ has a representation on each $\mathcal{H}^k(M)$ which gives a representation, say π on $\mathcal{H}$. Then it can be seen that $(C^\infty(M), \mathcal{H}, d + d^*)$ is a spectral triple and $d + d^*$ is called the Hodge Dirac operator. When M is compact, this spectral triple is of compact type.

Remark 2.2.6 Let us make it clear that by a 'classical spectral triple' we always mean the spectral triple obtained by the Dirac operator on the spinors (so, in particular, manifolds are assumed to be Riemannian spin manifolds), and not just any spectral triple on the commutative algebra $C^\infty(M)$.

Example 2.2.7 **The Noncommutative torus**

We recall from Sect. 1.1.1 that the noncommutative 2-torus $\mathcal{A}_\theta$ is the universal C^* algebra generated by two unitaries U and V satisfying $UV = e^{2\pi i\theta} VU$, where θ is a number in [0, 1].

There are two derivations d_1 and d_2 on $\mathcal{A}_\theta$ obtained by extending linearly the rule:

$$d_1(U) = U,\ d_1(V) = 0,$$

$$d_2(U) = 0,\ d_2(V) = V.$$

Then d_1 and d_2 are well defined on the following dense $*$-subalgebra of $\mathcal{A}_\theta$:

$$\mathcal{A}_\theta^\infty = \{ \sum_{m,n \in \mathbf{Z}} a_{mn} U^m V^n : \sup_{m,n} \left| m^k n^l a_{mn} \right| < \infty \ \text{ for all } k, l \text{ in } I\!N \}.$$

There is a faithful trace on $\mathcal{A}_\theta$ defined as follows:

$$\tau(\sum a_{mn} U^m V^n) = a_{00}.$$

Let $\mathcal{H} = L^2(\tau) \oplus L^2(\tau)$ where $L^2(\tau)$ denotes the GNS Hilbert space of $\mathcal{A}_\theta$ with respect to the state τ. We note that $\mathcal{A}_\theta^\infty$ is embedded as a subalgebra of $\mathcal{B}(\mathcal{H})$ by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

Now, we define $D = \begin{pmatrix} 0 & d_1 + i d_2 \\ d_1 - i d_2 & 0 \end{pmatrix}$.

Then, $(\mathcal{A}_\theta^\infty, \mathcal{H}, D)$ is a spectral triple of compact type. In particular, for $\theta = 0$, this coincides with the classical spectral triple on $C(\mathbb{T}^2)$.

Example 2.2.8 **Spectral triples on** $SU_\mu(2)$

In this example, we discuss the spectral triple on $SU_\mu(2)$ constructed by Chakraborty and Pal in [25]. We recall from Sect. 1.2.4 that by the symbols $t^n_{i,j}$, we will denote the (i, j)-th matrix element of the $(2n + 1)$ dimensional corepresentation of $SU_\mu(2)$. Moreover, e^n_{ij}'s will denote the normalized (with respect to the Haar state h) t^n_{ij}'s.

Then the spectral triple is given by $(\mathcal{O}(SU_\mu(2)), L^2(SU_\mu(2), h), D^{SU_\mu(2)})$, where $D^{SU_\mu(2)}$ is defined by

$$\begin{aligned} &D^{SU_\mu(2)}(e^n_{ij}) \\ &= (2n+1)e^n_{ij},\ n \neq i \\ &= -(2n+1)e^n_{ij},\ n = i. \end{aligned}$$

Example 2.2.9 **A class of spectral triples on the Podles' spheres**

We discuss the spectral triples on $S^2_{\mu c}$ discussed in [26].

Let $s = -c^{-\frac{1}{2}}\lambda_-, \ \lambda_\pm = \frac{1}{2} \pm (c + \frac{1}{4})^{\frac{1}{2}}$.

For all j belonging to $\frac{1}{2}\mathbb{N}$,

$u_j = (\alpha^* - s\gamma^*)(\alpha^* - \mu^{-1}s\gamma^*)......(\alpha^* - \mu^{-2j+1}s\gamma^*)$,

$w_j = (\alpha - \mu s\gamma)(\alpha - \mu^2 s\gamma)........(\alpha - \mu^{2j}s\gamma)$,

$u_{-j} = E^{2j} \triangleright w_j$,

$u_0 = w_0 = 1$,

$y_1 = (1 + \mu^{-2})^{\frac{1}{2}}(c^{\frac{1}{2}}\mu^2\gamma^{*2} - \mu\gamma^*\alpha^* - \mu c^{\frac{1}{2}}\alpha^{*2})$,

$N^l_{kj} = \left\| F^{l-k} \triangleright ({y_1}^{l-|j|}u_j)\right\|^{-1}$.

Define

$$v^l_{k,j} = N^l_{k,j}F^{l-k} \triangleright (y_1^{l-|j|}u_j), \ l \in \frac{1}{2}\mathbb{N}_0, \ \ j, k = -l, -l+1, ...l. \tag{2.2.1}$$

Let $\mathcal{M}_N$ be the Hilbert subspace of $L^2(SU_\mu(2))$ with the orthonormal basis $\{v^l_{m,N} : l = |N|, \ |N|+1, \ ...., \ m = -l, ...l\}$.

Set

$$\mathcal{H} = \mathcal{M}_{-\frac{1}{2}} \oplus \mathcal{M}_{\frac{1}{2}}.$$

Then it is easy to check that x_i keeps $\mathcal{H}$ for all $i \in \{-1, 0, 1\}$. In particular,

$$x_i.v^l_{m,N} = \alpha^-_i(l, m; N)v^{l-1}_{m+i,N} + \alpha^0_i(l, m; N)v^l_{m+i,N} + \alpha^+_i(l, m; N)v^{l+1}_{m+i,N}, \tag{2.2.2}$$

where $\alpha^-_i, \ \alpha^0_i, \ \alpha^+_i$ are some constants.

Thus, (2.2.2) defines a representation π of $S^2_{\mu,c}$ on $\mathcal{H}$.

We will often identify $\pi(S^2_{\mu,c})$ with $S^2_{\mu,c}$.

Finally by Proposition 7.2 of [26], the following Dirac operator D gives a spectral triple $(\mathcal{O}(S^2_{\mu,c}), \mathcal{H}, D)$ which we are going to work with :

$$D(v^l_{m,\pm\frac{1}{2}}) = (c_1 l + c_2)v^l_{m,\mp\frac{1}{2}}, \tag{2.2.3}$$

where c_1, c_2 are elements of $\mathbb{R}$, $c_1 \neq 0$.

2.2.2 The Noncommutative Space of Forms

We start this subsection by recalling the universal space of one forms corresponding to an algebra.

Proposition 2.2.10 *Given an algebra $\mathcal{B}$, there is a (unique upto isomorphism) $\mathcal{B} - \mathcal{B}$ bimodule $\Omega^1(\mathcal{B})$ and a derivation $\delta : \mathcal{B} \to \Omega^1(\mathcal{B})$ (that is, $\delta(ab) = \delta(a)b + a\delta(b)$ for all a, b in $\mathcal{B}$), satisfying the following properties:*

(i) $\Omega^1(\mathcal{B})$ is spanned as a vector space by elements of the form $a\delta(b)$ with a, b belonging to $\mathcal{B}$; and

(ii) for any $\mathcal{B}-\mathcal{B}$ bimodule E and a derivation $d : \mathcal{B} \to E$, there is an unique $\mathcal{B}-\mathcal{B}$ linear map $\eta : \Omega^1(\mathcal{B}) \to E$ such that $d = \eta \circ \delta$.

The bimodule $\Omega^1(\mathcal{B})$ is called the space of universal 1-forms an $\mathcal{B}$ and δ is called the universal derivation.

We can also introduce universal space of higher forms on $\mathcal{B}$, $\Omega^k(\mathcal{B})$, say, for $k = 2, 3, ...$, by defining them recursively as follows: $\Omega^{k+1}(\mathcal{B}) = \Omega^k(\mathcal{B}) \otimes_{\mathcal{B}} \Omega^1(\mathcal{B})$ and also set $\Omega^0(\mathcal{B}) = \mathcal{B}$.

Next, we briefly discuss the notion of the noncommutative Hilbert space of forms for a spectral triple of compact type. We refer to [27] (page 124 -127) and the references therein for more details.

Definition 2.2.11 A spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ of compact type is said to be Θ-summable if e^{-tD^2} is of trace class for all $t > 0$. A Θ-summable spectral triple is called finitely summable when there is some $p > 0$ such that $t^{\frac{p}{2}}\,\mathrm{Tr}(e^{-tD^2})$ is bounded on $(0, \delta]$ for some $\delta > 0$. The infimum of all such p, say p', is called the dimension of the spectral triple and the spectral triple is called p'-summable.

Remark 2.2.12 We remark that the definition of Θ-summability to be used in this book is stronger than the one in [14] (page 390, Definition 1.) in which a spectral triple is called Θ-summable if $\mathrm{Tr}(e^{-D^2}) < \infty$.

For a Θ-summable spectral triple, let $\sigma_\lambda(T) = \frac{\mathrm{Tr}(Te^{-\frac{1}{\lambda}D^2})}{\mathrm{Tr}(e^{-\frac{1}{\lambda}D^2})}$ for $\lambda > 0$. We note that $\lambda \mapsto \sigma_\lambda(T)$ is bounded.

Let

$$\tau_\lambda(T) = \frac{1}{\log\lambda}\int_a^\lambda \sigma_u(T)\frac{du}{u} \text{for } \lambda \geq a \geq e.$$

Now consider the quotient C^* algebra $\mathcal{B}_\infty = C_b([a,\infty))/C_0([a,\infty))$. Let for T in $\mathcal{B}(\mathcal{H})$, $\tau(T)$ in $\mathcal{B}_\infty$ be the class of $\lambda \to \tau_\lambda(T)$.

For any state ω on the C^* algebra B_∞, $Tr_\omega(T) = \omega(\tau(T))$ for all T in $\mathcal{B}(\mathcal{H})$ defines a functional on $\mathcal{B}(\mathcal{H})$. As we are not going to need the choice of ω in this book, we will suppress the suffix ω and simply write $\mathrm{Lim}_{t\to 0^+}\frac{\mathrm{Tr}(Te^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$ for $Tr_\omega(T)$. This is a kind of Banach limit because if $\lim_{t\to 0^+}\frac{\mathrm{Tr}(Te^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$ exists, then it agrees with the functional $\mathrm{Lim}_{t\to 0^+}$. Moreover, $\mathrm{Tr}_\omega(T)$ coincides (upto a constant) with the Dixmier trace (see Chapter IV, [14]) of the operator $T|D|^{-p}$ when the spectral triple has a finite dimension $p > 0$, where $|D|^{-p}$ is to be interpreted as the inverse of the restriction of $|D|^p$ on the closure of its range. In particular, this functional gives back the volume form for the classical spectral triple on a compact Riemannian manifold.

Let $\Omega^k(\mathcal{A}^\infty)$ be the space of universal k-forms on the algebra $\mathcal{A}^\infty$ which is spanned by $a_0\delta(a_1)\cdots\delta(a_k)$, a_i belonging to $\mathcal{A}^\infty$, where δ is as in Proposition 2.2.10. There is a natural graded algebra structure on $\Omega \equiv \bigoplus_{k\geq 0}\Omega^k(\mathcal{A}^\infty)$, which also has a natural involution given by $(\delta(a))^* = -\delta(a^*)$, and using the spectral triple, we get a $*$-representation $\Pi : \Omega \to \mathcal{B}(\mathcal{H})$ which sends $a_0\delta(a_1)\cdots\delta(a_k)$ to $a_0 d_D(a_1)\cdots d_D(a_k)$, where $d_D(a) = [D, a]$. Consider the state τ on $\mathcal{B}(\mathcal{H})$ given by, $\tau(X) = \mathrm{Lim}_{t\to 0^+} \frac{\mathrm{Tr}(Xe^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$, where Lim is as above. Using τ, we define a positive semi definite sesquilinear form on $\Omega^k(\mathcal{A}^\infty)$ by setting $\langle w, \eta\rangle = \tau(\Pi(w)^*\Pi(\eta))$. Let $K^k = \{w \in \Omega^k(\mathcal{A}^\infty) : \langle w,\ w\rangle = 0\}$, for $k \geq 0$, and $K^{-1} := (0)$. Let $\overline{\Omega^k_D}$ be the Hilbert space obtained by completing the quotient $\Omega^k(\mathcal{A}^\infty)/K^k$ with respect to the inner product mentioned above, and we define $\mathcal{H}^k_D := P_k^\perp \overline{\Omega^k_D}$, where P_k denotes the projection onto the closed subspace generated by $\delta(K^{k-1})$. The map $D' := d + d^* \equiv d_D + d_D^*$ on $\mathcal{H}_{d+d^*} := \bigoplus_{k\geq 0} \mathcal{H}^k_D$ has a self-adjoint extension (which is again denoted by $d + d^*$). Clearly, $\mathcal{H}^k_D$ has a total set consisting of elements of the form $[a_0\delta(a_1)\cdots\delta(a_k)]$, with a_i in $\mathcal{A}^\infty$ and where $[\omega]$ denotes the equivalence class $P_k^\perp(w + K^k)$ for ω belonging to $\Omega^k(\mathcal{A}^\infty)$. There is a $*$-representation $\pi_{d+d^*} : \mathcal{A} \to \mathcal{B}(\mathcal{H}_{d+d^*})$, given by $\pi_{d+d^*}(a)([a_0\delta(a_1)\cdots\delta(a_k)]) = [aa_0\delta(a_1)\cdots\delta(a_k)]$. Then it is easy to see that

Proposition 2.2.13 *$(\mathcal{A}^\infty, \mathcal{H}_{d+d^*}, d + d^*)$ is a spectral triple.*

Let us mention that for the classical spectral triple $(C^\infty(M), L^2(S), D)$ on a compact Riemannian spin manifold M, the above construction does give the usual Hilbert space of forms discussed in Sect. 2.1.1. Moreover, the volume form on $C^\infty(M)$ using $D' = d + d^*$ in place of D agrees with the usual volume form. It is enough to explain this for smooth functions supported in a small coordinate neighborhood on which the restriction of the spinor bundle S is trivial. Combining the local expressions (5.45), (5.48) and (5.49) in [15], one can easily see that D^2 has the following local expression:

$$D^2 = \Delta \otimes I_{\mathbb{C}^k} + A,$$

where A is a first order differential operator, $\Delta = -\sum_{i,j} g^{ij} \frac{\delta}{\delta x_i}\frac{\delta}{\delta x_j}$ is the Laplacian on the manifold, k is the dimension of the fiber of S, $\{x_1, x_2, ..., x_n\}$ are local coordinates, $((g_{ij}))$ is the Riemannian metric and $((g^{ij})) = ((g_{ij}))^{-1}$.

On the other hand, we can obtain the following local expression for $(D')^2$ on a suitable trivializing neighborhood for the bundle of forms:

$$(D')^2 = \mathcal{L} \otimes I_{\mathbb{C}^m},$$

where $\mathcal{L}$ is the Hodge Laplacian on M as in Sect. 2.1.2 and m is the dimension of the fiber of $\Lambda^* M$.

A direct calculation shows that $\mathcal{L} - \Delta$ is a first order differential operator. As $\mathcal{L}^{-\frac{n}{2}}$ and $\Delta^{-\frac{n}{2}}$ are of Dixmier trace class, it follows from the discussion in page 307 of [14] and the references cited there that

$$\mathrm{Tr}_\omega(M_f \mathcal{L}^{-\frac{n}{2}}) = \mathrm{Tr}_\omega(M_f \Delta^{-\frac{n}{2}}),$$

where M_f denotes the operator of multiplication by a smooth function f supported in a small enough coordinate neighborhood on which both S and $\Lambda^* M$ are trivial. Hence we have

$$\frac{\mathrm{Tr}_\omega(M_f (D')^{-\frac{n}{2}})}{\mathrm{Tr}_\omega((D')^{-\frac{n}{2}})} = \frac{\mathrm{Tr}_\omega(M_f (D^2)^{-\frac{n}{2}})}{\mathrm{Tr}_\omega((D^2)^{-\frac{n}{2}})}.$$

2.2.3 Laplacian in Noncommutative Geometry

Now we want to formulate and study an analog of the Hodge Laplacian in noncommutative geometry. We recall that in the classical case of a compact Riemannian manifold, $\mathcal{L} = -d_D^* d_D$ coincides with the Hodge Laplacian $-d^* d$ (restricted on the space of smooth functions), where d denotes the de-Rham differential. We need some mild technical assumptions on the spectral triple to define the associated Laplacian.

Definition 2.2.14 Let $(\mathcal{A}^\infty, \mathcal{B}(\mathcal{H}), D)$ be a Θ-summable spectral triple of compact type. Assume furthermore that it satisfies the following conditions:

(1) It is QC^∞, that is, $\mathcal{A}^\infty$ and $\{[D, a],\ a \in \mathcal{A}^\infty\}$ are contained in the domains of all powers of the derivation $[|D|, \cdot]$.

(2) Under condition (1), τ defined by $\tau(X) = \mathrm{Lim}_{t \to 0} \frac{\mathrm{Tr}(X e^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$ is a positive trace on the C^*-subalgebra generated by $\mathcal{A}^\infty$ and $\{[D, a] :\ a \in \mathcal{A}^\infty\}$. We assume that τ is also faithful on this subalgebra.

(3) The unbounded densely defined map d_D from $\mathcal{H}_D^0$ to $\mathcal{H}_D^1$ given by $d_D(a) = [D, a]$ for a in $\mathcal{A}^\infty$, is closable and let d_D also denote the closure.

(4) $\mathcal{L} := -d_D^* d_D$ has $\mathcal{A}^\infty$ in its domain.

Then, we call $\mathcal{L}$ the noncommutative Laplacian and $T_t = e^{t\mathcal{L}}$ the noncommutative heat semigroup. Moreover, the $*$-subalgebra of $\mathcal{A}^\infty$ generated by $\mathcal{A}_0^\infty$ will be denoted by $\mathcal{A}_0$.

Let us record the following observation.

Lemma 2.2.15 *Under the conditions of the Definition 2.2.14, then for $x \in \mathcal{A}^\infty$, we have $\mathcal{L}(x^*) = (\mathcal{L}(x))^*$.*

Proof It follows by simple calculation using the facts that τ is a trace and $d_D(x^*) = -(d_D(x))^*$ that

$$\begin{aligned}
&\tau(\mathcal{L}(x^*)^* y) \\
&= \tau(d_D(x) d_D(y)) = \tau(d_D(y) d_D(x)) = -\tau((d_D(y^*))^* d_D(x)) \\
&=< y^*, \mathcal{L}(x) >= \tau(y \mathcal{L}(x)) = \tau(\mathcal{L}(x) y),
\end{aligned}$$

for all $y \in \mathcal{A}^\infty$. By density of $\mathcal{A}^\infty$ in $\mathcal{H}_D^0$ (a) follows. □

It is well known that for compact Riemannian spin manifolds, the conditions (1) and (2) of Definition 2.2.14 are satisfied. On the other hand, we know from Sect. 2.1.2, (for example, Lemma 2.1.1) that the Hodge Laplacian on a compact Riemannian manifold satisfies the properties (3) and (4).

In the noncommutative case, the conditions (1) and (2) hold for many spectral triples including those coming from Rieffel deformations. The content of the next lemma is about the other conditions.

Lemma 2.2.16 *Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple of compact type and of finite dimension, say p. Suppose that for every element $a \in \mathcal{A}^\infty$, the map $\mathbb{R} \ni t \mapsto \alpha_t(X) := \exp(itD)X\exp(-itD)$ is differentiable at $t = 0$ in the norm-topology of $\mathcal{B}(\mathcal{H})$, where $X = a$ or $[D, a]$. Then the conditions (3) and (4) of Definition 2.2.14 are satisfied. Moreover, we have:*

(a) $\mathcal{L}$ maps $\mathcal{A}^\infty$ into the weak closure of $\mathcal{A}^\infty$ in $\mathcal{B}(\mathcal{H}_D^0)$.

(b) If $T_t = \exp(t\mathcal{L})$ maps $\mathcal{H}_D^0$ into $\mathcal{A}^\infty$ for all $t > 0$, then any eigenvector of $\mathcal{L}$ belongs to $\mathcal{A}^\infty$.

Proof We first observe that $\tau(\alpha_t(A)) = \tau(A)$ for all t and for all $A \in \mathcal{B}(\mathcal{H})$, since $\exp(itD)$ commutes with $|D|^{-p}$. If moreover, A belongs to the domain of norm-differentiability (at $t = 0$) of α_t, i.e., $\frac{\alpha_t(A)-A}{t} \to i[D, A]$ in operator-norm, then it follows from the property of the Dixmier trace that $\tau([D, A]) = \frac{1}{i}\lim_{t\to 0}\frac{\tau(\alpha_t(A))-\tau(A)}{t} = 0$. Now, since by assumption we have the norm- differentiability at $t = 0$ of $\alpha_t(A)$ for A belonging to the $*$-subalgebra (say $\mathcal{B}$) generated by $\mathcal{A}^\infty$ and $[D, \mathcal{A}^\infty]$, it follows that $\tau([D, A]) = 0 \; \forall A \in \mathcal{B}$. Let us now fix $a, b, c \in \mathcal{A}^\infty$ and observe that

$$
\begin{aligned}
&< a\, d_D(b), d_D(c) > \\
&= \tau((a\, d_D(b))^* d_D(c) > \\
&= -\tau([D, [D, b^*]a^*c]) + \tau([D, [D, b^*]a^*]c) \\
&= \tau([D, [D, b^*]a^*]c),
\end{aligned}
$$

using the fact that $\tau([D, [D, b^*]a^*c]) = 0$. This implies

$$
| < a\, d_D(b), d_D(c) > | \le \|[D, [D, b^*]a^*]\| \tau(c^*c)^{\frac{1}{2}} = \|[D, [D, b^*]a^*]\| \|c\|_2,
$$

where $\|c\|_2 = \tau(c^*c)^{\frac{1}{2}}$ denotes the L^2-norm of $c \in \mathcal{H}_D^0$. This proves that $a\, d_D(b)$ belongs to the domain of d_D^* for all $a, b \in \mathcal{A}^\infty$, so in particular d_D^* is dense, i.e., d_D is closable. Moreover, taking $a = 1$, we see that $d_D(\mathcal{A}^\infty) \subseteq \mathrm{Dom}(d_D^*)$, or in other words, $\mathcal{A}^\infty \subseteq \mathrm{Dom}(d_D^* d_D)$. This proves (3) and (4). The statement (a) can be proved along the line of Theorem 2.9, page 129, [27]. To prove (b), we note that if $x \in \mathcal{H}_D^0$ is an eigenvector of $\mathcal{L}$, say $\mathcal{L}(x) = \lambda x$ ($\lambda \in \mathbb{C}$), then we have $T_t(x) = e^{\lambda t}x$, hence $x = e^{-\lambda t}T_t(x) \in \mathcal{A}^\infty$. □

2.3 Quantum Group Equivariance in Noncommutative Geometry

We have already seen (Theorem 2.1.12) that the classical Dirac operator is equivariant with respect to the natural action of the group of orientation preserving Riemannian isometries. It is natural to explore similar equivariance of a spectral triple with respect to quantum group coactions. Let us begin by giving a precise definition of quantum group equivariance.

Definition 2.3.1 Consider a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ along with a coaction α of a CQG $\mathcal{Q}$ on the C^*-algebra A obtained by taking the norm closure of $\mathcal{A}^\infty$ in $\mathcal{B}(\mathcal{H})$. We say that $(\mathcal{A}^\infty, \mathcal{H}, D)$ is a $\mathcal{Q}$-equivariant spectral triple if there is a unitary corepresentation U of $\mathcal{Q}$ on $\mathcal{H}$ such that
(i) $\mathrm{ad}_U(.) = \alpha(.)$,
(ii) $\widetilde{U}(D \otimes I) = (D \otimes I)\widetilde{U}$.

It was not very easy to get examples of spectral triples, which are equivariant with respect to “a genuine (i.e., noncommutative as a C^* algebra) quantum group". In [25] (i.e., Example 2.2.8), the first example of an $SU_\mu(2)$-equivariant spectral triple was constructed. It was followed by the work of a number of mathematicians, see [26, 28–30] and the references therein. In the next two subsections, we show that the spectral triples of Examples 2.2.8 and 2.2.9 are indeed equivariant.

2.3.1 *The Example of $SU_\mu(2)$*

We deal with Example 2.2.8 here. Let U be the regular corepresentation of $SU_\mu(2)$ on $L^2(SU_\mu(2), h)$. Then $\mathrm{ad}_U(x) = \Delta(x)$ for all x in $SU_\mu(2)$. We recall from Example 2.2.8 the normalized vectors e^n_{ij}'s. Then $U(e^n_{ij}) = \sum_k \frac{1}{\|t^n_{ij}\| \|t^n_{ik}\|} e^n_{ik} \otimes t^n_{kj}$ from which it easily follows

Proposition 2.3.2 *([25]) The spectral triple* $(\mathcal{O}(SU_\mu(2), L^2(SU_\mu(2), h), D^{SU_\mu(2)})$ *of Example 2.2.8 is* $SU_\mu(2)$*-equivariant.*

2.3.2 *The Example of the Podles' Spheres*

Here, we consider the spectral triple constructed in [26] and explained in Example 2.2.9. We will use the notations of Example 2.2.9. From [26], we see that the vector spaces $\nu^l_{\pm\frac{1}{2}} = \mathrm{Span}\{v^l_{m,\pm\frac{1}{2}} : m = -l, ...l\}$ are $(2l+1)$ dimensional Hilbert spaces on which the $SU_\mu(2)$ corepresentation is unitarily equivalent to the standard l-th unitary irreducible corepresentation of $SU_\mu(2)$, that is, if the corepresentation is denoted by U_0, then $U_0(v^l_{i,\pm\frac{1}{2}}) = \sum v^l_{j,\pm\frac{1}{2}} \otimes t^l_{j,i}$ where $t^l_{i,j}$ denotes the matrix elements in the l-th unitary irreducible corepresentation of $SU_\mu(2)$.

We now recall Theorem 3.5 of [31].

Proposition 2.3.3 *Let R_0 be an operator on $\mathcal{H}$ defined by $R_0(v^n_{i,\pm\frac{1}{2}}) = \mu^{-2i\mp1} v^n_{i,\pm\frac{1}{2}}$. Then* $\mathrm{Tr}(R_0 e^{-tD^2}) < \infty$ (for all $t > 0$) *and one has*

$$(\tau_{R_0} \otimes \mathrm{id})(\widetilde{U_0}(x \otimes 1)\widetilde{U_0}^*) = \tau_{R_0}(x).1,$$

for all x in $\mathcal{B}(\mathcal{H})$, where $\tau_{R_0}(x) = \mathrm{Tr}(x R_0 e^{-tD^2})$.

We define a positive, unbounded operator R on $\mathcal{H}$ by $R(v^n_{i,\pm\frac{1}{2}}) = \mu^{-2i} v^n_{i,\pm\frac{1}{2}}$.

Proposition 2.3.4 ad_{U_0} *preserves the R-twisted volume. In particular, for x in $\pi(S^2_{\mu,c})$ and $t > 0$, we have $h(x) = \frac{\tau_R(x)}{\tau_R(1)}$, where* $\tau_R(x) := \mathrm{Tr}(x R e^{-tD^2})$, *and h denotes the restriction of the Haar state of $SU_\mu(2)$ to the subalgebra $S^2_{\mu,c}$, which is the unique $SU_\mu(2)$-invariant state on $S^2_{\mu,c}$.*

Proof It is enough to prove that τ_R is α_{U_0}-invariant. Let us denote by $P_{\frac{1}{2}}, P_{-\frac{1}{2}}$ the projections onto the closed subspaces generated by $\{v^l_{i,\frac{1}{2}}\}$ and $\{v^l_{i,-\frac{1}{2}}\}$, respectively. Moreover, let $\tau_\pm$ be the functionals defined by $\tau_\pm(x) = \mathrm{Tr}(x R_0 P_{\pm\frac{1}{2}} e^{-tD^2})$. Now observing that R_0, e^{-tD^2} and U_0 commute with $P_{\pm\frac{1}{2}}$ and using Proposition 2.3.3, we have, for x belonging to $\mathcal{B}(\mathcal{H})$,

$$\begin{aligned}
&(\tau_\pm \otimes \mathrm{id})(\alpha_{U_0}(x)) \\
&= (\mathrm{Tr} \otimes \mathrm{id})(\widetilde{U_0}(x \otimes 1)\widetilde{U_0}^*(R_0 P_{\pm\frac{1}{2}} e^{-tD^2} \otimes \mathrm{id})) \\
&= (\mathrm{Tr} \otimes \mathrm{id})(\widetilde{U_0}(x P_{\pm\frac{1}{2}} \otimes 1)\widetilde{U_0}^*(R_0 e^{-tD^2} \otimes \mathrm{id})) \\
&= (\tau_{R_0} \otimes \mathrm{id})(\alpha_{U_0}(x P_{\frac{1}{2}})) \\
&= \tau_{R_0}(x P_{\pm\frac{1}{2}}) \\
&= \tau_\pm(x).1,
\end{aligned}$$

that is, $\tau_\pm$ are α_{U_0}-invariant.

Thus, $x \mapsto \mathrm{Tr}(x R_0 P_{\pm\frac{1}{2}} e^{-tD^2})$ is invariant under α_{U_0}. Moreover, since we have $R P_{\pm\frac{1}{2}} = \mu^{\pm} R_0 P_{\pm\frac{1}{2}}$, the functional τ_R coincides with $\mu^{-1}\tau_+ + \mu\tau_-$, hence is α_{U_0}-invariant. □

Theorem 2.3.5 *The spectral triple described on the Podles' sphere $S^2_{\mu,c}$ as described in Example 2.2.9 is $SU_\mu(2)$ equivariant. If $\alpha : S^2_{\mu,c} \to S^2_{\mu,c} \otimes SO_\mu(3) \subseteq SU_\mu(2) \otimes SO_\mu(3)$ denotes the canonical coaction of $SO_\mu(3)$ on $S^2_{\mu,c}$ (Sect. 1.3.3) and U_0 is as above, then* $\mathrm{ad}_{U_0}(\pi(x)) = (\pi \otimes \mathrm{id})\alpha(x)$ *Moreover,* ad_{U_0} *preserves τ_R.*

Proof

$$\begin{aligned}(D \otimes \mathrm{id})U_0(v^l_{i,\pm\frac{1}{2}}) &= (D \otimes \mathrm{id})(\sum v^l_{j,\pm\frac{1}{2}} \otimes t^l_{j,i}) \\ &= (c_1 l + c_2) \sum v^l_{j,\mp\frac{1}{2}} \otimes t^l_{j,i} \\ &= (c_1 l + c_2) U_0(v^l_{i,\mp\frac{1}{2}}) \\ &= U_0 D(v^l_{i,\pm\frac{1}{2}}).\end{aligned}$$

Thus, the above spectral triple is equivariant w.r.t. the corepresentation U_0.

For the second statement, let U denote the right regular corepresentation of $SU_\mu(2)$ on $L^2(SU_\mu(2), h)$, so that $U_0 = U|_{\mathcal{H}}$. We already noted that the coaction α of $SU_\mu(2)$ is the restriction of the coproduct, that is, $\alpha(x) = U(x \otimes 1)U^*$ for $x \in S^2_{\mu,c} \subseteq \mathcal{B}(L^2(S^2_{\mu,c}))$. Now, $\pi(x) = x|_{\mathcal{H}}$, and we also observed that both x and U (hence U^*) leaves $\mathcal{H}$ invariant. Thus,

$\mathrm{ad}_{U_0}(\pi(x)) = U_0(\pi(x) \otimes \mathrm{id})U_0^* = (U(x \otimes \mathrm{id})U^*)|_{\mathcal{H} \otimes SO_\mu(3)} = \alpha(x)|_{\mathcal{H} \otimes SO_\mu(3)}$ $= (\pi \otimes \mathrm{id})(\alpha(x))$.

Finally, ad_{U_0} preserves τ_R by Proposition 2.3.4. □

2.3.3 Constructions from Coactions by Quantum Isometries

In this subsection, we shall briefly discuss the relevance of quantum isometry group to the problem of constructing quantum group equivariant spectral triples, which is important to understand the role of quantum groups in the framework of noncommutative geometry. There has been a lot of activity in this direction recently, see, for example, the articles by Chakraborty and Pal [25], Connes [32], Landi et al. [28], and the references therein. In the classical situation, there exists a natural unitary representation of the isometry group $G = \mathrm{ISO}(M)$ of a manifold M on the Hilbert space of forms, so that the operator $d + d^*$ (where d is the de-Rham differential operator) commutes with the representation. Indeed, $d + d^*$ is also a Dirac operator for the spectral triple given by the natural representation of $C^\infty(M)$ on the Hilbert space of forms, so we have a canonical construction of G-equivariant spectral triple. Our aim in this subsection is to generalize this to the noncommutative framework, by proving that $d_D + d_D^*$ is equivariant with respect to a canonical unitary corepresentation on the Hilbert space of 'noncommutative forms'.

Consider an admissible spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ and moreover, make the assumption of Lemma 2.2.16, i.e., assume that $t \mapsto e^{itD} x e^{-itD}$ is norm-differentiable at $t = 0$ for all x in the $*$-algebra $\mathcal{B}$ generated by $\mathcal{A}^\infty$ and $[D, \mathcal{A}^\infty]$.

Lemma 2.3.6 *In the notation of Lemma 2.2.16, we have the following (where $b, c \in \mathcal{A}^\infty$):*

$$d_D^*(d_D(b)c) = -\frac{1}{2}\left(b\mathcal{L}(c) - \mathcal{L}(b)c - \mathcal{L}(bc)\right). \tag{2.3.1}$$

Proof Denote by $\chi(b, c)$ the right hand side of Eq. (2.3.1) and fix any $a \in \mathcal{A}^\infty$. Using the facts the the functional τ is a faithful trace on the $*$-algebra $\mathcal{B}$, $\mathcal{L} = -d_D^* d_D$ and that $\tau([D, X]) = 0$ for any X in $\mathcal{B}$, we have,

$$\begin{aligned}
&\tau(a^*\chi(b, c)) \\
&= -\frac{1}{2}\{\tau(a^*b\mathcal{L}(c)) - \tau(ca^*\mathcal{L}(b)) - \tau(a^*\mathcal{L}(bc))\} \\
&= \frac{1}{2}\{\tau([D, a^*b][D, c]) - \tau([D, ca^*][D, b]) - \tau([D, a^*][D, bc])\} \\
&= \frac{1}{2}\{\tau(a^*[D, b][D, c]) - \tau([D, c]a^*[D, b]) - \tau(c[D, a^*][D, b]) - \tau([D, a^*][D, b]c)\} \\
&= -\tau([D, a^*][D, b]c) \\
&= \tau([D, a]^*[D, b]c) \\
&= \langle d_D(a), d_D(b)c\rangle \\
&= \tau(a^*(d_D^*(d_D(b)c))).
\end{aligned}$$

From this, we get the following by a simple computation:

$$\langle ad_D(b), a'd_D(b')\rangle = -\frac{1}{2}\tau(b^*\Psi(a^*a', b')), \tag{2.3.2}$$

for $a, b, a', b' \in \mathcal{A}^\infty$, and where $\Psi(x, y) := \mathcal{L}(x)y - x\mathcal{L}(y)$. Now, let us denote the quantum isometry group of the given spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ by $(\mathcal{G}, \Delta, \alpha)$. Let $\mathcal{A}_0$ denote the $*$-algebra generated by $\mathcal{A}_0^\infty$ and $\mathcal{G}_0$ denote the $*$-algebra of $\mathcal{G}$ generated by matrix elements of irreducible corepresentations. Clearly, $\alpha : \mathcal{A}_0 \to \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0$ is a Hopf-algebraic coaction of $\mathcal{G}_0$ on $\mathcal{A}_0$. Define a $\mathbb{C}$-bilinear map $\tilde{\Psi} : (\mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0) \times (\mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0) \to \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0$ by setting

$$\tilde{\Psi}((x \otimes q), (x' \otimes q')) := \Psi(x, x') \otimes (qq').$$

It follows from the relation $(\mathcal{L} \otimes \text{id}) \circ \alpha = \alpha \circ \mathcal{L}$ on $\mathcal{A}_0$ that

$$\tilde{\Psi}(\alpha(x), \alpha(y)) = \alpha(\Psi(x, y)). \tag{2.3.3}$$

We now define a linear map $\alpha^{(1)}$ from the linear span of $\{ad_D(b) : \ a, b \in \mathcal{A}_0\}$ to $\mathcal{H}_D^1 \otimes \mathcal{G}$ by setting

$$\alpha^{(1)}(ad_D(b)) := \sum_{i,j} a_i^{(1)} d_D(b_j^{(1)}) \otimes a_i^{(2)} b_j^{(2)},$$

where for any $x \in \mathcal{A}_0$ we write $\alpha(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)} \in \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0$ (summation over finitely many terms). We shall sometimes use the Sweedler convention of writing the above simply as $\alpha(x) = x^{(1)} \otimes x^{(2)}$. It then follows from the identities (2.3.2) and (2.3.3), and also the fact that $(\tau \otimes \text{id})(\alpha(a)) = \tau(a)1$ for all $a \in \mathcal{A}_0$ that

$$
\begin{aligned}
&\langle \alpha^{(1)}(a\, d_D(b)), \alpha^{(1)}(a'\, d_D(b'))\rangle_{\mathcal{G}} \\
&= -\frac{1}{2}(\tau \otimes \mathrm{id})(\alpha(b^*)\tilde{\Psi}(\alpha(a^*a'), \alpha(b'))) \\
&= -\frac{1}{2}(\tau \otimes \mathrm{id})(\alpha(b^*)\alpha(\Psi(a^*a', b'))) \\
&= -\frac{1}{2}(\tau \otimes \mathrm{id})(\alpha(b^*\Psi(a^*a', b'))) \\
&= -\frac{1}{2}\tau(b^*\Psi(a^*a', b'))1_{\mathcal{G}} \\
&= \langle a d_D(b), a' d_D(b')\rangle 1_{\mathcal{G}}.
\end{aligned}
$$

This proves that $\alpha^{(1)}$ is indeed well-defined and extends to a $\mathcal{G}$-linear isometry on $\mathcal{H}_D^1 \otimes \mathcal{G}$, to be denoted by $U^{(1)}$, which sends $(ad_D(b)) \otimes q$ to $\alpha^{(1)}(ad_D(b))(1 \otimes q)$, $a, b \in \mathcal{A}_0$, $q \in \mathcal{G}$. Moreover, since the linear span of $\alpha(\mathcal{A}_0^\infty)(1 \otimes \mathcal{G})$ is dense in $\mathcal{H}_D^0 \otimes \mathcal{G}$, it is easily seen that the range of the isometry $U^{(1)}$ is the whole of $\mathcal{H}_D^1 \otimes \mathcal{G}$, i.e., $U^{(1)}$ is a unitary. In fact, from its definition it can also be shown that $U^{(1)}$ is a unitary corepresentation of the compact quantum group $\mathcal{G}$ on $\mathcal{H}_D^1$.

In a similar way, we can construct unitary corepresentation $U^{(n)}$ of $\mathcal{G}$ on the Hilbert space of n-forms for any $n \geq 1$, by defining

$$
\begin{aligned}
&U^{(n)}((a_0 d_D(a_1) d_D(a_2) \ldots d_D(a_n)) \otimes q) \\
&= a_0^{(1)} d_D(a_1^{(1)}) \ldots d_D(a_n^{(1)}) \otimes (a_0^{(2)} a_1^{(2)} \ldots a_n^{(2)} q), \\
&(\text{where } a_i \in \mathcal{A}_0^\infty, q \in \mathcal{G}, \quad \text{and Sweedler convention is used}),
\end{aligned}
$$

and verifying that it extends to a unitary. We also denote by $U^{(0)}$ the unitary corepresentation $\tilde{\alpha}$ on $\mathcal{H}_D^0$ discussed before. Finally, we have a unitary corepresentation $U = \bigoplus_{n \geq 0} U^{(n)}$ of $\mathcal{G}$ on $\tilde{\mathcal{H}} := \bigoplus_n \mathcal{H}_D^n$, and also extend d_D as a closed densely defined operator on $\tilde{\mathcal{H}}$ in the obvious way, by defining $d_D(a_0 d_D(a_1) \ldots d_D(a_n)) = d_D(a_0) \ldots d_D(a_n)$. It is now straightforward to see the following:

Theorem 2.3.7 *The operator $D' := d_D + d_D^*$ is equivariant in the sense that $U(D' \otimes 1) = (D' \otimes 1)U$.*

We point out that there is a natural corepresentation π of $\overline{\mathcal{A}}$ on $\tilde{\mathcal{H}}$ given by $\pi(a)(a_0 d_D(a_1) \ldots d_D(a_n)) = a a_0 d_D(a_1) \ldots d_D(a_n)$, and $(\pi(\mathcal{A}^\infty), \tilde{\mathcal{H}}, D')$ is indeed a spectral triple, which is $\mathcal{G}$-equivariant.

Although the relation between spectral properties of D and D' is not clear in general, in many cases of interest (e.g., when there is an underlying type $(1, 1)$ spectral data in the sense of [27]) these two Dirac operators are closely related. As an illustration, consider the canonical spectral on the noncommutative 2-torus $\mathcal{A}_\theta$, which is discussed in some details in the next section. In this case, the Dirac operator D acts on $L^2(\mathcal{A}_\theta, \tau) \otimes \mathbb{C}^2$, and it can easily be shown (see [27]) that the Hilbert space of forms is isomorphic with $L^2(\mathcal{A}_\theta, \tau) \otimes \mathbb{C}^4 \cong L^2(\mathcal{A}_\theta) \otimes \mathbb{C}^2$; thus D' is essentially same as D in this case.

2.3.4 R-twisted Volume Form Coming from the Modularity of a Quantum Group

Let $(\mathcal{S}, \Delta)$ be a compact quantum group and $(\mathcal{A}^\infty, \mathcal{H}, D)$ be an $\mathcal{S}$-equivariant spectral triple with a unitary corepresentation V on $\mathcal{H}$ commuting with D. In this subsection, our aim is to show the existence of a densely defined positive functional on $\mathcal{B}(\mathcal{H})$, to be interpreted as a generalization of "volume form", which is kept invariant under ad_V.

The Hilbert space $\mathcal{H}$, on which D acts decomposes into finite dimensional eigenspaces $\mathcal{H}_k$ $(k \geq 1)$ of the operator D, i.e., $\mathcal{H} = \oplus_k \mathcal{H}_k$. Since D commutes with V, V preserves each of the $\mathcal{H}_k$'s and on each $\mathcal{H}_k$, V is a unitary corepresentation of the compact quantum group $\mathcal{Q}$. Then we have the decomposition of each $\mathcal{H}_k$ into the irreducibles, say

$$\mathcal{H}_k = \oplus_{\pi \in \mathcal{I}_k} \mathbb{C}^{d_\pi} \otimes \mathbb{C}^{m_{\pi,k}},$$

where $m_{\pi,k}$ is the multiplicity of the irreducible corepresentation of type π on $\mathcal{H}_k$ and $\mathcal{I}_k$ is some finite subset of $\mathrm{Rep}(\mathcal{Q})$. Since R commutes with V, R preserves direct summands of $\mathcal{H}_k$. Let $\{e_i^\pi : i = 1, 2, \cdots d_\pi\}$ be an orthonormal basis of $\mathbb{C}^{d_\pi}$ such that $V(e_i^\pi) = \sum_j e_j^\pi \otimes u_{ji}^\pi$.

Let $\mathcal{E}_D$ denote the WOT-dense $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by rank one operators of the form $|\xi >< \eta|$, where ξ, η are eigenvectors of D. We note that since V maps $\mathcal{H}_k$ into $\mathcal{H}_k \otimes_{\mathrm{alg}} \mathcal{S}_0$ for all k, ad_V will map $\mathcal{E}_D$ into $\mathcal{E}_D \otimes_{\mathrm{alg}} \mathcal{S}_0$.

With the above set up and notations, we give the following definition.

Definition 2.3.8 An R-twisted spectral data (of compact type) is given by a quadruplet $(\mathcal{A}^\infty, \mathcal{H}, D, R)$, where

1. $(\mathcal{A}^\infty, \mathcal{H}, D)$ is a spectral triple of compact type.
2. R a positive (possibly unbounded) invertible operator such that R commutes with D.

We shall also sometimes refer to $(\mathcal{A}^\infty, \mathcal{H}, D)$ as an R-twisted spectral triple.

Remark 2.3.9 We remark that in the above definition, we do not need the full strength of Definition 2.2 in [33].

Definition 2.3.10 The functional τ_R defined below on the weakly dense $*$-subalgebra $\mathcal{E}_D$ of $\mathcal{B}(\mathcal{H})$ will be called the R-twisted volume form:

$$\tau_R(x) = Tr(Rx), \quad x \in \mathcal{E}_D.$$

We now characterize those R for which ad_V preserves the functional τ_R.

Theorem 2.3.11 *Let $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ be an R-twisted spectral data of compact type which is equivariant with respect to a corepresentation V of a CQG $\mathcal{S}$ on $\mathcal{H}$. Then ad_V preserves the R-twisted volume form if and only if R is of the following form:*

$$R|_{\mathcal{H}_k} = \oplus_{\pi \in \mathcal{I}_k} F^\pi \otimes T_{\pi,k}, \tag{2.3.4}$$

for some $T_{\pi,k} \in \mathcal{B}(\mathbb{C}^{m_{\pi,k}})$, where F^π's are as in Sect. 1.2.2.

Proof Let $\{f_j^{\pi,k}\}_{j=1}^{m_{\pi,k}}$ be an orthonormal basis for $\mathbb{C}^{m_{\pi,k}}$. Then $\{e_i^\pi \otimes f_j^{\pi,k} : i = 1, 2, \cdots d_\pi, \ j = 1, \ldots m_{\pi,k}\}$ is an orthonormal basis for $\mathcal{H}_k$. As R commutes with D, it leaves $\mathcal{H}_k$ invariant. Let us write

$$R(e_i^\pi \otimes f_j^{\pi,k}) = \sum_{s,t} R^{\pi,k}(s, t, i, j) e_s \otimes f_t.$$

Let h denote the extension of the Haar state of $\mathcal{S}$ to a vector state on $\mathcal{B}(L^2(\mathcal{S}, h))$ given by $h(x) =< 1, x1 >$.

For a fixed π, k, denoting $e_i^\pi, \ f_j^{\pi,k}, \ R^{\pi,k}(s, t, i, j)$ by $e_i, \ f_j, \ R(s, t, i, j)$, respectively, and for a in $\mathcal{E}_D$, we have the following:

$$\begin{aligned}
(\tau_R \otimes h)\mathrm{ad}_V(a) &= \sum_{i,j} < V^*(e_i \otimes f_j \otimes 1_{\mathcal{Q}}), (a \otimes 1)V^* R(e_i \otimes f_j) > \\
&= \sum_{i,j,k,s,t,u} < e_k \otimes f_j \otimes (u_{ik}^\pi)^*, R(s, t, i, j) a(e_u \otimes f_t) \otimes (u_{su}^\pi)^* > \\
&= \sum_{i,j,k,s,t,u} \frac{R(s, t, i, j)}{M_\pi} < e_k \otimes f_j, a(e_u \otimes f_t) > \delta_{is} F^\pi(k, u) \\
&= \sum_{i,j,k,t,u} \frac{R(i, t, i, j)}{M_\pi} < e_k \otimes f_j, a(e_u \otimes f_t) > F^\pi(k, u).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\tau_R(a) &= Tr(a.R) \\
&= \sum_{i,j} < e_i \otimes f_j, aR(e_i \otimes f_j) > \\
&= \sum_{k,j,u,t} R(u, t, k, j) < e_k \otimes f_j, a(e_u \otimes f_t) > .
\end{aligned}$$

Now observe that if R is of the form given in the theorem, then $R(s, t, i, j) = F^\pi(i, s) T_{\pi,k}(j, t)$. Plugging this in the expressions for $(\tau_R \otimes h)\mathrm{ad}_V(a)$ and $\tau_R(a)$ obtained above, and using the fact that $M_\pi = \sum_i F^\pi(i, i)$, we get $(\tau_R \otimes h)\mathrm{ad}_V(a) = \tau_R(a)$. It follows easily that ad_V preserves τ_R.

We now prove the necessity part of the theorem. We note that $(\tau_R \otimes h)\mathrm{ad}_V(a) = \tau_R(a)$ implies:

$$\sum_{i,j,k,t,u} \frac{R(i,t,i,j)}{M_\pi} < e_k \otimes f_j, a(e_u \otimes f_t) > F^\pi(k,u)$$
$$= \sum_{k,j,u,t} R(u,t,k,j) < e_k \otimes f_j, a(e_u \otimes f_t) > . \qquad (2.3.5)$$

Now fix u_0, t_0 and consider $a \in \mathcal{B}(\mathcal{H})$ such that $a(e_{u_0} \otimes f_{t_0}) = e_p \otimes f_q$ and zero on the other basis elements. Then from (2.3.5), we get

$$\sum_{i,j,k} \frac{R(i,t_0,i,j)}{M_\pi} < e_k \otimes f_j, e_p \otimes f_q > F^\pi(k,u_0)$$
$$= \sum_{k,j} R(u_0,t_0,k,j) < e_k \otimes f_j, e_p \otimes f_q >,$$

which gives $\sum_i \frac{R(i,t_0,i,q)}{M_\pi} F^\pi(p,u_0) = R(u_0,t_0,p,q)$.

This proves that $R|_{\mathcal{H}_k} = \oplus_{\pi \in \mathcal{I}_k} F^\pi \otimes T_{\pi,k}$ with some $T_{\pi,k} \in \mathcal{B}(\mathbb{C}^{m_{\pi,k}})$ given by $T_{\pi,k}(t_0,q) = \sum_i \frac{R(i,t_0,i,q)}{M_\pi}$. □

As an immediate corollary, we get the following:

Proposition 2.3.12 *Let $R = \Pi_V(\phi_1) \in \mathcal{B}(\mathcal{H})$, where ϕ_1 is the functional defined in Proposition 1.2.19 and Π_V is as in Theorem 1.4.1. Suppose also that $L \in \mathcal{B}(\mathcal{H})$ is $(\mathcal{S}, V)$ equivariant. Then we have:*

a. R is a (possibly unbounded) positive operator with $\mathrm{Dom}(R)$ *containing the subspaces $\mathcal{H}_k$, $k \geq 1$.*

b. $RD = DR$.

c. ad_V *preserves the functional τ_R.*

Thus, given a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ (of compact type) which is $\mathcal{S}$-equivariant with respect to a corepresentation V of a CQG $\mathcal{S}$ on $\mathcal{H}$, we can always construct a positive (possibly unbounded) invertible operator R on $\mathcal{H}$ such that $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ is a twisted spectral data and ad_V *preserves the functional τ_R.*

Proof This follows from Theorem 2.3.11 as R is of the form (2.3.4) with $T_{\pi,k} = I$ for all π, k. □

Remark 2.3.13 If L in Proposition 2.3.12 is such that RL is trace class, then the functional χ is defined and bounded on $\mathcal{B}(\mathcal{H})$ and the conclusion of the proposition holds as well.

Remark 2.3.14 (a)When the spectral triple in question has a real structure as in Definition 2.2.3, there is a canonical choice of R (see Remark 3.3.3).

(b) When the Haar state of $\mathcal{S}$ is tracial, then it follows from the definition of R and Theorem 1.5 part 1. of [34] that R can be chosen to be I.

We record the following lemma for future use.

Lemma 2.3.15 *The* ad_V*-invariance of the functional* τ_R *on* $\mathcal{E}_D$ *is equivalent to the* ad_V*-invariance of the functional* $X \mapsto \mathrm{Tr}(XRe^{-tD^2})$ *on* $\mathcal{E}_D$ *for each* $t > 0$*. If, furthermore, the* R*-twisted spectral triple is* Θ*-summable in the sense that* Re^{-tD^2} *is trace class for every* $t > 0$*, then* ad_V *preserves the functional* $\mathcal{B}(\mathcal{H}) \ni x \mapsto \mathrm{Lim}_{t\to 0+} \frac{\mathrm{Tr}(xRe^{-tD^2})}{\mathrm{Tr}(Re^{-tD^2})}$*, where* Lim *is as defined in Sect. 2.2.2.*

Proof If W_λ denotes the eigenspace of D corresponding to the eigenvalue, say λ, it is clear that $\tau_R(X) = e^{t\lambda^2}\mathrm{Tr}(Re^{-tD^2}X)$ for all $X = |\xi><\eta|$ with ξ, η belonging to W_λ and for any $t > 0$. Thus, the ad_V-invariance of the functional τ_R on $\mathcal{E}_D$ is equivalent to the ad_V-invariance of the functional $X \mapsto \mathrm{Tr}(XRe^{-tD^2})$ on $\mathcal{E}_D$ for each $t > 0$. This can be argued as follows. Let ad_V be τ_R invariant on $\mathcal{E}_D$, that is, for all $|\xi><\eta|$ with ξ, η belonging to W_λ, $(\tau_R \otimes \mathrm{id})\mathrm{ad}_V(|\xi><\eta|) = \tau_R(|\xi><\eta|).1$ Therefore, $(\tau_R \otimes \mathrm{id})\mathrm{ad}_V(|\xi><\eta|) = \tau_R(|\xi><\eta|).1 = e^{t\lambda^2}Tr(Re^{-tD^2}|\xi><\eta|)$. On the other hand, $(\tau_R \otimes \mathrm{id})\mathrm{ad}_V(|\xi><\eta|) = e^{t\lambda^2}(Tr(Re^{tD^2}.) \otimes \mathrm{id})\mathrm{ad}_V(|\xi><\eta|)$. If the R-twisted spectral triple is Θ-summable, the above is also equivalent to the ad_V-invariance of the bounded normal functional $X \mapsto \mathrm{Tr}(XRe^{-tD^2})$ on the whole of $\mathcal{B}(\mathcal{H})$. In particular, this implies that ad_V preserves the functional $\mathcal{B}(\mathcal{H}) \ni x \mapsto \mathrm{Lim}_{t\to 0+} \frac{\mathrm{Tr}(xRe^{-tD^2})}{\mathrm{Tr}(Re^{-tD^2})}$.
□

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