Preface

Based on theorems of Picard and Borel, R. Nevanlinna published his paper [25] and evolved a theory that carries his name. The development of the theory into complex analysis in several variables was quite fast and a number of papers such as A. Bloch [26b], H. Cartan [33], and H. Weyl–J. Weyl [38] were published. Following H. Weyl–J. Weyl [38], the paper of A.L. Ahlfors [41] was published, and this work was succeeded by W. Stoll [53b], [54], being developed into the theory of meromorphic mappings from parabolic spaces into complex projective spaces.

S.-S. Chern studied the value distribution of holomorphic curves from the viewpoint of differential geometry, and published a number of papers; one of them was a joint paper, Bott–Chern [65], where they introduced a "connection" to the study of the value distribution theory and found a relation with characteristic classes. S.-S. Chern introduced with Osserman the Nevanlinna theoretic method into the study of Gauss' maps of minimal surfaces in Euclidean spaces. This approach was advanced much, especially by H. Fujimoto, and finally reached the solution of Gauss' map conjecture (H. Fujimoto [88a]).

From the late 1960s to the early 1970s the theory of Kobayashi hyperbolic manifolds and the value distribution theory of equidimensional holomorphic mappings were initiated (Kobayashi [67], [70]; Carlson–Griffiths [72]; Griffiths–King [73]), and the value distribution theory in several complex variables advanced considerably and was enlarged, so that it has broadened the border with more related subjects.

T. Ochiai [77] revived Bloch's paper (Bloch [26b]) and proved a lemma on holomorphic differentials, which is a counterpart to the lemma on logarithmic derivatives in Nevanlinna theory, and obtained the so-called the Bloch–Ochiai Theorem. In the study of the extension of the Bloch–Ochiai Theorem to the non-compact case, the relation with P. Deligne's logarithmic forms (Deligne [71], [74]) was found and the initial step to deal with varieties other than complex projective space began for the value distribution theory of holomorphic curves (Noguchi [77b]). As a result, the Bloch–Ochiai Theorem and the classical theorem of Borel were unified.

On the other hand, S. Lang formulated a conjecture of a higher dimensional version of Mordell's conjecture on rational points such that a Kobayashi hyperbolic algebraic manifold defined over a number field would have at most finitely many rational points (Lang's conjecture), and proposed a number of related conjectures to enhance the study of problems over function fields and of holomorphic curves (Mordell [22]; Lang [74], [86]). In 1987 P. Vojta [87], completing the viewpoint of Osgood [81], formulated an analogue between Nevanlinna theory and Diophantine approximation theory. He proposed the well-known Vojta conjecture, which has attracted the attention of many researchers.

As for monographs describing these developments of the study of Nevanlinna theory and Diophantine approximation theory, we may list, in order of year, e.g., Kobayashi [70], Wu [70], Stoll [70], [73], Griffiths [76], Stoll [77a], Lang [83], Shiffman [83], Ochiai–Noguchi [84], Shabat [85], Vojta [87], Lang [87], Noguchi–Ochiai [90], Lang [91], Fujimoto [93], Kobayashi [98], Ru [01] and Bombieri–Gubler [06].

On Kobayashi's hyperbolicity, his book [98] provides in great detail the results obtained up to the present. Fujimoto [93] describes the theory of Gauss' maps of minimal surfaces in a self-contained manner.

The core of Nevanlinna theory consists of two Main Theorems. The First Main Theorem is considered to be a non-compact version of Poincaré's duality, and we now have a satisfactory theory for it. In the present book the second chapter is devoted to it. On the other hand, there are not many cases where the Second Main Theorem is established for meromorphic mappings $f : \mathbf{C}^n \to M$ and divisors D on M; the following is essentially the full list:

(i) $n = 1$, $M = \mathbf{P}^m(\mathbf{C})$, D is a union of hyperplanes in general position (Cartan [33]; H. Weyl–J. Weyl [38]; Ahlfors [41]), Nochka's solution of Cartan's conjecture (Nochka [83]; Chen [90]).
(ii) $n \geqq 1$, $M = \mathbf{P}^m(\mathbf{C})$, D is a union of hyperplanes in general position (Stoll [53b], [54]; Chen [90]).
(iii) $n = 1$, $M = \mathbf{P}^m(\mathbf{C})$, D is a union of hypersurfaces in general position (Eremenko–Sodin [92]; Corvaja–Zannier [04b]; Ru [04]).
(iv) $n \geqq \dim M$ and f is differentiably non-degenerate (of maximal rank at some point) (Griffiths' theory, Carlson–Griffiths [72]; Griffiths–King [73]; Sakai [74a]; Shiffman [75]; Noguchi [76a]).
(v) M is an abelian or semi-abelian variety and D is an arbitrary divisor on it. When M is a semi-abelian variety, a compactification of M is taken so that the closure $\bar{D}$ is in general position (in a sense) (Noguchi–Winkelmann–Yamanoi [00], [02], [08]).
(vi) $M = \mathbf{P}^1(\mathbf{C})$, $n = 1$ and D is a moving divisor for which the counting functions are truncated to level one (K. Yamanoi [04d], [05], [06]).

There is no book among the list above mentioned that deals with all of the above cases. Because of the applications to the degeneracy problem and Kobayashi hyperbolicity, we deal here with the cases of (i), (ii), (iv), and (v), and describe the theory of the all Second Main Theorems in these cases from a uniformized viewpoint in a self-contained way (Chaps. 3, 4 and 6). For the other cases, we limit ourselves only to introducing some important results. The Kobayashi hyperbolicity will be discussed in Chap. 7, and we will describe some new results which were not mentioned in Kobayashi [98]. These are the first purpose of this book.

The second purpose is to describe and discuss the analogy between Nevanlinna theory and Diophantine approximation theory. The case of function fields is dealt with in Chap. 8, and the case of number fields in Chap. 9.

The readers we have in mind are graduate students and researchers who are interested in these subjects. For an easier understanding of the idea of the development of the theory we start with Nevanlinna theory for meromorphic functions on the one-dimensional complex plane. As prerequisites the basic theory of analytic functions in one variable, the very elementary part of analytic functions in several complex variables, and the terminologies of the theory of complex manifolds are assumed. Other materials which we use in the text without giving the proofs will be explained at least so that the readers are able to understand the contents.

We will briefly mention the content of each chapter in the sequel.

In Chap. 1 we prove the First and the Second Main Theorems in a self-contained manner for meromorphic functions on the Gaussian plane, which is the most basic part of Nevanlinna theory. This provides the prototype of the theory extended to the higher dimensional case.

In Chap. 2 plurisubharmonic functions are described and Jensen's formula is proved. Then we show the First Main Theorem for meromorphic mappings from the m-dimensional complex space $\mathbf{C}^m$ into a compact complex manifold with respect to holomorphic line bundles and coherent ideal sheaves of the structure sheaf. We will introduce a various fundamental properties of order functions. The content of this chapter serves the basic for the research of this subject, and we intended that the readers naturally master the fundamental method and knowledge of this subject.

In Chap. 3 we first extend the lemma on logarithmic derivatives to the case of several complex variables. We will give a simplified proof for it. Then we prove the Second Main Theorem by using the lemma on logarithmic derivatives, which is different to the metric method due to Griffiths et al. We also give some applications.

In Chap. 4 we deal with the theory of holomorphic curves from the complex plane, which we call "*entire curves*". We describe, following to Chen [90], the notion of the Nochka weights, which plays an important role in the proof of Cartan's conjecture, now Nochka's theorem. Lemma on logarithmic jet differentials is proved in a most generalized and uniformized form due to Noguchi [77b], [81b], [85b], [86], and Vitter [77]. Using this we prove an inequality of the Second Main Theorem type for entire curves in an algebraic variety, and then deduce the Logarithmic Bloch–Ochiai Theorem (Noguchi [77b], [81b]).

Chapter 5 is devoted to the semi-tori and their compactifications. The contents are the preparation for the next chapter.

In Chap. 6 we then establish the late result of the Second Main Theorem for entire curves in semi-abelian varieties (Noguchi–Winkelmann–Yamanoi [00], [02], [08]). From this Lang's conjecture for entire curves is derived, and more applications are given (Yamanoi [04b]; Noguchi–Winkelmann–Yamanoi [07], [13]; Winkelmann [11]; Corvaja–Noguchi [12]; Lu–Winkelmann [12]).

In Chap. 7 we describe the applications of the results of Chaps. 4 and 6 to the algebraic degeneracy problem of entire curves and the Kobayashi hyperbolicity problem. In particular, being related to the Kobayashi conjecture, the existence of an algebraic curve of degree 5 in $\mathbf{P}^2(\mathbf{C})$ whose complement is hyperbolic and hyperbolically embedded into $\mathbf{P}^2(\mathbf{C})$ is proved (Zaidenberg [89]). We construct explicit examples of hyperbolic hypersurfaces of $\mathbf{P}^n(\mathbf{C})$ for arbitrary n, and those whose complement are hyperbolic and hyperbolically embedded into $\mathbf{P}^n(\mathbf{C})$.

In Chap. 8 we describe Nevanlinna theory over function fields. This is motivated by the conjectures of S. Lang and P. Vojta. Nevanlinna theory over function fields is understood as an approximation theory of rational functions by rational functions. We prove an analogue of the so-called *abc*-conjecture of Masser and Oesterlé over function fields (cf. Oesterlé [88], Sect. 3; Granville–Tucker [02]).

In Chap. 9 we describe Diophantine approximation theory over number fields in an analogous way to Nevanlinna theory, based on Vojta's idea. As application we prove finiteness theorems of integral points. Readers will find analogies not only in conjectures or results but also even in proofs.

We would like to draw attention to Nevanlinna theory of p-adic analytic functions, started by Ha (Ha Hui Khoai [83]; Ha Hui Khoai–Tu [95]), which is not mentioned in this book.

There is a large undeveloped area in the relation between Nevanlinna theory of entire curves and Diophantine approximation theory, which is expected to be explored more. We hope that the present book will serve for readers an introduction to this subject.

In writing this book the authors received a valuable comment on Nochka's weights from Professor N. Toda, and helpful advice on Chap. 9 from Professor N. Hirata. Professors Y. Aihara and K. Yamanoi have given a number of invaluable comments all through the book. The authors express their sincere gratitude to all of them. Taking this opportunity, the authors thank Professors Mitsuru Ozawa, Wilhelm Stoll, Shoshichi Kobayashi, Hirotaka Fujimoto, Takushiro Ochiai, and Alan Huckleberry for their teachings, various suggestions and comments all through their research lives.

They also deeply thank the anonymous reviewers for their many helpful comments and suggestions, in particular, one who read Chap. 9 very carefully and suggested a number of valuable comments.

Tokyo — Junjiro Noguchi
Bochum — Jörg Winkelmann
February 2013

http://www.springer.com/978-4-431-54570-5

Nevanlinna Theory in Several Complex Variables and
Diophantine Approximation
Noguchi, J.; Winkelmann, J.
2014, XIV, 416 p. 6 illus., Hardcover
ISBN: 978-4-431-54570-5