Chapter 2

The Symplectic Group

This chapter is a review of the most basic concepts of the theory of the symplectic group, and of related concepts, such as symplectomorphisms or the machinery of generating functions.

We may well be witnessing the advent of a "symplectic revolution" in fundamental Science. In fact, since the late sixties there has been a burst of applications of symplectic techniques to mathematics and physics, and even to engineering or medical sciences (magnetic resonance imaging is a typical example). It seems on the other hand that it may be possible to recast a great deal of mathematics in symplectic terms: there is indeed a process of "symplectization of Science" as pointed out by Gotay and Isenberg [80].

Symplectic geometry differs profoundly from more traditional geometries (such as Euclidean geometry, or its refinement Riemannian geometry) because it appears somewhat counter-intuitive to the uninitiated. In symplectic geometry all vectors are "orthogonal" to themselves because the 'scalar product' is antisymmetric. As a consequence, the notion of length in a symplectic space does not make sense; but instead the notion of area does. For instance, in the plane $\mathbb{R}^2$, the standard symplectic form is (up to the sign) the determinant function: if $z = (x, p)$, $z' = (x', p')$ are two vectors in $\mathbb{R}^2$, then $\det(z, z') = xp' - x'p$ represents the oriented area of the parallelogram built on the vectors z, z'. In higher dimensions the situation is similar: the symplectic product of two vectors is the sum of the algebraic areas of the parallelograms built on the projections of these vectors on the conjugate planes. Symplectic geometry is thus an 'areal' type of geometry; this quality is actually reflected in recent, deep, theorems which express the fact that this 'two-dimensionality' has quite dramatic consequences for the behavior of Hamiltonian flows, which are much more rigid than was thought before the mid-1980s, when Gromov [87] proved very deep results in symplectic topology. Gromov was eventually awarded (2009) the Abel prize (the equivalent of the Nobel prize for mathematics) for his discoveries.

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2.1 Symplectic matrices

Recall that the "standard symplectic matrix" is $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where 0 and I are the $n \times n$ zero and identity matrices. we have $\det J = 1$ and $J^2 = I$, $J^T = J^{-1} = -J$.

2.1.1 Definition of the symplectic group

Definition 20. *The set of all symplectic matrices is denoted by* $\mathrm{Sp}(2n,\mathbb{R})$. Thus $S \in \mathrm{Sp}(2n,\mathbb{R})$ if and only if

$$S^T JS = SJS^T = J. \tag{2.1}$$

If S is symplectic then S^{-1} is also symplectic because

$$(S^{-1})^T JS^{-1} = -(SJS^{-1})^T = J$$

since $J^T = J^{-1} = -J$. The product of two symplectic matrices being obviously symplectic as well, symplectic matrices thus form a group; that group is denoted by $\mathrm{Sp}(2n,\mathbb{R})$ and is called the (real) *symplectic group*. The conditions (2.1) are actually redundant. In fact:

$$S \in \mathrm{Sp}(2n,\mathbb{R}) \Longleftrightarrow S^T JS = J \Longleftrightarrow SJS^T = J \tag{2.2}$$

as you are asked to prove in Exercise 21 below:

Exercise 21. Show that $S \in \mathrm{Sp}(2n,\mathbb{R})$ if and only $S^T \in \mathrm{Sp}(2n,\mathbb{R})$. [Hint: use the fact that $(S^{-1})^T JS^{-1} = J$].

The eigenvalues of a symplectic matrix are of a particular type:

Problem 22. (i) Show that the eigenvalues of a symplectic matrix occur in quadruples $(\lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1})$. [Hint: show that the characteristic polynomial P of a symplectic matrix is reflexive: $P(\lambda) = \lambda^{2n} P(\lambda^{-1})$.] (ii) Show that the determinant of a symplectic matrix is equal to 1. (iii) Show that the eigenvalues of a symplectic matrix S and those of its inverse S^{-1} are the same.

2.1.2 Symplectic block-matrices

It is often useful for practical purposes to use block-matrix notation and to write

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.3}$$

where the entries A, B, C, D are $n \times n$ matrices. Recalling that

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

one verifies by an explicit calculation, using the identities $S^TJS = J = SJS^T$, that this matrix is symplectic if and only the two following sets of equivalent conditions are satisfied:

$$A^TC,\, B^TD \text{ are symmetric, and } A^TD - C^TB = I, \tag{2.4}$$

$$AB^T,\, CD^T \text{ are symmetric, and } AD^T - BC^T = I. \tag{2.5}$$

Using the second set of conditions it follows that the inverse of a symplectic matrix S written in the form (2.3) is

$$S^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}. \tag{2.6}$$

Notice that in the case $n = 1$ the formula above reduces to the familiar

$$S^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

which is true for every 2×2 matrix $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\det(ad - bc) = 1$.

Exercise 23. Verify in detail the formulas (2.4), (2.5), (2.6) above.

Exercise 24. Show, using the conditions (2.4), (2.5) that S is symplectic if and only if S^T is.

Exercise 25. Show that if $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is symplectic, then $AA^T + BB^T$ is invertible. [Hint: calculate $(A+iB)(B^T+iA^T)$ and use the fact that $AB^T = BA^T$.]

2.1.3 The affine symplectic group

An interesting extension of $\mathrm{Sp}(2n,\mathbb{R})$ consists of the affine symplectic automorphisms. We denote by $\mathrm{T}(2n,\mathbb{R})$ the group of phase space translations: $T(z_0) \in \mathrm{T}(2n,\mathbb{R})$ is the mapping $z \longmapsto z + z_0$. Clearly $\mathrm{T}(2n,\mathbb{R})$ is isomorphic to $\mathbb{R}^n \oplus \mathbb{R}^n$ equipped with addition.

Definition 26. The affine (or inhomogeneous) symplectic group is the semi-direct product

$$\mathrm{ASp}(2n,\mathbb{R}) = \mathrm{Sp}(2n,\mathbb{R}) \ltimes \mathrm{T}(2n,\mathbb{R}).$$

Formally, the group law of the semi-direct product $\mathrm{ASp}(2n,\mathbb{R})$ is given by

$$(S,z)(S',z') = (SS', z + Sz');$$

this is conveniently written in matrix form as

$$\begin{pmatrix} S & z \\ 0_{1\times 2n} & 1 \end{pmatrix} \begin{pmatrix} S' & z' \\ 0_{1\times 2n} & 1 \end{pmatrix} = \begin{pmatrix} SS' & Sz' + z \\ 0_{1\times 2n} & 1 \end{pmatrix}. \tag{2.7}$$

One immediately checks that $\mathrm{ASp}(2n,\mathbb{R})$ is identified with the set of all affine transformations F of $\mathbb{R}^n \oplus \mathbb{R}^n$ such that F can be factorized as a product $F = ST(z)$ for some $S \in \mathrm{Sp}(2n,\mathbb{R})$ and $z \in \mathbb{R}^n \oplus \mathbb{R}^n$. Since translations are symplectomorphisms in their own right, it follows that $\mathrm{ASp}(2n,\mathbb{R})$ is the group of all affine symplectomorphisms of the symplectic space $(\mathbb{R}^n \oplus \mathbb{R}^n, \sigma)$. We note the following useful relations:

$$ST(z) = T(Sz)S, \quad T(z)S = ST(S^{-1}z).$$

2.2 Symplectic forms

We have defined the symplectic group in terms of matrices. It turns out that $\mathrm{Sp}(2n,\mathbb{R})$ can be defined intrinsically in terms of a general algebraic notion, that of symplectic form:

2.2.1 The notion of symplectic form

We begin with a general definition:

Definition 27. A bilinear form on $\mathbb{R}^n \oplus \mathbb{R}^n$ (or, more generally, on any even-dimensional real vector space) is called a "symplectic form" if it is antisymmetric and non-degenerate. The special antisymmetric bilinear form σ on $\mathbb{R}^n \oplus \mathbb{R}^n$ defined by

$$\sigma(z, z') = p \cdot x' - p' \cdot x \tag{2.8}$$

for $z = (x, p)$, $z' = (x', p')$ is symplectic; it is called the "standard symplectic form on $\mathbb{R}^n \oplus \mathbb{R}^n$".

The antisymmetry condition means that we have

$$\sigma(z, z') = -\sigma(z', z)$$

for all z, z' in $\mathbb{R}^{2n}$. Notice that the antisymmetry implies in particular that all vectors z are isotropic, that is:

$$\sigma(z, z) = 0.$$

The non-degeneracy condition means that the condition $\sigma(z, z') = 0$ for all $z \in \mathbb{R}^{2n}$ is equivalent to $z = 0$.

Definition (2.8) of the standard symplectic form can be rewritten in a convenient way using the symplectic standard matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where 0 and I are the $n \times n$ zero and identity matrices. In fact

$$\sigma(z, z') = Jz \cdot z' = (z')^T Jz. \tag{2.9}$$

Exercise 28. Show that the standard symplectic form is indeed non-degenerate.

Let s be a linear mapping $\mathbb{R}^n \oplus \mathbb{R}^n \longrightarrow \mathbb{R}^n \oplus \mathbb{R}^n$. The condition $\sigma(sz, sz') = \sigma(z, z')$ is equivalent to $S^T JS = J$ where S is the matrix of s in the canonical basis of $\mathbb{R}^n \oplus \mathbb{R}^n$ that is, to $S \in \mathrm{Sp}(2n, \mathbb{R})$. We can thus redefine the symplectic group by saying that it is the group of all linear automorphisms of $\mathbb{R}^n \oplus \mathbb{R}^n$ which preserve the standard symplectic form σ.

There are other more "exotic" symplectic forms which originate from physical problems (for instance from quantum gravity); here is one example that will be studied further when we discuss non-commutative quantum mechanics at this end of this book: set

$$\Omega = \begin{pmatrix} \hbar^{-1}\Theta & I \\ -I & \hbar^{-1}N \end{pmatrix}$$

where Θ and N are $n \times n$ real antisymmetric matrices, and I the $n \times n$ identity. One usually requires that Θ and N depend on $\hbar$ and that $\Theta = O(\hbar^2)$, $N = O(\hbar^2)$. From this viewpoint Ω can be viewed as perturbation of J: we have $\Omega = J + O(\hbar^2)$. One shows that if $\hbar$ is small enough then Ω is invertible. Since Ω is antisymmetric the formula

$$\omega(z, z') = z \cdot \Omega^{-1} z' = (\Omega^T)^{-1} z \cdot z'$$

defines a new symplectic form on $\mathbb{R}^n \oplus \mathbb{R}^n$ (see Dias and Prata [31]). Note that ω coincides with the standard symplectic form σ when $\Theta = N = 0$.

2.2.2 Differential formulation

There is another, slightly more abstract, way to define the standard symplectic form which has advantages if one has Hamiltonian mechanics on manifolds in mind. It consists in observing that we can view σ as an exterior two-form on $\mathbb{R}^n \oplus \mathbb{R}^n$, in fact:

$$\sigma = dp \wedge dx = \sum_{j=1}^{n} dp_j \wedge dx_j \tag{2.10}$$

where $dp_j \wedge dx_j$ is the wedge product of the coordinate one-forms dp_j and dx_j. This formula is a straightforward consequence of the relation

$$dp_j \wedge dx_j(x, p; x', p') = p_j x'_j - p'_j x_j.$$

With this identification the standard symplectic form is related to the Lebesgue volume form Vol on $\mathbb{R}^n \oplus \mathbb{R}^n$ by the formula

$$\mathrm{Vol} = (-1)^{n(n-1)/2} \frac{1}{n!} \underbrace{\sigma \wedge \sigma \wedge \cdots \wedge \sigma}_{n \text{ factors}}. \tag{2.11}$$

Using this approach one can express very concisely that a diffeomorphism f of $\mathbb{R}^n \oplus \mathbb{R}^n$ is a symplectomorphism:

$$f \in \mathrm{Symp}(2n, \mathbb{R}) \Longleftrightarrow f^*\sigma = \sigma$$

where $f^*\sigma$ is the pull-back of the two-form σ by the diffeomorphism f:

$$f^*\sigma(z_0)(z,z') = \sigma(f(z_0))Df(z_0)z, Df(z_0)z').$$

($Df(z_0)$ the Jacobian matrix at z_0.)

In particular one immediately sees that a symplectomorphism is volume-preserving since we then also have $f^*\,\mathrm{Vol} = \mathrm{Vol}$ in view of (2.11).

The language of differential form allows an elegant (and concise) reformulation of the previous definitions. For instance, part (i) of Theorem (9) can thus be re-expressed as

$$(\phi_t^H)^*\sigma = \sigma.$$

It tuns out that Hamilton's equations can be rewritten in a very neat and concise way using the notion of contraction of a differential form. They are in fact equivalent to the concise relation

$$\iota_{X_H}\sigma + d_z H = 0 \tag{2.12}$$

between the contraction of the symplectic form with the Hamilton field and the differential of the Hamiltonian; this is easily verified by writing this formula "in coordinates", in which case it becomes

$$\sigma(X_H(z,t),\cdot) + d_z H = 0. \tag{2.13}$$

Formula(2.12) is usually taken as the starting point of Hamiltonian mechanics on symplectic manifolds, which is a topic of great current interest.

It is quite easy to reconstruct a Hamiltonian function from its Hamilton vector field; in fact:

$$H(z,t) = H(0,t) - \int_0^1 \sigma(X_H(sz), z)ds. \tag{2.14}$$

This formula is an immediate consequence of the observation that we have, for fixed t,

$$\begin{aligned} H(z,t) - H(0,t) &= \int_0^1 \frac{d}{ds}H(sz,t)ds \\ &= \int_0^1 \partial_z H(sz,t)\cdot z ds \\ &= -\int_0^1 \sigma(X_H(sz), z)ds. \end{aligned}$$

Notice that formula (2.14) defines H up to the addition of a smooth function of t.

2.3 The unitary groups $U(n,\mathbb{C})$ and $U(2n,\mathbb{R})$

Let $U(n,\mathbb{C})$ denote the complex unitary group: $u \in U(n,\mathbb{C})$ if and only if $u \in \mathcal{M}(n,\mathbb{C})$ (the algebra of complex matrices of dimension n) and $u^*u = uu^* = I$ (the conditions $u^*u = I$ and $uu^* = I$ are actually equivalent).

2.3.1 A useful monomorphism

Writing the elements $Z \in \mathcal{M}(n,\mathbb{C})$ in the form $Z = A + iB$ where A and B are real matrices we define a mapping

$$\iota : \mathcal{M}(n,\mathbb{C}) \longrightarrow \mathcal{M}(2n,\mathbb{R})$$

by the formula:

$$\iota(A + iB) = \begin{pmatrix} A & -B \\ B & B \end{pmatrix}. \tag{2.15}$$

Lemma 29. *The mapping ι is an algebra monomorphism: ι is injective and $\iota(Z + Z') = \iota(Z) + \iota(Z')$, $\iota(\lambda Z) = \lambda\iota(Z)$ for $\lambda \in \mathbb{C}$, and $\iota(ZZ') = \iota(Z)\iota(Z')$.*

Proof. It is easy to verify that ι is an algebra homomorphism (we leave the direct calculations to the reader); that ι is injective immediately follows from its definition. □

We will see below that ι is an isomorphism of the unitary group onto a certain subgroup of the symplectic group.

2.3.2 Symplectic rotations

Let us prove the main result of this section; it identifies $U(n,\mathbb{C})$ with a subgroup of $\mathrm{Sp}(2n,\mathbb{R})$:

Proposition 30. *The restriction of the mapping*

$$\iota : \mathcal{M}(n,\mathbb{C}) \longrightarrow \mathcal{M}(2n,\mathbb{R}) \tag{2.16}$$

defined above is an isomorphism of $U(n,\mathbb{C})$ onto a subgroup $U(2n,\mathbb{R})$ of $\mathrm{Sp}(2n,\mathbb{R})$.

Proof. It follows from conditions (2.4), (2.5) for the entries of a symplectic matrix that the block matrix

$$U = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{2.17}$$

is in $U(2n,\mathbb{R})$ if and only if

$$AB^T = B^TA, \;\; AA^T + BB^T = I, \tag{2.18}$$

or, equivalently

$$A^TB = BA^T, \;\; A^TA + B^TB = I. \tag{2.19}$$

The equivalence of conditions (2.18) and (2.19) is proved by noting that $U \in U(2n,\mathbb{R})$ if and only if $U^T \in U(2n,\mathbb{R})$ which follows from the fact that the monomorphism (2.16) satisfies $\iota(u^*) = \iota(u)^T$ and that the unitary group is invariant under the operation of taking adjoints. □

Exercise 31. Show that $u \in U(2n,\mathbb{R})$ if and only if $UJ = JU$ and that

$$U(2n,\mathbb{R}) = \mathrm{Sp}(2n,\mathbb{R}) \cap O(2n,\mathbb{R}). \tag{2.20}$$

The identity above shows that $U(2n,\mathbb{R})$ (which is a copy of the unitary group) consists of *symplectic rotations*. It contains the group $O(n)$ of all symplectic matrices of the type

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \text{ with } AA^T = A^T A = I.$$

It is immediately verified that $O(n)$ is the image in $U(2n,\mathbb{R})$ of the orthogonal group $O(n,\mathbb{R})$ by the monomorphism ι.

2.3.3 Diagonalization and polar decomposition

A positive-definite matrix can always be diagonalized using an orthogonal matrix. When this matrix is in addition symplectic we can use a symplectic rotation to perform this diagonalization:

Proposition 32. *Let $S \in \mathrm{Sp}(2n,\mathbb{R})$ be positive definite (in particular $S = S^T$). There exists $U \in U(2n,\mathbb{R})$ such that $S = U^T DU$ where*

$$D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n; \lambda_1^{-1}, \ldots, \lambda_n^{-1})$$

where $\lambda_1, \ldots, \lambda_n$ are the n smallest eigenvalues of S.

Proof. The eigenvalues of a symplectic matrix occur in quadruples: if λ is an eigenvalue, then so are λ^{-1}, $\bar{\lambda}$, and $\bar{\lambda}^{-1}$ (Exercise 22). If $S > 0$ these eigenvalues occur in real pairs (λ, λ^{-1}) with $\lambda > 0$ and we can thus order them as follows:

$$\lambda_1 \leq \cdots \leq \lambda_n \leq \lambda_n^{-1} \leq \cdots \leq \lambda_1^{-1}.$$

Let now U be an orthogonal matrix such that $S = U^T DU$. We are going to show that $U \in U(2n,\mathbb{R})$. It suffices for this to show that we can write U in the form (2.17) with A and B satisfying (2.18). Let $e_1, \ldots, e_n$ be n orthonormal eigenvectors of U corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. Since $SJ = JS^{-1}$ (S is both symplectic and symmetric) we have, for $1 \leq k \leq n$,

$$SJe_k = JS^{-1}e_k = \frac{1}{\lambda_j} Je_k$$

hence $\pm Je_1,\dots,\pm Je_n$ are the orthonormal eigenvectors of U corresponding to the remaining n eigenvalues $1/\lambda_1,\dots,1/\lambda_n$. Write now the $2n\times n$ matrix $(e_1,\dots,e_n)$ as

$$(e_1,\dots,e_n)=\begin{pmatrix}A\\B\end{pmatrix}$$

where A and B are $n\times n$ matrices; we have

$$(-Je_1,\dots,-Je_n)=-J\begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}-B\\A\end{pmatrix}$$

hence U is indeed of the type

$$U=(e_1,\dots,e_n;-Je_1,\dots,-Je_n=\begin{pmatrix}A&-B\\B&A\end{pmatrix}.$$

The conditions (2.18) are satisfied since $U^TU=I$. □

The following consequence of the result above shows that one can take powers of symplectic matrices, and that these powers still are symplectic. In fact:

Corollary 33. *Let S be a positive definite symplectic matrix. Then:*

(i) *For every $\alpha\in\mathbb{R}$ there exists a unique $R\in\mathrm{Sp}(2n,\mathbb{R})$, $R>0$, $R=R^T$, such that $S=R^\alpha$. In particular $S^{1/2}\in\mathrm{Sp}(2n,\mathbb{R})$.*

(ii) *Conversely, if $R\in\mathrm{Sp}(2n,\mathbb{R})$ is positive definite, then $R^\alpha\in\mathrm{Sp}(2n,\mathbb{R})$ for every $\alpha\in\mathbb{R}$.*

Proof of (i). Set $R=U^TD^{1/\alpha}U$; then $R^\alpha=U^TDU=S$.

Proof of (ii). It suffices to note that we have

$$R^\alpha=(U^TDU)^\alpha=U^TD^\alpha U\in\mathrm{Sp}(2n,\mathbb{R}).$$ □

This result allows us to prove a polar decomposition result for the symplectic group. We denote by $\mathrm{Sym}_+(2n,\mathbb{R})$ the set of all symmetric positive definite real $2n\times 2n$ matrices.

Proposition 34. *For every $S\in\mathrm{Sp}(2n,\mathbb{R})$ there exists a unique $U\in U(2n,\mathbb{R})$ and a unique $R\in\mathrm{Sp}(2n,\mathbb{R})\cap\mathrm{Sym}_+(2n,\mathbb{R})$, such that $S=RU$ (resp. $S=UR$).*

Proof. The matrix $R=S^TS$ is symplectic and positive definite. Set $U=(S^TS)^{-1/2}S$; since $(S^TS)^{-1/2}\in\mathrm{Sp}(2n,\mathbb{R})$ in view of Corollary 33, we have $U\in\mathrm{Sp}(2n,\mathbb{R})$. On the other hand

$$UU^T=(S^TS)^{-1/2}SS^T(S^TS)^{-1/2}=I$$

so that we actually have

$$U\in\mathrm{Sp}(2n,\mathbb{R})\cap O(2n,\mathbb{R})=U(2n,\mathbb{R})$$

(cf. Exercise 31). That we can alternatively write $S = UR$ (with different choices of U and R) follows by applying the result above to S^T. The uniqueness statement follows from the generic uniqueness of polar decompositions. $\square$

We will see in Chapter 11, Subsection 11.3 that Proposition 34 can be refined by giving explicit formulas for the matrices R and U ("pre-Iwasawa factorization").

Exercise 35. Use the result above to prove that every $S \in \mathrm{Sp}(2n, \mathbb{R})$ has determinant 1.

One very important consequence of the results above is the connectedness of the symplectic group:

Corollary 36. *The symplectic group* $\mathrm{Sp}(2n, \mathbb{R})$ *is a connected Lie group.*

Proof. Let us set $\mathrm{Sp}_+(2n, \mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{Sym}_+(2n, \mathbb{R})$. In view of Proposition 34 above the mapping

$$f : \mathrm{Sp}(2n, \mathbb{R}) \longrightarrow \mathrm{Sp}_+(2n, \mathbb{R}) \times U(2n, \mathbb{R})$$

defined by $f(S) = RU$ is a bijection; both f and its inverse f^{-1} are continuous, hence f is a homeomorphism. Now $U(2n, \mathbb{R})$ is connected, and so is $\mathrm{Sp}_+(2n, \mathbb{R})$. It follows that $\mathrm{Sp}(2n, \mathbb{R})$ is also connected. $\square$

Exercise 37. Check that $\mathrm{Sp}_+(2n, \mathbb{R})$ is connected (use for instance Corollary 33).

2.4 Symplectic bases and Lagrangian planes

Symplectic bases in phase space are in a sense the analogues of orthonormal bases in Euclidean geometry.

2.4.1 Definition of a symplectic basis

Let δ_{ij} be the Kronecker index: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Definition 38. A set $\mathcal{B}$ of vectors

$$\mathcal{B} = \{e_1, \dots, e_n\} \cup \{f_1, \dots, f_n\}$$

of $\mathbb{R}^n \oplus \mathbb{R}^n$ is called a "*symplectic basis*" *of* $(\mathbb{R}^n \oplus \mathbb{R}^n, \sigma)$ if we have

$$\sigma(e_i, e_j) = \sigma(f_i, f_j) = 0, \quad \sigma(f_i, e_j) = \delta_{ij} \text{ for } 1 \leq i, j \leq n. \tag{2.21}$$

Exercise 39. Check that a symplectic basis is a basis in the usual sense.

An obvious example of a symplectic basis is the following: choose

$$e_i = (c_i, 0),\ \ e_i = (0, c_i)$$

where (c_i) is the canonical basis of $\mathbb{R}^n$. (For instance, if $n = 1$, $e_1 = (1, 0)$ and $f_1 = (0, 1)$.) These vectors form the canonical symplectic basis

$$\mathcal{C} = \{e_1, \ldots, e_n\} \cup \{f_1, \ldots, f_n\}$$

of $(\mathbb{R}^n \oplus \mathbb{R}^n, \sigma)$.

A very useful result is the following; it is a symplectic variant of the Gram–Schmidt orthonormalization procedure in Euclidean geometry. It also shows that there are (infinitely many) non-trivial symplectic bases:

Proposition 40. *Let A and B be two (possibly empty) subsets of $\{1, \ldots, n\}$. For any two subsets $\mathcal{E} = \{e_i : i \in A\}$, $\mathcal{F} = \{f_j : j \in B\}$ of the symplectic space $(\mathbb{R}^n \oplus \mathbb{R}^n, \sigma)$ such that the elements of $\mathcal{E}$ and $\mathcal{F}$ satisfy the relations*

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0\ ,\ \omega(f_i, e_j) = \delta_{ij} \text{ for } (i, j) \in A \times B, \tag{2.22}$$

there exists a symplectic basis $\mathcal{B}$ of $(\mathbb{R}^n \oplus \mathbb{R}^n, \sigma)$ containing these vectors.

For a proof we refer to de Gosson [67], §1.2.2.

Symplectic automorphisms take symplectic bases to symplectic bases: this is obvious from the definition. In fact, the symplectic group acts transitively on the set of all symplectic bases:

Exercise 41. Show that for any two symplectic bases $\mathcal{B}$ and $\mathcal{B}'$ there exists $S \in \mathrm{Sp}(2n, \mathbb{R})$ such that $\mathcal{B} = S(\mathcal{B}')$.

2.4.2 The Lagrangian Grassmannian

The group $\mathrm{Sp}(2n, \mathbb{R})$ not only acts on points of phase space $\mathbb{R}^n \oplus \mathbb{R}^n$ but also on subspaces of $\mathbb{R}^n \oplus \mathbb{R}^n$. Among these of particular interest are "Lagrangian planes":

Definition 42. A Lagrangian plane of the symplectic space $(\mathbb{R}^n \oplus \mathbb{R}^n, \sigma)$ is an n-dimensional linear subspace ℓ of $\mathbb{R}^n \oplus \mathbb{R}^n$ having the following property: if $(z, z') \in \ell \times \ell$ then $\sigma(z, z') = 0$. The set of all Lagrangian planes in $(\mathbb{R}^n \oplus \mathbb{R}^n, \sigma)$ is denoted by $\mathrm{Lag}(2n, \mathbb{R})$; it is called the Lagrangian Grassmannian of $(\mathbb{R}^n \oplus \mathbb{R}^n, \sigma)$.

Both "coordinate planes" $\ell_X = \mathbb{R}^n \times \{0\}$ and $\ell_P = \{0\} \times \mathbb{R}^n$ are Lagrangian, and so is the diagonal $\Delta = \{(x, x) : x \in \mathbb{R}^n\}$ of $\mathbb{R}^n \oplus \mathbb{R}^n$. If ℓ is a Lagrangian plane, so is $S\ell$ for every $S \in \mathrm{Sp}(2n, \mathbb{R})$: first ℓ and $S\ell$ have the same dimension n, and if $z_1 = Sz$ and $z_1' = Sz'$ are in $S\ell$ with z and z' in ℓ, then $\sigma(z_1, z_1') = \sigma(z, z') = 0$. In fact, we have the following much more precise result:

Proposition 43. *The group action*

$$\mathrm{Sp}(2n,\mathbb{R}) \times \mathrm{Lag}(2n,\mathbb{R}) \longrightarrow \mathrm{Lag}(2n,\mathbb{R})$$

defined by $(S,\ell) \longmapsto S\ell$ *is transitive. That is, for every pair* $(\ell,\ell') \in \mathrm{Lag}(2n,\mathbb{R}) \times \mathrm{Lag}(2n,\mathbb{R})$ *there exists* $S \in \mathrm{Sp}(2n,\mathbb{R})$ *such that* $\ell = S\ell'$.

Proof. Choose bases $\{e_1,\ldots,e_n\}$ and $\{e'_1,\ldots,e'_n\}$ of ℓ and ℓ' respectively. Since ℓ and ℓ' are Lagrangian planes we have $\sigma(e_i,e_j) = \sigma(e'_i,e'_j) = 0$ so in view of Proposition 40 we can find vectors $f_1,\ldots,f_n$ and $f'_1,\ldots,f'_n$ such that

$$\mathcal{B} = \{e_1,\ldots,e_n\} \cup \{f_1,\ldots,f_n\},$$
$$\mathcal{B}' = \{e'_1,\ldots,e'_n\} \cup \{f'_1,\ldots,f'_n\}$$

are symplectic bases of $(\mathbb{R}^n \oplus \mathbb{R}^n, \sigma)$. Defining $S \in \mathrm{Sp}(2n,\mathbb{R})$ by the condition $\mathcal{B} = S(\mathcal{B}')$ (see Exercise 41); we have $\ell = S\ell'$. □

Exercise 44. Show that the result above is still true if one replaces $\mathrm{Sp}(2n,\mathbb{R})$ by the unitary group $U(2n,\mathbb{R})$.

Here you are supposed to prove the following refinement of Proposition 43:

Problem 45. Two Lagrangian planes ℓ and ℓ' are said to be transversal if $\ell \cap \ell' = 0$; equivalently $\ell \oplus \ell' = \mathbb{R}^n \oplus \mathbb{R}^n$. Prove that $\mathrm{Sp}(2n,\mathbb{R})$ acts transitively on the set of all transversal Lagrangian planes (hint: use Proposition 40). Does the property remain true if we replace $\mathrm{Sp}(2n,\mathbb{R})$ by $U(2n,\mathbb{R})$?

The Lagrangian Grassmannian has a natural topology which makes it into a compact and connected topological space.

Proposition 46. *The Lagrangian Grassmannian* $\mathrm{Lag}(2n,\mathbb{R})$ *is homeomorphic to the coset space* $U(2n,\mathbb{R})/O(n)$ *where* $O(n)$ *is the image of* $O(n,\mathbb{R})$ *by the restriction of the embedding* $U(n,C) \longrightarrow U(2n,\mathbb{R})$. *Hence* $\mathrm{Lag}(2n,\mathbb{R})$ *is both compact and connected.*

Proof. $U(2n,\mathbb{R})$ acts transitively on $\mathrm{Lag}(2n,\mathbb{R})$ (Exercise 44); the isotropy subgroup of $\ell_P = \{0\} \times \mathbb{R}^n$ is precisely $O(n)$. It follows that $\mathrm{Lag}(2n,\mathbb{R})$ is homeomorphic to $U(2n,\mathbb{R})/O(n)$. Since $U(2n,\mathbb{R})/O(n)$ is trivially homeomorphic to $U(n,\mathbb{C})/O(n,\mathbb{R})$, and the projection $U(n,\mathbb{C}) \longrightarrow U(n,\mathbb{C})/O(n,\mathbb{R})$ is continuous, $\mathrm{Lag}(2n,\mathbb{R})$ is compact and connected because $U(n,\mathbb{C})$ has these properties. □

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