Preface

A basic problem in geometry is to find canonical metrics on smooth manifolds. Such metrics can be specified, for instance, by curvature conditions or extremality properties, and are expected to contain basic information on the topology of the underlying manifold. Constant curvature metrics on surfaces are such canonical metrics. Their distinguished role is emphasized by classical uniformization theory. A more recent characterization of these metrics describes them as critical points of the determinant functional for the Laplacian. The key tool here is Polyakov's variational formula for the determinant. In higher dimensions, however, it is necessary to further restrict the problem, for instance, to the search for canonical metrics in conformal classes. Here two metrics are considered to belong to the same conformal class if they differ by a nowhere vanishing factor. A typical question in that direction is the Yamabe problem ([165]), which asks for constant scalar curvature metrics in conformal classes.

In connection with the problem of understanding the structure of Polyakov type formulas for the determinants of conformally covariant differential operators in higher dimensions, Branson ([31]) discovered a remarkable curvature quantity which now is called Branson's Q-curvature. It is one of the main objects in this book.

Q-curvature is a scalar local Riemannian curvature invariant on manifolds of even dimension. On surfaces it coincides with Gauß curvature. On four-manifolds it first appeared in connection with the conformally covariant Paneitz operator. In this case, it is a certain linear combination of squared scalar curvature, the squared norm of Ricci curvature and the Laplacian of scalar curvature. On a manifold of dimension n, Q-curvature is an n^{th}-order curvature invariant. One of its remarkable properties is that its behaviour under conformal changes of the metric is governed by an n^{th}-order *linear* conformally covariant differential operator. In dimensions two and four, the respective operators are the Laplacian and the Paneitz operator. In higher dimensions, these operators are replaced by certain conformally covariant powers of the Laplacian (GJMS-operators) ([124]).

Besides their significance in conformal geometry, GJMS-operators also play an important role in physics. This is due to the fact that their definition extends to Lorentzian manifolds. They are common generalizations of the Yamabe operator and the conformally covariant powers of the wave operator on Minkowski space.

In recent years, new connections of Q-curvature with other parts of mathematics and theoretical physics have been discovered. Probably the most remarkable one is the relation to geometric scattering on asymptotically hyperbolic Einstein spaces. This is a relation in the spirit of the conjectural AdS/CFT-duality ([171]), which connects gravitation with gauge field theory, and which led to an outburst of activities in theoretical physics (for reviews see [1], [82]). In geometric analysis, recent efforts are being directed towards an understanding of the geometric significance of Q-curvature in dimension four, for instance, by studying Yamabe type problems and Q-curvature flows (for a review see [170]). Here one of the problems is to characterize the conformal classes which contain a metric with constant Q-curvature. Much less is known in higher dimensions.

Although Q-curvature is an intrinsic Riemannian curvature invariant, all known conceptual definitions in general dimension take one or another extrinsic point of view. The situation somewhat resembles Weyl's formula for the volume of a tube ([130]). This formula shows that the volume of a tube of a closed submanifold of Euclidean space is a polynomial in its radius, and the coefficients depend only on the intrinsic curvature of the submanifold. In particular, the Euler characteristic of the submanifold appears in the leading coefficient.

In the present book, we develop a new extrinsic point of view towards Q-curvature with the emphasis on general structural results. The guiding idea is to associate to a hypersurface $i : \Sigma \hookrightarrow M$ and a general background metric g on M certain one-parameter families of conformally covariant local operators which map functions on M to functions on Σ. Q-curvature and the GJMS-operator of the submanifold $(\Sigma, i^*(g))$ appear in the respective linear and constant coefficients of these families, and the fundamental transformation law of Q-curvature is a direct consequence of the covariance of the family. In particular, we introduce two specific constructions of conformally covariant families with such properties: the residue families and the tractor families.

The setting of residue families is more restricted, however. Here Σ is the boundary of M, and the background metric on M is a canonical extension of a given metric on Σ. Such situations arise in connection with conformally compact Einstein metrics and the Fefferman-Graham construction of an ambient metric. The closely related Poincaré-Einstein metrics associate to any conformal class on Σ a diffeomorphism class of conformally compact Einstein metrics on M with the given class as conformal infinity ([199]). The method of the ambient metric was introduced in [99] as a fundamental systematic construction of conformal invariants. During the last two decades the ambient metric had a major influence on the subject of conformal geometry. For full details see [96].

Poincaré-Einstein metrics are used in theoretical physics in connection with the speculative holographic principle in quantum gravity ([29], [227]). The bulk space/boundary duality between superstring theory on AdS-space and supersymmetric Yang-Mills theory on Minkowski space is regarded as a concrete manifestation of the principle.

In a pure gravity setting with homogeneous metrics, a related bulk/boundary duality is Helgason's well-known theory of Poisson transformations in harmonic analysis on symmetric spaces ([140]). Versions of that transform for conformally compact Einstein spaces play an important role in attempts to establish rigorous statements in the AdS/CFT-duality.

The residue families are defined by a certain residue construction, which has its origin in the spectral theory of Kleinian manifolds. This explains the name. These families can be regarded as local counterparts of the global scattering operator. They naturally lead to an understanding of Q-curvature of a metric on the boundary at infinity as part of a hologram of the associated Poincaré-Einstein metric in one higher dimension. More precisely, the *holographic formulas* describe Q-curvature in terms of holographic coefficients of the Poincaré-Einstein metric and its harmonic functions. Combining that relation between Q-curvature and residue families with structural properties of residue families (factorization relations), uncovers recursive structures among Q-curvatures and GJMS-operators. It is here where the lower order relatives Q_{2N} $(2N < n)$ of Branson's Q-curvature Q_n become important. All in all, the residue families are an effective tool for the systematic study of the interplay between the asymptotic geometry of Poincaré-Einstein metrics on bulk space and GJMS-operators (and Q-curvatures) of their conformal infinities.

The theory of the tractor families is an attempt to take a wider perspective. Here the conformal compactifications of Poincaré-Einstein metrics are replaced by arbitrary background metrics, and we extract the intrinsic Q-curvature of the submanifold using an appropriate extrinsic construction near Σ. That perspective leads to the notions of extrinsic and odd order Q-curvatures, which relate the subject of Q-curvature with conformal submanifold theory. The tractor families are defined in terms of the conformally invariant tractor calculus ([17]). A closely related construction was used in [40] in a different connection.

For certain classes of metrics, residue families and tractor families coincide. Such relations imply tractor formulas for GJMS-operators and Q-curvature, and will be termed *holographic duality*.

The new approach to Q-curvature grew out of results which relate the divisors of Selberg zeta functions to automorphic distributions. Such results are related to the dream of an interpretation of the Riemann-Weil explicit formula in analytic number theory as a version of a Lefschetz fixed point formula. The hope is that a cohomological interpretation may also contain a key to the Riemann hypothesis ([78]). In the same spirit, it was shown in [151] that, using Osborne's character formula, the Selberg trace formula can be regarded as a Lefschetz formula for the geodesic flow. This leads to characterizations of the divisors of Selberg zeta functions in terms of cohomology of Anosov foliations and representation theory. The basic principle is that the complex numbers which appear as zeros or poles of a zeta function are characterized by the non-vanishing of the Euler characteristics of associated complexes. Moreover, the values of the corresponding Euler characteristics yield the multiplicities. Equivariant Poisson transformations translate these results into characterizations in terms of group cohomology with values in distributions on the geodesic boundary of rank one symmetric spaces. The latter result can be regarded as a version of holography: the divisor, i.e., zeros and poles with multiplicities, of a zeta function, which is defined by the lengths of closed geodesics of a hyperbolic manifold, is completely characterized in terms of a theory which is formulated on a manifold of one dimension less. The cohomological objects which correspond to the zeros of the Riemann zeta function remain to be found, however.

The fascination of Q-curvature stems from its central role in the complex web of ideas outlined above. In this framework, we observe how classical and modern differential geometry, geometric analysis, harmonic analysis and theoretical physics meet each other.

Although in four dimensions the geometric meaning of Q-curvature has been studied intensively in recent years, there are only few results in higher dimensions. It would be pleasing if the perspectives and the structural insights presented here help to enter this unexplored field. Presently, the future role of Q-curvature is hard to predict, and it seems that we are now taking only the first steps towards its comprehension.

The reader will easily notice that the theory in this book has open ends on different levels. In addition to a number of explicitly formulated conjectures, there are results that are derived under conditions which certainly can be relaxed, and the full consequences of some arguments and constructions are not yet predictable. Moreover, the basic ideas should apply also in different contexts. We hope that this will motivate further investigations.

At first glance, it might seem that the text contains a jungle of complicated formulas. To some extent, this is typical for the subject. On the other hand, we believe that the ambitious reader finally will be delighted by the ways in which complex but beautiful formulas emerge from simple principles, albeit sometimes through non-trivial calculations. The disclosure of some of the hidden structures is one of the aims of this work.

First and foremost, the book is a research monograph presenting a new theory. On the other hand, we have attempted of a self-contained presentation of the material so that it should be accessible for non-specialists. Although we strictly concentrate on the development of new ideas, we necessarily touch upon many of the recent developments in conformal differential geometry. Therefore, the text may also serve as an informal introduction to the subject. We hope that we have succeeded in finding some balance between the presentation of structural ideas and the discussion of full (calculational) details. In particular, we also included proofs of some results which might be considered as well-known by specialists. But since the various fields which are touched upon here do not have a common folklore, proofs are given if required for the sake of a coherent presentation. Also, due to varying conventions, it was sometimes easier to supply proofs than to refer to the literature. The list of references is not representative for any of the numerous fields linked with the subject.

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