## Contents

Preface xi

Introduction xiii

I Background on Groups, Complexity, and Cryptography 1

1 Background on Public Key Cryptography 3
   1.1 From key establishment to encryption 4
   1.2 The Diffie-Hellman key establishment 5
   1.3 The ElGamal cryptosystem 6
   1.4 Authentication 7

2 Background on Combinatorial Group Theory 9
   2.1 Basic definitions and notation 9
   2.2 Presentations of groups by generators and relators 11
   2.3 Algorithmic problems of group theory 11
      2.3.1 The word problem 11
      2.3.2 The conjugacy problem 12
      2.3.3 The decomposition and factorization problems 12
      2.3.4 The membership problem 13
      2.3.5 The isomorphism problem 14
   2.4 Nielsen's and Schreier's methods 14
   2.5 Tietze's method 16
   2.6 Normal forms 17

3 Background on Computational Complexity 19
   3.1 Algorithms 19
      3.1.1 Deterministic Turing machines 19
      3.1.2 Non-deterministic Turing machines 20
      3.1.3 Probabilistic Turing machines 21
   3.2 Computational problems 21
      3.2.1 Decision and search computational problems 21
3.2.2 Size functions ........................................... 23
3.2.3 Stratification .......................................... 25
3.2.4 Reductions and complete problems ................. 26
3.2.5 Many-one reductions ................................... 27
3.2.6 Turing reductions ...................................... 27
3.3 The worst case complexity .................................. 28
3.3.1 Complexity classes .................................... 28
3.3.2 Class \textbf{NP} ........................................ 29
3.3.3 Polynomial-time many-one reductions and class \textbf{NP} .... 30
3.3.4 \textbf{NP}-complete problems .......................... 31
3.3.5 Deficiency of the worst case complexity ............ 33

II Non-commutative Cryptography

4 Canonical Non-commutative Cryptography .................. 37
4.1 Protocols based on the conjugacy search problem ......... 37
4.2 Protocols based on the decomposition problem ............ 39
4.2.1 “Twisted” protocol .................................... 40
4.2.2 Hiding one of the subgroups ............................ 41
4.2.3 Using the triple decomposition problem ............... 42
4.3 A protocol based on the factorization search problem ... 43
4.4 Stickel’s key exchange protocol ........................... 43
4.4.1 Linear algebra attack .................................. 45
4.5 The Anshel-AnsheL-Goldfeld protocol ....................... 47
4.6 Authentication protocols based on the conjugacy problem . 49
4.6.1 A Diffie-Hellman-like scheme ......................... 49
4.6.2 A Fiat-Shamir-like scheme ............................ 50
4.6.3 An authentication scheme based on the twisted conjugacy problem .................................... 51
4.7 Relations between different problems ....................... 52

5 Platform Groups ................................................. 55
5.1 Braid groups ............................................. 55
5.1.1 A group of braids and its presentation ................. 56
5.1.2 Dehornoy handle free form ............................ 59
5.1.3 Garside normal form .................................. 60
5.2 Thompson’s group ........................................ 61
5.3 Groups of matrices ....................................... 65
5.4 Small cancellation groups ................................ 67
5.4.1 Dehn’s algorithm ..................................... 67
5.5 Solvable groups .......................................... 68
5.5.1 Normal forms in free metabelian groups .............. 68
5.6 Artin groups ............................................. 71
## Contents

6 Using Decision Problems in Public Key Cryptography

6.1 The Shpilrain-Zapata scheme

6.1.1 The protocol

6.1.2 Pool of group presentations

6.1.3 Tietze transformations: elementary isomorphisms

6.1.4 Generating random elements in finitely presented groups

6.1.5 Isomorphism attack

6.1.6 Quotient attack

6.2 Public key encryption and encryption emulation attacks

III Generic Complexity and Cryptanalysis

7 Distributional Problems and the Average Case Complexity

7.1 Distributional computational problems

7.1.1 Distributions and computational problems

7.1.2 Stratified problems with ensembles of distributions

7.1.3 Randomized many-one reductions

7.2 Average case complexity

7.2.1 Polynomial on average functions

7.2.2 Average case behavior of functions

7.2.3 Average case complexity of algorithms

7.2.4 Average case vs worst case

7.2.5 Average case behavior as a trade-off

7.2.6 Deficiency of average case complexity

8 Generic Case Complexity

8.1 Generic Complexity

8.1.1 Generic sets

8.1.2 Asymptotic density

8.1.3 Convergence rates

8.1.4 Generic complexity of algorithms and algorithmic problems

8.1.5 Deficiency of the generic complexity

8.2 Generic- versus average case complexity

8.2.1 Comparing generic and average case complexities

8.2.2 When average polynomial time implies generic polynomial time

8.2.3 When generically easy implies easy on average
9 Generic Complexity of NP-complete Problems
  9.1 The linear generic time complexity of subset sum problem ....... 129
  9.2 A practical algorithm for subset sum problem ...................... 131
  9.3 3-Satisfiability ............................................. 131

IV Asymptotically Dominant Properties and Cryptanalysis 135
  10 Asymptotically Dominant Properties ............................................. 139
    10.1 A brief description .............................................. 139
    10.2 Random subgroups and generating tuples .............................. 141
    10.3 Asymptotic properties of subgroups ................................ 142
    10.4 Groups with generic free basis property ............................. 143
    10.5 Quasi-isometrically embedded subgroups .............................. 145

11 Length-Based and Quotient Attacks 149
  11.1 Anshel-Anshel-Goldfeld scheme ...................................... 149
    11.1.1 Description of the Anshel-Anshel-Goldfeld scheme ............ 149
    11.1.2 Security assumptions of the AAG scheme .......................... 150
    11.2 Length-based attacks ............................................. 152
      11.2.1 A general description ...................................... 152
      11.2.2 LBA in free groups ........................................ 155
      11.2.3 LBA in groups from $\mathcal{F}\beta_{exp}$ ............................. 156
    11.3 Computing the geodesic length in a subgroup ...................... 157
      11.3.1 Related algorithmic problems ................................ 158
      11.3.2 Geodesic length in braid groups ............................... 159
    11.4 Quotient attacks ................................................. 161
      11.4.1 Membership problems in free groups ......................... 162
      11.4.2 Conjugacy problems in free groups ............................ 164
      11.4.3 The MSP and SCSP* problems in groups with “good” quotients ................................................. 167

Bibliography 169

Abbreviations and Notation 179

Index 181