Preface

This book analyzes the existence and uniqueness of a generalized algebraic multiplicity for a general one-parameter family $\mathfrak{L}$ of bounded linear operators with Fredholm index zero at a value of the parameter λ_0 where $\mathfrak{L}(\lambda_0)$ is non-invertible. Precisely, given $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, two Banach spaces U and V over $\mathbb{K}$, an open subset $\Omega \subset \mathbb{K}$, and a point $\lambda_0 \in \Omega$, our admissible operator families are the maps

$$\mathfrak{L} \in \mathcal{C}^r(\Omega, \mathcal{L}(U, V)) \tag{1}$$

for some $r \in \mathbb{N}$, such that

$$\mathfrak{L}(\lambda_0) \in \mathrm{Fred}_0(U, V);$$

here $\mathcal{L}(U, V)$ stands for the space of linear continuous operators from U to V, and $\mathrm{Fred}_0(U, V)$ is its subset consisting of all Fredholm operators of index zero. From the point of view of its novelty, the main achievements of this book are reached in case $\mathbb{K} = \mathbb{R}$, since in the case $\mathbb{K} = \mathbb{C}$ and $r = 1$, most of its contents are classic, except for the axiomatization theorem of the multiplicity.

The book consists of three parts, and, although the necessary background to understand the book differs from each part, the entire book might become easily accessible to a multidisciplinary audience by establishing all its results in a finite-dimensional setting, which is not a serious restriction. Part I covers the classic finite-dimensional spectral theory, and assumes hardly any previous knowledge by the reader, except some basic facts in complex analysis. In fact, its contents are intended to be taught at an undergraduate level, as we have considerably tidied up and shortened most of the available material in the literature. Part II introduces a generalized concept of algebraic multiplicity, which substantially extends all previous available concepts, to cover the case of families $\mathfrak{L}$ as in (1). Actually, this objective is accomplished by adopting several different perspectives, each of them of great interest in its own right. For a comfortable reading of this part, some basic background in Functional Analysis is appropriate; essentially, the Hahn–Banach theorem and the most important properties of the Fredholm operators. Finally, Part III, which consists of a single chapter, introduces some pivotal concepts in nonlinear analysis (like the concept of nonlinear eigenvalue) and characterizes the nonlinear eigenvalues of $\mathfrak{L}$ through the value of $(-1)^{\mathfrak{m}[\mathfrak{L};\lambda_0]}$, where $\mathfrak{m}[\mathfrak{L}; \lambda_0]$ denotes the generalized algebraic multiplicity of $\mathfrak{L}$ at λ_0. Although it would be desirable to have a good knowledge of the topological degree for a comfortable reading of this part, this is far from necessary, because a self-contained introduction to the topological degree is provided therein. Going beyond would take us outside the general scope of this book.

A great proportion of Part II consists of versions and new findings of recent results attributable to the authors of this book and coworkers, except for the classic case when $\mathbb{K} = \mathbb{C}$ and $r = 1$, where most of the results are attributable to the Russian school. Among the most important results of Part II, there is the next axiomatization theorem of the multiplicity. We denote by $\mathcal{S}^{\infty}_{\lambda_0}(U)$ the set of functions of class $\mathcal{C}^{\infty}$ defined in a neighborhood of λ_0 with values in $\mathrm{Fred}_0(U, U)$.

Theorem 0.0.1. *There exists a unique map*

$$\mathfrak{m}[\cdot; \lambda_0] : \mathcal{S}^{\infty}_{\lambda_0}(U) \to [0, \infty]$$

satisfying the following axioms:

A1. *For all* $\mathfrak{L}, \mathfrak{M} \in \mathcal{S}^{\infty}_{\lambda_0}(U)$,

$$\mathfrak{m}[\mathfrak{L}\mathfrak{M}; \lambda_0] = \mathfrak{m}[\mathfrak{L}; \lambda_0] + \mathfrak{m}[\mathfrak{M}; \lambda_0].$$

A2. *There exists a rank one projection* $P_0 \in \mathcal{L}(U)$ *such that*

$$\mathfrak{m}[\mathfrak{L}; \lambda_0] = 1,$$

where

$$\mathfrak{L}(\lambda) = (\lambda - \lambda_0)P_0 + I_U - P_0, \qquad \lambda \in \mathbb{K}.$$

The value $\mathfrak{m}[\mathfrak{L}; \lambda_0]$ is called the *algebraic multiplicity* of $\mathfrak{L}$ at λ_0. Axiom A1 is known as the *product formula*, while Axiom A2 is nothing but a *normalization condition*, as will discussed in Chapter 6.

Before describing the properties of the multiplicity, we should recall the familiar concept of *order* of a zero of a function at a point. Let X be a $\mathbb{K}$-Banach space, $\Omega \subset \mathbb{K}$ an open set, $r \geq 0$ an integer or infinity, $\lambda_0 \in \Omega$, and $f \in \mathcal{C}^r(\Omega, X)$. If there exists an integer $0 \leq k \leq r$ such that $f^{j)}(\lambda_0) = 0$ for all $0 \leq j < k$, and $f^{k)}(\lambda_0) \neq 0$, then we define $\mathrm{ord}_{\lambda_0} f = k$. If $r = \infty$ and $f^{j)}(\lambda_0) = 0$ for all integers $j \geq 0$, then we define $\mathrm{ord}_{\lambda_0} f = \infty$. If $f^{j)}(\lambda_0) = 0$ for all $0 \leq j \leq r$, and f is not of class $\mathcal{C}^{r+1}$ in any neighborhood of λ_0, then we leave $\mathrm{ord}_{\lambda_0} f$ undefined.

Now we describe some properties of the multiplicity. For every $\mathfrak{L} \in \mathcal{S}^{\infty}_{\lambda_0}(U)$, the function $\mathfrak{m}[\cdot; \lambda_0]$ inherits, from Axioms A1 and A2, the following properties:

i) $\mathfrak{m}[\mathfrak{L}; \lambda_0] \in \mathbb{N} \cup \{\infty\}$.

ii) $\mathfrak{m}[\mathfrak{L}; \lambda_0] = 0$ if and only if $\mathfrak{L}(\lambda_0)$ is an isomorphism.

iii) $\mathfrak{m}[\mathfrak{L}; \lambda_0] < \infty$ if and only if there exists an integer $k \geq 0$ such that

$$(\lambda - \lambda_0)^k \mathfrak{L}(\lambda)^{-1}$$

exists and is bounded for λ in a perforated neighborhood of λ_0.

iv) If $\dim U < \infty$, then $\mathfrak{m}[\mathfrak{L};\lambda_0] = \mathrm{ord}_{\lambda_0} \det \mathfrak{L}$.

v) If $\mathbb{K} = \mathbb{C}$ and $U = V = \mathbb{C}^N$, then

$$\mathfrak{m}[\mathfrak{L};\lambda_0] = \mathrm{tr}\, \frac{1}{2\pi i} \int_\gamma \mathfrak{L}'(\lambda)\mathfrak{L}(\lambda)^{-1}\, d\lambda$$

for every Cauchy contour γ with inner domain D containing λ_0 such that $\mathfrak{L}^{-1}$ is holomorphic in $D \setminus \{\lambda_0\}$ and continuous in $\bar{D} \setminus \{\lambda_0\}$.

vi) For every $\mathbb{K}$-Banach space V and $\mathfrak{M} \in \mathcal{S}^\infty_{\lambda_0}(V)$,

$$\mathfrak{m}[\mathfrak{L} \oplus \mathfrak{M};\lambda_0] = \mathfrak{m}[\mathfrak{L};\lambda_0] + \mathfrak{m}[\mathfrak{M};\lambda_0].$$

vii) If $\mathfrak{L}$ is analytic, Ω is connected, and $\mathfrak{L}(\Omega)$ contains an isomorphism, then

$$\mathfrak{m}[\mathfrak{L};\lambda] < \infty \qquad \text{for all} \quad \lambda \in \Omega.$$

viii) If $A \in \mathcal{L}(U)$ and $\mathfrak{L}$ is the family defined by

$$\mathfrak{L}(\lambda) := \lambda I - A, \qquad \lambda \in \mathbb{K},$$

then

$$\mathfrak{m}[\mathfrak{L};\lambda_0] = \sup_{k\in\mathbb{N}} \dim N[(\lambda_0 I - A)^k].$$

ix) $\mathfrak{m}[\mathfrak{L};\lambda_0] < \infty$ if and only if there exist an open neighborhood $\tilde{\Omega} \subset \Omega$ of λ_0, a topological decomposition $U = U_0 \oplus U_1$ with $n := \dim U_0 < \infty$ and two maps $\mathfrak{E}, \mathfrak{F} \in \mathcal{S}^\infty_{\lambda_0}(U)$ such that $\mathfrak{E}(\lambda_0), \mathfrak{F}(\lambda_0)$ are isomorphisms and

$$\mathfrak{L}(\lambda) = \mathfrak{E}(\lambda)\, [\mathfrak{S}(\lambda) \oplus I_{U_1}]\, \mathfrak{F}(\lambda), \qquad \lambda \in \tilde{\Omega},$$

where I_{U_1} stands for the identity of U_1, and

$$\mathfrak{S}(\lambda) = \mathrm{diag}\left\{(\lambda-\lambda_0)^{\kappa_1}, \ldots, (\lambda-\lambda_0)^{\kappa_n}\right\}, \qquad \lambda \in \mathbb{K},$$

for some integers $\kappa_1 \geq \cdots \geq \kappa_n \geq 1$. Moreover, in such a case,

$$\mathfrak{m}[\mathfrak{L};\lambda_0] = \kappa_1 + \cdots + \kappa_n.$$

Owing to Theorem 0.0.1, each of the listed properties, and actually any further property of the multiplicity, must be a consequence of Axioms A1 and A2.

The concept of multiplicity for maps $\mathfrak{L}$ of the form (1), rather than for operators (as is the case in the context of the classic spectral theory) goes back to the Russian school; it began to be developed in the 1940s for holomorphic families, that is, for the special case when $\mathbb{K} = \mathbb{C}$ and $r = 1$ (see Keldyš [66, 67] for probably the first definition, and Markus & Sigal [95], Eni [28], Gohberg [42], Gohberg & Sigal [54], Gohberg, Kaashoek & Lay [44], as well as the references therein, for later generalizations and expositions). Most of the available underlying theories are based on generalized Jordan chains, as discussed in Chapter 7 of this book.

The concept of multiplicity in the case when $\mathbb{K} = \mathbb{R}$ came later, and was introduced independently through four rather different devices. By using generalized Jordan chains (see Sarreither [116] and later Rabier [110]), through a finite-dimensional reduction of the family $\mathfrak{L}$ so that it can be defined as the multiplicity of the determinant of the reduced family (see Ize [61]), by means of a polynomial factorization of $\mathfrak{L}$ (see Gohberg [42], Gohberg & Sigal [54], and Magnus [92]), as discussed in Chapter 5 of this book, or through a change of variable giving rise to a *transversal* family (see Esquinas & López-Gómez [30], Esquinas [29], and López-Gómez [82]), as discussed in Chapter 4.

According to Theorem 0.0.1, all these concepts of multiplicity must coincide as long as they satisfy Axioms A1 and A2, though the reader should be aware of the fact that obtaining the product formula may be fraught with higher technical difficulties than directly checking that they do coincide from their own definitions.

This book gives a complete description of all these (equivalent) constructions of the algebraic multiplicity. The reasons to bring them together in this book include:

a) all of them involve interesting constructions in their own right;
b) although equivalent, a number of multiplicity concepts depart from rather separated starting points and, consequently, their mutual connections are not clear;
c) some properties of the multiplicity are easier to prove through a particular definition than departing from another.

Nevertheless, the existence of a number of constructions of the multiplicity from different perspectives facilitates the derivation of its properties, both from the theoretical point of view and from the point of view of applications. Although there are some previous and partial comparisons between several different theories of algebraic multiplicity (see, for example, Esquinas [29], Fitzpatrick & Pejsachowicz [33], Rabier [110] and López-Gómez [82]), this book should provide the most complete monograph encompassing all those different constructions from the point of view of the axiomatization of Theorem 0.0.1.

These algebraic multiplicities have many applications in a number of separate areas of mathematics. Among them, the constructions of the algebraic multiplicity are used for accomplishing the following tasks:

- Finding optimal criteria for the change of the topological degree of $\mathfrak{L}(\lambda)$ as the parameter λ crosses an eigenvalue of $\mathfrak{L}$, within the context of local and global bifurcation theory (see Sarreither [116], Ize [61], Magnus [92], Esquinas & López-Gómez [31], Esquinas [29], López-Gómez [82], Rabier [110], Fitzpatrick & Pejsachowicz [34], López-Gomez & Mora-Corral [91]).

- Establishing local classification of operator families $\mathfrak{L}$ through the existence and uniqueness of the Smith canonical form (see Gohberg, Kaashoek & Lay [44]), as well as global spectral results for general operator families (see Gohberg, Kaashoek & Lay [44], Leiterer [78, 79], Gohberg, Lancaster & Rodman [50], López-Gómez [82]).
- Finding completeness criteria for eigenvectors and generalized eigenvectors of operators and operator families (see Keldyš [66, 67], Friedman & Shinbrot [35], Gohberg & Kreĭn [45, 46]).
- Analyzing the solution manifold of general classes of ordinary linear differential equations with constant coefficients, and boundary-value problems (see Gohberg, Lancaster & Rodman [50, 51], Rodman [114], Wloka, Rowley & Lawruk [127]), as well as difference equations with constant coefficients (see Gohberg, Lancaster & Rodman [50]).
- Studying interpolation of matrix families (see Rodman [114], Ball, Gohberg & Rodman [8]),
- Studying linear systems in control engineering (see Gohberg, Lancaster & Rodman [51], Rodman [114], Wyman, Sain, Conte & Perdon [129]).

In the sequel, we will outline the contents of each chapter of this book, with particular emphasis on the new findings originated by the authors' research in the field.

Chapter 1 is devoted to a proof of the Jordan theorem and a construction of the underlying canonical form. Chapter 2 provides a short self-contained introduction to operator calculus, yet in finite dimension. Then, the Dunford integral formula is introduced and the spectral mapping theorem is studied. Chapter 3 describes the construction of the Jordan spectral projections. The proofs are based upon the fact that the ascent of each eigenvalue equals its order as a pole of the resolvent operator. This methodology substantially differs from the standard one adopted in most classic textbooks on elementary finite-dimensional spectral theory, shortens the inherent difficulties for the general infinite-dimensional framework, and facilitates the teaching of these materials at an undergraduate level. Actually, this is the pivotal idea upon which the development of the subsequent abstract general theory of this book relies.

Part II and the general theory of multiplicities begin at Chapter 4, after introducing all the spaces and most common notation used in the rest of the book. Throughout Part II, $r \geq 1$ is an integer or infinity, $\mathbb{K}$ is either the real field or the complex field, $\Omega \subset \mathbb{K}$ is an open set containing λ_0, and $\mathfrak{L} \in \mathcal{C}^r(\Omega, \mathcal{L}(U,V))$ satisfies $\mathfrak{L}(\lambda_0) \in \mathrm{Fred}_0(U,V)$. Besides, the following notation for the derivatives at $\mathfrak{L}$ at λ_0

$$\mathfrak{L}_j := \frac{1}{j!}\mathfrak{L}^{j)}(\lambda_0), \qquad 0 \leq j < r+1,$$

is used everywhere, so that $\mathfrak{L}_j$ equals the jth coefficient of the Taylor development of $\mathfrak{L}$ at λ_0. The parameter value λ_0 is said to be an *eigenvalue* of $\mathfrak{L}$ when $\mathfrak{L}(\lambda_0)$ is not invertible. As $\mathfrak{L}(\lambda_0)$ is Fredholm of index zero, this amounts to

$$N[\mathfrak{L}(\lambda_0)] \neq \{0\}. \tag{2}$$

Let $\mathrm{Iso}(U, V)$ denote the set of linear topological isomorphisms from U to V. When $\mathfrak{L}(\lambda_0) \in \mathrm{Iso}(U, V)$, all results become trivial; thus, we will subsequently focus our attention on the case when (2) occurs.

The theory of algebraic multiplicity presented in Chapter 4 is based on the concept of transversality, which is a pivotal concept throughout this book; it goes back to Esquinas & López-Gómez [31].

Definition 0.0.2. Given an integer $1 \leq k \leq r$, it is said that λ_0 is a k-transversal eigenvalue of $\mathfrak{L}$ if

$$\bigoplus_{j=1}^{k} \mathfrak{L}_j(N[\mathfrak{L}_0] \cap \cdots \cap N[\mathfrak{L}_{j-1}]) \oplus R[\mathfrak{L}_0] = V,$$

and

$$\mathfrak{L}_k(N[\mathfrak{L}_0] \cap \cdots \cap N[\mathfrak{L}_{k-1}]) \neq \{0\}.$$

In such a case, the algebraic multiplicity of $\mathfrak{L}$ at λ_0, denoted by $\chi[\mathfrak{L}; \lambda_0]$, is defined through

$$\chi[\mathfrak{L}; \lambda_0] := \sum_{j=1}^{k} j \dim \mathfrak{L}_j(N[\mathfrak{L}_0] \cap \cdots \cap N[\mathfrak{L}_{j-1}]).$$

Although, at a first glance, Definition 0.0.2 might look shocking, it turns out that transversal eigenvalues enjoy a special structure that makes most of the involved operator calculations straightforward from an algorithmic point of view. The concept of multiplicity was originally introduced to capture the order of $\det \mathfrak{L}$ at λ_0 in the finite-dimensional setting, but this is far from being an elementary feature that can be briefly explained.

Another key concept of Part II, going back to López-Gómez [82], is the one introduced by the following definition.

Definition 0.0.3. Given an integer $k \geq 1$, it is said that λ_0 is a k-algebraic eigenvalue of $\mathfrak{L}$ when there exists an open neighborhood $\tilde{\Omega} \subset \Omega$ of λ_0 such that $\mathfrak{L}(\lambda) \in \mathrm{Iso}(U, V)$ for all $\lambda \in \tilde{\Omega} \setminus \{\lambda_0\}$, the values of the family

$$\lambda \mapsto (\lambda - \lambda_0)^k \mathfrak{L}(\lambda)^{-1}, \qquad \lambda \in \tilde{\Omega} \setminus \{\lambda_0\},$$

are bounded, but $(\lambda - \lambda_0)^{k-1} \mathfrak{L}(\lambda)^{-1}$ is unbounded for λ in every perforated neighborhood of λ_0. In such a case, we will write that $\lambda_0 \in \mathrm{Alg}_k(\mathfrak{L})$.

When $\mathfrak{L}$ is analytic, it turns out that $\lambda_0 \in \mathrm{Alg}_k(\mathfrak{L})$ if and only if λ_0 is a pole of $\mathfrak{L}^{-1}$ of order k. Consequently, this concept is well known in the analytic case. But, since there is no available concept of pole for non-meromorphic families, the concept of algebraic eigenvalue is new in the general $\mathcal{C}^r$ setting adopted in this book.

Chapter 9 shows that the concept of algebraic eigenvalue introduced by Definition 0.0.3 indeed provides us with a concept of pole for $\mathfrak{L}^{-1}$, thus considerably extending the classic concept of pole. Quite strikingly, it turns out that if $\lambda_0 \in \mathrm{Alg}_k(\mathfrak{L})$ and $r \geq 2k$, then the inverse map $\mathfrak{L}^{-1}$ possesses a Laurent asymptotic expansion at λ_0,

$$\mathfrak{L}(\lambda)^{-1} = \sum_{n=-k}^{-1} (\mathfrak{L}^{-1})_n (\lambda-\lambda_0)^n + o\left((\lambda-\lambda_0)^{-1}\right), \qquad \text{as} \quad \lambda \to \lambda_0,$$

as well as $\mathfrak{L}'\mathfrak{L}^{-1}$,

$$\mathfrak{L}'(\lambda)\mathfrak{L}(\lambda)^{-1} = \sum_{n=-k}^{-1} \mathfrak{M}_n (\lambda-\lambda_0)^n + o\left((\lambda-\lambda_0)^{-1}\right), \qquad \text{as} \quad \lambda \to \lambda_0,$$

and, in addition, according to the main theorem of López-Gómez & Mora-Corral [90],

$$\operatorname{tr} \mathfrak{M}_n = \begin{cases} 0 & \text{if} \quad -k \leq n \leq -2, \\ \chi[\mathfrak{L}; \lambda_0] & \text{if} \quad n = -1, \end{cases}$$

which is a deep generalization to the case when $\mathbb{K} = \mathbb{R}$ and $\mathfrak{L}$ is of class $\mathcal{C}^r$ for some $r \in \mathbb{N}$ of the following classic well-known result attributable to Gohberg [42], Markus & Sigal [95] and Gohberg & Sigal [54] (see also Gohberg, Goldberg & Kaashoek [43]). That generalization constitutes the bulk of Chapter 9.

Theorem 0.0.4. *Let $\Omega \subset \mathbb{C}$ be open and connected, and $\mathfrak{L} \in \mathcal{H}(\Omega, \mathcal{L}(U,V))$ such that $\mathfrak{L}(\Omega) \subset \mathrm{Fred}_0(U,V)$ and $\mathfrak{L}(\Omega) \cap \mathrm{Iso}(U,V) \neq \varnothing$. Then, the set*

$$\Sigma(\mathfrak{L}) := \left\{\lambda \in \Omega : N[\mathfrak{L}(\lambda)] \neq \{0\}\right\} \tag{3}$$

is discrete in Ω, and $\mathfrak{L}^{-1}$ is holomorphic in $\Omega \backslash \Sigma(\mathfrak{L})$. Moreover, for each $\lambda_0 \in \Sigma(\mathfrak{L})$, the meromorphic function $\mathfrak{L}^{-1}$ admits a Laurent development

$$\mathfrak{L}(\lambda)^{-1} = \sum_{n=-k}^{\infty} (\mathfrak{L}^{-1})_n (\lambda-\lambda_0)^n$$

for some integer $k \geq 1$, where $(\mathfrak{L}^{-1})_0 \in \mathrm{Fred}_0(V,U)$ and $(\mathfrak{L}^{-1})_n$ has finite rank for all $-k \leq n \leq -1$. Furthermore, the coefficients of the Laurent development of $\mathfrak{L}'\mathfrak{L}^{-1}$ at λ_0,

$$\mathfrak{L}'(\lambda)\mathfrak{L}(\lambda)^{-1} = \sum_{n=-k}^{\infty} \mathfrak{M}_n (\lambda-\lambda_0)^n,$$

have finite rank for every $n \in \{-k, \ldots, -1\}$, and their traces are given through

$$\operatorname{tr} \mathfrak{M}_n = \begin{cases} 0 & \textit{if} \quad -k \leq n \leq -2, \\ \mathfrak{m}[\mathfrak{L}; \lambda_0] & \textit{if} \quad n = -1. \end{cases}$$

In order to obtain our real non-analytic counterparts of Theorem 0.0.4, we need the previous analysis carried out in Chapters 4, 5 and 7.

The transversalization theorem of Chapter 4 establishes that if $\lambda_0 \in \mathrm{Alg}_k(\mathfrak{L})$ for some $0 \leq k < r+1$, then there exist a polynomial $\Phi : \mathbb{K} \to \mathcal{L}(U)$ with $\Phi(\lambda_0) = I_U$, and an integer $\nu = \nu(\Phi) \in \{1, \dots, k\}$, such that λ_0 is a ν-transversal eigenvalue of the product family

$$\mathfrak{L}^\Phi := \mathfrak{L}\Phi.$$

Moreover, $\nu(\Phi)$ and $\chi[\mathfrak{L}^\Phi; \lambda_0]$ are independent of Φ; therefore, one can extend the concept of multiplicity introduced in Definition 0.0.2 by setting

$$\chi[\mathfrak{L}; \lambda_0] := \chi[\mathfrak{L}^\Phi; \lambda_0]$$

if $\lambda_0 \in \mathrm{Alg}_k(\mathfrak{L})$, and

$$\chi[\mathfrak{L}; \lambda_0] := \infty$$

if $r = \infty$ and λ_0 is not an algebraic eigenvalue of $\mathfrak{L}$. Further, from the analysis of Chapter 5, it will become apparent that, in fact, if $\lambda_0 \in \mathrm{Alg}_k(\mathfrak{L})$, then $\nu(\Phi) = k$ for all transversalizing families of isomorphisms Φ, whether polynomial or not. When $\mathfrak{L}(\lambda_0)$ is an isomorphism, the multiplicity $\chi[\mathfrak{L}; \lambda_0]$ is defined as zero. Consequently, the multiplicity χ is finite if and only if λ_0 is an *algebraic value* of $\mathfrak{L}$, in the sense that either $\mathfrak{L}(\lambda_0) \in \mathrm{Iso}(U, V)$, or $\lambda_0 \in \mathrm{Alg}_k(\mathfrak{L})$ for some integer $1 \leq k < r+1$.

The main goal of Chapter 5 is to prove that this concept of multiplicity actually satisfies Axioms A1 and A2 of Theorem 0.0.1, thus providing us with the existence of the multiplicity $\mathfrak{m}$. The analysis of this chapter is based upon another independent concept of multiplicity, denoted by $\mu[\mathfrak{L}; \lambda_0]$, going back to Magnus [92], and defined through a certain polynomial factorization of the family $\mathfrak{L}$. Once some hidden connections between the algebraic invariants invoked in the definitions of μ and χ are established, it will follow that $\chi = \mu$ and that the following result is satisfied.

Theorem 0.0.5. *For every integer $1 \leq k \leq r$ there exist k finite-rank projections $P_1, \dots, P_k \in \mathcal{L}(U)$ and $\mathfrak{M}_k \in \mathcal{C}^{r-k}(\Omega, \mathcal{L}(U, V))$ such that*

$$\mathfrak{L}(\lambda) = \mathfrak{M}_k(\lambda)\,[(\lambda - \lambda_0)P_1 + I_U - P_1] \cdots [(\lambda - \lambda_0)P_k + I_U - P_k] \tag{4}$$

for all $\lambda \in \Omega$. Moreover, some of the following mutually exclusive options occur:

O1. $\mathfrak{L}(\lambda_0) \in \mathrm{Iso}(U, V)$*, and, in such a case,*

$$\chi[\mathfrak{L}; \lambda_0] = 0 \quad \textit{and} \quad P_1 = \cdots = P_k = 0.$$

O2. $\lambda_0 \in \mathrm{Alg}_\nu(\mathfrak{L})$ *for some $1 \leq \nu < r+1$, and, in such a case, $\mathfrak{M}_\nu(\lambda_0) \in \mathrm{Iso}(U, V)$, the projections $P_1, \dots, P_\nu$ are non-zero, and*

$$\chi[\mathfrak{L}; \lambda_0] = \sum_{i=1}^{\nu} \dim R[P_i].$$

O3. *There does not exist any integer* $1 \le \nu \le r$ *for which* $\lambda_0 \in \mathrm{Alg}_\nu(\mathfrak{L})$*, and, in such a case,* $\chi[\mathfrak{L};\lambda_0] = \infty$ *if* $r = \infty$*, whereas* $\chi[\mathfrak{L};\lambda_0]$ *remains undefined if* $r \in \mathbb{N}$*. Moreover, the projections* $P_1, \dots, P_k$ *are non-zero.*

Based on the Magnus factorization (4) guaranteed by Theorem 0.0.5, Chapter 6 proves Theorem 0.0.1 and some generalizations. Chapter 6 concludes by inferring Properties (iv), (v) and (viii) listed above from Theorem 0.0.1 and a number of variants of it.

Chapter 7 studies the classic concepts of Jordan chains, partial multiplicities and local Smith form. These are very old concepts going back to Smith [121] and Frobenius [36, 37], and characterize the local behavior of $\mathfrak{L}$ at λ_0. The starting point of Chapter 7 is the following classic result going back to Gohberg [41] (see Gohberg, Goldberg & Kaashoek [43], Gohberg & Sigal [54], Gohberg, Kaashoek & Lay [44] for later generalizations). Hereafter, $\mathcal{H}$ means holomorphic.

Theorem 0.0.6. *Suppose* $\mathbb{K} = \mathbb{C}$*, let* Ω *be an open and connected subset of* $\mathbb{C}$*, and let* $\mathfrak{L} \in \mathcal{H}(\Omega, \mathcal{L}(U,V))$ *satisfy* $\mathfrak{L}(\lambda_0) \in \mathrm{Fred}_0(U,V)$ *and* $\mathfrak{L}(\Omega) \cap \mathrm{Iso}(U,V) \neq \varnothing$*. Then, for every* $\lambda_0 \in \Omega$*, there exist an open neighborhood* $\tilde{\Omega} \subset \Omega$ *of* λ_0*, a topological decomposition* $U = U_0 \oplus U_1$ *with* $n := \dim U_0 = \dim N[\mathfrak{L}(\lambda_0)] < \infty$*, and three maps*

$$\mathfrak{E} \in \mathcal{H}(\tilde{\Omega}, \mathcal{L}(U,V)), \quad \mathfrak{F} \in \mathcal{H}(\tilde{\Omega}, \mathcal{L}(U)), \quad \mathfrak{S} \in \mathcal{H}(\tilde{\Omega}, \mathcal{L}(U_0))$$

such that $\mathfrak{E}(\lambda_0)$ *and* $\mathfrak{F}(\lambda_0)$ *are isomorphisms and*

$$\mathfrak{L}(\lambda) = \mathfrak{E}(\lambda)\,[\mathfrak{S}(\lambda) \oplus I_{U_1}]\,\mathfrak{F}(\lambda), \qquad \lambda \in \tilde{\Omega}, \tag{5}$$

where I_{U_1} *stands for the identity of* U_1 *and*

$$\mathfrak{S}(\lambda) = \mathrm{diag}\left\{(\lambda-\lambda_0)^{\kappa_1}, \dots, (\lambda-\lambda_0)^{\kappa_n}\right\}, \qquad \lambda \in \mathbb{K} \tag{6}$$

for some integers $\kappa_1 \ge \cdots \ge \kappa_n \ge 1$*.*

The factorization (5), with $\mathfrak{S}$ given through (6), is called the *local Smith form* of $\mathfrak{L}$ at λ_0. The integers $\kappa_1, \dots, \kappa_n$ of (6) must be uniquely determined, and are referred to as the *partial multiplicities* of $\mathfrak{L}$ at λ_0. One of the main results of Chapter 7 uses the transversalization theorem of Chapter 4 to show that

$$\mathfrak{m}[\mathfrak{L};\lambda_0] = \chi[\mathfrak{L};\lambda_0] = \sum_{j=1}^{n} \kappa_j.$$

Essentially, the proof of Theorem 0.0.6 is based on a preliminary finite-dimensional reduction relying on the property that $\mathfrak{L}(\lambda_0) \in \mathrm{Fred}_0(U,V)$, and on a further diagonalization algorithm based on the Gaussian elimination method. At each step of the underlying scheme one must make sure that the invariants of the transformed families remain unchanged, and this can be easily accomplished through the generalized Jordan chains of $\mathfrak{L}$ at λ_0.

In this context, the analysis of Chapter 7 reveals that the most appropriate device for characterizing the existence of a local Smith form is the concept of algebraic eigenvalue introduced by Definition 0.0.3, since it is a manageable and versatile invariant, intrinsic to the family $\mathfrak{L}$, which does not involve any constructive scheme, as occurs when using Jordan chains. Actually, the following sharp version of Theorem 0.0.6, valid for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, is satisfied; it constitutes the bulk of Chapter 7.

Theorem 0.0.7. *Suppose* $r \in \mathbb{N}$ *and* $\mathfrak{L} \in \mathcal{C}^r(\Omega, \mathcal{L}(U,V))$ *with* $\mathfrak{L}(\lambda_0) \in \mathrm{Fred}_0(U,V)$. *Then, the following assertions are equivalent:*

i) $\mathfrak{L}$ *admits a local Smith form at* λ_0*, in the sense that there exist an open neighborhood* $\tilde{\Omega} \subset \Omega$ *of* λ_0*, two families*

$$\mathfrak{E} \in \mathcal{C}(\tilde{\Omega}, \mathcal{L}(U,V)), \quad \mathfrak{F} \in \mathcal{C}(\tilde{\Omega}, \mathcal{L}(U))$$

such that $\mathfrak{E}(\lambda_0)$ *and* $\mathfrak{F}(\lambda_0)$ *are isomorphisms, and a topological decomposition* $U = U_0 \oplus U_1$ *with* $\dim U_0 = \dim N[\mathfrak{L}(\lambda_0)] = n$*, such that* (5) *and* (6) *hold true for some integers* $\kappa_1 \geq \cdots \geq \kappa_n$ *with* $\kappa_1 \leq r$ *and* $\kappa_n \geq 1$.

ii) $\lambda_0 \in \mathrm{Alg}_\nu(\mathfrak{L})$ *for some integer* $\nu \leq r$.

iii) *The length of every Jordan chain of* $\mathfrak{L}$ *at* λ_0 *is bounded above by* $\nu \leq r$.

iv) *The multiplicity* $\chi[\mathfrak{L}; \lambda_0]$ *is well defined and finite.*

Although some of the implications between the conditions stated in Theorem 0.0.7 were proved by Rabier [110], this characterization theorem is attributable to the authors.

Chapter 7 concludes with a short introduction to the theory of equivalence of families, as discussed, for example, by Gohberg, Kaashoek & Lay [44]. The pivotal feature of the underlying theory is the fact that the local Smith form of a family $\mathfrak{L}$ is the simplest representation of $\mathfrak{L}$ through pre- and post-multiplication by families of isomorphisms. As a by-product of Theorem 0.0.7, it becomes apparent that $\mathfrak{L}$ and $\mathfrak{M}$ are equivalent if and only if they possess the same local Smith form at λ_0.

Chapter 8 focuses the attention on analytic maps $\mathfrak{L} \in \mathcal{C}^\omega(\Omega, \mathcal{L}(U,V))$, and families of the form

$$\mathfrak{L}(\lambda) := \lambda I_U - A, \qquad \lambda \in \mathbb{C}, \tag{7}$$

for a given $A \in \mathcal{L}(U)$, in order to prove the following result (an early version of its first part can be found in Trofimov [124]).

Theorem 0.0.8. *Let* Ω *be an open and connected subset of* $\mathbb{K}$*, and* $\mathfrak{L} \in \mathcal{C}^\omega(\Omega, \mathcal{L}(U,V))$ *such that* $\mathfrak{L}(\Omega) \subset \mathrm{Fred}_0(U,V)$ *and* $\mathfrak{L}(\Omega) \cap \mathrm{Iso}(U,V) \neq \varnothing$. *Then, the spectrum* (3) *of* $\mathfrak{L}$ *in* Ω *is discrete in* Ω *and consists of algebraic eigenvalues of* $\mathfrak{L}$*, as discussed by Definition 0.0.3. Moreover, the map* $\mathfrak{L}^{-1}$ *is analytic in* $\Omega \setminus \Sigma(\mathfrak{L})$*, and exhibits a pole of order* $\nu \in \mathbb{N}$ *at every* $\lambda_0 \in \mathrm{Alg}_\nu(\mathfrak{L})$.

If, in addition, $\mathfrak{L}$ *has the form* (7)*, and* $\lambda_0 \in \mathrm{Alg}_\nu(\mathfrak{L})$*, then* ν *equals the algebraic ascent of* λ_0*, as an eigenvalue of* A*, and*

$$\mathfrak{m}[\mathfrak{L}; \lambda_0] = \chi[\mathfrak{L}; \lambda_0] = m_a(\lambda_0) = \dim N[(\lambda_0 I_U - A)^\nu].$$

Chapter 8 concludes by establishing the stability of the algebraic multiplicity for holomorphic families, as well as its homotopy invariance, within the spirit of the classic theorem of Rouché.

Chapter 10 shows that the Jordan theorem and the fundamental theorem of algebra are particular cases of the spectral theorem for monic matrix polynomials. By a matrix polynomial is meant a family $\mathfrak{L} \in \mathcal{H}(\mathbb{C}, \mathcal{L}(\mathbb{C}^N))$ of the form

$$\mathfrak{L}(\lambda) = \sum_{n=0}^{\ell} A_n \lambda^n, \qquad \lambda \in \mathbb{C},$$

where $A_0, \ldots, A_\ell \in \mathcal{L}(\mathbb{C}^N)$. It is said to be *monic* when $A_\ell = I_{\mathbb{C}^N}$. The results of this chapter give some new insights into the available results in the literature. Actually, they are intended as a first step to generalize the theory developed in this book to cover more abstract settings within the context of spectral theory.

Chapter 11 has an expository and bibliographical character. It introduces briefly some further developments of the multiplicity not previously treated in this book that are of interest in their own right.

Finally, Chapter 12 combines the linear theory developed in this book together with the topological degree to show that the algebraic multiplicity studied in this book provides an optimal algebraic invariant to characterize the nonlinear eigenvalues of a given family $\mathfrak{L}$. Thus, facilitating the transference of information from linear to nonlinear functional analysis.

It should be emphasized that eliminating the standard assumption that $\mathfrak{L}$ is holomorphic in Ω (through the concept of algebraic eigenvalue) gives rise to some deep versions of important classic results; for instance, the following striking version of the celebrated theorem of Riemann on removable singularities. Indeed, besides providing a particular case of the theorem of Riemann when $\mathbb{K} = \mathbb{C}$, it also covers the case $\mathbb{K} = \mathbb{R}$, where no previous result seems to be available. Its number of applications might be huge.

Theorem 0.0.9. *Let $r \in \mathbb{N} \cup \{\infty\}$, a point $\lambda_0 \in \Omega$, and $\mathfrak{L} \in \mathcal{C}^r(\Omega, \mathcal{L}(U, V))$ with $\mathfrak{L}(\lambda_0) \in \mathrm{Fred}_0(U, V)$. Suppose that, for some integer $0 \leq \nu \leq r$, the mapping*

$$\lambda \mapsto (\lambda - \lambda_0)^\nu \mathfrak{L}(\lambda)^{-1} \tag{8}$$

exists and is bounded in $\Omega \setminus \{\lambda_0\}$. Then there exists an open subset $\Omega' \subset \Omega$ with $\lambda_0 \in \Omega'$ for which the mapping

$$\lambda \mapsto (\lambda - \lambda_0)^\nu \mathfrak{L}(\lambda)^{-1}, \qquad \lambda \in \Omega' \setminus \{\lambda_0\},$$

admits an extension of class $\mathcal{C}^{r-\nu}$ to Ω'.

Consequently, to remove an isolated singularity for mappings of the form (8), it suffices to impose $\mathfrak{L}(\lambda_0) \in \mathrm{Fred}_0(U, V)$ and $\lambda_0 \in \mathrm{Alg}_\nu(\mathfrak{L})$ for some integer $\nu \leq r$; the holomorphy is not necessary. This result illustrates the strength of the general theory developed in this book.

Except Chapters 11 and 12, all chapters of this book end by proposing a list of exercises and some bibliographical comments. Some of the exercises are computational, but most of them ask the reader to provide alternative proofs of the results of the corresponding chapter, or to adapt the proofs already given in the chapter to obtain some related results.

Although we are indeed scientifically indebted to many people, in this occasion we are extremely honored to express our deepest gratitude to Professor I. Gohberg, the Editor of this series. It is not only that his pioneering works for meromorphic maps have influenced many chapters of this book, but also that his strong and lucid mathematical criticism during the final steps of its preparation have been tremendously encouraging for us.

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The authors

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López-Gómez, J.; Mora-Corral, C.

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