Chapter 2
Rings of Partial Differential Operators

Abstract In the ring of ordinary differential operators all ideals are principal. Consequently, the relation between an individual operator and the ideal that is generated by it is straightforward. The situation is different in rings of partial differential operators where in general ideals may have any number of generators, and only a Janet basis provides a unique representation. Therefore a more algebraic language is appropriate for dealing with partial differential operators and the ideals or modules they generate. It is introduced in the first section of this chapter. Subsequently it is applied for discussing certain properties of ideals in those rings of partial differential operators that are applied in later parts of this monograph. General references for this chapter are the books by Kolchin [37] or van der Put and Singer [71], or the article by Buium and Cassidy [8].

2.1 Basic Differential Algebra

This section summarizes some basic terminology from differential algebra that is used throughout this monograph. In order to study partial differential equations, rings of partial differential operators and modules over such rings are the proper concepts which are defined first.

A field $\mathscr{F}$ is called a *differential field* if it is equipped with a *derivation operator*. An operator δ on a field $\mathscr{F}$ is called a derivation operator if $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for all elements $a, b \in \mathscr{F}$. A field with a single derivation operator is called an *ordinary differential field*; if there is a finite set Δ containing several commuting derivation operators the field is called a *partial differential field*.

In this monograph rings of differential operators with derivative operators $\partial_x = \frac{\partial}{\partial x}$ and $\partial_y = \frac{\partial}{\partial y}$ with coefficients from some differential field are considered. Its elements have the form $\sum_{i,j} r_{i,j}(x,y)\partial_x^i \partial_y^j$; almost all coefficients $r_{i,j}$ are zero. The coefficient field is called the *base field*. If constructive and algorithmic methods

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are the main issue it is $\mathbb{Q}(x, y)$. However, in some places this is too restrictive and a suitable extension $\mathscr{F}$ of it may be allowed. The respective ring of differential operators is denoted by $\mathscr{D} = \mathbb{Q}(x, y)[\partial_x, \partial_y]$ or $\mathscr{D} = \mathscr{F}[\partial_x, \partial_y]$; if not mentioned explicitly, the exact meaning will be clear from the context.

The ring $\mathscr{D}$ is non-commutative, in general $\partial_x a = a\partial_x + \frac{\partial a}{\partial x}$ and similarly for the other variables; a is from the base field.

The following lemma is a simple consequence of this definition; it will be useful for understanding several details of the lattice structure in $\mathbb{Q}(x, y)[\partial_x, \partial_y]$. Its proof is considered in Exercise 2.1.

Lemma 2.1. *Two cases are distinguished for the commutativity of two first-order operators in the plane with coordinates x and y.*

(i) *Two operators $l_1 \equiv \partial_x + a_1\partial_y + b_1$ and $l_2 \equiv \partial_x + a_2\partial_y + b_2$ commute if*

$$(a_1 - a_2)_x + a_{1,y}a_2 - a_{2,y}a_1 = 0, \quad (b_1 - b_2)_x + b_{1,y}a_2 - b_{2,y}a_1 = 0.$$

(ii) *Two operators $l \equiv \partial_x + a\partial_y + b$ and $k \equiv \partial_y + c$ commute if*

$$(a + b)_y - c_x - ac_y = 0.$$

For an operator $L = \sum_{i+j\leq n} r_{i,j}(x, y)\partial_x^i\partial_y^j$ of order n the *symbol of* L is the homogeneous algebraic polynomial $symb(L) \equiv \sum_{i+j=n} r_{i,j}(x, y)X^iY^j$, X and Y algebraic indeterminates.

Let I be a left ideal which is generated by elements $l_i \in \mathscr{D}$, $i = 1, \ldots, p$. Then one writes $I = \langle l_1, \ldots, l_p\rangle$. Because right ideals are not considered in this monograph, sometimes I is simply called an ideal.

A m-dimensional left *vector module* $\mathscr{D}^m$ over $\mathscr{D}$ has elements $(l_1, \ldots, l_m)$, $l_i \in \mathscr{D}$ for all i. The sum of two elements of $\mathscr{D}^m$ is defined by componentwise addition; multiplication with a ring element l by $l(l_1, \ldots, l_m) = (ll_1, \ldots, ll_m)$.

The relation between left ideals in $\mathscr{D}$ or submodules of $\mathscr{D}^m$ on the one hand, and systems of linear pde's on the other is established as follows. Let $(z_1, \ldots, z_m)^T$ be an m-dimensional column vector of differential indeterminates such that $\partial_x z_i \neq 0$ and $\partial_y z_i \neq 0$. Then the product

$$(l_1, \ldots, l_m)(z_1, \ldots, z_m)^T = l_1z_1 + l_2z_2 + \ldots + l_mz_m \tag{2.1}$$

defines a linear differential polynomial in the z_i that may be considered as the left hand side of a partial differential equation; $z_1, \ldots, z_m$ are called the *dependent variables* or *functions*, depending on the *independent variables* x and y.

A $N \times m$ matrix $\{c_{i,j}\}$, $i = 1, \ldots, N$, $j = 1, \ldots, m$, $c_{i,j} \in \mathscr{D}$, defines a system of N linear homogeneous pde's

$$c_{i,1}z_1 + \ldots + c_{i,m}z_m = 0, \qquad i = 1, \ldots, N. \tag{2.2}$$

The $i-th$ equation of (2.2) corresponds to the vector

$$(c_{i,1}, c_{i,2}, \ldots, c_{i,m}) \in \mathscr{D}^m \quad \text{for} \quad i = 1, \ldots, N. \tag{2.3}$$

This correspondence between the elements of $\mathscr{D}^m$, the differential polynomials (2.1), and its corresponding pde's (2.2) allows to turn from one representation to the other whenever it is appropriate.

For $m = 1$ this relation becomes more direct. The elements $l_i \in \mathscr{D}$ are simply applied to a single differential indeterminate z. In this way the ideal $I = \langle l_1, l_2, \ldots \rangle$ corresponds to the system of pde's $l_1 z = 0$, $l_2 z = 0, \ldots$ for the single function z. Sometimes the abbreviated notation $Iz = 0$ is applied for the latter.

2.2 Janet Bases of Ideals and Modules

The generators of an ideal are highly non-unique; its members may be transformed in infinitely many ways by taking linear combinations of them or its derivatives over the base field without changing the ideal. This ambiguity makes it difficult to decide membership in an ideal or to recognize whether two sets of generators represent the same ideal. Furthermore, it is not clear what the solutions of the corresponding system of pde's are, if there are any. The same remarks apply to the vector-modules introduced above.

This was the starting point for Maurice Janet [30] early in the twentieth century to introduce a normal form for systems of linear pde's that has been baptized *Janet basis* in [60]. They are the differential analog to Gröbner bases of commutative algebra, originally introduced by Bruno Buchberger [7]; therefore they are also called *differential Gröbner bases*. Good introductions to the subject may be found in the articles by Oaku [50], Castro-Jiménez and Moreno-Frías [9], Plesken and Robertz [54] or Chap. 2 of Schwarz [61].

In order to generate a Janet basis, a ranking of derivatives must be defined. It is a total ordering such that for any derivatives δ, δ_1 and δ_2, and any derivation operator θ there holds $\delta \preceq \theta\delta$, and $\delta_1 \preceq \delta_2 \rightarrow \delta\delta_1 \preceq \delta\delta_2$. In this monograph lexicographic term orderings lex and graded lexicographic term orderings $grlex$ are applied. For partial derivatives of a single function their definition is analogous to the monomial orderings in commutative algebra. If $x \succ y$ is defined, derivatives up to order 3 in lex order are arranged like

$$\partial_{xxx} \succ \partial_{xxy} \succ \partial_{xx} \succ \partial_{xyy} \succ \partial_{xy} \succ \partial_x \succ \partial_{yyy} \succ \partial_{yy} \succ \partial_y \succ 1, \tag{2.4}$$

and in $grlex$ ordering

$$\partial_{xxx} \succ \partial_{xxy} \succ \partial_{xyy} \succ \partial_{yyy} \succ \partial_{xx} \succ \partial_{xy} \succ \partial_{yy} \succ \partial_x \succ \partial_y \succ 1. \tag{2.5}$$

For modules these orderings have to be generalized appropriately, e.g. the orderings TOP or POT of Adams and Loustaunau [2] may be applied.

The following convention will always be obeyed. In an individual operator or differential polynomial the terms are arranged decreasingly from left to right, i.e. the first term contains the highest derivative. A collection of such objects like the generators of an ideal or a module is arranged such that the leading terms do not increase. In particular, if the leading terms are pairwise different they will decrease from left to right, and from top to bottom. If the term order is not explicitly given it is assumed to be $grlex$ with $x \succ y$.

The most distinctive feature of a Janet basis is the fact that it contains all algebraic consequences for the derivatives in the ideal generated by its members explicitly. This is achieved by two basic operations, *reductions* and adding *integrability conditions*; the latter correspond to the S-pairs in commutative algebra.

An operator l_1 may be reduced w.r.t. another operator l_2 if the leading derivative of l_2 or a derivative thereof occurs in l_1. If this is true, its occurrence in l_1 may be removed by replacing it by the negative reductum of l_2 or its appropriate derivative. This process may be repeated until no further reduction is possible. This process will always terminate because in each step the derivatives in l_1 are lowered. The following example shows a single-step reduction of two operators.

Example 2.1. Let two operators l_1 and l_2 be given.

$$l_1 \equiv \partial_{xy} - \frac{x^2}{y^2}\partial_x - \frac{x-y}{y^2}, \quad l_2 \equiv \partial_x + \frac{1}{y}\partial_y + x.$$

The derivatives ∂_{xy} and ∂_x may be removed from l_1 with the result

$$\begin{aligned} Reduce(l_1, l_2) &= -\tfrac{1}{y}\partial_{yy} + \tfrac{1}{y^2}\partial_y - x\partial_y + \tfrac{x^2}{y^2}\big(\tfrac{1}{y}\partial_y + x\big) - \frac{x-y}{y^2} \\ &= -\tfrac{1}{y}\Big(\partial_{yy} + \tfrac{1}{y^2}(xy^3 - x^2 - y)\partial_y - \tfrac{1}{y}(x^3 - x + y)\Big). \end{aligned}$$

There are no further reductions possible. □

If a system of operators or differential polynomials is given, various reductions may be possible between pairs of its members. If all of them have been performed such that no further reduction is possible, the system is called *autoreduced*.

For an autoreduced system the integrability conditions have to be investigated. They arise if the same leading derivative occurs in two different members of the system or its derivatives. Upon subtraction, possibly after multiplication with suitable factors from the base field, the difference does not contain it any more. If it does not vanish after reduction w.r.t. the remaining members of the system, it is called an integrability condition that has to be added to the system. The following example shows this process.

Example 2.2. Consider the ideal

$$I = \Big\langle l_1 \equiv \partial_{xx} - \frac{1}{x}\partial_x - \frac{y}{x(x+y)}\partial_y,\ l_2 \equiv \partial_{xy} + \frac{1}{x+y}\partial_y,\ l_3 \equiv \partial_{yy} + \frac{1}{x+y}\partial_y \Big\rangle$$

in *grlex* term order with $x \succ y$. Its generators are autoreduced. If the integrability condition

$$l_{1,y} - l_{2,x} = \partial_{xy} + \frac{y}{2x+y}\partial_{yy}$$

is reduced w.r.t. to I, the new generator ∂_y is obtained. Adding it to the generators and performing all reductions, the given ideal is represented as $I = \langle \partial_{xx} - \frac{1}{x}\partial_x, \partial_y \rangle$. Its generators are autoreduced and the single integrability condition is satisfied. □

It may be shown that for any given system of operators or differential polynomials and a fixed ranking autoreduction and adding integrability conditions always terminates with a unique result; the proof may be found in the above quoted literature. Due to its fundamental importance a special term is introduced for it.

Definition 2.1 (Janet basis). For a given ranking an autoreduced system of differential operators is called a Janet basis if all integrability conditions reduce to zero.

If a system of operators or differential polynomials forms a Janet basis, it is a unique representation for the ideal or module it generates.

Due to its importance the following notation will be applied from now on. If the generators of an ideal or module are assured to be a Janet basis they are enclosed by a pair of $\langle\!\langle \ldots \rangle\!\rangle$. In general, if the Janet basis property is not known, the usual notation $\langle \ldots \rangle$ will be applied. According to this convention, in the preceding example the result may be written as $I = \langle\!\langle \partial_{xx} - \frac{1}{x}\partial_x, \partial_y \rangle\!\rangle$. By definition, a single element l is a Janet basis, i.e. there holds always $\langle l \rangle = \langle\!\langle l \rangle\!\rangle$. A system of operators or pde's with the property that all integrability conditions are satisfied is called *coherent*.

2.3 General Properties of Ideals and Modules

Just like in commutative algebra, the generators of an ideal in a ring of differential operators obey certain relations which are known as *syzygies*. Let a set of generators be $f = \{f_1, \ldots, f_p\}$ where $f_i \in \mathcal{D}$ for all i. Syzygies of f are relations of the form

$$d_{k,1} f_1 + \ldots + d_{k,p} f_p = 0$$

where $d_{k,i} \in \mathcal{D}$, $i = 1, \ldots p$, $k = 1, 2, \ldots$. The $(d_{k,1}, \ldots, d_{k,p})$ may be considered as elements of the module $\mathcal{D}^p$. The totality of syzygies generates a submodule.

Example 2.3. Consider the ideal $\langle f_1 \equiv \partial_x + a, f_2 \equiv \partial_y + b \rangle$ with the constraint $a_y = b_x$. The coherence condition for $\partial_x - a - f_1 = 0$ and $\partial_y + b - f_2 = 0$ yields $a\partial_y + a_y - f_{1,y} - b\partial_x - b_x + f_{2,x} = 0$. Reduction w.r.t. to the given generators and some simplification yields the single syzygy $(\partial_y + b) f_1 - (\partial_x + a) f_2 = 0$. □

Example 2.4. Consider the ideal

$$\left\langle\!\!\left\langle f_1 \equiv \partial_{xx} + \frac{4}{x}\partial_x + \frac{2}{x^2},\ f_2 \equiv \partial_{xy} + \frac{1}{x}\partial_y,\ f_3 \equiv \partial_{yy} + \frac{1}{y}\partial_y - \frac{x}{y^2}\partial_x - \frac{2}{y^2} \right\rangle\!\!\right\rangle.$$

The integrability condition for $\partial_{xx}+\frac{4}{x}\partial_x+\frac{2}{x^2}-f_1=0$ and $\partial_{xy}+\frac{1}{x}\partial_y-f_2=0$ yields upon reduction and simplification $f_{1,y}+f_{2,x}-\frac{3}{x}f_2=0$. Similarly from the last two elements $f_1-\frac{y^2}{x}f_{2,y}-\frac{y}{x}f_2+\frac{y^2}{x}f_{3,x}+\frac{y^2}{x^2}f_3=0$ is obtained. Autoreduction of these two equations yields the following two syzygies as the final answer

$$\big(\partial_{yy}+\tfrac{3}{y}\partial_y-\tfrac{x}{y^2}\partial_x-\tfrac{2}{y^2}\big)f_2-\big(\partial_{xy}+\tfrac{1}{x}\partial_y+\tfrac{2}{y}\partial_x+\tfrac{2}{xy}\big)f_3=0,$$

$$f_1-\big(\tfrac{y^2}{x}\partial_y+\tfrac{y}{x}\big)f_2+\big(\tfrac{y^2}{x}\partial_x+\tfrac{y^2}{x^2}\big)f_3=0. \qquad \square$$

Given any ideal I it may occur that it is properly contained in some larger ideal J with coefficients in the base field of I; then J is called a *divisor* of I. If the divisor J has the same differential type as I the latter is called *reducible*; if such a divisor does not exist it is called *irreducible*. If a divisor ideal of the same differential type does not exist even if a universal differential field is allowed for its coefficients, I is called *absolutely irreducible*. According to this definition an ideal may be irreducible, yet it may have divisors of lower differential type as the following example shows.

Example 2.5. Consider the operator L defined by $L\equiv\partial_{xx}+\frac{2}{x}\partial_x+\frac{y}{x^2}\partial_y-\frac{1}{x^2}$ of differential dimension $(1,2)$ (see Definition 2.2 below), i.e. its differential type is 1. The principal ideal $\langle L\rangle$ has the two divisors $l_1=\langle\!\langle\partial_x+\frac{1}{x},\partial_y-\frac{1}{y}\rangle\!\rangle$ and $l_2=\langle\!\langle\partial_x,\partial_y-\frac{1}{y}\rangle\!\rangle$, both of differential type 0; $l_1z=0$ has the solution $z=\frac{y}{x}$, $l_2z=0$ has the solution $z=y$. Both are also solutions of $Lz=0$. In Chap. 4 it will be shown that L is irreducible according to the above definition, and even absolutely irreducible. $\square$

The *greatest common right divisor* or *sum* of two ideals I and J is denoted by $Gcrd(I,J)=I+J$; it is the smallest ideal with the property that both I and J are contained in it. If they have the representation

$$I\equiv\langle f_1,\dots,f_p\rangle \quad\text{and}\quad J\equiv\langle g_1,\dots,g_q\rangle,$$

$f_i,g_j\in\mathscr{D}$ for all i and j, the sum is generated by the union of the generators of I and J (Cox et al. [13], page 191). The solution space of the equations corresponding to $Gcrd(I,J)$ is the intersection of the solution spaces of its arguments.

The *least common left multiple* or *left intersection* of two ideals I and J denoted by $Lclm(I,J)=I\cap J$; it is the largest ideal with the property that it is contained both in I and J. The solution space of $Lclm(I,J)z=0$ is the smallest space containing the solution spaces of its arguments. In the last two subsections of this chapter the properties of sum and intersection ideals will be discussed in some detail.[1]

[1] Some authors define it as the lowest element w.r.t. the term order in the ideal defined above [26].

Example 2.6. Consider the ideals $I = \langle\!\langle \partial_{yyy} + \frac{3}{y}\partial_{yy},\ \partial_x + \frac{y}{x}\partial_y \rangle\!\rangle$ and

$$J = \left\langle\!\!\left\langle \partial_{xx} + \frac{1}{x}\partial_x - \frac{1}{x^2},\ \partial_{xy} + \frac{1}{x}\partial_y + \frac{1}{y}\partial_x + \frac{1}{xy},\ \partial_{yy} + \frac{1}{y}\partial_y - \frac{1}{y^2} \right\rangle\!\!\right\rangle.$$

Applying the above specification, the $Gcrd$ and the $Lclm$ are

$$Gcrd(I,J) = \langle\!\langle \partial_{yy} + \tfrac{1}{y}\partial_y - \frac{1}{y^2},\ \partial_x + \tfrac{y}{x}\partial_y \rangle\!\rangle,$$

$$Lclm(I,J) = \langle\!\langle \partial_{yyy} + \tfrac{3}{y}\partial_{yy},\ \partial_{xx} - \frac{y^2}{x^2}\partial_{yy} + \tfrac{1}{x}\partial_x - \frac{y}{x^2}\partial_y,$$
$$\partial_{xy} + \tfrac{y}{x}\partial_{yy} + \tfrac{1}{y}\partial_x + \tfrac{2}{x}\partial_y.$$

It follows that $lc(H_I) = lc(H_J) = 3$, $lc(H_{I+J}) = 2$ and $lc(H_{I\cap J}) = 4$ in accordance with Sit's relation (2.8). In terms of solution spaces this result may be understood as follows. For $Iz = 0$ a basis of the solution space is $\{1, \frac{x}{y}, \frac{y}{x}\}$, and for $Jz = 0$ a basis is $\{\frac{1}{xy}, \frac{x}{y}, \frac{y}{x}\}$. A basis for their two-dimensional intersection space is $\{\frac{x}{y}, \frac{y}{x}\}$, it is the solution space of $Gcrd(I,J)z = 0$. □

Example 2.7. Consider the two ideals

$$I = \left\langle\!\!\left\langle \partial_x + \frac{1}{x}, \partial_y + \frac{1}{y} \right\rangle\!\!\right\rangle \text{ and } J = \left\langle\!\!\left\langle \partial_x + \frac{1}{x+y}, \partial_y + \frac{1}{x+y} \right\rangle\!\!\right\rangle.$$

Their one-dimensional solution spaces are generated by $\{\frac{1}{xy}\}$ and $\{\frac{1}{x+y}\}$ respectively. Then $Gcrd(I,J) = \langle 1 \rangle$ and

$$Lclm(I,J) = \left\langle\!\!\left\langle \partial_{yy} + \frac{2}{y}\frac{x+2y}{x+y}\partial_y + \frac{2}{y(x+y)},\ \partial_x - \frac{y^2}{x^2}\partial_y + \frac{x-y}{x^2} \right\rangle\!\!\right\rangle.$$

A basis for the solution space of $Lclm(I,J)z = 0$ is $\{\frac{1}{xy}, \frac{1}{x+y}\}$; it will be determined algorithmically in Example 3.5. □

For ordinary differential operators the exact quotient has been defined on page 3. Because all ideals of ordinary differential operators are principal, it is obtained by the usual division scheme. This is different in rings of partial differential operators and a proper generalization of the exact quotient is required. Let $I \equiv \langle f_1, \ldots, f_p \rangle \in \mathcal{D}$ and $J \equiv \langle g_1, \ldots, g_q \rangle \in \mathcal{D}$ be such that $I \subseteq J$, i.e. J is a *divisor* of I. The *exact quotient* is generated by

$$\{(e_{i,1}, \ldots, e_{i,q}) \in \mathcal{D}^q | e_{i,1}g_1 + \ldots + e_{i,q}g_q = f_i, i = 1, \ldots, p\}.$$

The *exact quotient module* $Exquo(I,J)$ is generated by

$$\{h = (h_1, \ldots, h_q) \in \mathcal{D}^q | h_1 g_1 + \ldots + h_q g_q \in I\}.$$

It generalizes the syzygy module of J; the latter is obtained for the special choice $I = 0$. If the elements of the exact quotient module are arranged as rows of a matrix with q columns, and the generators of J as elements of a q−dimensional vector, I may be represented as

$$I = Exquo(I, J)J. \tag{2.6}$$

This defines the juxtaposition of $Exquo(I, J)$ and J in terms of matrix multiplication; it generalizes the product representation $L = L_1 L_2$ of an operator L. In general, the result at the right hand side of (2.6) has to be transformed into a Janet basis in order to obtain the original generators $f_1, \ldots, f_p$.

Example 2.8. Consider again the ideals

$$I = \left\langle\!\left\langle \partial_{yy} + \frac{2}{y}\frac{x+2y}{x+y}\partial_y + \frac{2}{y(x+y)},\ \partial_x - \frac{y^2}{x^2}\partial_y + \frac{x-y}{x^2} \right\rangle\!\right\rangle$$

and $J = \langle\!\langle \partial_x + \frac{1}{x}, \partial_y + \frac{1}{y} \rangle\!\rangle$ of the preceding example; obviously $I \subset J$. Division yields $\left\{ \left(0, \partial_y + \frac{x+3y}{y(x+y)}\right), \left(1, -\frac{y^2}{x^2}\right) \right\}$. There is a single syzygy $\left(\partial_y + \frac{1}{y}, -\partial_x - \frac{1}{x}\right)$. The Janet basis representation for the exact quotient module is

$$Exquo(I, J) = \left\langle\!\left\langle \left(0, \partial_x + \frac{x-y}{x(x+y)}\right), \left(0, \partial_y + \frac{x+3y}{y(x+y)}\right), \left(1, -\frac{y^2}{x^2}\right) \right\rangle\!\right\rangle \subset \mathscr{D}^2.$$

Continued in Example 3.5. □

Example 2.9. Consider the ideal $I = \langle\!\langle \partial_{xx} + \frac{1}{x}\partial_x, \partial_{xy}, \partial_{yy} + \frac{1}{y}\partial_y \rangle\!\rangle$ in $grlex$, $x \succ y$ term order. The divisor $J = \langle\!\langle \partial_x, \partial_y \rangle\!\rangle$ yields the exact quotient module $Exquo(I, J) = \langle\!\langle \left(\partial_x + \frac{1}{x}, 0\right), (\partial_y, 0), (0, \partial_x), \left(0, \partial_y + \frac{1}{y}\right) \rangle\!\rangle$. It may be represented as the intersection of two maximal modules of order 1, i.e.

$$Exquo(I, J) = Lclm\Big(\langle\!\langle (1, 0), (0, \partial_x)\left(0, \partial_y + \tfrac{1}{y}\right) \rangle\!\rangle,$$
$$\langle\!\langle \left(\tfrac{x}{y}, 1\right), \left(\partial_x + \tfrac{1}{x}, 0\right), (\partial_y, 0) \rangle\!\rangle \Big). \qquad \square$$

In the next chapter it will be shown how to obtain their decomposition algorithmically, and how the exact quotient module may be applied for solving systems of linear pde's that are not completely reducible.

Consider $I \subseteq \mathscr{D}$, and denote by I_n the intersection of I with the $\mathscr{F}$-linear space of all derivatives of order not higher than n. Then according to Kolchin [37], see also Buium and Cassidy [8], and Pankratiev et al. [38], the Hilbert-Kolchin polynomial of I is defined by

$$H_I(n) \equiv \binom{n+k}{k} - \dim I_n; \tag{2.7}$$

k is the number of variables. The first term equals the number of all derivatives of order not higher than n. Hence, for sufficiently large n the value of $H_I(n)$ counts the number of derivatives of order not higher than n which is not in the ideal generated by the leading derivatives of the generators of I. The degree $deg(H_I)$ of H_I is called the *differential type* of I ([37], page 130; [8], page 602). Its leading coefficient $lc(H_I)$ is called the *typical differential dimension* of I, ibid. If $I, J \subseteq \mathscr{D}$ are two ideals, Sit [65], Theorem 4.1, has shown the important equality

$$lc(H_{I+J}) + lc(H_{I\cap J}) = lc(H_I) + lc(H_J) \tag{2.8}$$

for its typical differential dimensions.

According to Kolchin [36], $deg(H_I)$ and $lc(H_I))$ are differential birational invariants; their importance justifies the introduction of a special term for these quantities.

Definition 2.2. The pair $(deg(H_I), lc(H_I))$ for an ideal I is called the differential dimension of I, denoted by d_I [23].

For the solutions of the differential equations attached to any ideal or module of differential operators, these quantities have an important meaning [8, 36].

Theorem 2.1. *The differential type denotes the largest number of arguments occurring in any undetermined function of the general solution. The typical differential dimension means the number of functions depending on this maximal number of arguments.*

Apparently the differential dimension describes somehow the *"size"* of the solution space. In this terminology the differential dimension $(0, m)$ corresponds to a system of pde's with a finite-dimensional solution space of dimension m over the field of constants. This discussion shows that the differential dimension is the proper generalization of the dimension of a solution space to general systems of linear pde's.

Example 2.10. For the ideal $I = \langle\!\langle \partial_{xx} - \frac{1}{x}\partial_x, \partial_y \rangle\!\rangle$ of the preceding example, only the two derivatives 1 and ∂_x are not contained in the ideal generated by the leading derivatives. Thus $H_I = 2$ and $d_I = (0, 2)$. □

Example 2.11. Let the principal ideal $I = \langle \partial_x + a\partial_y + b \rangle$ be given. There are $\frac{1}{2}(n^2 + n)$ derivatives of order not higher than n containing at least a single derivative ∂_x. Therefore $H_I = n + 1$ and $d_I = (1, 1)$. □

Example 2.12. Consider the ideal $I = \langle\!\langle \partial_{xxx}, \partial_{xxy} \rangle\!\rangle$. The number of derivatives which are multiples of either leading term is $\frac{1}{2}(n-2)(n+1)$. Therefore $H_I = 2n+2$ and $d_I = (1, 2)$. □

By analogy with the well known *Landau symbol* of asymptotic analysis, the following notation will frequently be applied. Whenever in an expression terms of order lower than some fixed term τ are not relevant, they are collectively denoted by $o(\tau)$. This will frequently occur in lex term orderings where τ denotes the highest term involving a particular variable.

Another short hand notation concerns the generators of ideals or modules of differential operators. If only the number of generators and its leading derivatives are of interest, the abbreviated notation $\langle \ldots \rangle_{LT}$ will be used. For example, if an ideal of differential operators is generated by two elements with leading derivatives ∂_{xx} and ∂_{xy}, it is denoted by $\langle \partial_{xx}, \partial_{xy} \rangle_{LT}$. A principal ideal that is generated by a single generator with highest derivative ∂_{xxx} is abbreviated by $\langle \partial_{xxx} \rangle_{LT}$.

2.4 Differential Type Zero Ideals in $\mathbb{Q}(x, y)[\partial_x, \partial_y]$

Understanding the ideal structure in the ring $\mathbb{Q}(x, y)[\partial_x, \partial_y]$ will be of utmost importance when dealing with linear pde's and their decompositions. In the first place this means to characterize the ideals that are relevant in a certain context as detailed as possible. Ultimately this comes down to a partial classification of such ideals. Secondly, the relationships between individual ideals must be understood in order to utilize them for the solution procedure. For any two ideals this means deciding pairwise inclusion, determining their sum and their intersection.

To begin with, the range of ideals to be covered has to be delimited by suitable constraints. In particular they are selected with regard to solving differential equations that are of interest for some application. This requirement confines the derivatives to order not higher than three. In addition the differential dimension of an ideal turns out to be an important distinctive feature that may be applied for classification. Furthermore, some kind of completeness in the mathematical sense will be desirable.

Taking these considerations into account the subsequent three propositions describe certain "large" ideals close to the top of the complete lattice. The $grlex$ term order with $x \succ y$ is always applied. The first result concerns ideals of differential type zero; they correspond to systems of linear pde's with a finite-dimensional solution space. They will be denoted by $\mathbb{J}^{(0,k)}$ where $(0, k)$ means their differential dimension. This proceeding is only meaningful if the coherence conditions for the coefficients are explicitly known and it is assured that they are satisfied.

Proposition 2.1. *There are six types $\mathbb{J}^{(0,k)}$ of ideals with differential dimension $(0, k)$ and $k \leq 3$. The integrability conditions (IC's for short) are given for each case in terms of a Janet basis for the coefficients.*

$$\mathbb{J}^{(0,1)}: \ \langle \partial_x + a, \partial_y + b \rangle, \ \textit{integrability condition } a_y = b_x.$$

$$\mathbb{J}_1^{(0,2)}: \ \ \langle \partial_{yy} + a_1 \partial_y + a_2, \partial_x + b_1 \partial_y + b_2 \rangle,$$

integrability conditions

$$b_{2,yy} - 2b_{1,y}a_2 + b_{2,y}a_1 - a_{2,x} - a_{2,y}b_1 = 0,$$
$$b_{1,yy} - b_{1,y}a_1 + 2b_{2,y} - a_{1,x} - a_{1,y}b_1 = 0.$$

$$\mathbb{J}_2^{(0,2)}: \ \ \langle \partial_{xx} + a_1 \partial_x + a_2, \partial_y + b \rangle,$$

integrability conditions

$$b_x - \tfrac{1}{2}a_{1,y} = 0, \quad a_{2,y} - \tfrac{1}{2}a_{1,xy} - \tfrac{1}{2}a_{1,y}a_1 = 0.$$

$$\mathbb{J}_1^{(0,3)}: \quad \langle \partial_{yyy} + a_1\partial_{yy} + a_2\partial_y + a_3, \partial_x + b_1\partial_y + b_2 \rangle,$$

integrability conditions

$$\begin{aligned} & b_{2,yy} + \tfrac{1}{6}a_{1,xy} + \tfrac{1}{6}a_{1,yy}b_1 + b_{1,y}(\tfrac{1}{3}a_{1,y} + \tfrac{2}{9}a_1^2 - a_2) + \tfrac{1}{3}b_{2,y}a_1 \\ & \qquad + \tfrac{2}{9}a_{1,x}a_1 + \tfrac{2}{9}a_{1,y}b_1a_1 - \tfrac{1}{2}a_{2,x} - \tfrac{1}{2}a_{2,y}b_1 = 0, \\ & b_{1,yy} - \tfrac{1}{3}b_{1,y}a_1 + b_{2,y} - \tfrac{1}{3}a_{1,x} - \tfrac{1}{3}a_{1,y}b_1 = 0, \end{aligned}$$

$$\begin{aligned} & a_{1,xyy} + a_{1,yyy}b_1 + 2a_{1,xy}a_1 + a_{1,yy}(3b_{1,y} + 2b_1a_1) - 3a_{2,xy} - 3a_{2,yy}b_1 \\ & \quad + b_{1,y}(6a_{1,y}a_1 - 9a_{2,y} + \tfrac{4}{3}a_1^3 - 6a_1a_2 + 18a_3) \\ & \quad + a_{1,x}(2a_{1,y} + \tfrac{4}{3}a_1^2 - 2a_2) + 2a_{1,y}^2b_1 + a_{1,y}(\tfrac{4}{3}b_1a_1^2 - 2b_1a_2) \\ & \quad - 2a_{2,x}a_1 - 2a_{2,y}b_1a_1 + 6a_{3,x} + 6a_{3,y}b_1 = 0. \end{aligned}$$

$$\begin{aligned} \mathbb{J}_2^{(0,3)}: \ & \langle \partial_{xx} + a_1\partial_x + a_2\partial_y + a_3, \partial_{xy} + b_1\partial_x + b_2\partial_y + b_3, \\ & \partial_{yy} + c_1\partial_x + c_2\partial_y + c_3 \rangle, \end{aligned}$$

integrability conditions

$$\begin{gathered} b_{3,y} - c_{3,x} + a_3c_1 - b_1b_3 - b_2c_3 + b_3c_2 = 0, \\ b_{2,y} - c_{2,x} + a_2c_1 - b_1b_2 + b_3 = 0, \\ b_{1,y} - c_{1,x} + a_1c_1 - b_1^2 + b_1c_2 - b_2c_1 - c_3 = 0, \\ a_{3,y} - b_{3,x} - a_1b_3 - a_2c_3 + a_3b_1 + b_2b_3 = 0, \\ a_{2,y} - b_{2,x} - a_1b_2 + a_2b_1 - a_2c_2 + a_3 + b_2^2 = 0, \\ a_{1,y} - b_{1,x} - a_2c_1 + b_1b_2 - b_3 = 0. \end{gathered}$$

$$\mathbb{J}_3^{(0,3)}: \quad \langle \partial_{xxx} + a_1\partial_{xx} + a_2\partial_x + a_3, \partial_y + b \rangle,$$

integrability conditions

$$b_x - \tfrac{1}{3}a_{1,y} = 0, \quad a_{1,xy} + \tfrac{2}{3}a_{1,y}a_1 - a_{2,y} = 0,$$

$$a_{2,xy} - a_{1,y}(\tfrac{2}{3}a_{1,x} + \tfrac{2}{9}a_1^2 - a_2) + \tfrac{1}{3}a_{2,y}a_1 - 3a_{3,y} = 0.$$

Proof. Any ideal of differential dimension $(0, k)$ must contain generators with leading terms ∂_{x^i} and ∂_{y^j} with $i, j \geq 1$. From the listing (2.5) and the constraints $i, j \leq 3$ the following choices of leading derivatives are possible.

$$(\partial_x, \partial_y), \quad (\partial_{yy}, \partial_x), \quad (\partial_{xx}, \partial_y), \quad (\partial_{yyy}, \partial_x), \quad (\partial_{xx}, \partial_{xy}, \partial_{yy}), \quad (\partial_{xxx}, \partial_y).$$

They yield the six ideals given above. The integrability condition for $\mathbb{J}^{(0,1)}$ is obvious. For $\mathbb{J}_1^{(0,2)}$, differentiating $\partial_x + b_1\partial_y + b_2$ w.r.t. y and reducing the result w.r.t. $\partial_{yy} + a_1\partial_y + a_2$ yields

$$\partial_{xy} + (b_{1,y} + b_2 - a_1 b_1)\partial_y + b_{2,y} - a_1 b_1.$$

Deriving a second time w.r.t. y and performing all possible reductions leads to

$$\partial_{xyy} + (b_{1,yy} + 2b_{2,y} - a_{1,y}b_1 - 2a_1 b_{1,y} - a_1 b_2 - a_2 b_1 + a_1^2 b_1)\partial_y \\ + b_{2,yy} - a_{2,y}b_1 - 2a_2 b_{1,y} - a_2 b_2 + a_1 a_2 b_1.$$

Differentiating now the first generator $\partial_{yy} + a_1\partial_y + a_2$ w.r.t. x and applying the above expression with leading term ∂_{xy} for reduction yields

$$\partial_{xyy} + (a_{1,x} - a_1 b_{1,y} - a_1 b_2 - a_2 b_1 + a_1^2 b_1)\partial_y + a_{2,x} - a_1 b_{2,y} - a_2 b_2 + a_1 a_2 b_1.$$

Equating the coefficients of the two operators with leading term ∂_{xyy} leads to the given IC's after some simplifications. The calculations for the other ideals are similar and are therefore omitted. □

It should be noticed that the $IC's$ given in the above proposition are coherent. This is obvious due to the term order that has been chosen; they are linear in the leading derivatives with constant leading coefficient, and the leading derivatives contain pairwise different functions. The importance of the $IC's$ given in the above proposition will become clear later on, e.g. in proving the existence of certain divisors in Exercise 3.2.

2.5 Differential Type Zero Modules over $\mathbb{Q}(x, y)[\partial_x, \partial_y]$

In this subsection vector modules of differential dimension $(0, 1)$ and $(0, 2)$ in $\mathcal{D}^2$ are considered; they correspond to systems of linear pde's in two dependend variables.

Proposition 2.2. *There are seven types $\mathbb{M}^{(0,k)}$ of modules of differential dimension $(0, k)$ with $k = 1$ or $k = 2$ in $\mathcal{D}^2$. The integrability conditions are given for each case in terms of a Janet basis for its coefficients.*

$\mathbb{M}_1^{(0,1)} : \langle (1, a), (0, \partial_x + b), (0, \partial_y + c) \rangle$, *integrability condition* $b_y - c_x = 0$.

$\mathbb{M}_2^{(0,1)} : \langle (0, 1), (\partial_x + a, 0), (\partial_y + b, 0) \rangle$, *integrability condition* $a_y - b_x = 0$.

$\mathbb{M}_1^{(0,2)} : \langle (1, 0), (0, \partial_{yy} + a_1\partial_y + a_2), (0, \partial_x + b_1\partial_y + b_2) \rangle$

with the same integrability conditions as for ideal type $\mathbb{J}_1^{(0,2)}$.

$$\mathbb{M}_2^{(0,2)} : \langle (1, 0), (0, \partial_{xx} + a_1\partial_x + a_2), (0, \partial_y + b) \rangle$$

with the same integrability conditions as for ideal type $\mathbb{J}_2^{(0,2)}$.

$$\mathbb{M}_3^{(0,2)} : \langle (\partial_x + a_1, a_2), (\partial_y + b_1, b_2), (c_1, \partial_x + c_2), (d_1, \partial_y + d_2) \rangle$$

with integrability conditions

$$a_{1,y} - b_{1,x} - a_2 d_1 + b_2 c_1 = 0,$$
$$a_{2,y} - b_{2,x} - a_1 b_2 + a_2 b_1 - a_2 d_2 + b_2 c_2 = 0,$$

$$c_{1,y} - d_{1,x} + a_1 d_1 - b_1 c_1 + c_1 d_2 - c_2 d_1 = 0,$$
$$c_{2,y} - d_{2,x} + a_2 d_1 - b_2 c_1 = 0.$$

$$\mathbb{M}_4^{(0,2)} : \quad \langle (1, c), (0, \partial_{yy} + a_1\partial_y + b_2), (0, \partial_x + b_1\partial_y + b_2) \rangle$$

with the same integrability conditions as for ideal type $\mathbb{J}_4^{(0,2)}$.

$$\mathbb{M}_5^{(0,2)} : \quad \langle (1, c), (0, \partial_{xx} + a_1\partial_x + a_2), (0, \partial_y + b) \rangle$$

with the same integrability conditions as for ideal type $\mathbb{J}_5^{(0,2)}$.

Proof. For any component the generators must contain leading derivatives ∂_{x^i} and ∂_{y^j} with $i, j \geq 0$. For type $\mathbb{M}^{(0,1)}$ this allows the combination of leading terms $\{(1, 0), (0, \partial_x), (0, \partial_y)\}$ and $\{(0, 1), (\partial_x, 0), \partial_y, 0)\}$. For type $\mathbb{M}^{(0,2)}$ the possible leading terms are

$$\{(1, 0), (0, \partial_{xx}), (0, \partial_y)\}, \quad \{(1, 0), (0, \partial_{yy}), (0, \partial_x)\},$$
$$\{(\partial_x, 0), (\partial_y, 0), (0, \partial_x), (0, \partial_y)\},$$
$$\{(0, 1), (\partial_{xx}, 0), (\partial_y, 0)\}, \quad \{(0, 1), (\partial_{yy}, 0), (\partial_x, 0)\}.$$

The integrability conditions are obtained as in the preceding proposition. □

There is one more module of differential dimension $(0, 1)$ in $\mathscr{D}^3$ needed for the decompositions discussed in the next chapter; its type is defined to be $\mathbb{M}_3^{(0,1)}$ and is the subject of Exercise 2.8.

Example 2.13. The module

$$M \equiv \left\langle\!\!\left\langle \left(\partial_x - \frac{x^2 - 3y}{x(x^2 - y)}, -\frac{1}{x^2 - y}\right), (\partial_y, 0), \right.\right.$$
$$\left.\left.\left(\frac{4y(x^2 + y)}{x^2(x^2 - y)}, \partial_x - \frac{2(x^2 + y)}{x(x^2 - y)}\right)\left(-\frac{2x}{x^2 - y}, \partial_y + \frac{1}{x^2 - y}\right)\right\rangle\!\!\right\rangle$$

is of type $\mathbb{M}_3^{(0,2)}$; the coherence conditions may easily be tested by applying the explicit form as given in the above proposition without running a costly Janet basis algorithm. Continued in Example 3.4. □

2.6 Laplace Divisors $\mathbb{L}_{x^m}(L)$ and $\mathbb{L}_{y^n}(L)$

The origin of these ideals goes back to Laplace who introduced an iterative solution scheme for equations with leading derivative z_{xy}; it is described in Appendix C. Later on it was realized that this procedure is essentially equivalent to determining a so called *involutive system*. In more modern language this comes down to constructing a Janet basis for an ideal that is generated by the operator corresponding to the originally given equation, and an ordinary operator of fixed order involving exclusively derivatives w.r.t. x or y. Thus this important concept may be generalized to large classes of operators with a mixed leading derivative. Due to its origin the following definition is suggested.

Definition 2.3. Let L be a partial differential operator in the plane; define

$$\mathfrak{l}_m \equiv \partial_{x^m} + a_{m-1}\partial_{x^{m-1}} + \ldots + a_1\partial_x + a_0 \tag{2.9}$$

and

$$\mathfrak{k}_n \equiv \partial_{y^n} + b_{n-1}\partial_{y^{n-1}} + \ldots + b_1\partial_y + b_0 \tag{2.10}$$

be ordinary differential operators w.r.t. x or y; $a_i, b_i \in \mathbb{Q}(x, y)$ for all i; m and n are natural numbers not less than 2. Assume the coefficients a_i, $i = 0, \ldots, m-1$ are such that L and $\mathfrak{l}_m$ form a Janet basis. If m is the smallest integer with this property then $\mathbb{L}_{x^m}(L) \equiv \langle\!\langle L, \mathfrak{l}_m \rangle\!\rangle$ is called a Laplace divisor of L. Similarly, if b_j, $j = 0, \ldots, n-1$ are such that L and $\mathfrak{k}_n$ form a Janet basis and n is minimal, then $\mathbb{L}_{y^n}(L) \equiv \langle\!\langle L, \mathfrak{k}_n \rangle\!\rangle$ is called a Laplace divisor of L. Both Laplace divisors have differential dimension $(1, 1)$.

The possible existence of a Laplace divisor for operators of order 2 or 3 is investigated next.

Proposition 2.3. *Let the second-order partial differential operator*

$$L \equiv \partial_{xy} + A_1\partial_x + A_2\partial_y + A_3 \tag{2.11}$$

be given with $A_i \in \mathbb{Q}(x, y)$ for all i; m and n are natural numbers not less than 2.

(i) *If A_1, A_2 and A_3 satisfy the differential polynomial constructed in the proof below, there exists a Laplace divisor $\mathbb{L}_{x^m}(L) = \langle\!\langle L, \mathfrak{l}_m \rangle\!\rangle$.*
(ii) *If A_1, A_2 and A_3 satisfy the differential polynomial constructed in the proof below there exists a Laplace divisor $\mathbb{L}_{y^n}(L) = \langle\!\langle L, \mathfrak{k}_n \rangle\!\rangle$.*
(iii) *If there are two Laplace divisors $\mathbb{L}_{x^m}(L)$ and $\mathbb{L}_{y^n}(L)$, the operator L is completely reducible, $\langle L \rangle = Lclm\big(\mathbb{L}_{x^m}(L), \mathbb{L}_{y^n}(L)\big)$.*

Proof. The proof will be given for case (ii). If the operator (2.11) is derived repeatedly w.r.t. y, and the reductum is reduced in each step w.r.t. (2.11), $n-2$ expressions of the form

$$\partial_{xy^i} + p_i\partial_x + p_{i,i}\partial_{y^i} + p_{i,i-1}\partial_{y^{i-1}} + \ldots + p_{i,0} \tag{2.12}$$

for $2 \le i \le n-1$ are obtained. All coefficients p_i and $p_{i,j}$ of (2.12) are differential polynomials in the ring $\mathbb{Q}\{A_1, A_2, A_3\}$ with derivatives ∂_x and ∂_y. There is no reduction w.r.t. (2.10) possible. Deriving the last expression (2.12) corresponding to $i = n-1$ once more w.r.t. y and reducing the reductum w.r.t. both (2.11) and (2.10) yields

$$\begin{aligned}\partial_{xy^n} &+ p_n\partial_x + (p_{n,n-1} - p_{n,n}b_{n-1})\partial_{y^{n-1}}\\ &+(p_{n,n-2} - p_{n,n}b_{n-2})\partial_{y^{n-2}} + \ldots\\ &+(p_{n,1} - p_{n,n}b_1)\partial_y + p_{n,0} - p_{n,n}b_0.\end{aligned} \tag{2.13}$$

In the first derivative of (2.10) w.r.t. x

$$\begin{aligned}\partial_{xy^n} &+ b_{n-1,x}\partial_{y^{n-1}} + b_{n-2,x}\partial_{y^{n-2}} + \ldots + b_{1,x}\partial_y + b_{0,x}\\ &+b_{n-1}\partial_{xy^{n-1}} + b_{n-2}\partial_{xy^{n-2}} + \ldots + b_1\partial_{xy} + b_0\partial_x\end{aligned}$$

the terms containing derivatives of the form ∂_{xy^k} for $1 \le k \le n-1$ may be reduced w.r.t. (2.11) or (2.12) with the result

$$\begin{aligned}&\partial_{xy^n} + (b_{n-1,x} - p_{n-1,n-1}b_{n-1})\partial_{y^{n-1}}\\ &\quad+(b_{n-2,x} - p_{n-1,n-2}b_{n-1} - p_{n-2,n-2}b_{n-2})\partial_{y^{n-2}}\\ &\qquad\vdots \qquad\qquad\qquad\qquad \vdots\\ &+(b_{1,x} - p_{n-1,1}b_{n-1} - p_{n-2,1}b_{n-2}\ldots - p_{2,1}b_2 - A_2b_1)\partial_y\\ &\quad+b_{0,x} - p_{n-1,0}b_{n-1} - p_{n-2,0}b_{n-2} - \ldots - p_{2,0}b_2 - A_3b_1\\ &\quad+(b_0 - p_1b_{n-1} - p_{n-2}b_{n-2} - \ldots - p_2b_2 - A_1b_1)\partial_x.\end{aligned} \tag{2.14}$$

If this expression is subtracted from (2.13), the coefficients of the derivatives must vanish in order that (2.11) and (2.10) combined form a Janet basis. The resulting system of equations is

$$\begin{aligned}b_{n-1,x} + (p_{n,n} - p_{n-1,n-1})b_{n-1} - p_{n,n-1} &= 0,\\ b_{n-2,x} - p_{n-1,n-2}b_{n-1} + (p_{n,n} - p_{n-2,n-2})b_{n-2} - p_{n,n-2} &= 0,\\ \vdots \qquad\qquad\qquad \vdots\\ b_{1,x} - p_{n-1,1}b_{n-1} - \ldots + (p_{n,n} - A_2)b_1 - p_{n,1} &= 0,\\ b_{0,x} - p_{n-1,0}b_{n-1} - \ldots - A_3b_1 + p_{n,n}b_0 - p_{n,0} &= 0,\\ b_0 - p_{n-1}b_{n-1} - p_{n-2}b_{n-2} - \ldots - p_2b_2 - A_1b_1 &= 0.\end{aligned} \tag{2.15}$$

The last equation may be solved for b_0. Substituting it into the equation with leading term $b_{0,x}$, and eliminating the first derivatives $b_{j,x}$ for $j = 1, \ldots, n-1$ by means of the preceding equations, it may be solved for b_1. Proceeding in this way, due to the triangular structure, finally b_{n-1} is obtained from the equation with leading term $b_{n-2,x}$. Backsubstituting these results, all b_k are explicitly known. By construction they are all rational in the A_i and its derivatives, i.e. they are contained in the base field of L. Substituting them into the first equation a constraint for the coefficients A_i is obtained; it is the condition for the existence of the Laplace divisor.

The proof for case (i) is similar and is therefore omitted. In case (iii), the intersection ideal must divide both Laplace divisors. Due to the Janet basis property of the latter, its generators are linear combinations of L and $\mathfrak{l}_m$ as well as L and $\mathfrak{k}_n$. Transformation into a Janet basis yields the result given above. □

For low values of n and m the conditions for A_1, A_2 and A_3 for the existence of a Laplace divisor of the operator (2.11) may be given explicitly as shown next.

Corollary 2.1. *Let the second-order operator* $L \equiv \partial_{xy} + A_1\partial_x + A_2\partial_y + A_3$ *be given and define the quantities*

$$H_0 \equiv A_{1,x} + A_1A_2 - A_3, \quad K_0 = A_{2,y} + A_1A_2 - A_3,$$
$$H_1 \equiv H_{0,xy}H_0 - H_{0,x}H_{0,y} - H_0^2(2H_0 - K_0),$$
$$K_1 \equiv K_{0,xy}K_0 - K_{0,x}K_{0,y} - K_0^2(2K_0 - H_0),$$
$$H_2 \equiv H_{1,xy}H_1 - H_{1,x}H_{1,y} - H_1^2(3H_0 - 2K_0),$$
$$K_2 \equiv K_{1,xy}K_1 - K_{1,x}K_{1,y} - K_1^2(3K_0 - 2H_0).$$

Operators $\mathfrak{l}_m$ *and* $\mathfrak{k}_n$ *for* $m, n = 2, 3$ *may be constructed as follows.*
A divisor $\langle\!\langle L, \partial_{xx} + a_1\partial_x + a_0\rangle\!\rangle$ *exists if* $K_0 \neq 0$, $K_1 = 0$*; then*

$$a_1 = 2A_2 - \frac{K_{0,x}}{K_0}, \quad a_0 = A_{2,x} + A_2^2 - A_2\frac{K_{0,x}}{K_0}.$$

A divisor $\langle\!\langle L, \partial_{yy} + b_1\partial_y + b_0\rangle\!\rangle$ *exists if* $H_0 \neq 0$, $H_1 = 0$*; then*

$$b_1 = 2A_1 - \frac{H_{0,y}}{H_0}, \quad b_0 = A_{1,y} + A_1^2 - A_1\frac{H_{0,y}}{H_0}.$$

A divisor $\langle\!\langle L, \partial_{xxx} + a_2\partial_{xx} + a_1\partial_x + a_0\rangle\!\rangle$ *exists if* $K_0 \neq 0$, $K_1 \neq 0$, $K_2 = 0$*; then*

$$a_2 = 3A_2 - \frac{K_{1,x}}{K_1}, \quad a_1 = 3(A_{2,x} + A_2^2) - 2A_2\frac{K_{1,x}}{K_1} - \frac{K_{0,xx}}{K_0} + \frac{K_{0,x}}{K_0}\frac{K_{1,x}}{K_1},$$

$$a_0 = A_{2,xx} + 3A_2A_{2,x} + A_2^3 - (A_{2,x} + A_2^2)\frac{K_{1,x}}{K_1} - A_2\Big(\frac{K_{0,xx}}{K_0} - \frac{K_{0,x}}{K_0}\frac{K_{1,x}}{K_1}\Big).$$

A divisor $\langle\!\langle L, \partial_{yyy} + b_2\partial_{yy} + b_1\partial_y + b_0 \rangle\!\rangle$ *exists if* $H_0 \neq 0$, $H_1 \neq 0$, $H_2 = 0$*; then*

$$b_2 = 3A_1 - \frac{H_{1,y}}{H_1}, \quad b_1 = 3(A_{1,y} + A_1^2) - 2A_1\frac{H_{1,y}}{H_1} - \frac{H_{0,yy}}{H_0} + \frac{H_{0,y}}{H_0}\frac{H_{1,y}}{H_1},$$

$$b_0 = A_{1,yy} + 3A_1A_{1,y} + A_1^3 - (A_{1,y} + A_1^2)\frac{H_{1,y}}{H_1} - A_1\Big(\frac{H_{0,yy}}{H_0} - \frac{H_{0,y}}{H_0}\frac{H_{1,y}}{H_1}\Big).$$

This corollary and Proposition 2.3 show that the existence of a Laplace divisor may be decided for any fixed value of m or n. If the answer is affirmative, it may be constructed explicitly. However, in general there is no upper bound for m and n known. Only in special cases as in Example 2.14 below a general answer may be obtained.

The relation of this problem to the number of Laplace's iterations, and the relation between the quantities H_i and K_j to the Laplace invariants h_i and k_j is discussed in Appendix C.

The operators $\mathfrak{l}_m$ and $\mathfrak{k}_n$ given in the preceding corollary are reducible, they allow a first order right factor as shown next.

Corollary 2.2. *The operator* $\mathfrak{l}_m$ *defined by (2.9) allows the right factor* $\partial_x + A_2$*; the operator* $\mathfrak{k}_n$ *defined by (2.10) allows the right factor* $\partial_y + A_1$*.*

Proof. The proof will be given for the second case. Differentiating the operator $\partial_y + A_1$ $(i-1)$-times and reducing the reductum of the result w.r.t. to it yields

$$\partial_{y^i} + p_i(A_{1,y^{i-1}}, \ldots, A_{1,y}, A_1);$$

p_i is a differential polynomial of A_1. One more differentiation leads to

$$\partial_{y^{i+1}} + \Big(\frac{dp_i}{dy} - A_1 p_i\Big) = \partial_{y^{i+1}} + p_{i+1}(A_{1,y^i}, \ldots, A_{1,y}, A_1).$$

From this there follows the recursion

$$p_{i+1} = \frac{dp_i}{dy} - A_1 p_i \quad \text{for } i \geq 1, \quad p_1 = A_1. \tag{2.16}$$

The same recursion with the same initial condition is obtained for the coefficients p_i of Proposition 2.3 if (2.12) is differentiated once more w.r.t. y, i.e. the p_i are the same in either case.

If $\mathfrak{k}_n$ has the factor $\partial_y + A_1$ it may be reduced to zero w.r.t. it, i.e. it vanishes upon substitution of the above expressions $\partial_y^i + p_i$. This yields the condition

$$b_0 - p_{n-1}b_{n-1} - p_{n-2}b_{n-2} - \ldots - p_1 b_1 = 0$$

for the existence of the factor $\partial_y + A_1 = 0$ of $\mathfrak{k}_n$. Due to the above mentioned property of the p_i it is identical to the last equation of (2.15) which is obeyed due to the Janet basis property. □

Example 2.14. The operator $L \equiv \partial_{xy} + xy\partial_x - 2y$ has been considered by Imschenetzky [28]. The invariants are

$$H_0 = 3y,\ H_1 = -36y^3,\ H_2 = -6,480y^7 \text{ and } K_0 = 2y,\ K_1 = -4y^3,\ K_2 = 0.$$

According to Corollary 2.1 a divisor $\mathbb{L}_{y^k}(L)$ for $k \le 3$ does not exist. There follows $a_0 = a_1 = a_2 = 0$, hence there is the divisor $\mathbb{L}_{x^3}(L) = \langle\langle \partial_{xxx}, \partial_{xy} + xy\partial_x - 2y \rangle\rangle$.

In Exercise C.2 it will be shown that a divisor $\mathbb{L}_{y^k}(L)$ does not exist for any k by applying Laplace's iteration to the given operator. □

The syzygies for the generators L and $\mathfrak{l}_m$ or $\mathfrak{k}_n$ respectively of the Laplace divisors described in Proposition 2.3 are given next; they will be applied in later chapters for solving equations $Lz = 0$ allowing a Laplace divisor.

Corollary 2.3. *Let $L \equiv \partial_{xy} + A_1\partial_x + A_2\partial_y + A_3$ be such that a Laplace divisor $\mathbb{L}_{x^m}(L) = \langle\langle L, \mathfrak{l}_m \rangle\rangle$ exists. For $m \le 4$ there are the following syzygies.*

$$m = 2 : L_x - (A_2 - a_1)L = \mathfrak{l}_{2,y} + A_1\mathfrak{l}_2,$$

$$m = 3 : L_{xx} - (A_2 - a_2)L_x - (2A_{2,x} - A_2^2 + A_2a_2 - a_1)L = \mathfrak{l}_{3,y} + A_1\mathfrak{l}_3,$$

$$\begin{aligned} m = 4 : {} & L_{xxx} - (A_2 - a_3)L_{xx} - (3A_{2,x} - A_2^2 + A_2a_3 - a_2)L_x \\ & -(3A_{2,xx} - 5A_{2,x}A_2 + 2A_{2,x}a_3 + A_2^3 - A_2^2a_3 + A_2a_2 - a_1)L = \mathfrak{l}_{4,y} + A_1\mathfrak{l}_4. \end{aligned}$$

For any value of m the right hand side has the form $\mathfrak{l}_{m,y} + A_1\mathfrak{l}_m$.

For Laplace divisors $\mathbb{L}_{y^n}(L) = \langle\langle L, \mathfrak{k}_n \rangle\rangle$ and $n \le 4$ the syzygies are as follows.

$$n = 2 : L_y - (A_1 - b_1)L = \mathfrak{k}_{2,x} + A_2\mathfrak{k}_2,$$

$$n = 3 : L_{yy} - (A_1 - b_2)L_y - (2A_{1,y} - A_1^2 + A_1b_2 - b_1)L = \mathfrak{k}_{3,x} + A_2\mathfrak{k}_3$$

$$\begin{aligned} n = 4 : {} & L_{yyy} - (A_1 - b_3)L_{yy} - (3A_{1,y} - A_1^2 + A_1b_3 - b_2)L_y \\ & -(3A_{1,yy} - 5A_{1,y}A_1 + 2A_{1,y}b_3 + A_1^3 - A_1^2b_3 + A_1b_2 - b_1)L = \mathfrak{k}_{4,x} + A_2\mathfrak{k}_4. \end{aligned}$$

For any value of n the right hand side has the form $\mathfrak{k}_{n,y} + A_2\mathfrak{l}_n$.

Proof. The proof is given for the divisor $\mathbb{L}_{x^m}(L)$. As usual the term order $grlex$, $x \succ y$ is applied; the symbol $o(\tau)$, τ any derivative, is defined on page 29. Deriving L w.r.t. x up to order $m-1$, using the definition of L for substituting the non-leading mixed derivatives in each step, and the definition of $\mathfrak{l}_m$ for substituting the derivative ∂_{x^m}, the expression

$$\partial_{x^m y} + A_1\mathfrak{l}_m + o(\partial_{x^{m-1}}) = \sum_{i=0}^{m-1} p_i L_{x^i}$$

is obtained; the p_i are differential polynomials in A_2. Deriving $\mathfrak{l}_m$ once w.r.t. y and substituting all non-leading mixed derivatives by derivatives of L, an expression of the form

$$\partial_{x^m y} - \mathfrak{l}_{m,y} + O(\partial_{x^{m-1}}) = \sum_{i=0}^{m-1} q_i L_{x^i}$$

follows; the q_i are differential polynomials in A_1, A_2, A_3 and the a_j. Subtracting these expressions from each other leads to

$$\mathfrak{l}_{m,y} + A_1 \mathfrak{l}_m = \sum_{i=1}^{m-1} r_i L_{x^i}.$$

This is the desired syzygy; r_i is a differential polynomial in the coeffcients of L and $\mathfrak{l}_m$; due to the Janet basis property of the Laplace divisor all terms subsumed under the Landau symbol vanish. The proof for $\mathbb{L}_{y^n}(L)$ is similar. For low values of m and n the above syzygies are obtained explicitly. □

The ideals constructed in Proposition 2.3 may be generalized for higher order operators. The following proposition deals with Laplace Divisors for operators with leading derivative ∂_{xxy}.

Proposition 2.4. *Let an operator*

$$L \equiv \partial_{xxy} + A_1\partial_{xyy} + A_2\partial_{xx} + A_3\partial_{xy} + A_4\partial_{yy} + A_5\partial_x + A_6\partial_y + A_7 \quad (2.17)$$

be given; $\mathfrak{l}_m$ and $\mathfrak{k}_n$ are defined by (2.9) and (2.10 respectively). The following divisors may be constructed.

(i) If $m \geq 3$ is a natural number and the coefficients $A_1, \ldots, A_6$ satisfy two differential polynomials constructed below in the proof, there exists a Laplace divisor $\mathbb{L}_{x^m}(L) = \langle\!\langle L, \mathfrak{l}_m \rangle\!\rangle$.
(ii) If $n \geq 2$ is a natural number, $A_1 = A_4 = 0$ and the coefficients $A_1, \ldots, A_7$ satisfy in addition the differential polynomials constructed below in the proof, there exists a Laplace divisor $\mathbb{L}_{y^n}(L) = \langle\!\langle L, \mathfrak{k}_n \rangle\!\rangle$.
(iii) If there are two Laplace divisors $\mathbb{L}_{x^m}(L)$ and $\mathbb{L}_{y^n}(L)$, the operator L is completely reducible; then $\langle L \rangle = Lclm\big(\mathbb{L}_{x^m}(L), \mathbb{L}_{y^n}(L)\big)$.

Proof. It is given for case (i). If the operator L is derived k times w.r.t. x for $0 \leq k \leq m-2$, and in each step the reductum is reduced w.r.t. (2.17), the operator

$$\partial_{x^{k+2}y} + \sum_{j=0}^{k+2} P_{j,k}\partial_{x^j} + Q_k\partial_{xy} + R_k\partial_y \quad (2.18)$$

is obtained; $P_{j,k}$, Q_k and R_k are differential polynomials in the coefficients $A_1, \ldots, A_7$. There is no reduction w.r.t. (2.9) possible. For $k \leq 0 \leq m-3$, there is no reduction w.r.t. (2.9) possible either. However, if $k = m-2$ the highest derivative w.r.t. x in the reductum may be eliminated with the help of (2.9); the result is

$$\begin{aligned}
&\partial_{x^m y} + \textstyle\sum_{j=0}^{m} P_{j,m-2}\partial_{x^j} + Q_{m-2}\partial_{xy} + R_{m-2}\partial_y \\
&\qquad = \partial_{x^m y} + P_{m,m-2}\partial_{x^m} + \textstyle\sum_{j=0}^{m-1} P_{j,m-2}\partial_{x^j} + Q_{m-2}\partial_{xy} + R_{m-2}\partial_y \\
&\qquad = \partial_{x^m y} + \textstyle\sum_{j=0}^{m-1}(P_{j,m-2} - P_{m,m-2}a_{m-j})\partial_{x^j} + Q_{m-2}\partial_{xy} + R_{m-2}\partial_y .
\end{aligned} \tag{2.19}$$

Deriving $\mathfrak{l}_m$ as defined in (2.9) once w.r.t. y and assuming $a_0 = 1$ yields

$$\begin{aligned}
\Big(\sum_{j=0}^{m} a_{m-j}\,\partial_{x^j}\Big)_y &= \sum_{j=0}^{m} a_{m-j}\,\partial_{x^j y} + \sum_{j=0}^{m-1} a_{m-j,y}\,\partial_{x^j} \\
&= \partial_{x^m y} + \sum_{j=2}^{m-1} a_{m-j}\,\partial_{x^j y} + a_{m-1}\partial_{xy} + a_m\partial_y + \sum_{j=0}^{m-1} a_{m-j,y}\,\partial_{x^j} .
\end{aligned}$$

The terms in the first sum may be reduced w.r.t. (2.18) with the result

$$\begin{aligned}
\partial_{x^m y} - \sum_{j=2}^{m-1} a_{m-j}\Big(\sum_{i=0}^{j} P_{i,j-2}\partial_{x^i} + Q_{j-2}\partial_{xy} + R_{j-2}\partial_y\Big)& \\
+ a_{m-1}\partial_{xy} + a_m\partial_y + \sum_{j=0}^{m-1} a_{m-j,y}\,\partial_{x^j} .&
\end{aligned} \tag{2.20}$$

Equating the coefficients of this expression and the expression in the last line of (2.19) leads to the following system of equations.

$$a_{m-i,y} - \sum_{j=max(i,2)}^{m-1} a_{m-j}\,P_{i,j-2} + a_{m-i}\,P_{m,m-2} - P_{i,m-2} = 0, \tag{2.21}$$

$$a_{m-1} - \sum_{j=2}^{m-1} a_{m-j}\,Q_{j-2} - Q_{m-2} = 0,$$

$$a_m - \sum_{j=2}^{m-1} a_{m-j}\,R_{j-2} - R_{m-2} = 0.$$

The last two equations may be used to express a_m and a_{m-1} in terms of the variables with lower indices. Substituting these values into the system (2.21), and autoreducing it leads to a linear algebraic system comprising m equations for $a_1, a_2, \ldots, a_{m-2}$; they are rational in the coefficients of L. In order for a nontrivial solution to exist, two differential polynomials in the coefficients must be satisfied.

The calculations for case (ii) are similar and are therefore omitted. Case (iii) follows by the same reasoning as in Proposition 2.3. □

For later applications the syzygies for the above Laplace divisors are given next; the proof is similar as for Corollary 2.3 and is therefore omitted.

Corollary 2.4. *Let L be defined by*

$$L \equiv \partial_{xxy} + A_1\partial_{xyy} + A_2\partial_{xx} + A_3\partial_{xy} + A_4\partial_{yy} + A_5\partial_x + A_6\partial_y + A_7$$

and allow a Laplace divisor $\mathbb{L}_{x^m}(L) = \langle\!\langle L, \mathfrak{l}_m \rangle\!\rangle$. For $m \le 4$ there are the following syzygies.

$$\begin{aligned}
m = 3 : {} & L_x + (A_3 - a_2)L = \mathfrak{l}_{3,y} + A_2\mathfrak{l}_3,\\
m = 4 : {} & L_{xx} - A_1L_{xy} + (A_1A_2 - A_3 + a_3)L_x + A_1^2L_{yy}\\
& \quad -(2A_{1,x} - A_{1,y}A_1 + 2A_1^2A_2 - 2A_1A_3 + A_1a_3 + A_4)L_y\\
& \quad +(2A_{1,x}A_2 - A_{1,y}A_1A_2 + A_{2,x}A_1 - 2A_{2,y}A_1^2 - 2A_{3,x} + A_{3,y}A_1\\
& \quad\quad +2A_1^2A_2^2 - 3A_1A_2A_3 + A_1A_2a_3 + A_1A_5 + A_2A_4\\
& \quad\quad +A_3^2 - A_3a_3 - A_6 + a_2)L = \mathfrak{l}_{4,y} + A_2\mathfrak{l}_4.
\end{aligned}$$

For Laplace divisors $\mathbb{L}_{y^n}(L) = \langle\!\langle L, \mathfrak{k}_n \rangle\!\rangle$ and $n \le 3$ the syzygies are as follows.

$$\begin{aligned}
n = 2 : {} & L_y - (A_2 - b_1)L = \mathfrak{k}_{2,xx} + A_1\mathfrak{k}_{2,xy}\\
& \quad +(A_{1,y} - A_1A_2 + A_3)\mathfrak{k}_{2,x} + A_4\mathfrak{k}_{2,y} + (A_{4,y} - A_2A_4 + A_6)\mathfrak{k}_2,\\
n = 3 : {} & L_{yy} - (A_2 - b_2)L_y - (2A_{2,y} - A_2^2 + A_2b_2 - b_1)L = \mathfrak{k}_{3,xx} + A_1\mathfrak{k}_{3,xy}\\
& \quad +(2A_{1,y} - A_1A_2 + A_3)\mathfrak{k}_{3,x} + A_4\mathfrak{k}_{3,y} + (2A_{4,y} - b_{2,x}A_1 - A_2A_4 + A_6)\mathfrak{k}_3.
\end{aligned}$$

Finally, Laplace divisors for an operator with leading derivative ∂_{xyy} are considered.

Proposition 2.5. *Let an operator*

$$L \equiv \partial_{xyy} + A_1\partial_{xx} + A_2\partial_{xy} + A_3\partial_{yy} + A_4\partial_x + A_5\partial_y + A_6 \tag{2.22}$$

be given. The following divisors may be constructed.

(i) *If $n \ge 3$ is a natural number and the coefficients $A_1, \ldots, A_6$ satisfy the differential polynomials constructed below in the proof, there exists a Laplace divisor $\mathbb{L}_{y^n}(L) = \langle\!\langle L, \mathfrak{k}_n \rangle\!\rangle$.*

(ii) *If $m \geq 2$ is a natural number and the coefficients $A_1, \ldots A_6$ satisfy the differential polynomials constructed below in the proof, there exists a Laplace divisor $\mathbb{L}_{x^m}(L) = \langle\!\langle L, \mathfrak{l}_m \rangle\!\rangle$.*

(iii) *If there are two Laplace divisors $\mathbb{L}_{x^m}(L)$ and $\mathbb{L}_{y^n}(L)$, the operator L is completely reducible; there holds $\langle L \rangle = Lclm\big(\mathbb{L}_{x^m}(L), \mathbb{L}_{y^n}(L)\big)$.*

Proof. At first case (i) is considered. If (2.22) is derived repeatedly w.r.t. y, and the reductum is reduced in each step w.r.t. (2.22), $n-3$ expressions of the form

$$\partial_{xy^k} + R_{k,1}\partial_{xy} + R_{k,0}\partial_x + P_{k,k}\partial_{y^k} + P_{k,k-1}\partial_{y^{k-1}} + \ldots + P_{k,0} \tag{2.23}$$

for $3 \leq k \leq n-1$ are obtained. All coefficients $R_{k,l}$ and $P_{i,j}$ are differential polynomials in the ring $\mathbb{Q}\{A_1, \ldots, A_6\}$. There is no reduction w.r.t. (2.10) possible. Deriving the last expression once more w.r.t. y and reducing the reductum w.r.t. both (2.22) and (2.10) yields

$$\begin{aligned} \partial_{xy^n} + R_{n,1}\partial_{xy} + R_{n,0}\partial_x + (P_{n,n-1} - P_{n-1,n-1}b_1)\partial_{y^{n-1}} \\ +(P_{n,n-2} - P_{n-1,n-1}b_2)\partial_{y^{n-2}} + \ldots \\ +(P_{n,1} - P_{n-1,n-1}b_{n-1})\partial_y + P_{n,0} - P_{n-1,n-1}b_n. \end{aligned} \tag{2.24}$$

In the first derivative of (2.10) w.r.t. x

$$\begin{aligned} \partial_{xy^n} + b_{1,x}\partial_{y^{n-1}} + b_{2,x}\partial_{y^{n-2}} + \ldots + b_{n-1,x}\partial_y + b_{n,x} \\ +b_1\partial_{xy^{n-1}} + b_2\partial_{xy^{n-2}} + \ldots + b_{n-1}\partial_{xy} + b_n\partial_x \end{aligned}$$

the terms containing derivatives of the form ∂_{xy^k} may be reduced w.r.t. (2.23) or (2.22) with the result

$$\begin{aligned} \partial_{xy^n} &+ (b_{1,x} - P_{n-1,n-1}b_1)\partial_{y^{n-1}} \\ &+(b_{2,x} - P_{n-1,n-2}b_1 - P_{n-2,n-2}b_2)\partial_{y^{n-2}} \\ &\quad\vdots \qquad\qquad\qquad\qquad \vdots \\ &+(b_{n-1,x} - P_{n-1,1}b_1 - P_{n-2,1}b_2 \ldots - A_4 b_{n-2})\partial_y \\ &+b_{n,x} - P_{n-1,0}b_1 - P_{n-2,0}b_2 \ldots - A_5 b_{n-2} \\ &+(b_{n-1} - R_{n-1,1}b_1 - R_{n-2,1}b_2 - \ldots - A_1 b_{n-2})\partial_{xy} \\ &+(b_n - R_{n-1,0}b_1 - R_{n-2,0}b_2 - \ldots - A_3 b_{n-2})\partial_x \end{aligned}$$

where $Q_k, Q_{i,j} \in \mathbb{Q}\{A_1, \ldots, A_6\}$. If this expression is subtracted from (2.24), the coefficients of the derivatives must vanish in order that (2.22) and (2.10) form a

Janet basis. The resulting system of equations is

$$\begin{gathered}
b_{1,x} + (P_{n-1,n-2} - P_{n-1,n-1})b_1 - P_{n,n-1} = 0,\\
b_{2,x} - P_{n-1,n-2}b_1 + (P_{n-1,n-1} - P_{n-2,n-2})b_2 - P_{n,n-2} = 0,\\
\vdots \qquad\qquad \vdots\\
b_{n-1,x} - P_{n-1,1}b_1 - \ldots - A_4 b_{n-2} + P_{n-1,n-1}b_{n-1} - P_{n,1} = 0,\\
b_{n,x} - P_{n-1,0}b_1 - P_{n-2,0}b_2 - \ldots - A_5 b_{n-2} + P_{n-1,n-1}b_n - P_{n,0} = 0,\\
b_{n-1} - R_{n-1,1}b_1 - R_{n-2,1}b_2 - \ldots - A_1 b_{n-2} - R_{n,1} = 0,\\
b_n - R_{n-1,0}b_1 - R_{n-2,0}b_2 - \ldots - A_3 b_{n-2} - R_{n,0}.
\end{gathered}$$

The last two equations may be solved for b_n and b_{n-1}. Substituting these values into the equations with leading terms $b_{n,x}$ and $b_{n-1,x}$, and eliminating the first derivatives $b_{j,x}$ for $j = 1, \ldots, n-2$ by means of the preceding equations, they may be solved for b_{n-2} and b_{n-3}. Proceeding in this way, due to the triangular structure, all b_j are obtained in terms of b_i with $i < j$. Backsubstituting these results, all b_k are explicitly known; due to their linear occurrence they are rational in the coefficients of L. Substituting them into the remaining equation, a constraint for the coefficients $A_1, \ldots, A_6$ expressing the condition for the existence of a Janet basis comprising (2.22) and (2.10), is obtained.

For case (ii) expression (2.22) is derived repeatedly w.r.t. x, and the reductum is reduced in each step w.r.t. (2.22). In this way $m-2$ expressions of the form

$$\begin{aligned}
&\partial_{x^k yy} + P_{k,k}\partial_{x^k y} + P_{k,k-1}\partial_{x^{k-1} y} + \ldots + P_{k,2}\partial_{x^2 y}\\
&\qquad + Q_{k,k}\partial_{x^k} + Q_{k,k-1}\partial_{x^{k-1}} + \ldots + Q_{k,2}\partial_{x^2}\\
&\qquad + R_{k,1}\partial_{xy} + R_{k,2}\partial_{yy} + R_{k,3}\partial_x + R_{k,4}\partial_y + R_{k,5}
\end{aligned} \tag{2.25}$$

for $2 \le k \le m-1$ may be obtained. All coefficients $P_{i,j}$, $Q_{i,j}$ and $R_{i,j}$ are differential polynomials in the ring $\mathbb{Q}\{A_1, \ldots, A_6\}$. There is no reduction w.r.t. (2.9) possible. Deriving the last expression once more w.r.t. x, and reducing the reductum w.r.t. (2.22), (2.9) and its first derivative w.r.t. y yields a completely reduced expression with leading term $\partial_{x^m yy}$. Similarly, deriving (2.9) twice w.r.t. y, and performing all possible reductions w.r.t. (2.25) leads to another completely reduced expression with leading term $\partial_{x^m yy}$. Equating the coefficients of the lower order terms of these two expressions yields the following equations for the desired coefficients a_k, $1 \le k \le m$.

$$
\begin{aligned}
2a_{1,y} + (P_{m,m} - P_{m-1,m-1})a_1 - P_{m,m-1} = 0,\\
2a_{2,y} + (P_{m,m} - P_{m-1,m-2})a_1 - P_{m-2,m-2}a_2 - P_{m,m-2} = 0,\\
\vdots \qquad\qquad \vdots\\
2a_{m-2,y} - P_{m-1,2}a_1 - P_{m-2,2}a_2 - \ldots + (P_{m,m} - P_{2,2})a_{m-2} - P_{m,2} = 0,\\
2a_{m-1,y} + (P_{m,m} - A_1)a_{m-1} - \ldots - R_{m-2,1}a_2 - R_{m-1,1}a_1 - R_{m.1} = 0,\\
2a_{m,y} + P_{m,m}a_m - A_4a_{m-1} - \ldots - R_{m-2,4}a_2 - R_{m-1,4}a_1 - R_{m,4} = 0,\\
a_m - A_2a_{m-1} - \ldots - R_{m-2,2}a_2 - R_{m-1,2}a_1 - R_{m,2} = 0.
\end{aligned} \tag{2.26}
$$

In addition there are m equations of the general structure

$$
a_{k,yy} + pa_{k,y} + q_ka_k + q_{k-1}a_{k-1} + \ldots + q_2a_2 + q_1a_1 = 0 \tag{2.27}
$$

for $1 \leq k \leq m$. The system (2.26) has the same form as in case (i), therefore the coefficients a_k may be determined in a similar way, beginning with the highest coefficient a_m, then a_{m-1} and so on until a_1 is reached; due to the linearity of these equations the a_k are rational in the coefficients of L. Substituting them into the remaining equations (2.27) a set of constraints for $A_1, \ldots, A_6$ is obtained. If they are satisfied, (2.22) and (2.9) form a Janet basis. Case (iii) follows by the same reasoning as in Proposition 2.3. □

The syzygies for the above Laplace divisors are given next; the proof is again omitted because it is similar as for Corollary 2.3.

Corollary 2.5. *Let L be defined by (2.22) and allow a Laplace divisor $\mathbb{L}_{x^m}(L) = \langle\langle L, \mathfrak{l}_m \rangle\rangle$. For $m \leq 3$ there are the following syzygies.*

$$
\begin{aligned}
m = 2 :\ & L_x + (a_1 - A_3)L = \mathfrak{l}_{2,yy} + A_1\mathfrak{l}_{2,x} + A_2\mathfrak{l}_{2,y} + (A_{1,x} - A_1A_3 + A_4)\mathfrak{l}_2,\\
m = 3 :\ & L_{xx} + (a_2 - A_3)L_x + (a_1 - a_2A_3 - 2A_{3,x} + A_3^2)L\\
& \qquad = \mathfrak{l}_{3,yy} + A_1\mathfrak{l}_{3,x} + A_2\mathfrak{l}_{3,y} + (2A_{1,x} - A_1A_3 + A_4)\mathfrak{l}_3.
\end{aligned}
$$

For Laplace divisors $\mathbb{L}_{y^n}(L) = \langle\langle L, \mathfrak{k}_n \rangle\rangle$ and $n \leq 4$ the syzygies are as follows.

$$
\begin{aligned}
n = 3 :\ & L_y - (A_2 - b_2)L = \mathfrak{k}_{3,x} + A_3\mathfrak{k}_3,\\
n = 4 :\ & L_{yy} - A_1L_x + (b_3 - A_2)L_y\\
& \qquad -(2A_{2,y} + b_3A_2 - b_2 - A_1A_3 - A_2^2 + A_4)L = \mathfrak{k}_{4,x} + A_3\mathfrak{k}_4.
\end{aligned}
$$

2.7 The Ideals $\mathbb{J}_{xxx}$ and $\mathbb{J}_{xxy}$

Among the ideals involving only derivatives of order not higher than three those ideals that occur as generic intersection ideals of first-order operators are of particular importance; they are denoted by

$$\mathbb{J}_{xxx} \equiv \langle \partial_{xxx}, \partial_{xxy} \rangle_{LT} \quad \text{and} \quad \mathbb{J}_{xxy} \equiv \langle \partial_{xxy}, \partial_{xyy} \rangle_{LT}.$$

The subscripts of $\mathbb{J}$ denote the highest derivative occurring in the respective ideal. Both are generated by two third-order operators forming a Janet basis; their differential dimension is $(1, 2)$. For later use some of their properties are investigated next.

Lemma 2.2. *The ideal*

$$\begin{aligned}\mathbb{J}_{xxx} \equiv \langle L_1 \equiv\ & \partial_{xxx} + p_1\partial_{xyy} + p_2\partial_{yyy} + p_3\partial_{xx} + p_4\partial_{xy} + p_5\partial_{yy} + p_6\partial_x \\ & + p_7\partial_y + p_8, \\ L_2 \equiv\ & \partial_{xxy} + q_1\partial_{xyy} + q_2\partial_{yyy} + q_3\partial_{xx} + q_4\partial_{xy} + q_5\partial_{yy} + q_6\partial_x \\ & + q_7\partial_y + q_8\rangle\end{aligned}$$

is coherent if the coefficients of its generators obey the conditions

$$\begin{gathered}p_1 - q_2 + q_1^2 = 0, \quad p_2 + q_2q_1 = 0, \\ q_{2,y} - q_{1,x} - q_{1,y}q_1 + p_4 - p_3q_1 - q_5 + 2q_4q_1 + q_3q_2 - 2q_3q_1^2 = 0, \\ q_{2,x} + q_{1,y}q_2 - p_5 + p_3q_2 - q_5q_1 - q_4q_2 + 2q_3q_2q_1 = 0, \\ p_{3,y} - q_{3,x} + q_{3,y}q_1 - q_6 + q_4q_3 - q_3^2q_1 = 0, \\ p_{4,y} - q_{4,x} + q_{4,y}q_1 + p_6 + p_4q_3 - p_3q_4 - q_7 + q_6q_1 + q_4^2 - q_4q_3q_1 = 0, \\ p_{5,y} - q_{5,x} + q_{5,y}q_1 + p_7 + p_5q_3 - p_3q_5 + q_7q_1 + q_5q_4 - q_5q_3q_1 = 0, \\ p_{6,y} - q_{6,x} + q_{6,y}q_1 + p_6q_3 - p_3q_6 - q_8 + q_6q_4 - q_6q_3q_1 = 0, \\ p_{7,y} - q_{7,x} + q_{7,y}q_1 + p_8 + p_7q_3 - p_3q_7 + q_8q_1 + q_7q_4 - q_7q_3q_1 = 0, \\ p_{8,y} - q_{8,x} + q_{8,y}q_1 + p_8q_3 - p_3q_8 + q_8q_4 - q_8q_3q_1 = 0.\end{gathered}$$

A $grlex$ term order with $p_i \succ q_j$ for all i and j, $q_i \succ q_j$ and $p_i \succ p_j$ for $i < j$ is applied. There is a single syzygy between the generators.

$$L_{1,y} + q_3L_1 - L_{2,x} + q_1L_{2,y} - (p_3 + q_1q_3 - q_4)L_2 = 0. \tag{2.28}$$

The proof of this lemma is considered in Exercise 2.9. An example of an ideal $\mathbb{J}_{xxx}$ is given next.

Example 2.15. The ideal

$$\langle\!\langle L_1 \equiv \partial_{xxx} - \partial_{xyy} + 2x\partial_{xy} - 3(3x^2+1)\partial_x + 4(2x^2+1)\partial_y - 8x^3 - 24x,$$
$$L_2 \equiv \partial_{xxy} + \partial_{xyy} + x\partial_{xx} - (x^2+1)\partial_x - 4(2x^2+1)\partial_y - 8x^3 \rangle\!\rangle$$

is generated by a Janet basis. By the results of Chap. 6 it may be shown that both operators L_1 and L_2 do not have any first-order right factor. The single syzygy $L_{1,y} + xL_1 = L_{2,x} + xL_2$ is particularly simple in this case. □

Lemma 2.3. *The ideal*

$$\mathbb{J}_{xxy} \equiv \langle K_1 \equiv \partial_{xxy} + p_1\partial_{yyy} + p_2\partial_{xx} + p_3\partial_{xy} + p_4\partial_{yy} + p_5\partial_x + p_6\partial_y + p_7,$$
$$K_2 \equiv \partial_{xyy} + q_1\partial_{yyy} + q_2\partial_{xx} + q_3\partial_{xy} + q_4\partial_{yy} + q_5\partial_x + q_6\partial_y + q_7\rangle$$

is coherent if the coefficients of its generators obey the conditions

$$q_2 = 0, \quad p_1 + q_1^2 = 0,$$
$$p_{7,y} - q_{7,x} + q_{7,y}q_1 - p_2p_7 - p_3q_7 - p_7q_1q_2 + p_7q_3 - q_1q_3q_7 + q_4q_7 = 0,$$
$$p_{6,y} - q_{6,x} + q_{6,y}q_1 - p_2p_6 - p_3q_6 - p_6q_1q_2$$
$$+p_6q_3 + p_7 - q_1q_3q_6 + q_1q_7 + q_4q_6 = 0,$$
$$p_{5,y} - q_{5,x} + q_{5,y}q_1 - p_2p_5 - p_3q_5 - p_5q_1q_2$$
$$+p_5q_3 - q_1q_3q_5 + q_4q_5 - q_7 = 0,$$
$$p_{4,y} - q_{4,x} + q_{4,y}q_1 - p_2p_4 - p_3q_4 - p_4q_1q_2$$
$$+p_4q_3 + p_6 - q_1q_3q_4 + q_1q_6 + q_4^2 = 0,$$
$$p_{3,y} - q_{3,x} + q_{3,y}q_1 - p_2p_3 - p_3q_1q_2 + p_5 - q_1q_3^2 + q_1q_5 + q_3q_4 - q_6 = 0,$$
$$p_{2,y} - q_{2,x} + q_{2,y}q_1 - p_2^2 - p_2q_1q_2$$
$$+p_2q_3 - p_3q_2 - q_1q_2q_3 + q_2q_4 - q_5 = 0,$$
$$p_{1,y} - q_{1,x} + q_{1,y}q_1 - p_1p_2 - p_1q_1q_2$$
$$+p_1q_3 - p_3q_1 + p_4 - q_1^2q_3 + 2q_1q_4 = 0.$$

A $grlex$ term order with $p_i \succ q_j$ for all i and j, $q_i \succ q_j$ and $p_i \succ p_j$ for $i < j$ is applied. There is a single syzygy between the generators.

$$K_{1,y} - (p_2 + q_1q_2 - q_3)K_1 - K_{2,x} + q_1K_{2,y} - (p_3 + q_1q_3 - q_4)K_2 = 0. \quad (2.29)$$

The proof is also considered in Exercise 2.9.

2.8 Lattice Structure of Ideals in $\mathbb{Q}(x, y)[\partial_x, \partial_y]$

In any ring, commutative or not, its ideals form a lattice if the join operation is defined as the sum of ideals, and the meet operation as its intersection. In order to understand the structure of this lattice, these two operations have to be studied in detail. The basics of lattice theory required for this purpose may be found in the books by Grätzer [19] or Davey and Priestley [15].

The first result deals with a special case that guarantees the existence of a principal intersection ideal of first order operators.

Proposition 2.6. *Let L be a partial differential operator in x and y with leading term ∂_{x^n}, and let $l_i \equiv \partial_x + a_i\partial_y + b_i$, $i = 1, \ldots, n$, $a_i \neq a_j$ for $i \neq j$, be n right divisors of L. Then the intersection ideal generated by the l_i is principal and is generated by L.*

Proof. Let $I_i = \langle l_i \rangle$ for $1 \leq i \leq n$ and $I = I_1 \cap \ldots \cap I_n$ be the intersection ideal. For any $P \in I$, $symb(P)$ is divided by $\prod_{1 \leq i \leq n}(\partial_x + a_i\partial_y)$, considered as algebraic polynomial in ∂_x and ∂_y; therefore $ord_x(P) \geq n$. On the other hand, according to Sit's relation (2.8) on page 29, for the typical differential dimension there follows $dim(I) \leq n$. Hence if $I = \langle L \rangle$ is principal then $ord_x(L) = dim(I) = n$. Conversely let $P \in I$ and divide P by L with remainder, i.e. $P = QL + R$. Then $ord_x(R) < n$, therefore $R = 0$. Thus $I = \langle L \rangle$. □

The intersection ideals generated by two or three first-order operators in the plane are described in detail now.

Theorem 2.2. *Let the ideals $I_i = \langle \partial_x + a_i\partial_y + b_i \rangle$ for $i = 1, 2$ with $I_1 \neq I_2$ be given. Both ideals have differential dimension $(1, 1)$. There are three different cases for their intersection $I_1 \cap I_2$, all are of differential dimension $(1, 2)$.*

(i) If $a_1 \neq a_2$ and $\left(\frac{b_1 - b_2}{a_1 - a_2}\right)_x = \left(\frac{a_1b_2 - a_2b_1}{a_1 - a_2}\right)_y$ then

$$I_1 \cap I_2 = \langle \partial_{xx} \rangle_{LT} \text{ and } I_1 + I_2 = \left\langle \partial_x + \frac{a_1b_2 - a_2b_1}{a_1 - a_2}, \partial_y + \frac{b_1 - b_2}{a_1 - a_2} \right\rangle.$$

(ii) If $a_1 \neq a_2$ and $\left(\frac{b_1 - b_2}{a_1 - a_2}\right)_x \neq \left(\frac{a_1b_2 - a_2b_1}{a_1 - a_2}\right)_y$ then

$$I_1 \cap I_2 = \mathbb{J}_{xxx} \text{ and } I_1 + I_2 = \langle 1 \rangle.$$

(iii) If $a_1 = a_2 = a$ and $b_1 \neq b_2$ then

$$I_1 \cap I_2 = \langle \partial_{xx} \rangle_{LT} \text{ and } I_1 + I_2 = \langle 1 \rangle.$$

Case (ii) is the generic case for the intersection of two ideals I_1 and I_2.

Proof. The proof follows closely Grigoriev and Schwarz [22]. In accordance with Cox, Little and O'Shea [13], Theorem 11 on page 186, an auxiliary parameter u is introduced and the operators $u(\partial_x + a_1\partial_y + b_1)$ and $(1-u)(\partial_x + a_2\partial_y + b_2)$ are considered. In order to compute generators for the intersection ideal, a Janet basis with u as the highest variable has to be generated. To this end, computationally it is more convenient to find the Janet basis with respect to the differential indeterminate z and a new indeterminate $w = uz$ with $w \succ z$ in a lexicographic term ordering. The intersection ideal is obtained from the expressions not involving w; the sum ideal is obtained by substituting $z = 0$. This yields the differential polynomials

$$w_x + a_1 w_y + b_1 w \ \text{ and } \ w_x + a_2 w_y + b_2 w - z_x - a_2 z_y - b_2 z. \tag{2.30}$$

If $a_1 \neq a_2$ autoreduction leads to

$$\begin{aligned} w_x + \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2} w - \frac{a_1}{a_1 - a_2}(z_x + a_2 z_y + b_2 z), \\ w_y + \frac{b_1 - b_2}{a_1 - a_2} w + \frac{1}{a_1 - a_2}(z_x + a_2 z_y + b_2 z). \end{aligned} \tag{2.31}$$

Defining $U \equiv z_x + a_2 z_y + b_2 z$, the integrability condition between these two elements has the form

$$\begin{aligned} \left[\left(\frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}\right)_y - \left(\frac{b_1 - b_2}{a_1 - a_2}\right)_x\right] w - \frac{1}{a_1 - a_2} U_x - \frac{a_1}{a_1 - a_2} U_y \\ - \left[\left(\frac{1}{a_1 - a_2}\right)_x + \left(\frac{a_1}{a_1 - a_2}\right)_y + \frac{b_1}{a_1 - a_2}\right] U = 0. \end{aligned}$$

If the coefficient of w vanishes, the remaining expression has the leading term z_{xx} and is the lowest element of a Janet basis. The sum ideal is obtained from (2.31). This is case (i).

If the coefficient of w does not vanish, this expression may be applied to eliminate w in (2.31). It yields two expressions with leading derivatives U_{xxx} and U_{xxy} respectively; they correspond to an intersection ideal $\mathbb{J}_{xxx}$. The sum ideal is trivial. This is case (ii).

Finally, if $a_1 = a_2 = a$, autoreduction of (2.30) yields two expressions of the type $w + o(z_x)$ and $o(z_{xx})$ respectively; they correspond to an intersection ideal $\langle \partial_{xx} \rangle_{LT}$ and a trivial sum ideal. This is case (iii).

Case (ii) is the generic case because it does not involve any constraints for the coefficients of the generators of I_1 and I_2. □

The highest coefficients of the generators of the intersection ideal in case (ii) of the above theorem may be expressed explicitly in terms of the coefficients of the two first-order operators of the argument of the $Lclm$ as shown in Exercise 2.10; the expressions for the lower coefficients are too voluminous to be given explicitly. However, if an ideal is a priori known to be principal its single generator may be given explicitly in terms of the coefficients of its factors or the arguments of the intersection as shown next.

Corollary 2.6. *Let the second-order operator*

$$L \equiv \partial_{xx} + A_1\partial_{xy} + A_2\partial_{yy} + A_3\partial_x + A_4\partial_y + A_5$$

be given.

(i) *Let L have the representation $L = (\partial_x + a_2\partial_y + b_2)(\partial_x + a_1\partial_y + b_1)$. Then*

$$\begin{aligned} L = {} & \partial_{xx} + (a_1 + a_2)\partial_{xy} + a_1a_2\partial_{yy} + (b_1 + b_2)\partial_x \\ & + (a_{1,x} + a_{1,y}a_2 + a_1b_2 + a_2b_1)\partial_y + b_{1,x} + b_{1,y}a_2 + b_1b_2. \end{aligned} \tag{2.32}$$

(ii) *Let L have the representation $L = Lclm(\partial_x + a_2\partial_y + b_2, \partial_x + a_1\partial_y + b_1)$. If $a_1 \neq a_2$, then*

$$\begin{aligned} L = {} & \partial_{xx} + (a_1 + a_2)\partial_{xy} + a_1a_2\partial_{yy} \\ & + \Big(b_1 + b_2 - \frac{1}{a_1 - a_2}(a_{1,x} - a_{2,x} + a_{1,y}a_2 - a_{2,y}a_1)\Big)\partial_x \\ & + \Big(a_1b_2 + a_2b_1 - \frac{1}{a_1 - a_2}(a_{1,x}a_2 - a_{2,x}a_1 + a_{1,y}a_2^2 - a_{2,y}a_1^2)\Big)\partial_y \\ & + b_1b_2 + b_{2,x} + a_1b_{2,y} - \frac{b_2}{a_1 - a_2}\big(a_{1,x} - a_{2,x} + a_{1,y}a_2 - a_{2,y}a_1\big). \end{aligned} \tag{2.33}$$

If $a_1 = a_2 = a$ and $b_1 \neq b_2$ then

$$\begin{aligned} L = {} & \partial_{xx} + 2a\partial_{xy} + a^2\partial_{yy} + \left(b_1 + b_2 - \frac{(b_1 - b_2)_x}{b_1 - b_2} - a\frac{(b_1 - b_2)_y}{b_1 - b_2} \right) \partial_x \\ & + \left(a(b_1 + b_2) + a_x + aa_y - a\frac{(b_1 - b_2)_x}{b_1 - b_2} - a^2\frac{(b_1 - b_2)_y}{b_1 - b_2} \right) \partial_y \\ & + b_1b_2 - \frac{1}{b_1 - b_2}\big(b_{1,x}b_2 - b_{2,x}b_1 - a(b_{1,y}b_2 - b_{2,y}b_1)\big). \end{aligned} \tag{2.34}$$

Proof. In case (i), the representation of L is obtained by multiplication of the two first-order operators. In case (ii), reduction of L w.r.t. both first-order operators yields expressions in ∂_{yy}, ∂_y and a term without derivative. Its coefficients form a system of six linear algebraic equations for the coefficients of L. They may be solved for $A_1, \ldots, A_5$ with the given result if a constraint for a_1, b_1, a_2 and b_2 is satisfied. It is identical to the condition of case (i) in Theorem 2.2 assuring the existence of the principal intersection. □

This proof is simpler than that of the preceding theorem because the principality of the intersection ideal is part of the assumption.

There is a relation between the commutativity and the structure of the intersection ideal of two operators as shown next.

Lemma 2.4. *Two commuting operators* $l_1 \equiv \partial_x + a_1\partial_y + b_1$ *and* $l_2 \equiv \partial_x + a_2\partial_y + b_2$ *with* $a_1 \neq a_2$ *have a principal intersection ideal.*

Proof. The assertion is obvious from the representation

$$\left(\frac{b_1 - b_2}{a_1 - a_2}\right)_x - \left(\frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}\right)_y = \big((a_1 - a_2)_x + a_{1,y}a_2 - a_{2,y}a_1\big)\frac{b_1 - b_2}{(a_1 - a_2)^2}$$
$$+\big((b_1 - b_2)_x + b_{1,y}a_2 - b_{2,y}a_1\big)\frac{1}{a_1 - a_2}.$$

The left hand side is the condition of case (i) of Theorem 2.2. The coefficients of the fractions at the right hand side are just the conditions for commutativity given in Lemma 2.1. □

Example 2.16. Consider the two ideals

$$I_1 = \langle \partial_x + 1 \rangle \ \text{ and } \ I_2 = \langle \partial_x + (y+1)\partial_y \rangle,$$

both of differential dimension $(1, 1)$. The condition for case (i) of Theorem 2.2 is satisfied. Hence

$$Lclm(I_1, I_2) = \langle\!\langle \partial_{xx} + (y+1)\partial_{xy} + \partial_x + (y+1)\partial_y \rangle\!\rangle,$$
$$Gcrd(I_1, I_2) = \left\langle\!\!\left\langle \partial_x + 1, \partial_y - \frac{1}{y+1} \right\rangle\!\!\right\rangle;$$

their differential dimension is $(1, 2)$ and $(0, 1)$ respectively. □

Example 2.17. The two ideals $I_1 = \langle \partial_x + 1 \rangle$ and $I_2 = \langle \partial_x + x\partial_y \rangle$, both of differential dimension $(1, 1)$, do not satisfy the condition of case (i) of Theorem 2.2; furthermore $a_1 \neq a_2$. Therefore by case (ii) the intersection ideal is

$$Lclm(I_1, I_2) = \langle\!\langle \partial_{xxx} - x^2\partial_{xyy} + 3\partial_{xx} + (2x+3)\partial_{xy} - x^2\partial_{yy} + 2\partial_x + (2x+3)\partial_y,$$
$$\partial_{xxy} + x\partial_{xyy} - \tfrac{1}{x}\partial_{xy} + x\partial_{yy} - \tfrac{1}{x}\partial_x - \big(1 + \tfrac{1}{x}\big)\partial_y \rangle\!\rangle$$

of differential dimension $(1, 2)$; $Gcrd(I_1, I_2) = \langle 1 \rangle$. □

The above Theorem 2.2 does not cover the case that the two operators generating I_1 and I_2 have different leading derivatives; it is considered next.

Theorem 2.3. *Let the ideals* $I_1 = \langle \partial_x + a_1\partial_y + b_1 \rangle$ *and* $I_2 = \langle \partial_y + b_2 \rangle$ *be given,* $I_1 \neq I_2$. *There are two different cases for their intersection* $I_1 \cap I_2$.

(i) If $(b_1 - a_1 b_2)_y = b_{2,x}$ *then*

$$I_1 \cap I_2 = \langle \partial_{xy} \rangle_{LT} \text{ and } I_1 + I_2 = \langle \partial_x + b_1 - a_1 b_2, \partial_y + b_2 \rangle.$$

(ii) If the preceding case does not apply then

$$I_1 \cap I_2 = \mathbb{J}_{xxy} \text{ and } I_1 + I_2 = \langle 1 \rangle.$$

Proof. By a similar reasoning as in the preceding theorem, the differential polynomials $w_x + a_1 w_y + b_1 w$ and $w_y + b_2 w - z_y - b_2 z$ are obtained. Autoreduction yields

$$w_x + (b_1 - a_1 b_2)w + a_1 z_y + a_1 b_2 z, \quad w_y + b_2 w - z_y - b_2 z. \tag{2.35}$$

In order to make this into a Janet basis in a lex term order with $w \succ z$ and $x \succ y$, the single integrability condition has to be satisfied. Upon reduction w.r.t. (2.35) it assumes the form $[(b_1 - a_1 b_2)_y - b_{2,x}]w + o(z_{xy})$. If the coefficient of w vanishes, an expression with leading term z_{xy} remains; it corresponds to a principal intersection ideal. This is case (i).

If the coefficient of w does not vanish, this expression has to be applied to eliminate w from (2.35). The resulting polynomials with leading terms z_{xxy} and z_{xyy} correspond to the ideal in case (ii). □

The explicit expression for the generator of the principal intersection ideal of case (i) of Theorem 2.3 is determined in Exercise 2.11. In Exercise 2.12 the highest coefficients of the generators of the intersection ideal in case (ii) are determined.

All intersection ideals obtained in the above theorems are either principal, or they are generated by two operators in accordance with Stafford's theorem [67]. Furthermore, by construction the two generators form a Janet basis.

The above Theorem 2.2 and 2.3 are the key for understanding the decompositions of operators in the plane; in particular this is true for Blumberg's example that is considered later in Example 6.9 on page 128.

Intersecting three principal ideals generated by first order operators is much more involved as the next result shows.

Theorem 2.4. *Let the ideals $I_i = \langle \partial_x + a_i \partial_y + b_i \rangle$, $i = 1, 2, 3$ be given with $I_i \neq I_j$ for $i \neq j$. There are four different cases for their intersection $I_1 \cap I_2 \cap I_3$.*

(i) *Separable case $a_1 \neq a_2 \neq a_3$. If the constraints (2.41), (2.46) and (2.47) given below are satisfied the intersection is $I_1 \cap I_2 \cap I_3 = \langle \partial_{xxx} \rangle_{LT}$.*

(a) *If in addition $\dfrac{b_1 - b_2}{a_1 - a_2} = \dfrac{b_2 - b_3}{a_2 - a_3}$, there is a nontrivial sum ideal*

$$I_1 + I_2 + I_3 = \langle \partial_x + \frac{a_2 b_3 - a_3 b_2}{a_2 - a_3}, \partial_y + \frac{b_2 - b_3}{a_2 - a_3} \rangle.$$

(b) *If the condition of the preceding case does not apply the sum ideal is trivial, i.e. $I_1 + I_2 + I_3 = \langle 1 \rangle$.*

(ii) *Double root $a_1 = a_2 = a \neq a_3$, $b_1 \neq b_2$. If the two constraints (2.41) and (2.50) given below are satisfied, it follows that $I_1 \cap I_2 \cap I_3 = \langle \partial_{xxx} \rangle_{LT}$ and $I_1 + I_2 = \langle 1 \rangle$.*

(iii) *Triple root $a_1 = a_2 = a_3 = a$, $b_i \neq b_j$ for $i \neq j$. If condition (2.52) is satisfied, the intersection ideal is $\langle \partial_{xx} \rangle_{LT}$. If the latter condition is not satisfied, the intersection ideal is $\langle \partial_{xxx} \rangle_{LT}$. In either case $I_1 + I_2 + I_3 = \langle 1 \rangle$.*

(iv) *If the preceding three cases do not apply, the intersection ideal is generated by two operators. It may be $\langle \partial_{xxxx}, \partial_{xxxy} \rangle_{LT}$, $\langle \partial_{xxxxy}, \partial_{xxxyy} \rangle_{LT}$ or $\langle \partial_{xxxxyy}, \partial_{xxxyyy} \rangle_{LT}$.*

All intersection ideals have differential dimension $(1, 3)$ *except the first alternative in case* (iii) *with differential dimension* $(1, 2)$.

Proof. By a similar reasoning as in the preceding theorem the following differential polynomials are obtained.

$$\begin{gathered} w_{2,x} + a_3 w_{2,y} + b_3 w_2 + w_{1,x} + a_3 w_{1,y} + b_3 w_1 - z_x - a_3 z_y - b_3 z, \\ w_{2,x} + a_2 w_{2,y} + b_2 w_2, \quad w_{1,x} + a_1 w_{1,y} + b_1 w_1. \end{gathered} \tag{2.36}$$

At first $a_1 \neq a_2 \neq a_3$ is assumed. Autoreduction yields the system

$$w_{2,x} + \frac{a_2 b_3 - a_3 b_2}{a_2 - a_3} w_2 - a_2 \frac{a_1 - a_3}{a_2 - a_3} w_{1,y} - a_2 \frac{b_1 - b_3}{a_2 - a_3} w_1 + o(z_x), \tag{2.37}$$

$$w_{2,y} + \frac{b_2 - b_3}{a_2 - a_3} w_2 + \frac{a_1 - a_3}{a_2 - a_3} w_{1,y} + \frac{b_1 - b_3}{a_2 - a_3} w_1 + o(z_x), \tag{2.38}$$

$$w_{1,x} + a_1 w_{1,y} + b_1 w_1. \tag{2.39}$$

It has to be transformed into a Janet basis in lex term ordering with $w_2 \succ w_1 \succ z$ and $x \succ y$. There is a single integrability condition $(2.37)_y - (2.38)_x$. Upon reduction w.r.t. to (2.37), (2.38) and (2.39) the following expression is obtained.

$$\begin{aligned} &\left[\left(\frac{b_2 - b_3}{a_2 - a_3} \right)_x - \left(\frac{a_2 b_3 - a_3 b_2}{a_2 - a_3} \right)_y \right] w_2 - \frac{(a_1 - a_2)(a_1 - a_3)}{a_2 - a_3} w_{1,yy} \\ &\quad + \left[\left(\frac{a_1 - a_3}{a_2 - a_3} \right)_x - (a_1 - a_2)_y \frac{a_1 - a_3}{a_2 - a_3} + a_2 \left(\frac{a_1 - a_3}{a_2 - a_3} \right)_y \right. \\ &\qquad \left. - (a_1 - a_2) \frac{b_1 - b_3}{a_2 - a_3} - (b_1 - b_2) \frac{a_1 - a_3}{a_2 - a_3} \right] w_{1,y} \\ &\quad + \left[\left(\frac{b_1 - b_3}{a_2 - a_3} \right)_x + \left(a_2 \frac{b_1 - b_3}{a_2 - a_3} \right)_y - \frac{(b_1 - b_2)(b_1 - b_3)}{a_2 - a_3} \right. \\ &\qquad \left. - b_{1,y} \frac{a_1 - a_3}{a_2 - a_3} \right] w_1 + o(z_{xx}). \end{aligned} \tag{2.40}$$

At first it is assumed that the coefficient of w_2 vanishes, i.e.

$$\left(\frac{b_2 - b_3}{a_2 - a_3} \right)_x - \left(\frac{a_2 b_3 - a_3 b_2}{a_2 - a_3} \right)_y = 0. \tag{2.41}$$

As a consequence the terms of (2.37) and (2.38) involving w_2 do not change any more. Defining

$$\begin{aligned} P = {} & \frac{b_1 - b_2}{a_1 - a_2} + \frac{b_1 - b_3}{a_1 - a_3} + \frac{(a_1 - a_2)_y}{a_1 - a_2} \\ & - \frac{1}{a_1 - a_2} \left(\frac{(a_1 - a_3)_x}{a_1 - a_3} + a_2 \frac{(a_1 - a_3)_y}{a_1 - a_3} - \frac{(a_2 - a_3)_x}{a_2 - a_3} - a_2 \frac{(a_2 - a_3)_y}{a_2 - a_3} \right) \end{aligned} \tag{2.42}$$

and

$$\begin{aligned} Q = {} & \frac{1}{a_1 - a_2} \frac{b_1 - b_3}{a_1 - a_3} \left(\frac{(a_2 - a_3)_x}{a_2 - a_3} + a_2 \frac{(a_2 - a_3)_y}{a_2 - a_3} + b_1 - b_2 - a_{2,y} \right) \\ & - \frac{1}{a_1 - a_2} \left(\frac{(b_1 - b_3)_x}{a_1 - a_3} + a_2 \frac{(b_1 - b_3)_y}{a_1 - a_3} - b_{1,y} \right) \end{aligned} \tag{2.43}$$

and dividing (2.40) by the coefficient of $w_{1,yy}$, the result may be written as

$$w_{1,yy} + P w_{1,y} + Q w_1 + o(z_{xx}). \tag{2.44}$$

Combined with (2.39) it must form a Janet basis for w_1 and z. To this end, the single integrability condition between (2.39) and (2.44) has to be satisfied. If all reductions are performed it assumes the form

$$[P_x + (a_1 P - a_{1,y} - 2b_1)_y] w_{1,y} + [Q_x + a_{1,y} Q - b_{1,y} P + (a_1 Q - b_{1,y})_y] w_1 + o(z_{xxx}). \tag{2.45}$$

The first alternative in order to make this expression into a Janet basis combined with (2.39) and (2.44) is to require that the coefficients of $w_{1,y}$ and w_1 vanish, i.e.

$$P_x + (a_1 P - a_{1,y} - 2b_1)_y = 0 \tag{2.46}$$

and

$$Q_x + a_{1,y} Q - b_{1,y} P + (a_1 Q - b_{1,y})_y = 0. \tag{2.47}$$

The resulting expression has the leading term z_{xxx}. It is the lowest element of a Janet basis of the full system and does not contain w_1 or w_2. Hence, the intersection ideal corresponding to this alternative is principal and has the form $\langle \partial_{xxx} \rangle_{LT}$.

If (2.46) or (2.47) is not satisfied, (2.39), (2.44) and (2.45) combined must be transformed into a Janet basis. In either case, autoreduction yields in several steps a basis the two lowest members of which have leading terms not higher than z_{xxxxy} or z_{xxxyy} respectively. If (2.41) is not satisfied, w_2 may be eliminated from (2.37) and (2.38). The resulting system has leading terms $w_{1,xyy}$, $w_{1,yyy}$ and $w_{1,x}$. Again autoreduction yields after several steps two members with leading terms not higher than z_{xxxxyy} and z_{xxxyyy}. This completes case (i).

If $a_1 = a_2 = a \neq a_3$ and $b_1 \neq b_2$ the coefficient of $w_{1,yy}$ in (2.40) vanishes. If condition (2.41) holds, the relation

$$w_{1,y} + Rw_1 + o(z_{xx}) \tag{2.48}$$

follows with

$$R = -\frac{1}{b_1 - b_2}\left[\left(\frac{b_1 - b_3}{a - a_3}\right)_x + \left(a\frac{b_1 - b_3}{a - a_3}\right)_y - \frac{(b_1 - b_2)(b_1 - b_3)}{a - a_3}\right]. \tag{2.49}$$

The single integrability condition between (2.39) and (2.48) yields after two reductions an expression of the form $[R_x-(b_1-aR)_y]w_1+o(z_{xxx})$. If the coefficient of w_1 vanishes, i.e. if

$$R_x - (b_1 - aR)_y = 0, \tag{2.50}$$

an expression with leading term proportional to z_{xxx} remains. It leads to a principal ideal with leading derivative ∂_{xxx}. If it does not vanish, it may be used for eliminating $w_{1,x}$ from (2.39) and $w_{1,y}$ from (2.48). The resulting expressions contain the leading derivatives z_{xxxx} and z_{xxxy} respectively corresponding to the ideal $\langle \partial_{xxxx}, \partial_{xxxy}\rangle_{LT}$.

If condition (2.41) is not valid, w_2 may be eliminated from (2.37) and (2.38) using (2.40). If the result is reduced w.r.t. (2.39) and then autoreduced, two expressions with leading derivatives z_{xxxxy} and z_{xxxyy} are obtained. They correspond to an ideal $\langle \partial_{xxxxy}, \partial_{xxxyy}\rangle_{LT}$. This completes case (ii).

Assume now $a_1 = a_2 = a_3 = a$ and $b_i \neq b_j$ for $i \neq j$ in (2.36). A single autoreduction step yields

$$w_{2,x} + aw_{2,y} + b_2 w_2 + o(z_x), \quad w_{1,x} + aw_{1,y} + b_1 w_1, \quad w_2 + \frac{b_3 - b_1}{b_3 - b_2}w_1 + o(z_x).$$

The last element may be applied to eliminate w_2 from the first one with the result

$$\left[\frac{(b_1 - b_2)(b_1 - b_3)}{b_2 - b_3} - \left(\frac{b_3 - b_1}{b_3 - b_2}\right)_x - a\left(\frac{b_3 - b_1}{b_3 - b_2}\right)_y\right] w_1 + o(z_{xx}). \tag{2.51}$$

If the coefficient of w_1 vanishes the condition

$$\left(\frac{b_3 - b_1}{b_3 - b_2}\right)_x + a\left(\frac{b_3 - b_1}{b_3 - b_2}\right)_y = \frac{(b_1 - b_2)(b_1 - b_3)}{b_2 - b_3} \tag{2.52}$$

follows; the expression (2.51) has a leading term proportional to z_{xx} corresponding to a principal ideal $\langle \partial_{xx}\rangle_{LT}$. If it does not vanish it may be used to eliminate w_1 from the second relation. The result is an expression with leading term z_{xxx} corresponding to a principal ideal $\langle \partial_{xxx} \rangle_{LT}$ without further constraint. This completes case (iii).

If $a_i \neq a_j$ for $i \neq j$, $i, j = 1, 2, 3$, autoreduction of the system $l_i = \partial_x + a_i \partial_y + b_i$, $i = 1, 2, 3$, leads to the system

$$\frac{b_1 - b_2}{a_1 - a_2} - \frac{b_2 - b_3}{a_2 - a_3}, \tag{2.53}$$

$$\partial_x + \frac{a_2 b_3 - a_3 b_2}{a_2 - a_3}, \quad \partial_y + \frac{b_2 - b_3}{a_2 - a_3}. \tag{2.54}$$

In order that $\{l_1, l_2, l_3\}$ form a nontrivial Janet basis, the expression (2.53) must vanish. In addition the integrability condition for the system (2.54) must be satisfied which is identical to (2.41). This yields the $Gcrd$ in case (i). In cases (i) and (iii) at least for one pair of coefficients there holds $a_i = a_j$, $b_i \neq b_j$ which leads to a trivial $Gcrd$. □

These result will be illustrated now by a few examples. The reader is encouraged to reproduce them by using the software provided on the website `alltypes.de`; as a useful exercise the effect of small variations of the input operators may be studied.

Example 2.18. Let three operators be given by

$$l_1 \equiv \partial_x + \frac{y}{x}\partial_y + \frac{1}{x}, \quad l_2 \equiv \partial_x + \frac{y}{x}\partial_y - \frac{xy+1}{x^2 y - x}, \quad l_3 \equiv \partial_x + \frac{y}{x}\partial_y - \frac{x^2+1}{x^3 - x}.$$

Because $a_1 = a_2 = a_3$ case (iii) of Theorem 2.4 applies. Because the coefficients b_1, b_2 and b_3 satisfy (2.52), the intersection ideal is generated by

$$\partial_{xx} + \frac{2y}{x}\partial_{xy} + \frac{y^2}{x^2}\partial_{yy} + \frac{1}{x}\partial_x + \frac{y}{x^2}\partial_y - \frac{1}{x^2}. \qquad \square$$

Example 2.19. Consider the three operators

$$l_1 = \partial_x + 2, \quad l_2 = \partial_x + \partial_y + 1, \quad l_3 = \partial_x + 2\partial_y$$

with $a_i \neq a_j$ for $i \neq j$ and $\frac{b_1 - b_2}{a_1 - a_2} \neq \frac{b_2 - b_3}{a_2 - a_3}$, i.e. subcase (a) of case (i) of Theorem 2.4 applies. Their intersection is

$$Lclm(l_1, l_2, l_3) = \langle \partial_{xxx} + 3\partial_{xxy} + 2\partial_{xyy} + 3\partial_{xx} + 8\partial_{xy} + 4\partial_{yy} + 2\partial_x + 4\partial_y \rangle$$

and their sum ideal $Gcrd(l_1, l_2, l_3) = \langle\!\langle \partial_x + 2, \partial_y - 1 \rangle\!\rangle$. □

Example 2.20. Consider the three operators

$$l_1 = \partial_x + 2, \quad l_2 = \partial_x + \partial_y + 2, \quad l_3 = \partial_x + 2\partial_y$$

with $a_1 \neq a_2 \neq a_3$ and $\frac{b_1 - b_2}{a_1 - a_2} = 0 \neq \frac{b_2 - b_3}{a_2 - a_3} = -2$, i.e. subcase (b) of case (i) of Theorem 2.4 applies. Their intersection is

$$Lclm(l_1, l_2, l_3) = \langle \partial_{xxx} + 3\partial_{xxy} + 2\partial_{xyy} + 4\partial_{xx} + 10\partial_{xy} + 4\partial_{yy} + 4\partial_x + 8\partial_y \rangle$$

and their sum ideal $Gcrd(l_1, l_2, l_3) = \langle 1 \rangle$. □

The proof given above shows an additional feature of the intersection of three ideals. According to Theorem 2.2, condition (2.41) means that the ideals I_2 and I_3 have a principal intersection. If it is not satisfied, the subsequent Janet basis calculation leads into a branch which does not allow a principal intersection any more. Because this is true for all possible term orderings, the principality of the intersection of three ideals requires that the pairwise intersections of each pair be principal. However, the reverse is not true as may be seen from the following example.

Example 2.21. Let three principal ideals be given by

$$I_1 \equiv \langle \partial_x + \partial_y + x \rangle, \quad I_2 \equiv \langle \partial_x + x\partial_y + x \rangle \text{ and } I_3 \equiv \langle \partial_x + y\partial_y + x \rangle.$$

Any of its three pairwise intersections $I_i \cap I_j$, $i, j = 1, 2, 3$ is principal.

$$\begin{aligned}
Lclm(I_1, I_2) = \Big\langle\!\Big\langle & \partial_{xx} + (x+1)\partial_{xy} + x\partial_{yy} \\
& + \big(2x - \tfrac{1}{x-1}\big)\partial_x + \big(x^2 + x - \tfrac{1}{x-1}\big)\partial_y + x^2 - \tfrac{1}{x-1} \Big\rangle\!\Big\rangle, \\
Lclm(I_1, I_3) = \Big\langle\!\Big\langle & \partial_{xx} + (y+1)\partial_{xy} + y\partial_{yy} \\
& + \big(2x - \tfrac{1}{y-1}\big)\partial_x + \big(xy + x - \tfrac{1}{y-1}\big)\partial_y + x^2 + 1 - \tfrac{1}{y-1} \Big\rangle\!\Big\rangle, \\
Lclm(I_2, I_3) = \Big\langle\!\Big\langle & \partial_{xx} + (x+y)\partial_{xy} + xy\partial_{yy} \\
& + \big(2x + \tfrac{x-y}{y-1}\big)\partial_x + \big(x^2 - xy - \tfrac{x^2-y}{x-y}\big)\partial_y + x^2 + \tfrac{x^2-y}{x-y} \Big\rangle\!\Big\rangle,
\end{aligned}$$

yet the intersection of all three ideals is a rather complicated non-principal ideal, it is $I_1 \cap I_2 \cap I_3 = \langle \partial_{xxxx}, \partial_{xxxy} \rangle_{LT}$. □

The important property shown in this example is formulated as the next corollary.

Corollary 2.7. *In order that the intersection of the ideals* $I_i = \langle \partial_x + a_i\partial_y + b_i \rangle$, $i = 1, 2, 3$ *be principal it is necessary, but not sufficient, that their pairwise intersections are principal.*

Finally there remains the case of two operators with leading derivative ∂_y in the term ordering $grlex$ and $x \succ y$. The answer is given next.

Theorem 2.5. *Let the three ideals*

$$I_1 \equiv \langle \partial_y + b_1 \rangle, \quad I_2 \equiv \langle \partial_y + b_2 \rangle, \quad I_3 \equiv \langle \partial_x + a_3\partial_y + b_3 \rangle$$

be given with $I_i \neq I_j$ for $i \neq j$ and $a_3 \neq 0$. There are three different cases for their intersection $I_1 \cap I_2 \cap I_3$. All intersection ideals have differential dimension $(1,3)$.

(i) *If $b_{1,x} - (b_3 - a_3 b_1)_y = 0$ and $b_{2,x} - (b_3 - a_3 b_2)_y = 0$ the intersection ideal is principal of the form $\langle \partial_{xyy} \rangle_{LT}$.*

(ii) *If $b_{1,x} - (b_3 - a_3 b_1)_y = 0$ and $b_{2,x} - (b_3 - a_3 b_2)_y \neq 0$, or $b_{1,x} - (b_3 - a_3 b_1)_y \neq 0$ and $b_{2,x} - (b_3 - a_3 b_2)_y = 0$, the intersection ideal is not principal, it has the form $\langle \partial_{xyyy}, \partial_{xxyy} \rangle_{LT}$.*

(iii) *If $b_{1,x} - (b_3 - a_3 b_1)_y \neq 0$ and $b_{2,x} - (b_3 - a_3 b_2)_y \neq 0$ the intersection ideal is not principal, it has the form $\langle \partial_{xyyyy}, \partial_{xxyy} \rangle_{LT}$ of differential dimension* $(1,3)$.

Proof. By a similar reasoning as in the above theorems the following differential polynomials are obtained.

$$w_{1,x} + a_3 w_{1,y} + b_3 w_1 + w_{2,x} + a_3 w_{2,y} + b_3 w_2 - z_x - a_3 z_y - b_3 z,$$
$$w_{1,y} + b_1 w_1, \quad w_{2,y} + b_2 w_2.$$

By assumption the relation $b_1 \neq b_2$ is valid. The lex term order with $w_1 \succ w_2 \succ z$ and $x \succ y$ is always applied. Autoreduction yields the system

$$\begin{gathered} w_{1,x} + (b_3 - a_3 b_1) w_1 + w_{2,x} + (b_3 - a_3 b_2) w_2 - z_x - a_3 z_y - b_3 z, \\ w_{1,y} + b_1 w_1, \quad w_{2,y} + b_2 w_2. \end{gathered} \tag{2.55}$$

The single integrability condition between the first two members yields after reduction

$$\begin{aligned} &[(b_3 - a_3 b_1)_y - b_{1,x}] w_1 - (b_2 - b_1) w_{2,x} \\ &\qquad + [(b_3 - a_3 b_2)_y - b_{2,x} - (b_3 - a_3 b_2)(b_2 - b_1)] w_2 \\ &\qquad - z_{xy} - b_1 z_x - a_3 z_{yy} - (a_{3,y} + a_3 b_1 + b_3) z_y - (b_{3,y} + b_1 b_3) z. \end{aligned} \tag{2.56}$$

If the leading coefficient vanishes, i.e. if

$$b_{1,x} - (b_3 - a_3 b_1)_y = 0; \tag{2.57}$$

upon autoreduction the full system is

$$
\begin{aligned}
w_{1,x} &+(b_3 - a_3 b_1)w_1 - \frac{1}{b_2 - b_1}[b_{2,x} - (b_3 - a_3 b_2)_y]w_2 \\
&-\frac{1}{b_2 - b_1}[z_{xy} + b_2 z_x + a_3 z_{yy} + (a_{3,y} + a_3 b_2 + b_3)z_y \\
&+(b_{3,y} + b_2 b_3)z], \quad w_{1,y} + b_1 w_1, \\
w_{2,x} &+ \frac{1}{b_2 - b_1}[b_{2,x} - (b_3 - a_3 b_2)_y + (b_3 - a_3 b_2)(b_2 - b_1)]w_2 \\
&+\frac{1}{b_2 - b_1}[z_{xy} + b_1 z_x + a_3 z_{yy} + (a_{3,y} + a_3 b_1 + b_3)z_y \\
&+(b_{3,y} + b_1 b_3)z], \quad w_{2,y} + b_2 w_2.
\end{aligned}
$$

The two integrability conditions for this system are satisfied if

$$
b_{2,x} - (b_3 - a_3 b_2)_y = 0. \tag{2.58}
$$

In addition a differential polynomial for z with leading derivative z_{xyy} has to be satisfied. This is case (i) yielding a principal intersection ideal. If (2.58) is not satisfied, the integrability condition for the last two equations of (2.8) is an expression of the form $z + o(z_{xyy})$. Reduction w.r.t. it yields a Janet basis the two lowest polynomials of which have leading derivatives z_{xyyy} and z_{xxyy}; this is case (ii). If neither (2.57) or (2.58) are satisfied, the two lowest polynomials of the resulting Janet basis have leading derivatives z_{xyyyy} and z_{xxyy}; this is case (iii). □

Subsequently three examples are given covering the three cases of the above theorem; they show that these alternatives do actually exist. Due to the Janet basis calculations involved the coefficients of the given operators I_1, I_2 and I_3 have to be fairly simple in order to obtain a manageable problem.

Example 2.22. The three operators $\partial_y + x$, $\partial_y + x + \frac{1}{x}$ and $\partial_x + \frac{y}{x}\partial_y + 2y$ have the principal intersection

$$
\begin{aligned}
\partial_{xyy} &+ \big(2x + \tfrac{1}{x}\big)\partial_{xy} + (x^2+1)\partial_x + \tfrac{y}{x}\partial_{yyy} + \big(4y + \tfrac{2}{x} + \tfrac{y}{y^2}\partial_{yy}\big) \\
&+\big(5xy + 6 + \tfrac{3y}{x} + \tfrac{1}{x^2}\big)\partial_y + 2xy + 4x + 2y + \tfrac{2}{x}
\end{aligned}
$$

corresponding to case (i) of the above theorem. □

Example 2.23. The non-vanishing coefficients of the operators $\partial_y + x$, $\partial_y + x + y$ and $\partial_x + \frac{y}{x}\partial_y + 2y$ are $b_1 = x$, $b_2 = x + y$, $b_3 = \frac{y}{x}$ and $a_3 = \frac{1}{x}$. They satisfy (2.57) but not (2.58). Therefore case (ii) of the above theorem applies. Their intersection ideal $\langle \partial_{xyyy}, \partial_{xxyy} \rangle_{LT}$ is not principal; its two generators are too voluminous to be given here. □

Example 2.24. The coefficients of the operators $\partial_y + \frac{1}{x}$, $\partial_y + \frac{1}{y}$ and $\partial_x + \frac{y}{x}\partial_y + 2y$ satisfy neither (2.57) nor (2.58); thus by case (iii) their intersection ideal is not principal; it is generated by two huge operators with leading derivatives ∂_{xyyyy} and ∂_{xxyy}. □

2.9 Exercises

Exercise 2.1. Prove Lemma 2.1 on page 22.

Exercise 2.2. What does the relation (2.8) on page 29 mean for ideals of differential dimension $(0, k)$?

Exercise 2.3. Determine the third-order terms of the generators of the intersection ideal in case (ii) of Theorem 2.2 on page 47 explicitly.

Exercise 2.4. The same problem for the generator of the principal ideal in case (i) of Theorem 2.3.

Exercise 2.5. Solve the equations $l_i z_i = 0$ with l_i, $i = 1, 2, 3$ from Example 2.19 and explain the solution of $Gcrd(l_1, l_2, l_3)z = 0$ in terms of the solutions of these three equations.

Exercise 2.6. Let three operators $l_i \equiv \partial_x + a_i\partial_y + b_i$ with constant coefficients be given, $a_i \neq a_j$ for $i \neq j$. Determine the general expressions for $Lclm(l_1, l_2, l_3)$ and $Gcrd(l_1, l_2, l_3)$ and distinguish subcases (a) and $b)$ of case (i) in Theorem 2.4.

Exercise 2.7. Determine the Hilbert-Kolchin polynomial H_I and the differential dimension d_I for the ideal $I = \langle \partial_{xxxxxx}, \partial_{xxxxyy} \rangle_{LT}$.

Exercise 2.8. Assume a module of differential dimension $(0, 1)$ in $\mathscr{D}^3$ is generated by the four elements $(\partial_x + a_1, a_2, a_3)$, (∂_y, b_2, b_3), $(c, 1, 0)$ and $(d, 0, 1)$. Determine the coherence conditions for its coefficients such that they form a Janet basis of a type $\mathbb{M}_3^{(0,1)}$ module defined on page 33.

Exercise 2.9. Prove Lemma 2.2 and Lemma 2.3 on page 45 and page 46 respectively.

Exercise 2.10. Determine explicit expressions for the coefficients p_1, p_2, q_1 and q_2 of the intersection ideal for case (ii) of Theorem 2.2; the notation is the same as in Lemma 2.2 on page 45.

Exercise 2.11. Determine the generator of the intersection ideal of $\partial_x + a_1\partial_y + b_1$ and $\partial_y + b_2$ if case (i) of Theorem 2.3 applies.

Exercise 2.12. Determine explicit expressions for the coefficients p_1, p_2, q_1 and q_2 of the intersection ideal for case (ii) of Theorem 2.3; the notation is the same as in Lemma 2.3 on page 45.

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