Evidence Logics with Relational Evidence

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Abstract. We introduce a family of logics for reasoning about *relational evidence*: evidence that involves an ordering of states in terms of their relative plausibility. We provide sound and complete axiomatizations for the logics. We also present several evidential actions and prove soundness and completeness for the associated dynamic logics.

Keywords: Evidence logic · Dynamic epistemic logic · Belief revision

Dynamic evidence logics [2,14–17] are logics for reasoning about the evidence and evidence-based beliefs of agents in a dynamic environment. Evidence logics are concerned with scenarios in which an agent collects several pieces of evidence about a situation of interest, from a number of sources, and uses this evidence to form and revise her beliefs about this situation. The agent is typically uncertain about the actual state of affairs, and as a result takes several alternative descriptions of this state as possible (these descriptions are typically called *possible worlds* or *possible states*). The existing evidence logics, i.e., *neighborhood evidence logics* (NEL) [2,14–17], have the following features:

1. *All evidence is 'binary'.* Each piece of evidence is modeled as a set of possible states. This set indicates which states are good candidates for the actual state, and which ones are not, according to the source. Hence the name binary; every state is either a good candidate ('in'), or a bad candidate ('out').
2. *All evidence is equally reliable.* The agent treats all evidence pieces on a par. There is no explicit modeling of the relative *reliability* of pieces of evidence.
3. *One procedure to combine evidence.* The logics developed so far study the evidence and beliefs held by an agent relying on one specific procedure for combining evidence.

This work presents a family of dynamic evidence logics which we call *relational evidence logics* (REL). Relational evidence logics aim to contribute to the existing work on evidence logic as follows.

1. *Relax the assumption that all evidence is binary.* This is accomplished by modeling pieces of evidence by *evidence relations*. Evidence relations are preorders over the set of possible states. The ordering is meant to represent the

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relative plausibility of states based on the evidence. While a special type of evidence relation – *dichotomous order* – can be used to model binary evidence, less 'black-and-white' forms of evidence can also be encoded in REL models.

2. *Model levels of evidence reliability.* In general, not all evidence is equally reliable. Expert advice and gossip provide very different grounds for belief, and a rational agent should weight the evidence that it is exposed to accordingly. To model evidence reliability, we equipped our models with *priority orders*, i.e., orderings of the family of evidence relations according to their relative reliability. Priority orders were introduced in [1], and have already been used in other DEL logics (see, e.g. [9,12]). Here, we use them to model the relative reliability of pieces of evidence.
3. *Explore alternative evidence aggregation rules.* Our evidence models come equipped with an aggregator, which merges the available evidence relations into a single relation representing the combined plausibility of the possible states. The beliefs of the agent are then defined on the basis of this combined plausibility order. By focusing on different classes of evidence models, given by their underlying aggregator, we can then compare the logics of belief arising from different approaches to combining evidence.

1 Relational Evidence Models

Relational evidence. We call *relational evidence* any type of evidence that induces an ordering of states in terms of their relative plausibility. A suitable representation for relational evidence, which we adopt, is given by the class of *preorders*. We call preorders representing relational evidence, *evidence relations*, or *evidence orders*. As is well-known, preorders can represent several meaningful types of orderings, including those that feature incomparable or tied alternatives.

Definition 1 (Preorder). *A* preorder *is a binary relation that is reflexive and transitive. We denote the set of all preorders on a set* X *by* $Pre(X)$*. For a preorder* R *on* X *and an element* $x \in X$*, we define the following associated notions:* $R[x] := \{y \in X \mid Rxy\}$*;* $R^{<} := \{(x,y) \in X^2 \mid Rxy \text{ and } Ryx\}$*;* $R^{\sim} := \{(x,y) \in X^2 \mid Rxy \text{ and } \neg Ryx\}$*.*

Evidence reliability. In general, not all sources are equally trustworthy, so an agent combining evidence may be justified in giving priority to some evidence items over others. As suggested in [17], a next reasonable step in evidence logics is modeling levels of reliability of evidence. One general format for this is given by the *priority graphs* of [1], which have already been used extensively in dynamic epistemic logic (see, e.g., [9,12]). In this work, we will use the related, yet simpler format of a 'priority order', as used in [5,6], to represent hierarchy among pieces of evidence. Our definition of a priority order is as follows:

Definition 2 (Priority order). *Let* $\mathscr{R}$ *be a family of evidence orders over* W*. A priority order for* $\mathscr{R}$ *is a preorder* $\preceq$ *on* $\mathscr{R}$*. For* $R, R' \in \mathscr{R}$*,* $R \preceq R'$ *reads as: "the evidence order* R' *has at least the same priority as evidence order* R*".*

Intuitively, priority orders tell us which pieces of evidence are more reliable according to the agent. They give the agent a natural way to break stalemates when faced with inconsistent evidence.

Evidence aggregators. We are interested in modeling a situation in which an agent integrates evidence obtained from multiple sources to obtain and update a combined plausibility ordering, and forms beliefs based on this ordering. When we consider relational evidence with varying levels of priority, a natural way model the process of evidence combination is to define a function that takes as input a family of evidence orders $\mathscr{R}$ together with a priority order $\preceq$ defined on them, and combines them into a plausibility order. The agent's beliefs can then be defined in terms of this output.

Definition 3 (Evidence aggregator). *Let W be a set of alternatives. Let $\mathcal{W}$ be the set of preorders on W. An* evidence aggregator *for W is a function Ag mapping any preordered family $P = \langle \mathscr{R}, \preceq \rangle$ to a preorder $Ag(P)$ on W, where $\emptyset \notin \mathscr{R} \subseteq \mathcal{W}$ and $\preceq$ is a preorder on $\mathscr{R}$. $\mathscr{R}$ is seen here as a family of evidence orders over W, $\preceq$ as a priority order for $\mathscr{R}$, and $Ag(P)$ as an evidence-based plausibility order on W.*

At first glance, our definition of an aggregator may seem to impose mild constraints that are met by most natural aggregation functions. However, as it is well-known, the output of some common rules, like the majority rule, may not be transitive (thus not a preorder), and hence they don't count as aggregators. A specific aggregator that *does* satisfy the constraints is the *lexicographic rule*. This aggregator was extensively studied in [1], where it was shown to satisfy several nice aggregative properties. The definition of the aggregator is the following:

Definition 4. *The (anti-)lexicographic rule is the aggregator* lex *given by*

$$(w, v) \in \mathsf{lex}(\langle \mathscr{R}, \preceq \rangle) \textit{ iff } \forall R' \in \mathscr{R}\ (R'wv\ \vee\ \exists R \in \mathscr{R}(R' \prec R \wedge R^{<}wv))$$

Intuitively, the lexicographic rule works as follows. Given a particular hierarchy $\preceq$ over a family of evidence $\mathscr{R}$, aggregation is done by giving priority to the evidence orders further up the hierarchy in a compensating way: the agent follows what all evidence orders agree on, if it can, or follows more influential pieces of evidence, in case of disagreement. Other well-known aggregators that satisfy the constraints, but don't make use of the priority structure, are the intersection rule (defined below), or the Borda rule.

Definition 5. *The intersection rule is the aggregator $Ag_\cap$ given by $(w, v) \in Ag_\cap(\langle \mathscr{R}, \preceq \rangle)$ iff $(w, v) \in \bigcap \mathscr{R}$.*

The models. Having defined relational evidence and evidence aggregators, we are now ready to introduce relational evidence models.

Definition 6 (Relational evidence model). *Let* P *be a set of propositional variables. A* relational evidence model *(REL model, for short) is a tuple*

$M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ where W is a non-empty set of states*; $\langle \mathscr{R}, \preceq \rangle$ is an ordered family of evidence, where: $\mathscr{R}$ is a set of evidence orders on W with $W^2 \in \mathscr{R}$ and $\preceq$ is a priority order for $\mathscr{R}$; $V : \mathsf{P} \to 2^W$ is a valuation function; Ag is an evidence aggregator for W.*

$W^2 \in \mathscr{R}$ is called the *trivial evidence order*. It represents the evidence stating that "the actual state is in W", which is taken to be always available to the agent as a starting point. $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ is called an *f-model* iff $Ag = f$.

Syntax and semantics. We now introduce a *static* language for reasoning about relational evidence, which we call $\mathscr{L}$. In [2], this language is interpreted over NEL models (there, the language is called $\mathscr{L}_{\forall \Box \Box_0}$).

Definition 7 ($\mathscr{L}$). *Let P be a countably infinite set of propositional variables. The language $\mathscr{L}$ is defined by:*

$$\varphi ::= \ p \mid \neg \varphi \mid \varphi \wedge \varphi \mid \Box_0 \varphi \mid \Box \varphi \mid \forall \varphi \quad (p \in \mathsf{P})$$

The intended interpretation of the modalities is as follows. $\Box_0 \varphi$ reads as: 'the agent has basic, factive evidence for φ'; $\Box \varphi$ reads as: 'the agent has combined, factive evidence for φ'. The language $\mathscr{L}$ is interpreted over REL models as follows.

Definition 8 (Satisfaction). *Let $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ be an REL model and $w \in W$. The satisfaction relation $\models$ between pairs (M, w) and formulas $\varphi \in \mathscr{L}$ is defined as follows (the propositional clauses are as usual):*

$M, w \models \Box_0 \varphi$ *iff there is $R \in \mathscr{R}$ such that, for all $v \in W, Rwv$ implies $M, v \models \varphi$*
$M, w \models \Box \varphi$ *iff for all $v \in W, Ag(\langle \mathscr{R}, \preceq \rangle)wv$ implies $M, v \models \varphi$*
$M, w \models \forall \varphi$ *iff for all $v \in W, M, v \models \varphi$*

Definition 9 (Truth map). *Let $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ be a REL model. We define a truth map $[\![\cdot]\!]_M : \mathscr{L} \to 2^W$ given by: $[\![\varphi]\!]_M = \{w \in W \mid M, w \models \varphi\}$.*

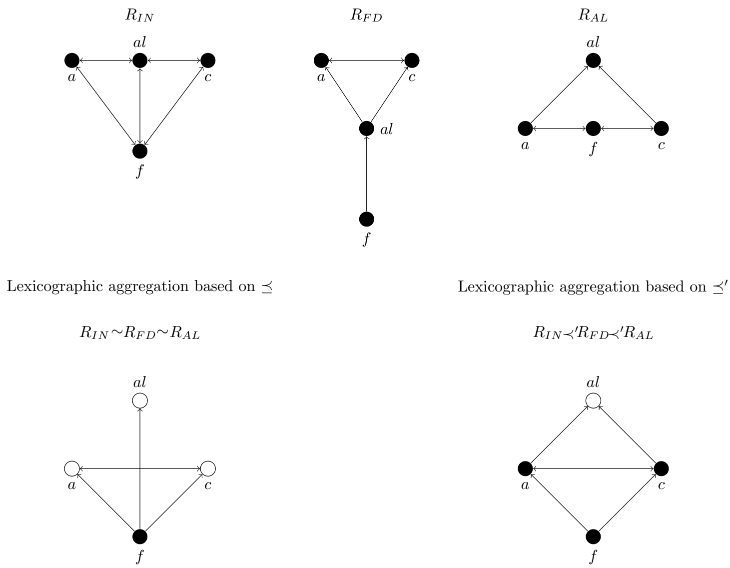
Next, we introduce some definable notions of evidence and belief over REL models, illustrated below with an example. Fix a model $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$.

Basic (factive) Evidence. We say that a piece of evidence $R \in \mathscr{R}$ *supports* φ at $w \in W$ iff $R[w] \subseteq [\![\varphi]\!]_M$. That is, every world that is at least as plausible as w under R satisfies φ. Using this notion of support, we say that the agent has basic, factive evidence for φ at $w \in W$ if there is a piece of evidence $R \in \mathscr{R}$ that supports φ at w. That is: 'the agent has basic evidence for φ at $w \in W$' iff $\exists R \in \mathscr{R}(R[w] \subseteq [\![\varphi]\!]_M)$ iff $M, w \models \Box_0 \varphi$. We also have a non-factive version of this notion, which says that the agent has basic evidence for φ if there is a piece of evidence R that supports φ at *some* state, i.e.: 'the agent has basic evidence for φ (at any state)' iff $\exists w(\exists R \in \mathscr{R}(R[w] \subseteq [\![\varphi]\!]_M))$ iff $M, w \models \exists \Box_0 \varphi$. We can also have a *conditional* version of basic evidence: 'the agent has basic, factive evidence for ψ at w, conditional on φ being true'. Putting $\Box_0^\varphi \psi := \Box_0(\varphi \to \psi)$, we have: 'the agent has basic, factive evidence for ψ at w, conditional on φ being true' iff $\exists R \in \mathscr{R}(\forall v(Rwv \Rightarrow (v \in [\![\varphi]\!]_M \Rightarrow v \in [\![\psi]\!]_M)))$ iff $M, w \models \Box_0^\varphi \psi$. The notion of conditional evidence reduces to that of plain evidence when $\varphi = \top$.

Aggregated (factive) Evidence. We propose a notion of aggregated evidence based on the output of the aggregator: the agent has aggregated, factive evidence for φ at $w \in W$ iff $Ag(\langle \mathscr{R}, \preceq \rangle)[w] \subseteq [\![\varphi]\!]_M$ iff $M, w \models \Box\varphi$. The non-factive version of the previous notion is as follows: the agent has aggregated evidence for φ (at any state) iff $\exists w(Ag(\langle \mathscr{R}, \preceq \rangle)[w] \subseteq [\![\varphi]\!]_M)$ iff $M, w \models \exists\Box\varphi$. As we did with basic evidence, we can define a conditional notion of aggregated evidence in φ by putting $\Box^{\varphi}\psi := \Box(\varphi \to \psi)$. The unconditional version is given by $\varphi = \top$.

Evidence-Based Belief. The notion of belief we will work with is based on the agent's plausibility order, which in REL models corresponds to the output of the aggregator. As we don't require the plausibility order to be converse-well founded, it may have no maximal elements, which means that Grove's definition of belief may yield inconsistent beliefs. For this reason, we adopt a usual generalization of Grove's definition, which defines beliefs in terms of truth in all 'plausible enough' worlds (see, e.g., [3,16]). Putting $B\varphi := \forall\Diamond\Box\varphi$, we have: the agent believes φ (at any state) iff $\forall w(\exists v((w, v) \in Ag(\langle \mathscr{R}, \preceq \rangle)$ and $Ag(\langle \mathscr{R}, \preceq \rangle)[v] \subseteq [\![\varphi]\!]_M))$ iff $M, w \models \forall\Diamond\Box\varphi$. That is, the agent believes φ iff for every state $w \in W$, we can always find a more plausible state $v \in [\![\varphi]\!]_M$, all whose successors are also in $[\![\varphi]\!]_M$. When the plausibility relation is indeed converse well-founded, this notion of belief coincides with Grove's one, while ensuring consistency of belief otherwise. We can also define a notion of conditional belief. Putting $B^{\varphi}\psi := \forall(\varphi \to \Diamond(\varphi \to (\Box\varphi \to \psi)))$, we have: 'the agent believes ψ conditional on φ iff $\forall w(w \in [\![\varphi]\!]_M \Rightarrow \exists v(Ag(\langle \mathscr{R}, \preceq \rangle)$ wv and $v \in [\![\varphi]\!]_M$ and $Ag(\langle \mathscr{R}, \preceq \rangle)[v] \cap [\![\varphi]\!]_M \subseteq [\![\psi]\!]_M))$ iff $M, w \models B^{\varphi}\psi$. As before, this conditional notion reduces to that of absolute belief when $\varphi = \top$.

Example 1 (The diagnosis). Consider an agent seeking medical advice on an ongoing health issue. To keep thing simple, assume that there are four possible diseases: asthma (a), allergy (al), cold (c), and flu (f). This can be described by a set W consisting of four possible worlds, $\{w_a, w_{al}, w_c, w_f\}$ and a set of atomic formulas $\{a, al, c, f\}$ (each true at the corresponding world). The agent consults three sources, a medical intern (IN), a family doctor (FD) and an allergist (AL). The doctors inspect the patient, observing fairly non-specific symptoms: cough, no fever, and some inconclusive swelling at an allergen test spot. Given the non-specificity of the symptoms, the doctors can't single out a condition that best explains all they observed. Instead, comparing the diseases in terms of how well they explain the observed symptoms, and drawing on their experience, each doctor arrives at a ranking of the possible diseases. Let us denote by R_{IN}, R_{FD} and R_{AL} the evidence orders representing the judgment of the intern, family doctor and allergist, respectively, which we assume to be as depicted below. If the agent has no information about how reliable each doctor is, she may just trust them all equally. We can model this by a priority order $\preceq$ over the evidence orders $R_{IN} \sim R_{FD} \sim R_{AL}$ that puts all evidence as equally likely. On the other hand, if the agent knows that the intern is the least experienced of the doctors, she may consider his evidence as strictly less reliable than the one provided by the other doctors. Similarly, if the allergist has a strong reputation, the agent may wish to give the allergist's judgment strict priority over the rest. We can model this by a different priority order $\preceq'$ given by $R_{IN} \prec' R_{FD} \prec' R_{AL}$ (note that this is meant to be reflexive and transitive). If, e.g., the agent uses the lexicographic rule, we arrive at the following scenarios, with different aggregated evidence depending on the priority order used:



The best candidates for the actual disease, in each case, are depicted in white. Note that, e.g., the agent has basic evidence for $a \vee al \vee c$, but she doesn't have evidence for f. Moreover, in the scenario based on $\preceq'$, the agent believes that the allergy is the actual disease, but she doesn't in the scenario based on $\preceq$.

A PDL language for relational evidence. Later in this work, we will discuss *evidential actions* by which the agent, upon receiving a new piece of relational evidence, revises its existing body of evidence. To encode syntactically the evidence pieces featured in evidential actions, we will enrich our basic language $\mathscr{L}$ with formulas that stand for specific evidence relations. A natural way to introduce relation-defining expressions, in a modal setting such as ours, is to employ suitable program expressions from Propositional Dynamic Logic (PDL). We will follow this approach, augmenting $\mathscr{L}$ with PDL-style *evidence programs* that define pieces of relational evidence. As evidence orders are preorders, we will employ a set of program expressions whose terms are guaranteed to always define preorders. An natural fragment of PDL meeting this condition is the one provided by programs of the form π^*, which always define the reflexive transitive closure of some relation.

Definition 10 (Evidence programs). *The set Π has all program symbols π defined as follows:*

$$\pi ::= A \mid ?\varphi \mid \pi \cup \pi \mid \pi ; \pi \mid \pi^*$$

where $\varphi \in \mathscr{L}$. Here A denotes the universal program, while the rest of the programs have their usual PDL *meanings (see, e.g., [10]). We call $\Pi_* := \{\pi^* \mid \pi \in \Pi\}$ the set of* evidence programs.

To interpret evidence programs in REL models, we extend the truth map:

Definition 11 (Truth map). *Let $M = \langle W, \langle \mathscr{R}, \prec \rangle, V, Ag \rangle$ be an REL model. We define an extended truth map $[\![\cdot]\!]_M : \mathscr{L} \cup \Pi \to 2^W \cup 2^{W^2}$ given by: $[\![\varphi]\!]_M = \{w \in W \mid M, w \models \varphi\}$; $[\![A]\!]_M = W^2$; $[\![?\varphi]\!]_M = \{(w,w) \in W^2 \mid w \in [\![\varphi]\!]^M\}$; $[\![\pi \cup \pi']\!]_M = [\![\pi]\!]_M \cup [\![\pi']\!]_M$; $[\![\pi;\pi']\!]_M = [\![\pi]\!]_M \circ [\![\pi']\!]_M$; $[\![\pi^*]\!]_M = [\![\pi]\!]^*_M$.*

Some Examples of Definable Evidence Programs. Here are some natural types of relational evidence that can be constructed with programs from Π_*.

Dichotomous evidence. For a formula φ, let $\pi_\varphi := (A; ?\varphi) \cup (?\neg\varphi; A; ?\neg\varphi)$. π_φ puts the φ worlds strictly above the $\neg\varphi$ worlds, and makes every world equally plausible within each of these two regions. It is easy to see that π_φ always defines a preorder, and therefore $(\pi_\varphi)^*$ is an evidence program equivalent to π_φ (Fig. 1).

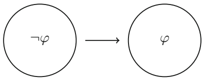


Fig. 1. The dichotomous order defined by π_φ

Totally ordered evidence. Several programs can be used to define total preorders. For example, for formulas $\varphi_1, \ldots, \varphi_n$, we can define the program

$$\begin{aligned}\pi_{\varphi_1,\ldots,\varphi_n} := \; & (A; ?\varphi_1) \cup (?\neg\varphi_1; A; ?\neg\varphi_1; ?\varphi_2) \\ & \cup (?\neg\varphi_1; \neg\varphi_2; A; ?\neg\varphi_1; ?\neg\varphi_2; ?\varphi_3) \\ & \cup \ldots \\ & \cup (?\neg\varphi_1; \ldots; ?\neg\varphi_n; A; ?\neg\varphi_1; \ldots; ?\neg\varphi_{n-1}; ?\varphi_n) \cup (?\top)\end{aligned}$$

This type of program, described in [18], puts the φ_1 worlds above everything else, the $\neg\varphi_1 \wedge \varphi_2$ worlds above the $\neg\varphi_1 \wedge \neg\varphi_2$ worlds, and so on, and the $\neg\varphi_1 \wedge \neg\varphi_2 \wedge \cdots \wedge \neg\varphi_{n-1} \wedge \varphi_n$ above the $\neg\varphi_1 \wedge \neg\varphi_2 \wedge \cdots \wedge \neg\varphi_n$ worlds. $\pi^t(\varphi_1, \ldots, \varphi_n)$ always defines a preorder, so the evidence program $(\pi^t(\varphi_1, \ldots, \varphi_n))^*$ is equivalent to it (Fig. 2).

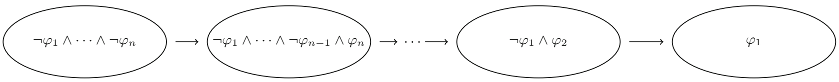


Fig. 2. The total preorder defined by $\pi_{\varphi_1,\ldots,\varphi_n}$

Incompletely ordered evidence. Several programs can be used to define evidence orders featuring incomparabilities. To illustrate this, let us consider the program $\pi_{\varphi\wedge\psi} := (A; ?\varphi \wedge \psi) \cup (?\neg\varphi \wedge \neg\psi; A; ?\varphi \vee \psi) \cup (?\neg\varphi \wedge \psi; A; ?\neg\varphi \wedge \psi) \cup (?\varphi \wedge \neg\psi; A; ?\varphi \wedge \neg\psi) \cup (?\top)$. As depicted in Fig. 3, this program puts the $\varphi \wedge \psi$ worlds above everything else, the $\neg\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ as incomparable 'second-best' worlds, and the $\neg\varphi \wedge \neg\psi$ below everything else. As with the other programs $\pi_{\varphi\wedge\psi}$ always defines a preorder, so $(\pi_{\varphi\wedge\psi})^*$ is an equivalent evidence program.

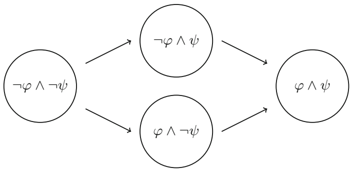


Fig. 3. The incomplete preorder defined by $\pi_{\varphi\wedge\psi}$

2 The Logics of $Ag_\cap$-Models and lex-Models

We initiate here our logical study of the statics of belief and evidence in the REL setting. We first zoom into two specific classes of REL models, the classes of $Ag_\cap$-models and lex-models, and study the static logics for belief and evidence based on these models. In particular, we introduce systems $\mathsf{L}_\cap$ and $\mathsf{L}_{\mathsf{lex}}$ that axiomatize the class of $Ag_\cap$-models and the class of lex-models, respectively. (To simplify notation, we write $\cap$-models instead of $Ag_\cap$-models hereafter). In later sections, we will 'zoom out' and study the class of all REL models. Our choice to study $\cap$ and lex models in some detail is motivated as follows. The class of $\cap$-models is interesting because it links our relational evidence setting back to the NEL setting that inspired it. Indeed, as we show right below, given any NEL model with finitely many pieces of evidence, we can always find a $\cap$-model that is modally equivalent to it (with respect to language $\mathscr{L}$). This $\cap$-model represents binary evidence in a relational way, thereby encoding the same information presented in the NEL model. lex-models, on the other hand, provide a good study case for the REL setting, as they exemplify its main novel features: non-binary evidence and reliability-sensitive aggregation. We recall here the definition of a NEL model to compare them to $\cap$-models. The definition given for these models follows the one in [2]. For a more general notion, see [15], where the models we consider are called *uniform* models.

Definition 12 (Neighborhood evidence model). *A* neighborhood evidence model *is a tuple $M = \langle W, E_0, V\rangle$ where: W is a non-empty set of states; $E_0 \subseteq \mathscr{P}(W)$ is a family of basic evidence sets, such that $\emptyset \notin E_0$ and $W \in E_0$; $V : \mathsf{P} \to \mathscr{P}(W)$ is a valuation function. A model is called* feasible *if E_0 is finite. A* body of evidence *is a family $F \subseteq E_0$ such that every non-empty finite subfamily*

$F' \subseteq F$ is consistent, i.e., $\bigcap F' \neq \emptyset$. A piece of combined evidence *is any non-empty intersection of finitely many pieces of basic evidence. We denote by $E := \{\bigcap F \mid F \subseteq E_0, |F| \in \mathbb{N}\}$ the family of all combined evidence.*

Definition 13 (Satisfaction). *Let $M = \langle W, E_0, V\rangle$ be an* NEL *model and $w \in W$. The satisfaction relation $\models$ between pairs (M, w) and formulas $\varphi \in \mathscr{L}$ is:*

$M, w \models \Box_0\varphi$ *iff there is $e \in E_0$ such that $w \in e \subseteq [\![\varphi]\!]_M$*
$M, w \models \Box\varphi$ *iff there is $e \in E$ such that $w \in e \subseteq [\![\varphi]\!]_M$*
$M, w \models \forall\varphi$ *iff $W = [\![\varphi]\!]_M$*

We now present a way to 'transform' a NEL model into a matching REL model. To do that, we first encode binary evidence, the type of evidence considered in NEL models, as relational evidence.

Definition 14. *Let W be a set. For each $e \subseteq W$, we denote by R_e the relation given by: $(w, v) \in R_e$ iff $w \in e \Rightarrow v \in e$.*

That is, R_e is a preorder with at most two indifference classes (i.e., a dichotomous weak order) of 'good' and 'bad' candidates for the actual state, which puts all the 'bad' candidates strictly below the 'good' ones (Fig. 4).

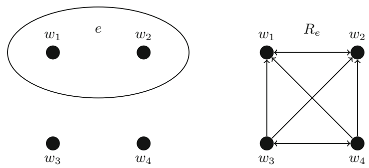


Fig. 4. A piece of binary evidence, represented as an evidence set e (left) and as a dichotomous evidence order R_e (right).

Having fixed this connection between evidence sets and evidence orders, we can now consider a natural way to transform every NEL into a $\cap$-model in which each evidence order is dichotomous. To fix this link, we define a mapping between NEL and REL models.

Definition 15. *Let Rel be a map from* NEL *to* REL *models given by:*

$$\langle W, E_0, V\rangle \mapsto \langle Rel(W), \langle Rel(E_0), \preceq\rangle, Rel(V), Ag_\cap\rangle$$

where $Rel(W) := W$, $Rel(V) := V$ $Rel(E_0) := \{R_e \mid e \in E_0\}$ and $\preceq = Rel(E_0)^2$.

We can then observe that feasible NEL models and their images under *Rel* are modally equivalent, in the sense of having point-wise equivalent modal theories.

Proposition 1. *Let $M = \langle W, E_0, V\rangle$ be a feasible* NEL *model. For any $\varphi \in \mathscr{L}$ and any $w \in W$, we have: $M, w \models \varphi$* iff *$Rel(M), w \models \varphi$.*

That is, feasible NEL models can be seen as 'special cases' of REL models in which all evidence is dichotomous and equally reliable. As the following proposition shows, the modal equivalence result does not extend to non-feasible **NEL** models. This is because, in models with infinitely many pieces of evidence, the notion of combined evidence presented in [2] differs from the one proposed here for **REL** models. To clarify this, consider a **NEL** model $M = \langle W, E_0, V\rangle$. Recall that the agent has combined evidence for a proposition φ at w if there is a *finite* body of evidence whose combination contains w and supports φ, i.e., if there is some finite $F \subseteq E_0$ such that $w \in \bigcap F \subseteq [\![\varphi]\!]_M$. Suppose M is a non-feasible model in which we have $w \in \bigcap E_0 \subseteq [\![\varphi]\!]_M$, while no finite family $F \subseteq E_0$ is such that $w \in \bigcap F \subseteq [\![\varphi]\!]_M$. That is, the combination of *all* the evidence supports φ at w, but no combination of a finite subfamily of E_0 does. In a **NEL** model like this, the agent does *not* have combined evidence for φ at w. That is, $M, w \not\models \Box\varphi$. However, our proposed notion of aggregated evidence for **REL** models is based on combining *all* the available evidence, and as a result in $Rel(M)$ the agent does have aggregated evidence for φ (i.e., $Rel(M), w \models \Box\varphi$). A concrete example of such a model is $M = \langle W, E_0, V\rangle$ with $W = \mathbb{N}$, $E_0 = \{\mathbb{N} \setminus \{2n+1\} \mid n \in \mathbb{N}\}$ and $V(p) = \{2n \mid n \in \mathbb{N}\}$. It is easy to verify that $M, 0 \not\models p$, while $Rel(M), 0 \models p$. *The proofs for all the results presented in this paper can be found in an extended version of it that will constitute the basis for a journal version. This extended version can be found in* [4].

Proposition 2. *Non-feasible **NEL** models need not be modally equivalent to their images under Rel. In particular, the left-to-right direction of Proposition 1 holds for non-feasible evidence models, but the right-to-left direction doesn't: there are non-feasible neighborhood models M s.t. $Rel(M), w \models \Box\psi$ but $M, w \not\models \Box\psi$.*

Having motivated our interest in ∩-models via their connection to neighborhood evidence logics, we now focus again on the static logics of ∩- and lex-models. Table 1 lists the axioms and rules in $\mathsf{L}_\cap$ and $\mathsf{L}_{\mathsf{lex}}$.

As stated in Theorem 1, these two systems completely axiomatize the logics of ∩ and lex models, respectively.

Theorem 1. *$\mathsf{L}_\cap$ and $\mathsf{L}_{\mathsf{lex}}$ are sound and strongly complete with respect to ∩-models and lex-models, respectively.*

Evidence dynamics for ∩-models. Having established the soundness and completeness of the static logics, we now turn to evidence dynamics, starting with ∩-models. In line with the work on NEL, we consider update, evidence addition and evidence upgrade actions for ∩-models. As the intersection rule is insensitive to the priority order, when we consider ∩-models, it is convenient to treat the models as if they came with a family of evidence orders $\mathscr{R}$ only, instead of an ordered family $\langle \mathscr{R}, \preceq\rangle$. Accordingly, hereafter we will write ∩-models as follows: $M = \langle W, \mathscr{R}, V, Ag_\cap\rangle$. Let us fix a ∩-model $M = \langle W, \mathscr{R}, V, Ag_\cap\rangle$, some proposition $P \subseteq W$ and some evidence order $R \in Pre(W)$.

Table 1. The systems $\mathsf{L}_\cap$ and $\mathsf{L}_{\mathsf{lex}}$

Axioms and inference rules	System(s)
All tautologies of propositional logic	both
$\mathsf{S5}$ axioms for $\forall$, $\mathsf{S4}$ axioms for $\Box$, axiom $\mathsf{4}$ for $\Box_0$	both
$\forall\varphi \to \Box_0\varphi$	both
$(\Box_0\varphi \wedge \forall\psi) \to \Box_0(\varphi \wedge \forall\psi)$	$\mathsf{L}_\cap$
$(\Box_0\varphi \wedge \forall\psi) \leftrightarrow \Box_0(\varphi \wedge \forall\psi)$	$\mathsf{L}_{\mathsf{lex}}$
$\Box_0\varphi \to \Box\varphi$	$\mathsf{L}_\cap$
Axioms T and N for $\Box_0$	$\mathsf{L}_{\mathsf{lex}}$
$\forall\varphi \to \Box\varphi$	$\mathsf{L}_{\mathsf{lex}}$
Modus ponens	both
Necessitation Rule for $\bullet \in \{\forall, \Box\}$: from φ infer $\bullet\varphi$	both
Monotonicity Rule for $\Box_0$: from $\varphi \to \psi$ infer $\Box_0\varphi \to \Box_0\psi$	both

Update. We first consider updates that involve learning a new fact P with absolute certainty. Upon learning P, the agent rules out all possible states that are incompatible with it. For REL models, this means keeping only the worlds in $[\![P]\!]_M$ and restricting each evidence order accordingly.

Definition 16 (Update). *The model* $M^{!P} = \langle W^{!P}, \mathscr{R}^{!P}, V^{!P}, Ag_\cap^{!P}\rangle$ *has* $W^{!P} := P$, $\mathscr{R}^{!P} := \{R \cap P^2 \mid R \in \mathscr{R}\}$, $Ag_\cap^{!P} := Ag_\cap$ *restricted to* P, *and for all* $p \in \mathsf{P}$, $V^{!P}(p) := V(p) \cap P$.

Evidence addition. Unlike update, which is standardly defined in terms of an incoming proposition $P \subseteq W$, our proposed notion of evidence addition for $\cap$-models involves accepting a new piece of *relational evidence* R from a trusted source. That is, relational evidence addition consists of adding a new piece of relational evidence $R \subseteq Pre(W)$ to the family $\mathscr{R}$.

Definition 17 (Evidence addition). *The model* $M^{+R} = \langle W^{+R}, \mathscr{R}^{+R}, V^{+R}, Ag_\cap^{+R}\rangle$ *has* $W^{+R} := W$, $\mathscr{R}^{+R} := \mathscr{R} \cup \{R\}$, $V^{+R} := V$ *and* $Ag_\cap^{+R} := Ag_\cap$.

Evidence upgrade. Finally, we consider an action of upgrade with a piece of relational evidence R. This upgrade action is based on the notion of *binary lexicographic merge* from Andréka et al. [1].

Definition 18 (Evidence upgrade). *The model* $M^{\Uparrow R} = \langle W^{\Uparrow R}, \mathscr{R}^{\Uparrow R}, V^{\Uparrow R}, Ag_\cap^{\Uparrow R}\rangle$ *has* $W^{\Uparrow R} := W$, $\mathscr{R}^{\Uparrow R} := \{R^< \cup (R \cap R') \mid R' \in \mathscr{R}\}$, $V^{\Uparrow R} := V$ *and* $Ag_\cap^{\Uparrow R} := Ag_\cap$.

Intuitively, this operation modifies each existing piece of evidence R' with R following the rule: "keep whatever R and R' agree on, and where they conflict, give priority to R". To encode syntactically the evidential actions described above, we present extensions of $\mathscr{L}$, obtained by adding to $\mathscr{L}$ dynamic modalities for update, evidence addition and evidence upgrade. The modalities for update will be standard, i.e., modalities of the form $[!\varphi]\psi$. The new formulas of the form $[!\varphi]\psi$ are used to express the statement: "ψ is true after φ is publicly announced".

Definition 19 **($\mathscr{L}^!$).** *The language $\mathscr{L}^!$ is defined recursively by:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box\varphi \mid \forall\varphi \mid [!\varphi]\varphi \quad (p \in \mathsf{P})$$

Satisfaction for formulas $[!\varphi]\psi \in \mathscr{L}^!$ is given by: $M, w \models [!\varphi]\psi$ iff $M, w \models \varphi$ implies $M^{![\![\varphi]\!]_M}, w \models \psi$. For the remaining actions, we extend $\mathscr{L}$ with dynamic modalities of the form $[+\pi]\psi$ for addition and $[\Uparrow \pi]\psi$ for upgrade, where the symbol π occurring inside the modality is an evidence program.

Definition 20 **($\mathscr{L}^\bullet$).** *Let $\bullet \in \{+, \Uparrow\}$. The language $\mathscr{L}^\bullet$ is defined by:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box\varphi \mid \forall\varphi \mid [\bullet\pi^*]\varphi \quad (p \in \mathsf{P})$$
$$\pi ::= A \mid ?\varphi \mid \pi \cup \pi \mid \pi;\pi \mid \pi^*$$

The new formulas of the form $[+\pi]\varphi$ are used to express the statement: "φ is true after the evidence order defined by π is added as a piece of evidence", while the $[\Uparrow \pi]\varphi$ are used to express: "φ is true after the existing evidence is upgraded with the relation defined by π". We extend the satisfaction relation $\models$ to cover formulas of the form $[\bullet\pi]\varphi$ as follows: for a formula $[\bullet\pi]\varphi \in \mathscr{L}^\bullet$, we have $M, w \models [\bullet\pi]\varphi$ iff $M^{\bullet[\![\pi]\!]_M}, w \models \varphi$.

The next natural step is to introduce proof systems for the languages $\mathscr{L}^!$, $\mathscr{L}^+$ and $\mathscr{L}^\Uparrow$ with respect to $\cap$-models. A standard approach to obtain soundness and completeness proofs is via a reductive analysis, appealing to *reduction axioms*. We refer to [11] for an extensive explanation of this technique. Taking this route, we obtained complete proof systems for the dynamic logics. The reduction axioms and the completeness proofs can be found in the Extended version of this paper (see Definitions 12, 13 and 21, Lemma 1 and Theorem 2 therein).

Theorem 2. *There exist proof systems for $\mathscr{L}^!$, $\mathscr{L}^+$ and $\mathscr{L}^\Uparrow$ that are sound and complete with respect to $\cap$-models.*

Evidence dynamics for lex-models. We now have a first look at the dynamics of evidence over lex models. In the REL setting, evidential actions can be seen as complex actions involving two possible transformations on the initial model: (i) modifying the stock of evidence, $\mathscr{R}$, perhaps by adding a new evidence relation R to it, or modifying the existing evidence with R; and (ii) updating the priority order, $\preceq$, e.g. to 'place' a new evidence item where it fits, according to its reliability. We may also have actions involving evidence, not about the world, but about evidence itself or its sources (sometimes called 'higher-order evidence' [7]), which trigger a reevaluation of the priority order without changing the stock of evidence (for instance, upon learning that a specific source is less reliable than we initially thought, we may want to lower the priority of the evidence provided by this source). To illustrate the type of actions that can be explored in this setting, here we study an action of *prioritized addition* over lex models. For the sake of generality, we describe this action over REL models.

Prioritized addition. Let $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag\rangle$ be a REL model and $R \in Pre(W)$ a piece of relational evidence. The prioritized addition of R adds R to the set of available evidence $\mathscr{R}$, giving the highest priority to the new evidence.

Definition 21 (Prioritized addition). *The model* $M^{\oplus R} = \langle W^{\oplus R}, \langle \mathscr{R}^{\oplus R}, \preceq^{\oplus R}\rangle, V^{\oplus R}, Ag^{\oplus R}\rangle$ *has* $W^{\oplus R} := W$, $\mathscr{R}^{\oplus R} := \mathscr{R} \cup \{R\}$, $V^{\oplus R} := V$, $Ag^{\oplus R} := Ag$ *and* $\preceq^{\oplus R} := \preceq \cup \{(R', R) \mid R' \in \mathscr{R}\}$.

To encode this action, we add formulas $[\oplus\pi]\varphi$, used to express the statement that φ is true after the prioritized addition of the evidence order defined by π.

Definition 22 ($\mathscr{L}^{\oplus}$). *The language* $\mathscr{L}^{\oplus}$ *is given by:*

$$\begin{aligned}
\varphi &::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box\varphi \mid \forall\varphi \mid [\oplus\pi^*]\varphi \quad (p \in \mathsf{P})\\
\pi &::= A \mid ?\varphi \mid \pi \cup \pi \mid \pi;\pi \mid \pi^*
\end{aligned}$$

Satisfaction for formulas $[\oplus\pi]\varphi \in \mathscr{L}^{\oplus}$ is given by: $M, w \models [\oplus\pi]\varphi$ iff $M^{\oplus [\![\pi]\!]_M}, w \models \varphi$. As we did with the dynamic extensions presented for actions in ∩-models, we wish to obtain a matching proof system for our dynamic language $\mathscr{L}^{\oplus}$. We do this via reduction axioms; the axioms and the completeness proof can be found in the Extended version of this paper (see Definition 29 and Theorem 3 there).

Theorem 3. *There exists a proof system for* $\mathscr{L}^{\oplus}$ *that is sound and complete with respect to* lex *models.*

3 The Logic of REL Models

In this section, we briefly study the logic of evidence and belief based on some abstract aggregator. That is, instead of fixing an aggregator, we are now interested in reasoning about the beliefs that an agent would form, based on her evidence, *irrespective* of the aggregator used. With respect to dynamics, we will focus on the action of *prioritized addition* introduced for lex-models, considering an *iterated* version of prioritized addition, defined with a (possibly empty) sequence of evidence orders $\boldsymbol{R} = \langle R_1, \ldots, R_n\rangle$ as input.

Definition 23 (Iterated prioritized addition). *Let* $M = \langle W, \langle \mathscr{R}, \preceq\rangle, V, Ag\rangle$ *be a REL model and* $\boldsymbol{R} = \langle R_1, \ldots, R_n\rangle$ *be a sequence of evidence orders. The model* $M^{\oplus R} = \langle W^{\oplus \boldsymbol{R}}, \langle \mathscr{R}^{\oplus \boldsymbol{R}}, \preceq^{\oplus \boldsymbol{R}}\rangle, V^{\oplus \boldsymbol{R}}, Ag^{\oplus \boldsymbol{R}}\rangle$ *has* $W^{\oplus \boldsymbol{R}} := W$, $\mathscr{R}^{\oplus \boldsymbol{R}} := \mathscr{R} \cup \{R_i \mid i \in \{1, \ldots n\}\}$, $V^{\oplus \boldsymbol{R}} := V$, $Ag^{\oplus \boldsymbol{R}} := Ag$ *and*

$$\begin{aligned}
\preceq^{\oplus \boldsymbol{R}} := \preceq &\cup \{(R, R_1) \mid R \in \mathscr{R}\} \cup \{(R, R_2) \mid R \in \mathscr{R} \cup \{R_1\}\}\\
&\cup \ldots\\
&\cup \{(R, R_n) \mid R \in \mathscr{R} \cup \{R_j \mid j \in \{1, \ldots, n-1\}\}\}
\end{aligned}$$

That is, first R_1 is added as the highest priority evidence, then R_2 is added as the highest priority evidence, on top of every other evidence (including R_1), and so on, up to R_n. When $\boldsymbol{R}$ has one element, we get the basic notion of addition.

Syntax and semantics. To pre-encode part of the dynamics of iterated prioritized addition, we will modify our basic language $\mathscr{L}$ with *conditional modalities* of the form $\Box^{\boldsymbol{\pi}}$, where $\boldsymbol{\pi}$ is a finite, possibly empty sequence of evidence programs $\pi_1, \ldots, \pi_n$. The intended interpretation of $\Box^{\boldsymbol{\pi}}\varphi$ is "the agent would have aggregated evidence for φ, after the iterated prioritized addition of $\boldsymbol{\pi}$".

Definition 24 **($\mathscr{L}_c$).** *The language $\mathscr{L}_c$ is defined as follows:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box^{\boldsymbol{\pi}}\varphi \mid \forall\varphi \quad (p \in \mathsf{P})$$
$$\pi ::= A \mid ?\varphi \mid \pi \cup \pi \mid \pi;\pi \mid \pi^*$$

where $\boldsymbol{\pi}$ is a (possibly empty) finite sequence of evidence programs (i.e. $*$-programs).

Notation 1. We abuse the notation for the truth map $[\![\cdot]\!]_M$ and write $[\![\boldsymbol{\pi}]\!]_M$ to denote $\langle [\![\pi_1]\!]_M, \ldots, [\![\pi_n]\!]_M \rangle$, where $\boldsymbol{\pi} = \langle \pi_1, \ldots, \pi_n \rangle$.

As we allow $\boldsymbol{\pi}$ to be empty, $\Box^{\boldsymbol{\pi}}$ reduces to the $\Box\varphi$ from $\mathscr{L}$ when $\boldsymbol{\pi}$ is the empty sequence, giving us a fully *static* sub-language. Satisfaction for formulas $\Box^{\boldsymbol{\pi}}\varphi \in \mathscr{L}_c$ is given by: $M, w \models \Box^{\boldsymbol{\pi}}\varphi$ iff $Ag(\langle \mathscr{R}^{\oplus[\![\boldsymbol{\pi}]\!]_M}, \preceq^{\oplus[\![\boldsymbol{\pi}]\!]_M} \rangle)[w] \subseteq [\![\varphi]\!]_M$. Next, we introduce a complete proof system for the language with conditional modalities (proof of completeness in the Extended Version).

Definition 25 **(L_c).** *The system L_c includes the same axioms and inference rules as $\mathsf{L}_{\mathsf{lex}}$, with axioms and inference rules for $\Box$ in $\mathsf{L}_{\mathsf{lex}}$ applying to $\Box^{\boldsymbol{\pi}}$ in L_c.*

Theorem 4. *L_c is sound and strongly complete with respect to REL models.*

Evidence dynamics for REL models. Having established the soundness and completeness of the static logic, we now turn to evidence dynamics, focusing on prioritized evidence addition. To encode prioritized addition, we add formulas of the form $[\oplus\boldsymbol{\pi}]\varphi$, used to express the statement that φ is true after the prioritized addition of the sequence of evidence orders defined by $\boldsymbol{\pi}$.

Definition 26 **($\mathscr{L}_c^{\oplus}$).** *The language $\mathscr{L}_c^{\oplus}$ is given by:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box^{\pi}\varphi \mid \forall\varphi \mid [\oplus\boldsymbol{\pi}]\varphi \quad (p \in \mathsf{P})$$
$$\pi ::= A \mid ?\varphi \mid \pi \cup \pi \mid \pi;\pi \mid \pi^*$$

where $\boldsymbol{\pi}$ is a (possibly empty) finite sequence of evidence programs (i.e. $*$-programs).

The satisfaction for these formulas is given by $M, w \models [\oplus\boldsymbol{\pi}]\varphi$ iff $M^{\oplus[\![\boldsymbol{\pi}]\!]_M}, w \models \varphi$. A complete system for $\mathscr{L}_c^{\oplus}$ can be found in the extended version of this paper (see Definition 36).

Theorem 5. *There is a proof system for $\mathscr{L}_c^{\oplus}$ that is complete w.r.t REL models.*

4 Conclusions and Future Work

We have presented evidence logics that use a novel, non-binary representation for evidence and consider reliability-sensitive forms of evidence aggregation. Here are a few avenues for future research. *Additional aggregators*: we studied two of them. An interesting extension to this work involves developing logics based on other well-known rules. *Additional actions*: in a setting with ordered evidence, evidence actions are complex transformations, both of the stock of evidence and the priority order. For the lexicographic case, we studied a form of prioritized addition. More general actions, e.g., transforming the priority order (re-evaluation of reliability) without affecting the stock of evidence, can be explored. *Probabilistic evidence*: we moved from the binary evidence case to the relational evidence case. *Probabilistic opinion pooling* [8] and *pure inductive logic* [13] study the aggregation of probability functions, but a dynamic-logic analysis is missing.

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