Chapter 3
Polynomial Rings

Locally nilpotent derivations are useful if rather elusive objects. Though we do not have them at all on "majority" of rings, when we have them, they are rather hard to find and it is even harder to find all of them or to give any qualitative statements. We do not know much even for polynomial rings.

Leonid Makar-Limanov, *Introduction* to [282]

This chapter investigates locally nilpotent derivations in the case B is a polynomial ring in a finite number of variables over a field k of characteristic zero. Equivalently, we are interested in the algebraic actions of $\mathbb{G}_a$ on $\mathbb{A}_k^n$.

3.1 Variables, Automorphisms, and Gradings

If $B = k^{[n]}$ for $n \geq 0$, then there exist $x_1, \ldots, x_n \in B$ such that $B = k[x_1, \ldots, x_n]$. Note that B cannot be generated over k by fewer than n elements Any such set $\mathbf{x} = \{x_1, \ldots, x_n\}$ is called a **system of variables** or a **coordinate system** for B over k. Any subset $\{x_1, \ldots, x_i\}$ is called a **partial system of variables** for B $(1 \leq i \leq n)$. A polynomial $f \in B$ is called a **variable** or **coordinate function** for B if and only if f belongs to some system of variables for B.

The group $\text{Aut}_k(B)$ of algebraic k-automorphisms of $B = k^{[n]}$ is called the **general affine group** or **affine Cremona group** in dimension n, and is denoted $GA_n(k)$. This group may be viewed as an infinite-dimensional algebraic group over k. See [239].

G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*,
Encyclopaedia of Mathematical Sciences 136,
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3.1.1 Linear Maps and Derivations

Let V be a vector space of finite dimension n over k. The symmetric algebra $S(V)$ is isomorphic to B as a k-algebra and is $\mathbb{N}$-graded: $S(V) = \bigoplus_{d\geq 0} S^d(V)$, where $S^d(V)$ is the vector space of **n-forms of degree d** in $S(V)$. In particular, $S^0(V) = k$ and $S^1(V) = V$. If $B = k[x_1, \ldots, x_n]$, then identifying V in $S(V)$ with the vector space $kx_1 \oplus \cdots \oplus kx_n$ in B gives an isomorphism $\alpha : S(V) \to B$.

A linear operator $L : V \to V$ induces both a k-algebra endomorphism $\phi_L : S(V) \to S(V)$ and a k-derivation $D_L : S(V) \to S(V)$. These, in turn, give $\alpha\phi_L\alpha^{-1} \in \text{End}_k(B)$ and $\alpha D_L \alpha^{-1} \in \text{Der}_k(B)$. Any $\phi \in \text{End}_k(B)$ arising in this way is a **linear endomorphism** of B, and any $D \in \text{Der}_k(B)$ arising in this way is a **linear derivation** of B. Given $D \in \text{Der}_k(B)$, D is **linearlizable** if D is conjugate to a linear derivation by some element of $GA_n(k)$.

Note that both ϕ_L and D_L are homogeneous functions of degree 0 relative to the $\mathbb{N}$-grading of $S(V)$. In addition, observe that, if $I : V \to V$ is the identity, then $\phi_I : B \to B$ is the identity, whereas D_I is the **Euler derivation**: $D_I(f) = df$ for $f \in S^d(V)$. We also have:

Proposition 3.1 *A linear derivation of B is locally finite. If $D \in \text{Der}_k(B)$ is a linear derivation induced by the linear operator $L : V \to V$, then D is locally nilpotent if and only if L is nilpotent.*

Proof Suppose that $D \in \text{Der}_k(B)$ is linear, where $D = \beta D_L \beta^{-1}$ for some isomorphism $\beta : S(V) \to B$ and some linear operator L on V. By *Lemma 1.5*, D_L is locally finite, and therefore D is locally finite. In addition, *Corollary 1.11* implies that D_L is locally nilpotent if and only if L is nilpotent. Therefore, D is locally nilpotent if and only if L is nilpotent. □

If $L \in GL(V)$, then $\alpha\phi_L\alpha^{-1} \in GA_n(k)$, and this gives an algebraic embedding $GL(V) \to GA_n(k)$. The image of $GL(V)$ is denoted by $GL_n(k)$, the **general linear group** of order n, and elements of $GL_n(k)$ are called **linear automorphisms** of B. Suppose that $G \subset GL(V)$ is an algebraic subgroup. An algebraic representation $\rho : G \to GA_n(k)$ is **linearizable** if and only if ρ factors through a representation $\gamma : GL(V) \to GA_n(k)$, i.e., $\rho = \gamma\iota$, where $\iota : G \to GL(V)$ is the inclusion.

Example 3.2 Let $G \subset GL_2(k)$ act on $V = k^2$. Then G acts on the symmetric algebra $S(V) = \oplus_{d\geq 0} S^d(V) = k^{[2]}$ and this action restricts to each homogeneous summand $V_d = S^d(V) = k^{d+1}$. If $S(V) = k[x, y]$, then V_d has basis $x^i y^{d-i}$, $0 \leq i \leq d$, and is called the **G-module of binary forms of degree d**.

3.1.2 Triangular and Tame Automorphisms

Given a coordinate system $B = k[x_1, \ldots, x_n]$, an automorphism $F \in GA_n(k)$ is given by $F = (F_1, \ldots, F_n)$, where $F_i = F(x_i) \in B$. The **triangular automorphisms** or **Jonquières automorphisms** are those of the form $F = (F_1, \ldots, F_n)$, where

$F_i \in k[x_1, \ldots, x_i]$.[1] The triangular automorphisms form a subgroup, denoted $BA_n(k)$, which is the generalization of the Borel subgroup in the theory of finite-dimensional representations. Note that the subgroup $BA_n(k)$ depends on the underlying coordinate system.

The **tame subgroup** of $GA_n(k)$ is the subgroup generated by $GL_n(k)$ and $BA_n(k)$. Its elements are called **tame automorphisms** relative to the coordinate system $\mathbf{x}$. It is known that for $n \leq 2$, every element of $GA_n(k)$ is tame (see *Chap. 4*), whereas non-tame automorphisms exist in $GA_3(k)$. See [382, 383].

As to gradings of polynomial rings, we are mainly interested in $\mathbb{Z}^m$-gradings for some $m \geq 1$. In particular, suppose $B = k[x_1, \ldots, x_n]$. Given a homomorphism $\alpha : \mathbb{Z}^n \to \mathbb{Z}^m$ for $m \geq 1$, define the function $\deg_\alpha$ on the set of monomials by $\deg_\alpha(x_1^{e_1} \cdots x_n^{e_n}) = \alpha(e_1, \ldots, e_n)$. This defines a $\mathbb{Z}^m$-grading $B = \bigoplus_{i \in \mathbb{Z}^m} B_i$. For example, if $\alpha : \mathbb{Z}^n \to \mathbb{Z}$ is defined by $\alpha(e_1, \ldots, e_n) = \sum e_i$, then the induced grading is called the **standard $\mathbb{Z}$-grading** of B, relative to $\mathbf{x}$. Likewise, if $\alpha(e_1, \ldots, e_n) = e_1$, then B is graded according to its usual degree relative to x_1.

Remark 3.3 By considering the Jordan normal form of its defining matrix, we see that any linear $\mathbb{G}_a$-action on $\mathbb{A}^n$ is conjugate to a linear triangular $\mathbb{G}_a$-action. In addition, it is well-known that an action of a linear algebraic group G on $\mathbb{A}^n$ can be extended to a linear action on some larger affine space $\mathbb{A}^N$. Therefore, any algebraic $\mathbb{G}_a$-action on $\mathbb{A}^n$ extends to a linear triangular $\mathbb{G}_a$-action on some larger affine space $\mathbb{A}^N$.

3.2 Derivations of Polynomial Rings

3.2.1 Definitions

Let $B = k^{[n]}$. Given $D \in \mathrm{Der}_k(B)$, define the **corank** of D to be the maximum integer i such that there exists a partial system of variables $\{x_1, \ldots, x_i\}$ of B contained in $\ker D$. In other words, the corank of D is the maximal number of variables within the same system annihilated by D. Denote the corank of D by $\mathrm{corank}(D)$. The **rank** of D is $\mathrm{rank}(D) = n - \mathrm{corank}(D)$. By definition, the rank and corank are invariants of D, in the sense that these values do not change after conjugation by an element of $GA_n(k)$. The rank and corank were first defined in [159].

[1] Ernest Jean Philippe Fauque de Jonquières (1820–1901) was a career officer in the French navy, achieving the rank of vice-admiral in 1879. He learned advanced mathematics by reading works of Poncelet, Chasles, and other geometers. In 1859, he introduced the planar transformations $(x, y) \to \left(x, \frac{a(x)y+b(x)}{c(x)y+d(x)}\right)$, where $ad - bc \neq 0$. These were later studied by Cremona.

A k-derivation D of B is said to be **rigid** when the following condition holds: If $\text{corank}(D) = i$, and if $\{x_1, \ldots, x_i\}$ and $\{y_1, \ldots, y_i\}$ are partial systems of variables of B contained in $\ker D$, then $k[x_1, \ldots, x_i] = k[y_1, \ldots, y_i]$. This definition is due to Daigle [69].

We say that $D \in \text{Der}_k(B)$ is **quasi-linear** if and only if there exists a coordinate system $\mathbf{x} = (x_1, \ldots, x_n)$ and matrix $M \in \mathcal{M}_n(\ker D)$ such that $D\mathbf{x} = \mathbf{x}M$, where $D\mathbf{x} = (Dx_1, \ldots, Dx_n)$. Note that D is locally nilpotent if and only if M is a nilpotent matrix, since $D^i\mathbf{x} = \mathbf{x}M^i$. A family of quasi-linear locally nilpotent derivations is discussed in *Sect. 3.9.3*.

$D \in \text{Der}_k(B)$ is a **triangular derivation** of B if and only if $Dx_i \in k[x_1, \ldots, x_{i-1}]$ for $i = 2, \ldots, n$ and $Dx_1 \in k$. Note that triangularity depends on the choice of coordinates on B. By a **triangularizable derivation** of B we mean any $D \in \text{Der}_k(B)$ which is triangular relative to some system of coordinates on B, i.e., conjugate to a triangular derivation. As we will see, the triangular derivations form a large and important class of locally nilpotent derivations of polynomial rings. Several of the main examples and open questions discussed below involve triangular derivations.

For polynomial rings, other natural categories of derivations to study include the following: Let $D \in \text{Der}_k(B)$ for $B = k[x_1, \ldots, x_n] = k^{[n]}$.

1. D is a **monomial derivation** if each image Dx_i is a monomial in $x_1, \ldots, x_n$.
2. D is an **elementary derivation** if, for some j with $1 \leq j \leq n$, $Dx_i = 0$ for $1 \leq i \leq j$, and $Dx_i \in k[x_1, \ldots, x_j]$ if $j + 1 \leq i \leq n$.
3. D is a **nice derivation**[2] if $D^2 x_i = 0$ for each i.

These definitions depend on the underlying coordinate system. Note that any nice derivation is locally nilpotent, and that any elementary derivation is both triangular and nice. We also have:

Proposition 3.4 ([243]) *For the polynomial ring $B = k[x_1, \ldots, x_n] = k^{[n]}$, every monomial derivation $D \in \text{LND}(B)$ is triangular relative to some ordering of $x_1, \ldots, x_n$.*

Proof We may assume, with no loss of generality, that:

$$\deg_D(x_1) \leq \deg_D(x_2) \leq \cdots \leq \deg_D(x_n)$$

Given i, write $Dx_i = a x_1^{e_1} \cdots x_n^{e_n} \neq 0$ for $a \in k$ and $e_i \geq 0$. If $Dx_i \neq 0$, then $\deg_D(x_i) - 1 = \sum_{j=1}^{n} e_j \deg_D(x_j)$. Due to the ordering above, this is only possible if $e_j = 0$ for $j \geq i$. Therefore, $Dx_i \in k[x_1, \ldots, x_{i-1}]$ for every i. □

We will see that triangular monomial derivations provide us with important examples.

[2] Van den Essen gives a more exclusive definition of a nice derivation. See [142], 7.3.12.

3.2.2 Partial Derivatives

To each system of variables $\mathbf{x} = (x_1, \ldots, x_n)$ on the polynomial ring $B = k[\mathbf{x}]$ we associate a corresponding system of **partial derivatives** ∂_{x_i} relative to $\mathbf{x}$, $1 \le i \le n$. In particular, $\partial_{x_i} \in \mathrm{Der}_k(B)$ is defined by $\partial_{x_i}(x_j) = \delta_{ij}$ (Kronecker delta). Another common notation for ∂_{x_i} is $\frac{\partial}{\partial x_i}$. Given $f \in B$, let $f_{x_i} = \partial_{x_i} f$.

Note that ∂_{x_i} is locally nilpotent for each i, since $B = A[x_i]$ for $A = k[x_1, \ldots, \widehat{x_i}, \ldots, x_n]$, and $\partial_{x_i}(A) = 0$. Note also that the meaning of ∂_{x_i} depends on the entire system of variables to which x_i belongs. For example, in the two-dimensional case, $k[x, y] = k[x, y+x]$, and $\partial_x(y+x) = 1$ relative to (x, y), whereas $\partial_x(y+x) = 0$ relative to $(x, y + x)$. In general, we will say $D \in \mathrm{LND}(B)$ is a **partial derivative** if and only if there exists a system of coordinates $(y_1, \ldots, y_n)$ on B relative to which $D = \partial_{y_1}$.

It is easy to see that, as a B-module, $\mathrm{Der}_k(B)$ is freely generated by $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$, and that this is a basis of commuting derivations. In particular, given $D \in \mathrm{Der}_k(B)$:

$$D = \sum_{1 \le i \le n} D(x_i)\partial_{x_i}$$

To verify this expression for D, it suffices to check equality for each x_i, and this is obvious. Note that the rank of D is the minimal number of partial derivatives needed to express D in this form. Thus, elements of $\mathrm{Der}_k(B)$ having rank one are precisely those of the form $f\partial_{x_1}$ for $f \in B$, relative to some system of coordinates $(x_1, \ldots, x_n)$ for B.

Example 3.5 On the polynomial ring $B = k[x_1, \ldots, x_n] = k^{[n]}$, define the derivation:

$$D = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$$

If $N = \prod_{i=1}^{n-1} i^i$, then

$$W_D(x_1^{n-1}, \ldots, x_n^{n-1}) = N \cdot \det \begin{pmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & \cdots & x_n^{n-2} \\ \vdots & & \vdots \\ x_1 & \cdots & x_n \\ 1 & \cdots & 1 \end{pmatrix} = N \cdot \prod_{i>j} (x_i - x_j) \ ,$$

i.e., the Vandermonde determinant of $x_1, \ldots, x_n$ may be realized as a Wronskian. □

The partial derivatives ∂_{x_i} also extend (uniquely) to the field $K = k(x_1, \ldots, x_n)$ by the quotient rule, although they are no longer locally nilpotent on all K:

$$\mathrm{Nil}(\partial_{x_i}) = k(x_1, \ldots, \widehat{x_i}, \ldots, x_n)[x_i] \ .$$

In this case, we see that $\mathrm{Der}_k(K)$ is a vector space over K of dimension n, with basis $\partial_{x_1}, \ldots, \partial_{x_n}$. More generally:

Proposition 3.6 *If L is a field of finite transcendence degree n over k, then* $\mathrm{Der}_k(L)$ *is a vector space over L of dimension n.*

Proof Suppose $k \subset k(x_1, \ldots, x_n) \subset L$ for algebraically independent x_i, and set $K = k(x_1, \ldots, x_n)$. Suppose $D \in \mathrm{Der}_k(L)$ and $t \in L$ are given, and let $P \in K[T] = K^{[1]}$ be the minimal polynomial of t over k. Suppose $P(T) = \sum_i a_i T^i$ for $a_i \in K$. Then $0 = D(P(t)) = P'(t)Dt + \sum_i D(a_i)t^i$. Since $P'(t) \neq 0$, this implies

$$Dt = -(P'(t))^{-1} \sum_i D(a_i)t^i$$

meaning that D is completely determined by its values on K. Conversely, this same formula shows that every $D \in \mathrm{Der}_k(K)$ can be uniquely extended to L.

In particular, the partial derivatives ∂_{x_i} extend uniquely to L. If $f \in K$ and $D \in \mathrm{Der}_k(L)$, then $Df = \partial_{x_1} f Dx_1 + \cdots + \partial_{x_n} f Dx_n$. We conclude that

$$\mathrm{Der}_k(L) = \mathrm{span}_L\{\partial_{x_1}, \ldots, \partial_{x_n}\} .$$

If $a_1\partial_{x_1} + \cdots a_n\partial_{x_n} = 0$ for $a_i \in L$, then evaluation at x_i shows that $a_i = 0$. Therefore, the partial derivatives are linearly independent over L, and the dimension of $\mathrm{Der}_k(L)$ equals n. □

Proposition 3.7 (Multivariate Chain Rule) *Suppose $D \in \mathrm{Der}_k(K)$ for $K = k(x_1, \ldots, x_n)$, and $f_1, \ldots, f_m \in K$. Then for any $g \in k(y_1, \ldots, y_m) = k^{(m)}$:*

$$D\big(g(f_1, , , f_m)\big) = \frac{\partial g}{\partial y_1}(f_1, \ldots, f_m) \cdot Df_1 + \cdots + \frac{\partial g}{\partial y_m}(f_1, \ldots, f_m) \cdot Df_m$$

Proof By the product rule, it suffices to assume $g \in k[y_1, \ldots, y_m]$. In addition, by linearity, it will suffice to show the formula in the case g is a monomial: $g = y_1^{e_1} \cdots y_m^{e_m}$ for $e_1, \ldots, e_m \in \mathbb{N}$.

From the product rule and the univariate chain rule, we have that:

$$\begin{aligned}\frac{\partial}{\partial x_j}(f_1^{e_1} \cdots f_m^{e_m}) &= \sum_i f_1^{e_1} \cdots \widehat{f_i^{e_i}} \cdots f_m^{e_m} \cdot \partial_{x_j}(f_i^{e_i}) \\ &= \sum_i e_i f_1^{e_1} \cdots f_i^{e_i - 1} \cdots f_m^{e_m} \cdot (f_i)_{x_j}\end{aligned}$$

Since $D = Dx_1\partial_{x_1} + \cdots + Dx_n\partial_{x_n}$, we have:

$$\begin{aligned}
D(f_1^{e_1}\cdots f_m^{e_m}) &= \sum_j \partial_{x_j}(f_1^{e_1}\cdots f_m^{e_m})\cdot Dx_j \\
&= \sum_j\sum_i (f_i)_{x_j}\cdot e_i f_1^{e_1}\cdots f_i^{e_i-1}\cdots f_m^{e_m}\cdot Dx_j \\
&= \sum_i\sum_j (f_i)_{x_j}\cdot e_i f_1^{e_1}\cdots f_i^{e_i-1}\cdots f_m^{e_m}\cdot Dx_j \\
&= \sum_i e_i f_1^{e_1}\cdots f_i^{e_i-1}\cdots f_m^{e_m}\sum_j (f_i)_{x_j}\cdot Dx_j \\
&= \sum_i e_i f_1^{e_1}\cdots f_i^{e_i-1}\cdots f_m^{e_m}\cdot Df_i \\
&= \sum_i \frac{\partial g}{\partial y_i}(f_1^{e_1}\cdots f_m^{e_m})\cdot Df_i
\end{aligned}$$

□

The use of partial derivatives also allows us to describe homogeneous decompositions of derivations relative to G-gradings of $B = k[x_1,\ldots,x_n] = k^{[n]}$.

Proposition 3.8 (See Prop. 5.1.14 of [142]) *Let G be an abelian group and $B = \bigoplus_{g\in G} B_g$ a G-grading such that x_i is G-homogeneous for $1 \le i \le n$. Every nonzero $D \in \mathrm{Der}_k(B)$ admits a unique decomposition $D = \sum_{g\in G} D_g$, where $D_g \in \mathrm{Der}_k(B)$ is G-homogeneous of degree g and $D_g = 0$ for all but finitely many $g \in G$.*

Proof There exist $f_1,\ldots,f_n \in B$ such that $D = \sum f_i\partial_{x_i}$. Since each x_i is G-homogeneous, each partial derivative ∂_{x_i} is a G-homogeneous derivation of B. Each coefficient function f_i admits a decomposition into G-homogeneous summands; suppose $f_i = \sum_{g\in G}(f_i)_g$. Then each summand $f_i\partial_{x_i}$ can be decomposed as a finite sum of G-homogeneous derivations, namely, $f_i\partial x_i = \sum_{g\in G}(f_i)_g\partial_{x_i}$. Therefore, $D = \sum_{i,g}(f_i)_g\partial_{x_i}$, and by gathering terms of the same degree, the desired result follows. □

Example 3.9 Let $G = \mathbb{Z}_2$ and define a G-grading on $\mathbb{C}[x,y,z] = \mathbb{C}^{[3]}$ by declaring that x, y, z are G-homogeneous with $\deg_G x = \deg_G z = 0$ and $\deg_G y = 1$. Then $\partial_x, \partial_y, \partial_z$ are G-homogeneous with $\deg_G \partial_x = \deg_G \partial_z = 0$ and $\deg_G \partial_y = 1$. Define $D \in \mathrm{LND}(\mathbb{C}[x,y,z])$ by:

$$D = (-3z^2)\partial_x + (3iz^2)\partial_y + 2(x - iy)\partial_z$$

Then $D = D_0 + D_1$ for $D_0 = (-3z^2)\partial_x + 2(x-iy)\partial_z$ and $D_1 = (3iz^2)\partial_y$. Note that, if $f = x^2 + y^2 + z^3$, then $\deg_G f = 0$ and $Df = 0$.

The preceding example can be used to show that *Proposition 3.8* does not generalize to affine rings: Let $B = \mathbb{C}[x,y,z]/(f)$ and let $\delta \in \mathrm{LND}(B)$ be the quotient derivation defined by D. In particular, $\delta \neq 0$. Since f is G-homogeneous, B inherits a non-trivial G-grading. However, it is shown in [83], Prop. 6.5, that $\Delta = 0$ for any G-homogeneous $\Delta \in \mathrm{LND}(B)$.

On the other hand, recall that when the group G is totally ordered and B is a G-graded affine k-domain, then any nonzero $D \in \mathrm{LND}(B)$ induces a nonzero G-homogeneous element of $\mathrm{LND}(B)$; see *Sect. 1.1.5*.

3.2.3 Jacobian Derivations

Let $B = k[x_1,\ldots,x_n] = k^{[n]}$. The **jacobian matrix** of $f_1,\ldots,f_m \in B$ is the $m \times n$ matrix of partial derivatives:

$$\mathcal{J}(f_1,\ldots,f_m) := \frac{\partial(f_1,\ldots,f_m)}{\partial(x_1,\ldots,x_n)} = \big((f_i)_{x_j}\big)$$

Note that the jacobian matrix depends on the underlying system of coordinates. When $m = n$, the **jacobian determinant** of $f_1,\ldots,f_n \in B$ is $\det \mathcal{J}(f_1,\ldots,f_n) \in B$.

Suppose $k[y_1,\ldots,y_m] = k^{[m]}$, and let $F : k[y_1,\ldots,y_m] \to k[x_1,\ldots,x_n]$ be a k-algebra homomorphism. Then the jacobian matrix of F is $\mathcal{J}(F) = \mathcal{J}(f_1,\ldots,f_m)$, where $f_i = F(y_i)$, and the jacobian determinant of F is $\det \mathcal{J}(F)$.[3] In addition, suppose $A = (a_{ij})$ is a matrix with entries a_{ij} in $k[y_1,\ldots,y_m]$. Then $F(A)$ denotes the matrix $\big(F(a_{ij})\big)$ with entries in $k[x_1,\ldots,x_m]$.

Given k-algebra homomorphisms

$$k[z_1,\ldots,z_l] \xrightarrow{G} k[y_1,\ldots,y_m] \xrightarrow{F} k[x_1,\ldots,x_n]$$

the **chain rule for jacobian matrices** is

$$\mathcal{J}(F \circ G) = F(\mathcal{J}(G)) \cdot \mathcal{J}(F)$$

where $(\cdot)$ on the right denotes matrix multiplication. This follows from the multivariate chain rule above. Note that if $\mathcal{J}(G)$ is a square matrix, then we have:

$$\det F\,(\mathcal{J}(G)) = F\,(\det \mathcal{J}(G))$$

Observe that the standard properties of determinants imply that:

det $\mathcal{J}$ is a k-derivation of B in each one of its arguments.

[3] Some authors use DF to denote the jacobian matrix of F, but we prefer to reserve D for derivations.

In particular, suppose $f_1, \ldots, f_{n-1} \in B$ are given, and set $\mathbf{f} = (f_1, \ldots, f_{n-1})$. Then $\mathbf{f}$ defines $\Delta_{\mathbf{f}} \in \mathrm{Der}_k(B)$ via:

$$\Delta_{\mathbf{f}}(g) := \det \mathcal{J}(f_1, \ldots, f_{n-1}, g) \quad (g \in B)$$

$\Delta_{\mathbf{f}}$ is called the **jacobian derivation** of B determined by $\mathbf{f}$.

Observe that the definitions of jacobian matrices and jacobian derivations also extend to the rational function field $K = k(x_1, \ldots, x_n)$.

If $F = (f_1, \ldots, f_n)$ is a system of variables for B, then:

$$\det \mathcal{J}(F) = \det \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)} = \Delta_{\mathbf{f}}(f_n) \in k^*$$

This is easily seen from the chain rule: By definition, F admits a polynomial inverse F^{-1}, and $I = FF^{-1}$ implies that

$$1 = \det\big(F(\mathcal{J}(F^{-1})) \cdot \mathcal{J}(F)\big) = \det F(\mathcal{J}(F^{-1})) \det \mathcal{J}(F)$$

meaning $\det \mathcal{J}(F)$ is a unit of B.

In the other direction lurks the famous **Jacobian Conjecture**, which can be formulated in the language of derivations: Suppose $\mathbf{f} = (f_1, \ldots, f_{n-1})$ for $f_i \in B$.

> If $\Delta_{\mathbf{f}}$ has a slice s, then $k[f_1, \ldots, f_{n-1}, s] = B$. Equivalently, if $\Delta_{\mathbf{f}}$ has a slice, then $\Delta_{\mathbf{f}}$ is locally nilpotent and $\ker \Delta_{\mathbf{f}} = k[f_1, \ldots, f_{n-1}]$.

See van den Essen [142], Chap. 2, for further details about the Jacobian Conjecture.

Let $B = \bigoplus_{i \in \mathbb{Z}} B_i$ be the standard $\mathbb{Z}$-grading relative to $(x_1, \ldots, x_n)$. Given a system of variables $F = (f_1, \ldots, f_n)$ for B, write $F = \sum_{i \in \mathbb{Z}} F_i$, where $F_i = ((f_1)_i, \ldots, (f_n)_i)$. It is easy to check that $\det \mathcal{J}(F) = \det \mathcal{J}(F_1) \in k^*$. It follows that F_1 is a linear system of variables for B. We have thus shown:

$$F \in GA_n(k) \quad \Rightarrow \quad F_1 \in GL_n(k) \tag{3.1}$$

Following are several lemmas about jacobian derivations, which will be used to prove certain properties of locally nilpotent derivations of polynomial rings.

Lemma 3.10 *Suppose $K = k(x_1, \ldots, x_n) = k^{(n)}$. Given $f_1, \ldots, f_{n-1} \in K$, set $\mathbf{f} = (f_1, \ldots, f_{n-1})$ and consider $\Delta_{\mathbf{f}} \in \mathrm{Der}_k(K)$.*

(a) *$\Delta_{\mathbf{f}} = 0$ if and only if $f_1, \ldots, f_{n-1}$ are algebraically dependent.*
(b) *If $\Delta_{\mathbf{f}} \neq 0$, then $\ker \Delta_{\mathbf{f}}$ is the algebraic closure of $k(f_1, \ldots, f_{n-1})$ in K.*
(c) *For any $g \in K$, $\Delta_{\mathbf{f}}(g) = 0$ if and only if $f_1, \ldots, f_{n-1}, g$ are algebraically dependent.*

Proof (Following [276]) To prove part (a), suppose $f_1, \ldots, f_{n-1}$ are algebraically dependent. Let $P(t)$ be a polynomial with coefficients in the field $k(f_2, \ldots, f_{n-1})$ of minimal degree such that $P(f_1) = 0$. Then:

$$0 = \Delta_{(P(f_1), f_2, \ldots, f_{n-1})} = P'(f_1)\Delta_{(f_1, f_2, \ldots, f_{n-1})} = P'(f_1)\Delta_{\mathbf{f}}$$

By minimality of degree, $P'(f_1) \neq 0$, so $\Delta_{\mathbf{f}} = 0$.

Conversely, suppose $f_1, \ldots, f_{n-1}$ are algebraically independent, and choose $f_n \in K$ transcendental over $k(f_1, \ldots, f_{n-1})$. Then for each i, x_i is algebraic over $k(f_1, \ldots, f_n)$, and there exists $P_i \in k[y_1, \ldots, y_{n+1}] = k^{[n+1]}$ such that $P_i(f_1, \ldots, f_n, x_i) = 0$. Now $\partial P_i/\partial y_{n+1} \neq 0$, since otherwise P_i gives a relation of algebraic dependence for $f_1, \ldots, f_n$. We may assume the degree of P_i is minimal in y_{n+1}, so that $\partial P_i/\partial y_{n+1}$ is nonzero when evaluated at $(f_1, \ldots, f_n, x_i)$.

By the chain rule, for each i and each j,

$$0 = \partial_{x_j} P_i(f_1, \ldots, f_n, x_i) = \sum_{1 \le s \le n} (P_i)_s (f_s)_{x_j} + (P_i)_{n+1}(x_i)_{x_j}$$

where $(P_i)_s$ denotes $\frac{\partial P_i}{\partial y_s}(f_1, \ldots, f_n, x_i)$. In matrix form, this becomes

$$0 = \begin{pmatrix} (P_i(f_1, \ldots, f_n, x_i))_{x_1} \\ \vdots \\ (P_i(f_1, \ldots, f_n, x_i))_{x_n} \end{pmatrix} = M \begin{pmatrix} (P_i)_1 \\ \vdots \\ (P_i)_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ (P_i)_{n+1} \\ \vdots \\ 0 \end{pmatrix}$$

where $M = \mathcal{J}(f_1, \ldots, f_n)$. Let $e_i = (0, \ldots, 1, \ldots 0) \in K^n$ be the standard basis vectors $(1 \le i \le n)$. The image of M as a linear operator on K^n is spanned by $(P_1)_{n+1}e_1, \ldots, (P_n)_{n+1}e_n$, and since $(P_i)_{n+1} \neq 0$ for each i, we conclude that M is surjective. Therefore, $\det M = \Delta_{\mathbf{f}}(f_n) \neq 0$. So part (a) is proved.

To prove (b), note first that, under the hypothesis $\Delta_{\mathbf{f}} \neq 0$, part (a) implies $f_1, \ldots, f_{n-1}$ are algebraically independent. This means that the transcendence degree of $k(f_1, \ldots, f_{n-1})$ equals $n - 1$. Since $k(f_1, \ldots, f_{n-1}) \subset \ker \Delta_{\mathbf{f}}$, we have that $\ker \Delta_{\mathbf{f}}$ is the algebraic closure of $k(f_1, \ldots, f_{n-1})$ in K.

To prove (c), suppose first that $f_1, \ldots, f_{n-1}, g$ are algebraically independent. Then $f_1, \ldots, f_{n-1}$ are algebraically independent, and $\ker \Delta_{\mathbf{f}}$ is an algebraic extension of $k(f_1, \ldots, f_{n-1})$. Since g is transcendental over $k(f_1, \ldots, f_{n-1})$, it is also transcendental over $\ker \Delta_{\mathbf{f}}$. Therefore, $\Delta_{\mathbf{f}}(g) \neq 0$.

Conversely, suppose that $f_1, \ldots, f_{n-1}, g$ are algebraically dependent. If $f_1, \ldots, f_{n-1}$ are algebraically independent, the same argument used above shows that $g \in \ker \Delta_{\mathbf{f}}$. And if $f_1, \ldots, f_{n-1}$ are algebraically dependent, then $\Delta_{\mathbf{f}}$ is the zero derivation, by part (a). □

Lemma 3.11 (Lemma 6 of [276]) *Suppose* $K = k(x_1,\ldots,x_n) = k^{(n)}$ *and* $D \in \mathrm{Der}_k(K)$ *has* $\mathrm{tr.deg}_k(\ker D) = n-1$. *Then for any set* $\mathbf{f} = (f_1,\ldots,f_{n-1})$ *of algebraically independent elements of* $\ker D$, *there exists* $a \in K$ *such that* $D = a\Delta_{\mathbf{f}}$.

Proof First, $\ker D = \ker \Delta_{\mathbf{f}}$, since each is equal to the algebraic closure of $k(f_1,\ldots,f_{n-1})$ in K. Choose $g \in K$ so that $Dg \neq 0$. Define $a = Dg(\Delta_{\mathbf{f}} g)^{-1}$. Then $D = a\Delta_{\mathbf{f}}$ when restricted to the subfield $k(f_1,\ldots,f_{n-1},g)$. Since $Dg \neq 0$, g is transcendental over $\ker D$, hence also over $k(f_1,\ldots,f_{n-1})$. Thus, K is an algebraic extension of $k(f_1,\ldots,f_{n-1},g)$. By *Proposition 1.14* we conclude that $D = a\Delta_{\mathbf{f}}$ on all of K. □

Lemma 3.12 (Lemma 7 of [276]) *For* $n \geq 2$, *let* $K = k(x_1,\ldots,x_n) = k^{(n)}$. *Given* $f_1,\ldots,f_{n-1} \in K$ *algebraically independent, set* $\mathbf{f} = (f_1,\ldots,f_{n-1})$. *If* $\mathbf{g} = (g_1,\ldots,g_{n-1})$ *for* $g_i \in \ker\Delta_{\mathbf{f}}$, *then there exists* $a \in \ker\Delta_{\mathbf{f}}$ *such that* $\Delta_{\mathbf{g}} = a\Delta_{\mathbf{f}}$.

Proof If $\Delta_{\mathbf{g}} = 0$, we can take $a = 0$. So assume $\Delta_{\mathbf{g}} \neq 0$, meaning that $g_1,\ldots,g_{n-1}$ are algebraically independent. In particular, $g_i \notin k$ for each i.

Since $\mathrm{tr.deg.}_k\ker\Delta_{\mathbf{f}} = n-1$, the elements $f_1,\ldots,f_{n-1},g_1$ are algebraically dependent. Let $P \in k[T_1,\ldots,T_n] = k^{[n]}$ be such that $P(\mathbf{f},g_1) = 0$. The notation P_i will denote the partial derivative $\partial P/\partial T_i$. Then we may assume that $P_n(\mathbf{f},g_1) \neq 0$; otherwise replace P by P_n. Likewise, by re-ordering the f_i if necessary, we may assume that $P_1(\mathbf{f},g_1) \neq 0$. It follows that:

$$
\begin{aligned}
0 &= \Delta_{(P(\mathbf{f},g_1),f_2,\ldots,f_{n-1})} \\
&= \sum_{1\leq i\leq n-1} P_i(\mathbf{f},g_1)\Delta_{(f_i,f_2,\ldots,f_{n-1})} + P_n(\mathbf{f},g_1)\Delta_{(g_1,f_2,\ldots,f_{n-1})} \\
&= P_1(\mathbf{f},g_1)\Delta_{(f_1,f_2,\ldots,f_{n-1})} + P_n(\mathbf{f},g_1)\Delta_{(g_1,f_2,\ldots,f_{n-1})}
\end{aligned}
$$

Thus, $\Delta_{(g_1,f_2,\ldots,f_{n-1})} = a\Delta_{\mathbf{f}}$ for some nonzero $a \in \ker\Delta_{\mathbf{f}}$.

If $n = 2$ we are done. Otherwise $n \geq 3$, and we may assume inductively that for some i with $1 \leq i \leq n-2$ we have

$$\Delta_{(g_1,\ldots,g_i,f_{i+1}\cdots f_{n-1})} = b\Delta_{\mathbf{f}}$$

for some nonzero $b \in \ker\Delta_{\mathbf{f}}$. Then $g_1,\ldots,g_i,f_{i+1}\ldots,f_{n-1}$ are algebraically independent, since the derivation they define is nonzero. Choose $Q \in k[T_1,\ldots,T_n]$ with $Q(g_1,\ldots,g_i,f_{i+1},\ldots,f_{n-1},g_{i+1}) = 0$, noting that $Q_n \neq 0$ (otherwise Q is a dependence relation for $g_1,\ldots,g_i,f_{i+1},\ldots,f_{n-1}$). By re-ordering the f_i if necessary, we may assume that $Q_{i+1}(g_1,\ldots,g_i,f_{i+1},\ldots,f_{n-1},g_{i+1}) \neq 0$. As above, we have

$$
\begin{aligned}
0 &= \Delta_{(g_1,\ldots,g_i,Q(*),f_{i+2},\ldots,f_{n-1})} \\
&= Q_{i+1}(*)\Delta_{(g_1,\ldots,g_i,f_{i+1},\ldots,f_{n-1})} + Q_n(*)\Delta_{(g_1,\ldots,g_i,g_{i+1},f_{n+2},\ldots,f_{n-1})} \\
&= Q_{i+1}(*)\cdot b\Delta_{\mathbf{f}} + Q_n(*)\Delta_{(g_1,\ldots,g_i,g_{i+1},f_{n+2},\ldots,f_{n-1})}
\end{aligned}
$$

where $(*)$ denotes the input $(g_1,\ldots,g_i,f_{i+1},\ldots,f_{n-1},g_{i+1})$. Therefore,

$$\Delta_{(g_1,\ldots,g_{i+1}f_{i+2}\cdots,f_{n-1})} = c\Delta_{\mathbf{f}}$$

for some nonzero $c \in \ker \Delta_{\mathbf{f}}$. By induction, the proof is complete. □

If $K = k(x_1,\ldots,x_n) = k^{(n)}$ and $D \in \mathrm{Der}_k(K)$, define the **divergence** of D by:

$$\mathrm{div}(D) = \sum_i \partial_{x_i}(Dx_i)$$

Lemma 3.13 *If $\Delta_{\mathbf{f}}$ is a jacobian derivation of $k^{(n)}$, then* $\mathrm{div}(\Delta_{\mathbf{f}}) = 0$.

Proof Given x_i, *Proposition 2.60* implies that:

$$\partial_{x_i}\big(\Delta_{\mathbf{f}}(x_i)\big) = \sum_{j=1}^{n} \Delta_{(f_1,\ldots,(f_j)_{x_i},\ldots,f_{n-1})}(x_i)$$

Therefore:

$$\mathrm{div}(\Delta_{\mathbf{f}}) = \sum_{1\le i,j\le n} \Delta_{(f_1,\ldots,(f_j)_{x_i},\ldots,f_{n-1})}(x_i)$$

Expanding these determinants, we see that

$$\mathrm{div}(\Delta_{\mathbf{f}}) = \sum_{\sigma\in S_n} \mathrm{sign}(\sigma)(f_1)_{y_1}(f_2)_{y_2}\cdots(f_j)_{y_jy_n}\cdots(f_{n-1})_{y_{n-1}}$$

where $\sigma = (y_1,\ldots,y_n)$ is a permutation of $(x_1,\ldots,x_n)$. Since $(f_j)_{y_jy_n} = (f_j)_{y_ny_j}$, terms corresponding to $(y_1,\ldots,y_j,\ldots,y_n)$ and $(y_1,\ldots,y_n,\ldots,y_j)$ cancel each other out, their signs being opposite. Therefore, the entire sum is 0. □

An additional fact about jacobian derivations is due to Daigle. It is based on the following result; the reader is referred to the cited paper for its proof.

Proposition 3.14 (Cor. 3.10 of [70]) *Let $f_1,\ldots,f_m \in B = k[x_1,\ldots,x_n] = k^{[n]}$ be given. Set $A = k[f_1,\ldots,f_m]$ and $M = \mathcal{J}(f_1,\ldots,f_m)$. Suppose $I \subset B$ is the ideal generated by the $d\times d$ minors of M, where d is the transcendence degree of A over k. If A is factorially closed in B, then* $\mathrm{height}(I) > 1$.

Corollary 3.15 (Cor. 2.4 of [70]) *Suppose $f_1,\ldots,f_{n-1} \in B = k[x_1,\ldots,x_n] = k^{[n]}$ are algebraically independent, and set $\mathbf{f} = (f_1,\ldots,f_{n-1})$. If $k[f_1,\ldots,f_{n-1}]$ is a factorially closed subring of B, then $\Delta_{\mathbf{f}}$ is irreducible, and* $\ker \Delta_{\mathbf{f}} = k[f_1,\ldots,f_{n-1}]$.

Proof Since $\Delta_{\mathbf{f}} \neq 0$, we have that $\ker \Delta_{\mathbf{f}}$ is equal to the algebraic closure of $k[f_1,\ldots,f_{n-1}]$ in B. By hypothesis, $k[f_1,\ldots,f_{n-1}]$ is factorially closed, hence also algebraically closed in B. Therefore $\ker \Delta_{\mathbf{f}} = k[f_1,\ldots,f_{n-1}]$.

Let I be the ideal generated by the image of $\Delta_{\mathbf{f}}$, namely,

$$I = (\Delta_{\mathbf{f}}(x_1), \ldots, \Delta_{\mathbf{f}}(x_n)) \ .$$

Since the images $\Delta_{\mathbf{f}}(x_i)$ are precisely the $(n-1) \times (n-1)$ minors of the jacobian matrix $\mathcal{J}(f_1, \ldots, f_{n-1})$, the foregoing proposition implies that height$(I) > 1$. Therefore, I is contained in no principal ideal other than B itself, and $\Delta_{\mathbf{f}}$ is irreducible. □

This, of course, has application to the locally nilpotent case, as we will see. However, not all derivations meeting the conditions of this corollary are locally nilpotent. For example, it was pointed out in *Chap. 1* that $k[x^2 - y^3]$ is factorially closed in $k[x, y] = k^{[2]}$, but is not the kernel of any locally nilpotent derivation of $k[x, y]$.

Another key fact about Jacobians is given by van den Essen.

Proposition 3.16 (1.2.9 of [142]) *Let k be a field of characteristic zero and let $F = (F_1, \ldots, F_n)$ for $F_i \in k[x_1, \ldots, x_n] = k^{[n]}$. Then the rank of $\mathcal{J}(F)$ equals* $\text{tr.deg}_k k(F)$.

Here, the **rank** of the jacobian matrix is defined to be the maximal order of a nonzero minor of $\mathcal{J}(F)$.

Remark 3.17 It was observed that the jacobian determinant of a system of variables in a polynomial ring is always a unit of the base field. This fact gives a method to construct locally nilpotent derivations of polynomial rings, as follows. Let $B = k[x_1, \ldots, x_n] = k^{[n]}$ for $n \geq 2$. Given i with $1 \leq i \leq n-1$, let $K = k(x_1, \ldots, x_i)$, and suppose $f_{i+1}, \ldots, f_{n-1} \in B$ satisfy $K[x_{i+1}, \ldots, x_n] = K[f_{i+1}, \ldots, f_{n-1}, g]$ for some $g \in B$. Define $D \in \text{Der}_k(B)$ by

$$D = \Delta_{(x_1, \ldots, x_i, f_{i+1}, \ldots, f_{n-1})}$$

and let E denote the extension of D to $K[x_{i+1}, \ldots, x_n]$. Since $E(f_j) = 0$ for each j and $E(g) \in K^*$, it follows that E is locally nilpotent. Therefore, D (being a restriction of E) is also locally nilpotent.

Example 3.18 Let $B = \mathbb{C}[x, y, z, u] = \mathbb{C}^{[4]}$, and define:

$$p = yu + z^2 \ , \ v = xz + yp \ , \ w = x^2u - 2xzp - yp^2$$

The **Vénéreau polynomials** are $f_n := y + x^n v$, $n \geq 1$. The preceding remark can be used to prove that f_n is an x-variable of B when $n \geq 3$.

First, define a $\mathbb{C}(x)$-derivation θ of $\mathbb{C}(x)[y, z, u]$ by

$$\theta y = 0 \ , \ \theta z = x^{-1}y \ , \ \theta u = -2x^{-1}z$$

noting that $\theta p = 0$. Then:

$$y = \exp(p\theta)(y)\ ,\ v = \exp(p\theta)(xz) \quad \text{and} \quad w = \exp(p\theta)(x^2u)$$

It follows that, for all $n \geq 1$:

$$\mathbb{C}(x)[y, z, u] = \mathbb{C}(x)[y, v, w] = \mathbb{C}[f_n, v, w]$$

Next, assume $n \geq 3$, and define a derivation d of B by $d = \Delta_{(x,v,w)}$. Since $\mathbb{C}(x)[y, v, w] = \mathbb{C}(x)[y, z, u]$, it follows from the preceding remark that d is locally nilpotent. And since $dx = dv = 0$, we have that $x^{n-3}vd$ is also locally nilpotent. In addition, it is easily checked that $dy = x^3$. Therefore:

$$\exp(x^{n-3}vd)(x) = x \quad \text{and} \quad \exp(x^{n-3}vd)(y) = y + x^{n-3}vd(y) = y + x^n v = f_n$$

Set $P_n = \exp(x^{n-3}vd)(z)$ and $Q_n = \exp(x^{n-3}vd)(u)$. Then $\mathbb{C}[x, f_n, P_n, Q_n] = \mathbb{C}[x, y, z, u]$.

The Vénéreau polynomials are further explored in *Sect. 10.3* below.

3.2.4 Homogenizing a Derivation

Suppose $B = k[x_1, \ldots, x_n] = k^{[n]}$, and $D \in \mathrm{Der}_k(B)$ is given, $D \neq 0$. Set $A = \ker D$. Write $Dx_i = f_i(x_1, \ldots, x_n)$ for $f_i \in B$, and set $d = \max_i \deg(Dx_i)$, where degrees are taken relative to the standard $\mathbb{Z}$-grading of B. The **homogenization** of D is the derivation $D^H \in \mathrm{Der}_k(B[w])$ defined by

$$D^H(w) = 0 \quad \text{and} \quad D^H(x_i) = w^d f_i(\tfrac{x_1}{w}, \ldots, \tfrac{x_n}{w})$$

where w is an indeterminate over B. Note that D^H is homogeneous of degree $d - 1$, relative to the standard grading of $B[w]$, and $D^H \bmod (w - 1) = D$ as derivations of B. In addition, if D is (standard) homogeneous to begin with, then $D^H(x_i) = Dx_i$ for every i.

In order to give further properties of D^H relative to D, we first extend D to the derivation $\mathcal{D} \in \mathrm{Der}_k(B[w, w^{-1}])$ defined by $\mathcal{D}b = Db$ for $b \in B$, and $\mathcal{D}w = 0$. Note that $\ker \mathcal{D} = A[w, w^{-1}]$, and that if $D \in \mathrm{LND}(B)$, then $\mathcal{D} \in \mathrm{LND}(B[w, w^{-1}])$.

Next, define $\alpha \in \mathrm{Aut}_k(B[w, w^{-1}])$ by $\alpha(x_i) = \frac{x_i}{w}$ and $\alpha(w) = w$, noting that $\alpha\mathcal{D}\alpha^{-1} \in \mathrm{Der}_k(B[w, w^{-1}])$. In particular:

$$\alpha\mathcal{D}\alpha^{-1}(x_i) = \alpha\mathcal{D}(wx_i) = w\alpha(Dx_i) = wf_i(\tfrac{x_1}{w}, \ldots, \tfrac{x_n}{w})$$

Therefore, $w^{d-1} \cdot \alpha\mathcal{D}\alpha^{-1}(x_i) = D^H(x_i)$, that is, D^H equals the restriction of $w^{d-1}\alpha\mathcal{D}\alpha^{-1}$ to $B[w]$. From this we conclude that D^H has the following properties.

1. D^H is homogeneous of degree $d-1$ in the standard $\mathbb{Z}$-grading of $B[w]$.
2. $\ker(D^H) = \ker(\alpha\mathcal{D}\alpha^{-1}) \cap B[w] = \alpha(A[w, w^{-1}]) \cap B[w]$
3. If $p : B[w] \to B$ is evaluation at $w = 1$, then $p(\ker D^H) = \ker D$.
4. If D is irreducible, then D^H is irreducible.
5. If $D \in \text{LND}(B)$, then $D^H \in \text{LND}_w(B[w])$.

Since $D^H \equiv D$ modulo $(w-1)$, the assignment $D \mapsto D^H$ is an injective function from $\text{LND}(B)$ into the subset of standard homogeneous elements of $\text{LND}_w(B[w])$. This is not, however, a bijective correspondence, since D^H will never be of the form wE for $E \in \text{LND}_w(B[w])$.

Homogenizations are used in *Chap. 8* to calculate kernel elements of D, where property (3) above is especially important.

3.2.5 Other Base Rings

Observe that many of the definitions given for $k^{[n]}$ naturally generalize to the rings $A^{[n]}$ for non-fields A. In this case, we simply include the modifier **over** A. For example, if $B = A[x_1, \dots, x_n]$, we refer to **variables of B over A** as those $f \in B$ such that $B = A[f]^{[n-1]}$. Likewise, **partial derivatives over A**, **jacobian derivations over A**, **linear derivations over A**, and **triangular derivations over A** are defined as elements of $\text{Der}_A(B)$ in the obvious way.

For example, let A be a commutative k-domain and $A[x_1, \dots, x_n] = A^{[n]}$. Given $D \in \text{Der}_A(A[x_1, \dots, x_n])$, the **divergence** of D over A is defined by

$$\text{div}_A(D) = \sum_{i=1}^{n} \partial_{x_i}(Dx_i)$$

where $\partial_{x_i}(A) = 0$ and $\partial_{x_i}(x_j) = \delta_{ij}$ for each i, j. Nowicki [333] defines D to be **special** if $\text{div}_A(D) = 0$. When D is locally nilpotent, we have:

Proposition 3.19 ([142], Prop. 1.3.51; [22], Prop. 2.8) $\text{div}_A(D) = 0$ *for every* $D \in \text{LND}_A(A[x_1, \dots, x_n])$.

3.3 Locally Nilpotent Derivations of Polynomial Rings

One fundamental fact about locally nilpotent derivations of polynomial rings is the following, which is due to Makar-Limanov (Lemma 8 of [276]).

Theorem 3.20 (Makar-Limanov Theorem) *Let $D \in \mathrm{LND}(B)$ be irreducible, where $B = k^{[n]}$. Let $f_1, \ldots, f_{n-1}$ be algebraically independent elements of* $\ker D$*, and set $\mathbf{f} = (f_1, \ldots, f_{n-1})$. Then there exists $a \in \ker D$ such that $\Delta_{\mathbf{f}} = aD$. In particular, $\Delta_{\mathbf{f}} \in \mathrm{LND}(B)$.*

In case $n \leq 3$, even stronger properties hold; see *Theorem 5.9* below.

The proof below follows that of Makar-Limanov, using the lemmas proved earlier concerning jacobian derivations.

Proof Let S be the set of nonzero elements of $A = \ker D$, and let K be the field $S^{-1}A$. Then D extends to a locally nilpotent derivation $S^{-1}D$ of $S^{-1}B$. By *Principle 13*, we have that $K = \ker(S^{-1}D)$, and $S^{-1}B = K[r] = K^{[1]}$ for some local slice r of D. Therefore $(S^{-1}B)^* = K^*$.

Extend D to a derivation D' on all $\mathrm{frac}(B)$ via the quotient rule. (Note: D' is not locally nilpotent.) From *Corollary 1.29*, we have that $\ker D' = K$.

By *Lemma 3.11*, there exists $\eta \in \mathrm{frac}(B)$ such that $D' = \eta\Delta_{\mathbf{f}}$. Note that $\Delta_{\mathbf{f}}$ restricts to a derivation of B.

Suppose $\eta = b/a$ for $a, b \in B$ with $\gcd(a, b) = 1$. Write $\Delta_{\mathbf{f}} = c\delta$ for $c \in B$ and irreducible $\delta \in \mathrm{Der}_k(B)$. Then $aD = bc\delta$, and by *Proposition 2.3* we have that $(a) = (bc)$. Since $\gcd(a, b) = 1$, this means $b \in B^*$, so we may just as well assume $b = 1$. Therefore, $\Delta_{\mathbf{f}} = aD$. The key fact to prove is that $a \in \ker D$.

Let $g_1, \ldots, g_n \in S^{-1}B$ be given, and consider the jacobian determinant $\det \mathcal{J}(g_1, \ldots, g_n) \in \mathrm{frac}(B)$. We claim that $\det \mathcal{J}(g_1, \ldots, g_n)$ is contained in the principal ideal $aS^{-1}B$ of $S^{-1}B$.

Since $S^{-1}B = K[r]$, each g_i can be written as a finite sum $g_i = \sum a_{ij}r^j$ for $a_{ij} \in K$ and $j \geq 0$. Therefore, $\det \mathcal{J}(g_1, \ldots, g_n)$ is a sum of functions of the form $\det \mathcal{J}(a_1r^{e_1}, \ldots, a_nr^{e_n})$ for $a_i \in K$ and $e_i \geq 0$. By the product rule, for each i we also have:

$$\det \mathcal{J}(a_1r^{e_1}, \ldots, a_nr^{e_n}) = \\ a_i \det \mathcal{J}(a_1r^{e_1}, \ldots, r^{e_i}, \ldots, a_nr^{e_n}) + r^{e_i} \det \mathcal{J}(a_1r^{e_1}, \ldots, a_i, \ldots, a_nr^{e_n})$$

So $\det \mathcal{J}(g_1, \ldots, g_n)$ may be expressed as a sum of functions of the form $q \det \mathcal{J}(b_1, \ldots, b_n)$, where $q \in S^{-1}B$, and either $b_i \in K$ or $b_i = r^{e_i}$ for $e_i \geq 1$. If every $b_i \in K$, then $b_1, \ldots, b_n$ are linearly dependent, and this term will be zero. Likewise, if $b_i = r^{e_i}$ and $b_j = r^{e_j}$ for $i \neq j$, then $b_1, \ldots, b_n$ are linearly dependent, and this term is zero. Therefore, by re-ordering the b_i if necessary, any nonzero summand $q \det \mathcal{J}(b_1, \ldots, b_n)$ is of the form $q \det \mathcal{J}(a_1, \ldots, a_{n-1}, r^e) = q\Delta_{\mathbf{a}}(r^e)$, where $q \in S^{-1}B$, $a_i \in K$, $\mathbf{a} = (a_1, \ldots, a_{n-1})$, and $e \geq 1$. By *Lemma 3.12*, there exists $h \in \ker \Delta_{\mathbf{f}} = K$ such that $\Delta_{\mathbf{a}} = h\Delta_{\mathbf{f}}$ for some $h \in K$. In particular, $\Delta_{\mathbf{a}}$ restricts to $S^{-1}B$. Since $\Delta_{\mathbf{f}}(y) \in aB$ for all $y \in B$, it follows that $q\Delta_{\mathbf{a}}(r^e) \in ahS^{-1}B = aS^{-1}B$ (since h is a unit). Since $\det \mathcal{J}(g_1, \ldots, g_n)$ is a sum of such functions, we conclude that $\det \mathcal{J}(g_1, \ldots, g_n) \in aS^{-1}B$ for *any* $g_1, \ldots, g_n \in S^{-1}B$, as claimed.

In particular, if $B = k[x_1, \ldots, x_n]$, then $1 = \det \mathcal{J}(x_1, \ldots, x_n) \in aS^{-1}B$, implying that $a \in (S^{-1}B)^* = K^*$. But this means $a \in B \cap K = \ker D$. □

Makar-Limanov generalized this result in [282] to give a description of the locally nilpotent derivations of any commutative affine $\mathbb{C}$-domain. He writes that his goal is "to give a standard form for an lnd on the affine domains. This form is somewhat analogous to a matrix representation of a linear operator" (p. 2). The theorem he proves is the following.

Theorem 3.21 (Generalized Makar-Limanov Theorem) *Let I be a prime ideal of $B = \mathbb{C}^{[n]}$, and let R be the factor ring B/I, with standard projection $\pi : B \to R$. Given $D \in \text{LND}(R)$, there exist elements $f_1, \ldots, f_{n-1} \in B$ and nonzero elements $a, b \in R^D$ such that, for every $g \in B$:*

$$aD(\pi(g)) = b\pi(\det \mathcal{J}(f_1, \ldots, f_{n-1}, g))$$

Another way to express the conclusion of this theorem is that $aD = b\Delta_{\mathbf{f}}/I$, where $\mathbf{f} = (f_1, \ldots, f_{n-1})$. The reader is referred to Makar-Limanov's paper for the general proof.

The Makar-Limanov Theorem implies the following.

Corollary 3.22 (Prop. 1.3.51 of [142]) *If $B = k^{[n]}$ and $D \in \text{LND}(B)$, then* $\text{div}(D) = 0$.

Proof Choose algebraically independent $f_1, \ldots, f_{n-1} \in \ker D$. There exists an irreducible $\delta \in \text{LND}(B)$ and $c \in \ker D$ such that $D = c\delta$. According to the theorem above, there also exists $a \in \ker D$ such that $a\delta = \Delta_{\mathbf{f}}$. Therefore, $D = (c/a)\Delta_{\mathbf{f}}$, so by the product rule, together with *Lemma 3.13*, we have:

$$\text{div}(D) = (c/a)\text{div}(\Delta_{\mathbf{f}}) + \sum_i \partial_{x_i}(c/a)\Delta_{\mathbf{f}}(x_i) = 0 + \Delta_{\mathbf{f}}(c/a) = 0$$

□

The next two results are due to Daigle.

Lemma 3.23 (Prop. 1.2 of [70]) *Let B be a commutative k-domain and A a subalgebra such that B has transcendence degree 1 over A. If $D, E \in \text{Der}_A(B)$, then there exist $a, b \in B$ for which $aD = bE$.*

Proof Let $K = \text{frac}(A)$ and $L = \text{frac}(B)$. By *Proposition 3.6*, the dimension of $\text{Der}_K(L)$ as a vector space over L is equal to one. Therefore, if S is the set of nonzero elements of B, then $S^{-1}D$ and $S^{-1}E$ are linearly dependent over K, and consequently $aD = bE$ for some $a, b \in B$. □

Proposition 3.24 (Cor. 2.5 of [70]) *Suppose $B = k^{[n]}$, and $D \in \text{LND}(B)$ has* $\ker D \cong k^{[n-1]}$. *If* $\ker D = k[f_1, \ldots, f_{n-1}]$ *and* $\mathbf{f} = (f_1, .., f_{n-1})$, *then $\Delta_{\mathbf{f}}$ is irreducible and locally nilpotent, and $D = a\Delta_{\mathbf{f}}$ for some $a \in \ker D$.*

Proof Let $A = \ker D$. Since A is factorially closed, the fact that $\Delta_{\mathbf{f}}$ is irreducible follows from *Corollary 3.15* above. By *Lemma 3.23*, there exist $a, b \in B$ such that $bD = a\Delta_{\mathbf{f}}$, since D and $\Delta_{\mathbf{f}}$ have the same kernel. We may assume $\gcd(a, b) = 1$. Then $\Delta_{\mathbf{f}}B \subset bB$, implying that b is a unit. So we may assume $b = 1$. The fact that $\Delta_{\mathbf{f}}$ is locally nilpotent and $a \in A$ now follows from *Principle 7*. □

In the other direction, we would like to know whether, if $\mathbf{f} = (f_1, \ldots, f_{n-1})$ for $f_i \in B$, the condition that $\Delta_{\mathbf{f}}$ is irreducible and locally nilpotent always implies $\ker \Delta_{\mathbf{f}} = k[f_1, \ldots, f_{n-1}]$. But this is a hard question. For example, the truth of this property for $n = 3$ would imply the truth of the two-dimensional Jacobian Conjecture!

To see this, we refer to Miyanishi's Theorem in *Chap. 5*, which asserts that the kernel of any nonzero locally nilpotent derivation of $k^{[3]}$ is isomorphic to $k^{[2]}$. Suppose $A = k[f, g]$ is the kernel of a locally nilpotent derivation of $k^{[3]}$. Let $u, v \in k[f, g]$ have the property that $\det \frac{\partial(u,v)}{\partial(f,g)}$ is a nonzero constant. We have

$$\Delta_{(u,v)} = \det \frac{\partial(u,v)}{\partial(f,g)} \Delta_{(f,g)}$$

which we know to be irreducible and locally nilpotent. If the above property were true, it would follow that $A = \ker \Delta_{(u,v)} = k[u, v]$.

Proposition 3.25 (Lemma 3 of [159]) *Suppose $B = k^{[n]}$ and $D \in \mathrm{Der}_k(B)$ is linear relative to the coordinate system $(x_1, \ldots, x_n)$ on B. Let V be the vector space $V = kx_1 \oplus \cdots \oplus kx_n$. Then* rank$(D)$ *equals the rank of D as a linear operator on V.*

Proof Suppose that corank$(D) = m$, and let η denote the nullity of D as a linear operator on V. Let $F = (f_1, \ldots, f_n)$ be a system of variables on B for which $f_1, \ldots, f_m \in \ker D$. Suppose that the standard $\mathbb{N}$-grading of B is given by $B = \oplus_{j \in \mathbb{N}} B_j$ and let $f_i = \sum_{j \in \mathbb{N}} (f_i)_j$. Since $Df_1 = \cdots = Df_m = 0$, we also have $D(f_1)_1 = \cdots = D(f_m)_1 = 0$. By (3.1), $(f_1)_1, \ldots, (f_m)_1$ are linearly independent. It follows that $\eta \geq m$.

Conversely, let $v_1, \ldots, v_\eta \in V$ be linearly independent vectors annihilated by D. Since $(v_1, \ldots, v_\eta)$ is a partial system of variables on B, it follows that $\eta \leq m$. □

In his thesis, Wang [414] (Lemma 2.3.5) gives the equivalent statement: With the notation and hypotheses of the proposition above:

$$\dim_k(V \cap \ker D) = \mathrm{corank}(D)$$

3.4 Slices in Polynomial Rings

The general topic of slices for locally nilpotent derivations is covered in *Chap. 10*. For polynomial rings, we have the following basic result.

Proposition 3.26 *Suppose $B = k^{[n]}$ and $D \in \mathrm{LND}(B)$ has $Ds = 1$ for $s \in B$.*

(a) *s is a variable of $B[w] = k^{[n+1]}$.*

(b) *If $B/sB = k^{[n-1]}$, then D is a partial derivative.*

Proof Let $A = \ker D \subset B$. By the Slice Theorem, $B = A[s]$ and $\pi_s(B) = A$, where π_s is the Dixmier map defined by s. Let $B[w] = B^{[1]}$ and extend D to $D^* \in$

LND($B[w]$) by setting $D^*w = 0$. Then $\ker D^* = A[w]$. Since w is transcendental over A, we have $A[w] \cong A[s] = B = k^{[n]}$. So there exist $g_1, \ldots, g_n \in B[w]$ such that $A[w] = k[g_1, \ldots, g_n]$. Therefore,

$$B[w] = A[s][w] = A[w][s] = k[g_1, \ldots, g_n, s] = k^{[n+1]}$$

and s is a variable of $B[w]$.

In addition, we have that $A \cong B/sB$ by the Slice Theorem. Thus, if $B/sB = k^{[n-1]}$, then $B = A[s]$ implies that s is a variable of B. □

Note that the condition of part (b) holds if s is a variable. Part (a) appears as part of the proof of Thm. 1.2 in [283]. But it clearly deserves to be highlighted. A crucial question is:

If $s \in B$ is a variable of $B[w]$, does it follow that follow that s is a variable of B?

A negative answer to this question would imply a negative solution to either the Embedding Problem or Cancellation Problem. A potential example of such phenomena is provided by the Vénéreau polynomial $f_1 \in \mathbb{C}^{[4]}$: This is known to be a variable of $\mathbb{C}^{[5]}$, but it is an open question whether it is a variable of $\mathbb{C}^{[4]}$. See *Chap. 10* for details.

In summary, suppose $D \in$ LND($k^{[n]}$) has a slice s. Then:

1. $\ker D$ is n-generated.
2. The trivial extension D^* of D to $k^{[n+1]}$ has $\ker D^*$ is n-generated.
3. If s is a variable of $k^{[n]}$, then $\ker D$ is $(n-1)$-generated.

The following result concerns systems of local slices in a ring; δ_{ij} denotes the Kronecker delta.

Proposition 3.27 *B is a commutative k-domain. Suppose that there exist $D_1, \ldots, D_n \in$ LND(B) and $s_1, \ldots, s_n \in B$ such that, for $1 \le i \le n$:*

1. *$[D_i, D_j] = 0$*
2. *$D_i s_j = \delta_{ij}$*

Then $B = A[s_1, \ldots, s_n] = A^{[n]}$, where $A = \bigcap_{1 \le i \le n} \ker D_i$.

Proof We proceed by induction on n. The case $n = 1$ follows from the Slice Theorem. Assume that $n \ge 2$ and that $B = C[s_1, \ldots, s_{n-1}] = C^{[n-1]}$, where $C = \bigcap_{1 \le i \le n-1} \ker D_i$. Note that $s_n \in C$. In addition, since $[D_i, D_n] = 0$ for $1 \le i \le n$, it follows that D_n restricts to C. Since $D_n s_n = 1$, the Slice Theorem implies $C = A[s_n] = A^{[1]}$. Therefore, $B = A[s_1, \ldots, s_n] = A^{[n]}$. □

As an application, we have the following.

Corollary 3.28 *Let $A = k[x_1, \ldots, x_n] = k^{[n]}$, and let $\bar{k}$ be the algebraic closure of k. If $y_1, \ldots, y_n \in A$ are such that $\bar{k}[x_1, \ldots, x_n] = \bar{k}[y_1, \ldots, y_n]$, then $k[x_1, \ldots, x_n] = k[y_1, \ldots, y_n]$.*

Proof Since $A = k[x_1,\ldots,x_n]$, we have:

$$\bar{A} := \bar{k} \otimes_k A = \bar{k}[x_1,\ldots,x_n] = \bar{k}[y_1,\ldots,y_n] = \bar{k}^{[n]}$$

Define the jacobian derivations $D_1,\ldots,D_n$ of $\bar{A}$ by:

$$D_i f = \frac{\partial(y_1,\ldots,\hat{y}_i,\ldots,y_n,f)}{\partial(x_1,\ldots,x_n)} \quad (f \in \bar{A})$$

Note that D_i restricts to A. If $c_i = D_i y_i$, then $c_i \in \bar{k}^* \cap A$ for each i; see *Sect. 3.2.3*. Since k is algebraically closed in A, we have $\bar{k}^* \cap A = k^*$, so $c_i \in k^*$. In addition, $D(c_i^{-1}y_i) = 1$ and $D_i y_j = 0$ for $i \neq j$, and $[D_i, D_j] = 0$ for every i,j. Since each $c_i^{-1}y_i$ belongs to A, *Proposition 3.27* implies that $A = k[c_1^{-1}y_1,\ldots,c_n^{-1}y_n] = k[y_1,\ldots,y_n]$. ☐

3.5 Triangular Derivations and Automorphisms

Fix a coordinate system $B = k[x_1,\ldots,x_n]$. Define subgroups $H_i, K_i \subset BA_n(k)$, $i = 1,\ldots,n$, by:

$$H_i = \{h \in BA_n(k) | h(x_j) = x_j, 1 \le j \le n-i\}$$
$$K_i = \{g \in BA_n(k) | g(x_j) = x_j, i+1 \le j \le n\} = BA_i(k)$$

Then for each i, K_i acts on H_i by conjugation, and $BA_n(k) = H_i \rtimes K_{n-i}$.

Proposition 3.29 *Suppose $B = k^{[n]}$ and $D \in \mathrm{Der}_k(B)$ is triangular in some coordinate system. Then $D \in \mathrm{LND}(B)$. In addition, if $n \ge 2$, then $\mathrm{rank}(D) \le n-1$.*

Proof We argue by induction on n for $n \ge 1$, the case $n = 1$ being obvious. For $n \ge 2$, note that since D is triangular, D restricts to a triangular derivation of $k[x_1,\ldots,x_{n-1}]$. By induction, D is locally nilpotent on this subring. In particular, $Dx_n \in k[x_1,\ldots,x_{n-1}] \subset \mathrm{Nil}(D)$, which implies $x_n \in \mathrm{Nil}(D)$. Therefore, D is locally nilpotent on all B.

Now suppose $n \ge 2$. If $Dx_1 = 0$ we are done, so assume $Dx_1 = c \in k^*$. Choose $f \in k[x_1]$ so that $Dx_2 = f'(x_1)$. Then $D(cx_2 - f(x_1)) = 0$, and $cx_2 - f(x_1)$ is a triangular variable of B. ☐

We next describe the factorization of triangular automorphisms into unipotent and semi-simple factors. (See [123] for a related result.)

Proposition 3.30 *Every triangular automorphism of $k^{[n]}$ is of the form $\exp T \circ L$, where L is a diagonal matrix and T is a triangular derivation.*

Proof If $F \in BA_n(k)$, then $F \circ L$ is unipotent triangular for some diagonal matrix L. So it suffices to assume F is unipotent, i.e., of the form

$$F = (x_1, x_2 + f_2(x_1), x_3 + f_3(x_1,x_2), \ldots, x_n + f_n(x_1,\ldots,x_{n-1}))$$

for polynomials f_i. We show by induction on n that the map $F - I = (0, f_2, \ldots, f_n)$ is locally nilpotent, the case $n = 1$ being obvious. (Observe that $(F - I)(c) = 0$ for $c \in k$.)

Let $A = k[x_1, \ldots, x_{n-1}]$, and suppose by induction that $F - I$ restricts to a locally nilpotent map on A. Then it suffices to show that $F - I$ is nilpotent at every polynomial of the form ax_n^t $(a \in A)$. One easily obtains the formula:

$$(F - I)^m(ax_n^t) = (F - I)^m(a)x_n^t + (\text{lower } x_n \text{ terms})$$

By induction, $(F - I)^m(a) = 0$ for $m \gg 0$. Since the x_n-degree is thus lowered, we eventually obtain $(F - I)^M(ax_n^t) = 0$ for $M \gg 0$. It follows that $F - I$ is locally nilpotent on all B. Thus, *Proposition 2.57* implies $F = \exp D$ for $D = \log(I + (F - I)) \in \text{LND}(B)$. □

Observe that, for triangular derivations D_1, D_2 of $B = k^{[n]}$, $D_1 + D_2$ is again triangular, hence locally nilpotent. In general, however, the triangular derivations D_1 and D_2 do not commute, and $\exp D_1 \exp D_2 \neq \exp(D_1 + D_2)$. Nonetheless, the product on the left is an exponential automorphism.

Corollary 3.31 *If D_1 and D_2 are triangular k-derivations of $B = k^{[n]}$, then there exists a triangular k-derivation E of B such that:*

$$\exp D_1 \exp D_2 = \exp E$$

Proof Since $\exp D_1 \exp D_2$ is triangular, it equals $\exp E \circ L$ for triangular E and diagonal L; see *Proposition 3.30*. It is clear that in this case $L = I$ (identity). □

See also the proof of Cor. 3 in [123].

The main theorem of this section is the following.

Theorem 3.32 *If $F \in BA_n(k)$ has finite order, then there exists $L \in GL_n(k)$ and a triangular $D \in \text{LND}(B)$ such that $F = \exp(-D)L\exp D$.*

The linearizability of finite-order triangular automorphisms was first proved by Ivanenko in [219]. The proof presented below makes use of exponential automorphisms to give a shorter demonstration. Whether a general element of finite order in $GA_n(k)$ can be linearized remains an open problem.

The proof of the theorem is based on the following more general fact.

Proposition 3.33 *Let R be a UFD containing k, let $D \in \text{LND}(R)$, and let $\lambda \in \text{Aut}_k(R)$ have finite order $m \geq 2$. Set $A = \ker D$ and $\gamma = \exp D \circ \lambda$. Suppose the following properties hold.*

1. *$\lambda(a) \in A$ for all $a \in A$.*
2. *$\lambda(a) = a$ for all $a \in A^*$.*
3. *γ has finite order m*

Then there exists $E \in \text{LND}(R)$ such that $\ker E = A$ and $\gamma = \exp(-E)\lambda \exp E$.

Proof Write $D = f\Delta$ for irreducible $\Delta \in \text{LND}(R)$ and $f \in A$. Since $\ker \Delta = \ker(\lambda^{-1}\Delta\lambda) = A$ by hypothesis (1), we conclude from *Principle 12*, together with the fact that R is a UFD and Δ is irreducible, that $\lambda^{-1}\Delta\lambda = c\Delta$ for some $c \in A^*$. By hypothesis, $\lambda(c) = c$, and thus $\lambda^{-i}\Delta\lambda^i = c^i\Delta$ for each $i \in \mathbb{Z}$. It follows that for each $i \in \mathbb{Z}$, $\lambda^{-i}D\lambda^i = \lambda^{m-i}(f)c^i\Delta$. In particular, $D = \lambda^{-m}D\lambda^m = c^mD$, so $c^m = 1$.

Set $E = g\Delta$ for undetermined $g \in A$. Then:

$$\begin{aligned}\exp(-E)\lambda\exp(E) = (\exp D)\lambda &\Leftrightarrow \exp(-E)\exp(\lambda E\lambda^{-1}) = \exp D\\ &\Leftrightarrow \exp((\lambda(g)c^{-1} - g)\Delta) = \exp(f\Delta)\end{aligned}$$

So we need to solve for $g \in A$ which satisfies the equation $f = c^{-1}\lambda(g) - g$. We find a solution $g \in \mathrm{span}_{k[c]}\{f, \lambda(f), \lambda^2(f), \ldots, \lambda^{m-1}(f)\} \subset A$. (Note that $k[c]$ is a field.)

First, if $\gamma_i := \lambda^{-i}(\exp D)\lambda^i$, then:

$$1 = \gamma^m = (\exp D \circ \lambda)^m = \gamma_m\gamma_{m-1}\cdots\gamma_2\gamma_1$$

Since $\gamma_i = \exp(\lambda^{m-i}(f)c^i\Delta)$, it follows that

$$\exp(h\Delta) = 1 \quad \text{for} \quad h = \sum_{i=1}^{m}\lambda^{m-i}(f)c^i$$

Therefore, $h = 0$, and we may eliminate $\lambda^{m-1}(f)$ from the spanning set above.

Next, for undetermined coefficients $a_i \in k[c]$, consider $g = a_1f + a_2\lambda(f) + \cdots + a_{m-1}\lambda^{m-2}(f)$. Then $c^{-1}\lambda(g) - g$ equals:

$$-a_1f + (c^{-1}a_1 - a_2)\lambda(f) + \cdots + (c^{-1}a_{m-2} - a_{m-1})\lambda^{m-2}(f) + c^{-1}a_{m-1}\lambda^{m-1}(f)$$

Since $h = 0$, we have that $c^{-1}a_{m-1}\lambda^{m-1}(f)$ equals:

$$-c^{-2}a_{m-1}f - c^{-3}a_{m-1}\lambda(f) - \cdots - c^{-(m-1)}a_{m-1}\lambda^{m-3}(f) - a_{m-1}\lambda^{m-2}(f)$$

Combining these gives that $c^{-1}\lambda(g) - g$ equals:

$$\begin{aligned}&(-a_1 - c^{-2}a_{m-1})f + (c^{-1}a_1 - a_2 - c^{-3}a_{m-1})\lambda(f) + \cdots\\ &\cdots + (c^{-1}a_{m-3} - a_{m-2} - c^{-(m-1)}a_{m-1}\lambda^{m-3}(f) + (c^{-1}a_{m-2} - 2a_{m-1})\lambda^{m-2}(f)\end{aligned}$$

So we need to solve for a_i such that $M(a_1, a_2, \ldots, a_{m-1})^T = (1, 0, \ldots, 0)^T$ for:

$$M = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & -c^{-2}\\ c^{-1} & -1 & 0 & \cdots & 0 & -c^{-3}\\ 0 & c^{-1} & -1 & \cdots & 0 & -c^{-4}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & -c^{m-1}\\ 0 & 0 & 0 & \cdots & c^{-1} & -2 \end{pmatrix}_{(m-1)\times(m-1)}$$

It is easily checked that $|M| \neq 0$. For example, replace row 2 by c^{-1}(row 1) + (row 2); then replace row 3 by c^{-1}(row 2) + (row 3); and so on. Eventually, we obtain the non-singular upper-triangular matrix:

$$N = \begin{pmatrix} -1 & 0 & 0 & \cdots & -c^{-2} \\ 0 & -1 & 0 & \cdots & -2c^{-3} \\ 0 & 0 & -1 & \cdots & -3c^{-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -m \end{pmatrix}$$

Therefore, we can solve for g, and thereby conjugate γ to λ. □

Proof of Theorem 3.32 Let m be the order of F. We have that $BA_n(k) = H_1 \rtimes K_{n-1}$, so we can write $F = hg$ for $g \in K_{n-1}$ and $h \in H_1$. Then $1 = F^m = (gh)^m = g^m h'$ for some $h' \in H_1$, which implies $g^m = h' = 1$. By induction, there exists a triangular derivation D with $Dx_n = 0$ and $\tilde{g} := \exp(-D)g\exp D \in GL_n(k) \cap K_{n-1}$. Thus, $\exp(-D)F\exp D = \tilde{h}\tilde{g}$ for $\tilde{h} := \exp(-D)h\exp D \in H_1$. So it suffices to assume from the outset that $F = hg$ for linear $g \in K_{n-1}$ and $h \in H_1$.

If $h = (x_1, \dots, x_{n-1}, ax_n + f(x_1, \dots, x_{n-1}))$, then:

$$h = \exp\left(f\frac{\partial}{\partial x_n}\right) \circ (x_1, \dots, x_{n-1}, ax_n)$$

Thus, $F = \exp(f\frac{\partial}{\partial x_n})L$, where $L = (x_1, \dots, x_{n-1}, ax_n)g \in GL_n(k)$. Note that L restricts to $\ker(\frac{\partial}{\partial x_n}) = k[x_1, \dots, x_{n-1}]$. By *Proposition 3.33*, the theorem now follows. □

3.6 Group Actions on $\mathbb{A}^n$

3.6.1 Terminology

Given $f \in B = k^{[n]}$, the variety in $\mathbb{A}^n$ defined by f will be denoted by $\mathcal{V}(f)$. Likewise, if $I \subset B$ is an ideal, the variety defined by I is $\mathcal{V}(I)$.

The group of algebraic automorphisms of $\mathbb{A}^n$ is anti-isomorphic to $GA_n(k)$, in the sense that $(F_1 \circ F_2)^* = F_2^* \circ F_1^*$ in $GA_n(k)$ when F_1 and F_2 are automorphisms of $\mathbb{A}^n$. Thus, we identify these two groups with one another.

If an algebraic k-group G acts algebraically on affine space $X = \mathbb{A}^n$, we also define the **rank** of the G-action exactly as rank was defined for a derivation, i.e., the least integer $r \geq 0$ for which there exists a coordinate system $(x_1, \dots, x_n)$ on $k[X]$ such that $k[x_{r+1}, \dots, x_n] \subset k[X]^G$.

The G-action on $X = \mathbb{A}^n$ is a **linear action** if and only if G acts by linear automorphisms. The action is a **triangular action** if and only if G acts by triangular automorphisms. And the action is a **tame action** if and only if G acts by tame automorphisms. Similarly, the action is **linearizable** if it is conjugate to a linear action, and **triangularizable** if it is conjugate to a triangular action.

The case in which the ring of invariants is a polynomial ring over k is important. For example, if H is a normal subgroup of G, and if $k[X]^H = k^{[m]}$ for some m, then G/H acts on the affine space $\mathbb{A}^m$ defined by $k[X]^H$, and this action can be quite interesting. This is the idea behind the main examples of *Chaps. 7* and *10* below.

Following are some particulars when the group $\mathbb{G}_a$ acts on affine space. Let a $\mathbb{G}_a$-action on $\mathbb{A}^n$ be given by

$$\rho : \mathbb{G}_a \times \mathbb{A}^n \to \mathbb{A}^n \quad \text{where} \quad \rho(t, \mathbf{x}) = (F_1(t, \mathbf{x}), \ldots, F_n(t, \mathbf{x}))$$

for functions F_i, and $\mathbf{x} = (x_1, \ldots, x_n)$ for coordinate functions x_i on $\mathbb{A}^n$.

- ρ is algebraic if and only if $F_i \in k[t, x_1, \ldots, x_n] \cong k^{[n+1]}$ for each i.
- ρ is linear if and only if each F_i is a linear polynomial in $x_1, \ldots, x_n$ over $k[t]$.
- ρ is triangular if and only if $F_i \in k[t, x_1, \ldots, x_i]$ for each i.
- ρ is **quasi-algebraic** if and only if $F_i(t_0, \mathbf{x}) \in k[x_1, \ldots, x_n]$ for each $t_0 \in k$ and each i. (See [387].)
- If $k = \mathbb{C}$, then ρ is **holomorphic** if and only if each F_i is a holomorphic function on $\mathbb{C}^{n+1}$.

Note that $\exp(tD)$ is a linear algebraic $\mathbb{G}_a$-action if and only if D is a linear locally nilpotent derivation (i.e., given by a nilpotent matrix), and $\exp(tD)$ is a triangular $\mathbb{G}_a$-action if and only if D is a triangular derivation. In [398], Suzuki classified the quasi-algebraic and holomorphic $\mathbb{C}^+$-actions on $\mathbb{C}^2$, and the holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^2$.

3.6.2 Translations

The simplest algebraic $\mathbb{G}_a$-action on $X = \mathbb{A}^n$ is a **translation**, meaning that for some system of coordinates $(x_1, \ldots, x_n)$, the action is given by

$$t \cdot (x_1, \ldots, x_n) = (x_1 + t, x_2, \ldots, x_n) = \exp(t\partial_{x_1}) \ .$$

Clearly, a translation is fixed-point free, and admits a geometric quotient: $X/\mathbb{G}_a = X/\!\!/\mathbb{G}_a \cong \mathbb{A}^{n-1}$.

In case $n = 1$, the locally nilpotent derivations of $k[x]$ are those of the form $c\frac{d}{dx}$ for some $c \in k$ (*Principle 8*). So translations are the only algebraic $\mathbb{G}_a$-actions on the affine line: $t \cdot x = x + tc$.

3.6.3 Planar Actions

The simplest linear $\mathbb{G}_a$-action on the plane comes from the standard representation of $\mathbb{G}_a$ on $V = \mathbb{A}^2$ via matrices:

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \quad (t \in k)$$

The algebraic quotient $V /\!/ \mathbb{G}_a$ is a line $\mathbb{A}^1$. If $\pi : V \to V /\!/ \mathbb{G}_a$ is the quotient map, then the fiber $\pi^{-1}(\lambda)$ over any $\lambda \in V /\!/ \mathbb{G}_a$ is the line $x = \lambda$, which is a single orbit if $\lambda \neq 0$, and a line of fixed points if $\lambda = 0$. In this case, the geometric quotient $V/\mathbb{G}_a$ does not exist.

More generally, a triangular action on $\mathbb{A}^2$ is defined by

$$t \cdot (x, y) = (x, y + tf(x)) = \exp(tD)$$

for any $f(x) \in k[x]$, where $D = f(x)\partial_y$. In case $k = \mathbb{C}$, define a planar $\mathbb{G}_a$-action by the orthogonal matrices

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad (t \in \mathbb{C}) \ .$$

This is not an algebraic action, although it is quasi-algebraic, locally finite, and holomorphic. It is the exponential of the locally finite derivation $x\partial_y - y\partial_x$ on $\mathbb{C}[x, y]$.

3.6.4 Theorem of Deveney and Finston

Deveney and Finston showed the following fundamental property of invariant rings for $\mathbb{G}_a$-actions on affine spaces.

Theorem 3.34 ([101]) *Over the ground field $\mathbb{C}$, the quotient field of the ring of invariants of an algebraic action of $\mathbb{G}_a$ on $\mathbb{A}^n$ $(n \geq 1)$ is ruled. Equivalently, if $D \in \mathrm{LND}(\mathbb{C}^{[n]})$ and $A = \ker D$, then* $\mathrm{frac}(A)$ *is a ruled field.*

Suppose that $D \in \mathrm{LND}(\mathbb{C}^{[n]})$ is given, where $1 \leq n \leq 4$ and $D \neq 0$. Then $\ker D$ is a polynomial ring: The case $n = 1$ is true because $\ker D$ is an algebraically closed subring of $\mathbb{C}$; the case $n = 2$ follows from results in *Chap. 4*; and the case $n = 3$ is the content of Miyanishi's Theorem in *Chap. 5*.

When $n = 4$, there are kernels which are not polynomial rings; see *Sect. 3.8* below for examples. However, these kernels are rational over $\mathbb{C}$. To see this, let $A = \ker D$ and let $s \in B$ be a local slice. By the Deveney and Finston result,

$\mathrm{frac}(A) = L^{(1)}$ for a subfield $L \subset A$. Since $B_{Ds} = A_{Ds}^{[1]}$ we have:

$$\left(\mathbb{C}^{(2)}\right)^{(2)} = \mathrm{frac}(\mathbb{C}^{[4]}) = (\mathrm{frac}(A))^{(1)} = \left(L^{(1)}\right)^{(1)} = L^{(2)}$$

We can now invoke the cancellation theorem for fields to conclude that $L \cong \mathbb{C}^{(2)}$, and therefore $\mathrm{frac}(A) = \mathbb{C}^{(3)}$.

In this way, Deveney and Finston obtain the following corollary.

Corollary 3.35 *Over the ground field $\mathbb{C}$, the quotient field of the ring of invariants of an algebraic action of $\mathbb{G}_a$ on $\mathbb{A}^4$ is rational. Equivalently, if $D \in \mathrm{LND}(\mathbb{C}^{[4]})$ is nonzero and $A = \ker D$, then $\mathrm{frac}(A) \cong_{\mathbb{C}} \mathbb{C}^{(3)}$.*

3.6.5 Proper and Locally Trivial $\mathbb{G}_a$-Actions

Proper $\mathbb{G}_a$-actions on complex affine varieties were studied in the 1976 paper of Fauntleroy and Magid [151], with particular attention to surfaces. This paper, together with the examples of Winkelman given in [421], motivated a series of papers on the subject dating from 1994 by Deveney and Finston [103–110] and by Deveney, Finston and Gehrke [111]. These papers study proper and locally trivial $\mathbb{G}_a$-actions on $\mathbb{C}^n$.

Suppose that $B = \mathbb{C}^{[n]}$ and $D \in \mathrm{LND}(B)$, let $A = \ker D$ and let

$$\sigma : \mathbb{G}_a \times \mathbb{C}^n \to \mathbb{C}^n$$

denote the $\mathbb{G}_a$-action on $\mathbb{C}^n$ associated to D. In addition, let $B[t] = B^{[1]}$ and extend D to $B[t]$ by $Dt = 0$. The following result is from [111], Thm. 2.3.

Theorem 3.36 (Properness Criterion) *σ is proper if and only if:*

$$B[\exp(tD)B] = B[t]$$

Moreover, a proper $\mathbb{G}_a$-action on $\mathbb{C}^n$ is fixed-point free and its topological orbit space is Hausdorff.

The same paper also characterizes the locally trivial actions, as follows (see [111], Thm. 2.5, Thm. 2.8).

Theorem 3.37 (Local Triviality Criterion) *The following conditions are equivalent.*

1. *σ is locally trivial.*
2. *σ is proper and B is a flat extension of A.*
3. $\mathrm{pl}(D) \cdot B = B$

In [103], Deveney and Finston asked if the ring of invariants for a locally trivial $\mathbb{G}_a$-action on $\mathbb{C}^n$ is finitely generated. In [110], they gave an affirmative answer to this question.

Theorem 3.38 ([110], Thm. 2.1) *Let X be a factorial affine variety over $\mathbb{C}$. For any locally trivial $\mathbb{G}_a$-action on X, the invariant ring $\mathbb{C}[X]^{\mathbb{G}_a}$ is finitely generated as a $\mathbb{C}$-algebra.*

Any fixed-point free $\mathbb{G}_a$-action on $\mathbb{C}^2$ or $\mathbb{C}^3$ is a translation, due to Rentschler and Kaliman, respectively (see *Chaps. 4* and *5*). In higher dimensions this is no longer the case. The examples in *Sect. 3.8* below show that there are fixed-point free $\mathbb{G}_a$-actions on $\mathbb{C}^4$ which are not proper; proper $\mathbb{G}_a$-actions on $\mathbb{C}^5$ which are not locally trivial; and locally trivial $\mathbb{G}_a$-actions on $\mathbb{C}^5$ which are not globally trivial. Each of these examples is triangular. In [221], Question 2, Jorgenson asked: Is there a triangular $\mathbb{G}_a$-action on $\mathbb{C}^4$ that is locally trivial but not equivariantly trivial? Recently, Dubouloz, Finston and Jaradat showed the following, which gives a negative answer to this question.

Theorem 3.39 ([130]) *A proper triangular $\mathbb{G}_a$-action on $\mathbb{C}^4$ is a translation.*

It is an open question whether every proper $\mathbb{G}_a$-action on $\mathbb{C}^4$ is a translation.

3.7 $\mathbb{G}_a$-Actions Relative to Other Group Actions

A special property belonging to a $\mathbb{G}_a$-action is, in many cases, equivalent to the condition that that the action can be embedded in a larger algebraic group action. For example, homogeneity for $\mathbb{Z}$-gradings equates to an action of $\mathbb{G}_a \rtimes \mathbb{G}_m$. Another important condition to consider is symmetry. The symmetric group S_n acts naturally on the polynomial ring $k[x_1,\ldots,x_n]$ by permutation of the variables x_i.

In the first case, suppose $D \in \mathrm{LND}(B)$ is homogeneous of degree d relative to some $\mathbb{Z}$-grading of B, where B is any affine k-domain. This is equivalent to giving an algebraic action of the group $\mathbb{G}_a \rtimes \mathbb{G}_m$ on $X = \mathrm{Spec}(B)$, where the action of $\mathbb{G}_m$ on $\mathbb{G}_a = \mathrm{Spec}(k[x])$ is given by $t \cdot x = t^d x$. This is further equivalent to giving $D \in \mathrm{LND}(B)$ and an action $\mathbb{G}_m \to \mathrm{Aut}_k(B)$, $t \to \lambda_t$, such that $\lambda_t^{-1} D \lambda_t = t^d D$ for all t. The homogeneous polynomials $f \in B_i$ are the semi-invariants $f \in B$ for which $t \cdot f = t^i f$ $(t \in \mathbb{G}_m)$.

Proposition 3.40 *Under the hypotheses above, if $s \in \mathbb{G}_m$ has finite order m not dividing d, then $\exp D \circ \lambda_s$ is conjugate to λ_s. In particular,*

$$(\exp D \circ \lambda_s)^m = 1 \ .$$

Proof

$$\exp\left(\tfrac{s^d}{1-s^d}D\right)(\exp D)\lambda_s \exp\left(-\tfrac{s^d}{1-s^d}D\right) = \lambda_s$$

□

In particular, this result shows that any action of a finite cyclic group on $k^{[n]}$ of the form given in the proposition can be embedded in a $\mathbb{G}_m$-action.

The second result of this section is about kernels of homogeneous derivations.

Proposition 3.41 *Suppose* $D \in \mathrm{LND}(B)$*,* $D \neq 0$*, is homogeneous relative to some* $\mathbb{N}$*-grading* $\oplus_{i\in\mathbb{N}} B_i$ *of* $B = k^{[n]}$*. If* $\ker D$ *is a polynomial ring and* $B_0 \cap \ker D = k$*, then* $\ker D = k[g_1, \dots, g_{n-1}]$ *for homogeneous* g_i*.*

This is immediately implied by the following more general fact about positive $\mathbb{Z}$-gradings, which is due to Daigle.

Proposition 3.42 (Lemma 7.6 of [68]) *Let* $A = k^{[r]}$ *for* $r \geq 1$ *and let* $A = \bigoplus_{i\in\mathbb{Z}} A_i$ *be a positive* $\mathbb{Z}$*-grading. If* $A = k[f_1, \dots, f_m]$ *for homogeneous* $f_i \in A$*, then there is a subset* $\{g_1, \dots, g_r\}$ *of* $\{f_1, \dots, f_m\}$ *with* $A = k[g_1, \dots, g_r]$*.*

Proof By *Corollary 3.28*, it suffices to assume that the field k is algebraically closed.

Let $M = \bigoplus_{i>0} A_i$. Then M is an ideal, and since $A_0 = k$, it is a maximal ideal of A. Since A is a polynomial ring, there exist $X_1, \dots, X_r \in A$ so that $A = k[X_1, \dots, X_r]$ and $M = (X_1, \dots, X_r)$. We may assume, without loss of generality, that $f_i \in M$ for $1 \leq i \leq m$.

Consider a subset $\{g_1, \dots, g_s\}$ of $\{f_1, \dots, f_m\}$ satisfying $A = k[g_1, \dots, g_s]$ and minimal with respect to this property; in particular, $\deg g_i > 0$ for all i. Let $R = k[T_1, \dots, T_s] = k^{[s]}$ with positive $\mathbb{Z}$-grading $R = \bigoplus_{i\in\mathbb{Z}} R_i$ determined by $\deg T_i = \deg g_i$. Then the surjective k-homomorphism $e : R \to A$, $e(\varphi) = \varphi(g_1, \dots, g_s)$, is homogeneous of degree zero, and $\ker e$ is a homogeneous ideal.

If $\mathfrak{m} = (T_1, \dots, T_s)$, then $e(\mathfrak{m}) \subset M$ and $e(\mathfrak{m}^2) \subset M^2$. We thus have a well-defined mapping of k-vector spaces $\bar{e} : R/\mathfrak{m}^2 \to A/M^2$, where $\{1, \bar{T}_1, \dots, \bar{T}_s\}$ is a basis of $R/\mathfrak{m}^2$ and $\{1, \bar{X}_1, \dots, \bar{X}_r\}$ is a basis of A/M^2.

Given $F \in \ker e$, write $F = \sum F_i$ for $F_i \in R_i$. Since $\ker e$ is a homogeneous ideal, $F_i \in \ker e$ for all i. In particular, $F_0 \in \ker e$. Since $R_0 = k$ by hypothesis, we see that $F_0 \in k$. But e is a k-map, so $F_0 = 0$. It follows that $\ker e \subset \mathfrak{m}$.

If $F \notin \mathfrak{m}^2$, then $F_i \notin \mathfrak{m}^2$ for some $i \geq 1$. Therefore, there exist $c_1, \dots, c_r \in k$ not all 0 such that $F_i \equiv c_1T_1 + \cdots + c_rT_r (\mathrm{mod}\, \mathfrak{m}^2)$. By degree considerations, it follows that, if $c_j \neq 0$ and $F_i = c_1T_1 + \cdots + c_rT_r + G$ for $G \in \mathfrak{m}^2$, then $G \in k[T_1, \dots, T_{j-1}, T_{j+1}, \dots, T_r]$. Therefore:

$$F_i - c_jT_j \in k[T_1, \dots, T_{j-1}, T_{j+1}, \dots, T_s]$$

But then

$$\varphi(F_i - c_jT_j) = -c_jg_j \in k[g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_s]$$

contradicting the minimality of $\{g_1, \dots, g_s\}$.

Therefore, $\ker e \subset \mathfrak{m}^2$. Consequently, there is a well-defined surjection $A = R/\ker e \to R/\mathfrak{m}^2$, which implies that, if $P_i \in R$ is such that $e(P_i) = X_i$, $1 \leq i \leq r$, then $R/\mathfrak{m}^2$ has basis $\{1, \bar{P}_1, \dots, \bar{P}_r\}$. It follows that $r = s$. □

Corollary 3.43 *If $B = k^{[n]}$ and if $\mathbb{G}_m$ acts algebraically on $\mathbb{A}^n$ in such a way that $B^{\mathbb{G}_m} = k$, then the action is linearizable. Equivalently, for any positive $\mathbb{Z}$-grading of B there exists a system of homogeneous variables for B.*

Proof The action induces a $\mathbb{Z}$-grading of B for which elements of B_i are semi-invariants of weight i. In particular, $B_0 = B^{\mathbb{G}_m}$. If $f \in B_i$ and $g \in B_j$ for $i < 0$ and $j > 0$, then $f^j g^{-i} \in B_0$, a contradiction. Therefore, we can assume any non-constant semi-invariant has strictly positive weight. So the grading on B induced by the $\mathbb{G}_m$-action is an $\mathbb{N}$-grading: $B = \oplus_{i\in\mathbb{N}} B_i$.

Suppose $B = k[x_1, \ldots, x_n]$. Given i ($1 \le i \le n$), we can write $x_i = \sum_{j\in\mathbb{N}} f_{ij}$, where $f_{ij} \in B_j$. So B is generated as a k-algebra by finitely many homogeneous polynomials f_{ij}. By the preceding result, there exist homogeneous $g_1, \ldots, g_n \in B$ such that $B = k[g_1, \ldots, g_n]$, i.e., $(g_1, \ldots, g_n)$ is a system of semi-invariant variables for B. □

Next, let $B = k^{[n]}$ and consider the standard action of the symmetric group S_n on B relative to coordinates $(x_1, \ldots, x_n)$. Define $D \in \mathrm{Der}_k(B)$ to be **fully symmetric** if and only if $D\sigma = \sigma D$ for each $\sigma \in S_n$. To give $D \in \mathrm{LND}(B)$ fully symmetric is equivalent to giving an algebraic action of $\mathbb{G}_a \times S_n$ on B or on $\mathbb{A}^n$.

Example 3.44 $E = \sum_{i=1}^n \partial_{x_i}$ is fully symmetric and locally nilpotent, and $\ker E = k[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n]$. Note that E is a partial derivative. If $f \in \ker E \cap B^{S_n}$, then fE is also fully symmetric and locally nilpotent.

Proposition 3.45 *Let $\mathbb{Z}_2$ act on $B = k[x_1, \ldots, x_n]$ by transposing x_1 and x_2, and fixing $x_3, \ldots, x_n$. If $D \in$ LND(B) commutes with this $\mathbb{Z}_2$-action, then $D(x_1 - x_2) = 0$.*

Proof Let $\tau \in \mathbb{Z}_2$ transpose x_1 and x_2, fixing $x_3, \ldots, x_n$, and let $Dx_1 = F(x_1, x_2)$ for $F \in k[x_3, \ldots, x_n]^{[2]}$. Then $Dx_2 = D(\tau x_1) = \tau D x_1 = F(x_2, x_1)$. This implies:

$$D(x_1 - x_2) = F(x_1, x_2) - F(x_2, x_1) \in (x_1 - x_2)B$$

By *Corollary 1.23*, we conclude that $D(x_1 - x_2) = 0$. □

Now suppose D is a fully symmetric locally nilpotent derivation. Then $D(x_i - x_j) = 0$ for all i, j, so $k[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n] \subset \ker D$. Consequently, the derivations fE above are the only fully symmetric locally nilpotent derivations.

Corollary 3.46 *If $D \in \mathrm{LND}(B)$ is fully symmetric and $D \neq 0$, then* $\mathrm{rank}(D) = 1$.

Remark 3.47 The conclusion of *Proposition 3.42* may fail to hold for more general polynomial algebras. For instance, we saw in *Example 1.27* that if $U = \mathbb{R}[x_1, x_2]/(x_1^2 + x_2^2 - 1)$ and $B = U[y_1, y_2]/(x_1 y_1 + x_2 y_2)$, then $B = U[s] = U^{[1]}$. But we also have $B = U[x_1 s, x_2 s]$, a homogeneous system of generators when $\deg x_i = 0$ and $\deg y_i = 1$ for each i, whereas $B \neq U[x_i s]$ for $i = 1, 2$.

3.8 Some Important Early Examples

This section illustrates the fact that the triangular derivations of polynomial rings already provide a rich source of examples.

In 1972, Nagata [325] published an example of a polynomial automorphism of $\mathbb{A}^3$ which, he conjectured, is not tame. Later, Bass embedded Nagata's automorphism as an element of a one-parameter subgroup of polynomial automophisms of $\mathbb{A}^3$, gotten by exponentiating a certain non-linear locally nilpotent derivation of $k[x, y, z]$. It was known at the time that every unipotent group of polynomial automorphisms of the plane is triangular in some coordinate system (see *Chap. 4*). In sharp contrast to the situation for the plane, Bass showed that the subgroup he constructed could not be conjugated to the triangular subgroup. Then Popov generalized Bass's construction to produce non-triangularizable $\mathbb{G}_a$-actions on $\mathbb{A}^n$ for every $n \geq 3$. These discoveries initiated the exploration of a new world of algebraic representations $\mathbb{G}_a \hookrightarrow GA_n(k)$.

Note that for some of the examples below, we exhibit, without explanation, the kernel of the derivation under consideration. Methods for calculating these kernels are discussed in *Chap. 8* below.

3.8.1 Bass's Example ([12], 1984)

The example of Bass begins with the linear derivation of $k[x, y, z]$ given by $\Delta = x\partial_y + 2y\partial_z$. Then $\ker \Delta = k[x, F]$, where $F = xz - y^2$. Note that $D := F\Delta$ is also a locally nilpotent derivation of $k[x, y, z]$, and the corresponding $\mathbb{G}_a$-action on $\mathbb{A}^3$ is:

$$\alpha_t := \exp(tD) = (x, y + txF, z + 2tyF + t^2xF^2)$$

Nagata's automorphism is α_1. The fixed point set of this action is the cone $F = 0$, which has an isolated singularity at the origin. On the other hand, Bass observed that any triangular automorphism $(x, y + f(x), z + g(x, y))$ has a cylindrical fixed point set, i.e., defined by $f(x) = g(x, y) = 0$, which (if non-empty) has the form $C \times \mathbb{A}^1$ for some variety C. In general, an affine variety X is called a **cylindrical variety** if $X = Y \times \mathbb{A}^1$ for some affine variety Y. Since a cylindrical variety can have no isolated singularities, it follows that α_t cannot be conjugated into $BA_3(k)$ relative to the coordinate system (x, y, z).

3.8.2 Popov's Examples ([344], 1987)

Generalizing Bass's approach, Popov pointed out that the fixed-point set of any triangular $\mathbb{G}_a$-action on $\mathbb{A}^n$ is a cylindrical variety, whereas the hypersurface defined by a non-degenerate quadratic form is not a cylindrical variety. So to produce non-triangularizable examples in higher dimensions, it suffices to find $D \in \text{LND}(k^{[n]})$ such that $\ker D$ contains a non-degenerate quadratic form h; then $\exp(thD)$ is a non-triangularizable $\mathbb{G}_a$-action. In even dimensions, let $B = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$, and define D by:

$$Dx_1 = 0,\ Dx_2 = x_1,\ Dx_3 = x_2,\ \ldots, Dx_n = x_{n-1}$$
$$Dy_1 = y_2,\ Dy_2 = y_3,\ \ldots, Dy_{n-1} = y_n,\ Dy_n = 0$$

Then D is a triangular (linear) derivation, and $Dh = 0$ for the non-degenerate quadratic form $h = \sum_{i=1}^{n}(-1)^{i+1}x_iy_i$. For odd dimensions at least 5, start with D above, and extend D to $k[x_1, \ldots, x_n, y_1, \ldots, y_n, z]$ by $Dz = 0$. Then $h + z^2$ is a non-degenerate quadratic form annihilated by D.

3.8.3 Smith's Example ([386], 1989)

At the conclusion of his paper, Bass asked whether the $\mathbb{G}_a$-action he gave on $\mathbb{A}^3$ is **stably tame**, i.e., whether the action becomes tame when extended trivially to $\mathbb{A}^4$. M. Smith gave a positive answer to this question by first showing the following.

Lemma 3.48 (Smith's Formula) *Let $D \in \text{LND}(B)$ for $B = k^{[n]}$ and let $f \in \ker D$ be given. Extend D to $B[w]$ by $Dw = 0$, and define $\tau \in GA_{n+1}(k)$ by $\tau = \exp(f\partial_w)$. Then:*

$$\exp(fD) = \tau^{-1}\exp(-wD)\tau\exp(wD)$$

Proof Since τ fixes B, $\tau D = D\tau$, so $\tau^{-1}(-wD)\tau = \tau^{-1}(-w)D = (f - w)D$. Applying the exponential now gives:

$$\begin{aligned}\exp(fD)\exp(-wD) &= \exp((f-w)D)\\ &= \exp\big(\tau^{-1}(-wD)\tau\big)\\ &= \tau^{-1}\exp(-wD)\tau\end{aligned}$$

□

Applying this lemma with $f = tF$ and $D = \Delta$ from Bass's example yields the following tame factorization for the example of Bass-Nagata. For $t \in \mathbb{G}_a$:

$$\begin{aligned}\exp(tD) &= (x, y + txF, z + 2tyF + t^2xF^2, w)\\ &= (x, y, z, w - tF) \circ (x, y - wx, z - 2wy + w^2x, w)\\ &\quad \circ(x, y, z, w + tF) \circ (x, y + wx, z + 2wy + w^2x, w)\end{aligned}$$

Lemma 3.49 *This $\mathbb{G}_a$-action on $\mathbb{A}^4$ is not triangularizable.*

Proof Note first that the rank of D on $k^{[4]}$ is clearly 2. Let $X \subset \mathbb{A}^4$ be the set of fixed points. Then $X = C \times \mathbb{A}^1$ for a singular cone C, and the singularities of X form a line. Suppose $k[x, y, z, w] = k[a, b, c, d]$ and that D is triangular in the latter system of coordinates, with $Da = 0$ and $Db \in k[a]$. The ideal defining X is (Db, Dc, Dd), and thus $X \subset \mathcal{V}(Db)$. If $Db \neq 0$, this is a union of parallel coordinate hyperplanes, implying $X \subset H$ for a coordinate hyperplane H. Since this is clearly impossible, $Db = 0$. We also have $X \subset \mathcal{V}(Dc)$, where $Dc \in k[a, b]$. If $Dc \neq 0$, this implies $X = Y \times \mathbb{A}^2$, where Y is a component of the curve in Spec$(k[a, b])$ defined by Dc. But this also cannot occur, since then the singularities of X would be of dimension 2. Thus, $Dc = 0$. But this would imply that the rank of D is 1, a contradiction. Therefore, D extended to $k[x, y, z, w]$ cannot be conjugated to a triangular derivation by any element of $GA_4(k)$. □

So in dimension 4 (and likewise in higher dimensions), there exist $\mathbb{G}_a$-actions which are tame but not triangularizable. It is an important open question whether every tame $\mathbb{G}_a$-action on $\mathbb{A}^3$ can be triangularized. It goes to the structure of the tame subgroup. Shestakov and Umirbaev [382, 383] have shown that the Nagata automorphism α_1 above is *not* tame as an element of $GA_3(k)$, thus confirming the conjecture of Nagata. In [428], Wright gives a structural description of $TA_3(k)$ as an amalgamation of three of its subgroups.

3.8.4 Winkelmann's Example 1 ([421], 1990)

In this groundbreaking paper, Winkelmann investigates $\mathbb{C}^+$-actions on $\mathbb{C}^n$ which are fixed-point free, motivated by questions about their quotients. In dimension 4, he defines $\exp(tD)$, where D is the triangular derivation on $B = \mathbb{C}[x, y, z, w]$ defined by:

$$Dx = 0\,,\ \ Dy = x\,,\ \ Dz = y\,,\ \ Dw = y^2 - 2xz - 1$$

$\exp(tD)$ defines a free algebraic $\mathbb{C}^+$-action on $\mathbb{C}^4$, but the orbit space (geometric quotient) is not Hausdorff in the natural topology (Lemma 8). In particular, D is not a partial derivative, i.e., the action is not a translation, since both the geometric and algebraic quotient for a translation of $\mathbb{C}^4$ is $\mathbb{C}^3$. Winkelmann calculates this kernel explicitly: $\ker D = \mathbb{C}[x, f, g, h]$, where:

$$f = y^2 - 2xz\,,\ \ g = xw + (1-f)y \quad \text{and} \quad xh = g^2 - f(1-f)^2$$

In particular, $\ker D$ is the coordinate ring of a singular hypersurface in $\mathbb{C}^4$. This implies $\text{rank}(D) = 3$, since if the rank were 1 or 2, the kernel would be a polynomial ring (see *Chap. 4*).

Let $B[t] = B^{[1]}$ and consider the subring:

$$R = B[\exp(tD)B] = B[tx, ty + \tfrac{1}{2}t^2x, t(f-1)]$$

If $R = B[t]$, then setting $y = 1$ and $z = w = 0$ shows:

$$\mathbb{C}[x, tx, t + \tfrac{1}{2}t^2x] = \mathbb{C}[x, t]$$

However:

$$\mathbb{C}[x, tx, t + \tfrac{1}{2}t^2x] \cong \mathbb{C}[X, Y, Z]/(XZ - Y - \tfrac{1}{2}Y^2)$$

This ring is evidently not a UFD, and is therefore not isomorphic to $\mathbb{C}^{[2]}$, a contradiction. Therefore, $R \neq B[t]$ and the $\mathbb{G}_a$-action defined by D is not proper.

In [388], Snow gives the similar example

$$Ex = 0\,,\ Ey = x\,,\ Ez = y\,,\ Ew = 1 + y^2$$

and also provides a simple demonstration that the topological quotient is non-Hausdorff (Example 3.5). (It is easy to show that D and E are conjugate.) In [142], van den Essen considers E, and indicates that E does not admit a slice, a condition which is *a priori* independent of the fact that the corresponding quotient is not an affine space (Example 9.5.25). And in [111], Sect. 3, Deveney, Finston, and Gehrke consider E as well, showing that the associated $\mathbb{C}^+$-action $\exp(tE)$ on $\mathbb{C}^4$ is not proper.

3.8.5 Winkelmann's Example 2 ([421], 1990)

On $B = \mathbb{C}[u, v, x, y, z] = \mathbb{C}^{[5]}$, define the triangular derivation F by:

$$Fu = Fv = 0\,,\ Fx = u\,,\ Fy = v\,,\ Fz = 1 + (vx - uy)$$

Then $Fx, Fy, Fz \in \ker F$ and $(Fx, Fy, Fz) = (1)$, which implies $\exp(tF)$ is a locally trivial $\mathbb{C}^+$-action on $\mathbb{C}^5$. The kernel of F is presented in [111], namely:

$$\ker F = \mathbb{C}[u, v, vx - uy, x + x(vx - uy) - uz, y + y(vx - uy) - vz]$$

To see that the associated $\mathbb{C}^+$-action on $\mathbb{C}^5$ is not globally trivial, note that F is homogeneous of degree 0 relative to the $\mathbb{C}^*$-action $(\lambda u, \lambda^{-1}v, \lambda x, \lambda^{-1}y, z)$, $\lambda \in \mathbb{C}^*$. We thus have an action of $\mathbb{C}^+ \times \mathbb{C}^*$ on $\mathbb{C}^5$. The invariant ring of the $\mathbb{C}^*$-action is $B_0 = \mathbb{C}[uv, xy, vx, uy, z]$, the ring of degree-0 elements. Therefore F restricts to B_0. If F has a slice in B, then by homogeneity there exists a slice $s \in B_0$. But the ideal generated by the image of F restricted to B_0 equals $(vx+uy, uv, 1+vx-uy)$, which does not contain 1, meaning F has no slice in B_0. (The fixed-point set of the induced $\mathbb{C}^+$-action on $\mathrm{Spec}(B_0)$ is of dimension one.) Therefore, F has no slice in B.

3.8.6 Example of Deveney and Finston ([104], 1995)

Define δ on $B = \mathbb{C}[u, v, x, y, z] = \mathbb{C}^{[5]}$ by:

$$\delta u = \delta v = 0 \ , \ \delta x = u \ , \ \delta y = v \ , \ \delta z = 1 + uy^2$$

The authors show that $\exp(t\delta)$ is a proper $\mathbb{C}^+$-action on $\mathbb{C}^5$. To see this, let $B[t] = B^{[1]}$ and consider the subring:

$$R = B[\exp(t\delta)B] = B[tu, tv, t(1 + vy^2) + t^2uvy + \tfrac{1}{3}t^3uv^2]$$

Then $R = B[t]$, since:

$$t = \big(t(1 + vy^2) + t^2uvy + \tfrac{1}{3}t^3uv^2\big) - \big((tu)y^2 + (tu)(tv)y + \tfrac{1}{3}(tu)(tv)^2\big)$$

Therefore δ defines a proper action. Deveney and Finston show that $\ker\delta$ is isomorphic to the ring

$$\mathbb{C}[u_1, u_2, u_3, u_4, u_5]/(u_2u_5 - u_1^2u_4 - u_3^3 - 3u_1u_3)$$

which is the coordinate ring of a singular hypersurface $Y \subset \mathbb{C}^5$. If $p : \mathbb{C}^5 \to Y$ is the quotient morphism, then fibers of p over singular points of Y are two-dimensional, which implies that the action is not locally trivial.

3.9 Homogeneous Dependence Problem

In a remarkable paper [184] dating from 1876, Paul Gordan and Max Nöther investigated the vanishing of the Hessian determinant of an algebraic form, using the language of systems of differential operators. In particular, the question they consider is the following. Suppose $h \in \mathbb{C}[x_1, \ldots, x_n]$ is a homogeneous polynomial whose Hessian determinant is identically zero:

$$\det\left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right)_{ij} = 0$$

Does it follow that h is degenerate, i.e., that $h \in \mathbb{C}[Tx_1, \ldots, Tx_{n-1}]$ for some $T \in GL_n(\mathbb{C})$? They prove that the answer is yes when $n = 3$ and $n = 4$, and garner some partial results for the case $n = 5$.

In the course of their proof, the authors consider changes of coordinates involving a parameter $\lambda \in \mathbb{C}$:

> Die Functionen $\Phi(x)$, gebildet für die Argumente $x + \lambda\xi$, sind unabhängig von λ:
>
> $$\Phi(x + \lambda\xi) = \Phi(x) \ . \quad \text{(p. 550)}$$ [4]

Here, x denotes a vector of coordinates $(x_1, \ldots, x_n)$, and ξ a vector of homogeneous polynomials. In modern terms, the association $\lambda \cdot x = x + \lambda\xi$ gives a $\mathbb{C}^+$-action on $\mathbb{C}^n$ (where $\lambda \in \mathbb{C}$), and the functions Φ are its invariants. The authors continue:

> Ist eine solche ganze Function Φ das Product zweier ganzen Functionen
>
> $$\Phi = \phi(x) \cdot \psi(x)$$
>
> so sind auch die Factoren selbst Functionen Φ. (p. 551)[5]

We recognize this as the property that the ring of invariants of a $\mathbb{C}^+$-action is factorially closed. In effect, Gordan and Nöther studied an important type of $\mathbb{C}^+$-action on $\mathbb{C}^n$, which we will now describe in terms of derivations.

Let $B = k[x_1, \ldots, x_n] = k^{[n]}$, and let $D \in \text{LND}(B)$ be given, $D \neq 0$. The **Homogeneous Dependence Problem** for locally nilpotent derivations asks:

> If D is standard homogeneous and has the property that $D^2 x_i = 0$ for each i, is the rank of D always strictly less than n? Equivalently, does there exist a linear form $L \in B$ with $DL = 0$, i.e., are the images Dx_i linearly dependent?

For such a derivation D, note that the $\mathbb{G}_a$-action is simply

$$\exp(tD) = (x_1 + tDx_1, \ldots, x_n + tDx_n)$$

and these are precisely the kinds of coordinate changes considered by Gordan and Nöther. Note also that, given i:

$$D \circ \exp D(x_i) = D(x_i + Dx_i) = Dx_i + D^2 x_i = Dx_i$$

On the other hand, $Dx_i \in \ker D$ means that $\exp D(Dx_i) = Dx_i$. Therefore, D and $\exp D$ commute. This in turn implies that, if we write $F = \exp D = x + H$, where $x = (x_1, \ldots, x_n)$ and $H = (Dx_1, \ldots, Dx_n)$, then $H \circ H = 0$. Herein lies the connection to the work of Gordan and Nöther.

In their paper, Gordan and Nöther effectively proved that the answer to the Homogeneous Dependence Problem is yes when $n = 3$ or $n = 4$. In fact, they

[4] "The functions $\Phi(x)$, constructed for the arguments $x + \lambda\xi$, are independent of λ."

[5] "If such an entire function Φ is a product of two entire functions $\Phi = \phi(x)\psi(x)$, then so also are the factors themselves functions Φ."

showed that in these cases there exist two independent linear forms, L and M, with $DL = DM = 0$, which implies that the rank of D is 1 when $n = 3$, and at most 2 when $n = 4$.

In the modern era, Wang proved in his 1999 thesis (Prop. 2.4.4) that if $D \in \mathrm{LND}(k[x_1, x_2, x_3])$ has the property that $D^2 x_i = 0$ for each i, then $\mathrm{rank}(D) \leq 1$ [414, 415]. So in the case of dimension 3, the homogeneity condition can be removed. A short proof of Wang's result is given in *Chap. 5* below. Wang further proved that, in dimension 4, the rank of a homogeneous derivation having $D^2 x_i = 0$ for each i could not equal 3 (Lemma 2.5.2). Then in 2000, Derksen constructed an example of such a derivation D in dimension 8 whose rank is 7, thereby showing that the stronger result of Gordan and Nöther (i.e., that the kernel contains two independent linear forms) does not generalize. In 2004, de Bondt found a way to construct counterexamples to the Homogeneous Dependence Problem in all dimensions $n \geq 6$ by using derivations of degree 4. So the Homogeneous Dependence Problem remains open only for the case $n = 5$. The examples of Derksen and de Bondt are discussed below.

At the time of their work, neither Wang nor Derksen seems to have been aware of the paper of Gordan and Nöther. Rather, it is an example of an important question resurfacing. The Gordan-Nöther paper was brought to the author's attention by van den Essen, and its existence was made known to him by S. Washburn. Van den Essen was interested in its connections to his study of the Jacobian Conjecture; see [33–35, 147] for a discussion of these connections, and some positive results for this conjecture. The article of DeBondt [93] gives a modern proof of the results of Gordan and Nöther, in addition to some partial results in dimension 5.

3.9.1 Construction of Examples

We construct, for each $N \geq 8$, a family of derivations D of the polynomial ring $k[x_1, \ldots, x_N]$ with the property that $D^2 x_i = 0$ for each i. The example of Derksen belongs to this family.

Given $m \geq 1$, let $B = k[s_1, \ldots, s_m] = k^{[m]}$ and let $\delta \in \mathrm{LND}(B)$ be such that $\delta^2 s_i = 0$ for each i (possibly $\delta = 0$). Let $u \in B^\delta = \ker \delta$ be given ($u \neq 0$). Extend δ to $B[t] = B^{[1]}$ by setting $\delta t = 0$.

Next, given $n \geq 3$, choose an $n \times n$ skew-symmetric matrix M with entries in $B[t]^\delta$, i.e., $M \in \mathcal{M}_n(B[t]^\delta)$ and $M^T = -M$. Also, let $\mathbf{v} \in (B[t]^\delta)^n$ be a nonzero vector in the kernel of M.

Next, let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, and z be indeterminates over $B[t]$, so that $B[t, \mathbf{x}, \mathbf{y}, z] = k^{[m+2n+2]}$. Note that $m + 2n + 2 \geq 9$. Extend δ to a locally nilpotent derivation of this larger polynomial ring by setting:

$$\delta \mathbf{x} = u\mathbf{v} \ , \ \delta \mathbf{y} = \mathbf{x}M \ , \ \delta z = u^{-1}\delta(\langle \mathbf{x}, \mathbf{y} \rangle)$$

Here, it is understood that for vectors $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, the statement $\delta\mathbf{a} = \mathbf{b}$ means $\delta a_i = b_i$ for each i. In addition, $\langle \mathbf{a}, \mathbf{b} \rangle$ denotes the inner product of $\mathbf{a}$ and $\mathbf{b}$. Observe the **product rule for inner products**::

$$\delta(\langle \mathbf{a}, \mathbf{b} \rangle) = \langle \delta\mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{a}, \delta\mathbf{b} \rangle$$

It is clear from the definition that $\delta^2\mathbf{x} = 0$. In addition:

$$\delta^2\mathbf{y} = \delta(\mathbf{x}M) = (\delta\mathbf{x})M = (u\mathbf{v})M = u(\mathbf{v}M) = 0$$

Further, since M is skew-symmetric, we have $0 = \langle \mathbf{x}, \mathbf{x}M \rangle = \langle \mathbf{x}, \delta\mathbf{y} \rangle$. Therefore:

$$\delta(\langle \mathbf{x}, \mathbf{y} \rangle) = \langle \delta\mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \delta\mathbf{y} \rangle = \langle \delta\mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{x}, \delta\mathbf{y} \rangle \in \ker\delta$$

It follows that δz is a well-defined polynomial (since u divides $\delta\mathbf{x}$), and $\delta^2 z = 0$. In addition, if $F = uz - \langle \mathbf{x}, \mathbf{y} \rangle$, then $\delta F = 0$.

Since F does not involve t, the kernel element $t - F$ is a variable. It follows that

$$B[t, \mathbf{x}, \mathbf{y}, z]/(t - F) = B[\mathbf{x}, \mathbf{y}, z] = k^{[m+2n+1]}$$

and that the derivation $D := \delta \bmod (t - F)$ has the property that $D^2\mathbf{x} = D^2\mathbf{y} = D^2 z = 0$.

3.9.2 Derksen's Example

This example appears in [142], 7.3, Exercise 6. It uses the minimal values $m = 1$ and $n = 3$ from the construction above, so that $m + 2n + 1 = 8$. Derksen found this example by considering the exterior algebra associated to three linear derivations.

First, let δ be the zero derivation of $B = k[s] = k^{[1]}$, and choose $u = s$. The extension of δ to $k[s, t]$ is also zero. Choose:

$$\mathbf{v} = \begin{pmatrix} t^2 \\ s^2 t \\ s^4 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & s^4 & -s^2 t \\ -s^4 & 0 & t^2 \\ s^2 t & -t^2 & 0 \end{pmatrix}$$

With these choices, we get the derivation D on the polynomial ring

$$k[s, x_1, x_2, x_3, y_1, y_2, y_3, z] = k^{[8]}$$

defined by $Ds = 0$,

$$D\mathbf{x} = \begin{pmatrix} sF^2 \\ s^3F \\ s^5 \end{pmatrix}, \quad D\mathbf{y} = \begin{pmatrix} 0 & s^4 & -s^2F \\ -s^4 & 0 & F^2 \\ s^2F & -F^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and $Dz = F^2y_1 + s^2Fy_2 + s^4y_3$, where F is the quadratic form $F = sz - (x_1y_1 + x_2y_2 + x_3y_3)$.

Observe that D is homogeneous, of degree 4. To check that s is the only linear form in the kernel of D (up to scalar multiples), let V_i denote the vector space of forms of degree i in these 8 variables, and let $W \subset V_5$ denote the subspace generated by the monomials appearing in the image of $D : V_1 \to V_5$. Then it suffices to verify that the linear map $D : V_1 \to W$ has a one-dimensional kernel, and this is easily done with standard methods of linear algebra. We conclude that the rank of D is 7. □

3.9.3 De Bondt's Examples

Theorem 3.50 ([92]; [93], Cor. 3.3) *For $n \geq 3$, let*

$$B = k^{[2n]} = k[x_1, y_1, \ldots, x_n, y_n]$$

and define $D \in \mathrm{Der}_k(B)$ by

$$Dx_i = fgx_i - g^2y_i \quad \text{and} \quad Dy_i = f^2x_i - fgy_i$$

where $f = x_1y_2 - x_2y_1$ and $g = x_1y_3 - x_3y_1$. Then:

(a) *D is standard homogeneous of degree 4*
(b) $f, g \in \ker D$
(c) $D^2x_i = D^2y_i = 0$ *for each i*
(d) $\mathrm{rank}(D) = 2n$

Proof Let $R = k[a, b] = k^{[2]}$ and let $N \in \mathcal{M}_2(R)$ be given by:

$$N = \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix}$$

Then $N^2 = 0$.

Let $\mathcal{B} = R[x_1, y_1, \ldots, x_n, y_n] = k^{[2n+2]}$. Define R-linear $\mathcal{D} \in \text{LND}_R(\mathcal{B})$ by:

$$\mathcal{D} = \begin{pmatrix} N & 0 & \cdots & 0 \\ 0 & N & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N \end{pmatrix}_{2n\times 2n}$$

Then for each i, we have:

$$\mathcal{D}x_i = abx_i - b^2y_i \ , \ \mathcal{D}y_i = a^2x_i - aby_i \ , \quad \text{and} \quad \mathcal{D}^2x_i = \mathcal{D}^2y_i = 0$$

In addition, for every pair i, j, we have

$$\mathcal{D}(x_iy_j) = x_i(a^2x_j - aby_j) + y_j(abx_i - b^2y_i) = a^2x_ix_j - b^2y_iy_j = \mathcal{D}(x_jy_i)$$

which implies $x_iy_j - x_jy_i \in \ker \mathcal{D}$ for each pair i, j.

Set $f = x_1y_2 - x_2y_1$ and $g = x_1y_3 - x_3y_1$. The crucial observation is that f and g are kernel elements not involving a or b. Thus, $(a - f, b - g, x_1, \ldots, y_n)$ is a triangular system of coordinates on $\mathcal{B}$. If $I \subset \mathcal{B}$ is the ideal $I = (a - f, b - g)$, then $B := \mathcal{B} \bmod I$ is isomorphic to $k^{[2n]}$, and we may take $B = k[x_1, y_1, \ldots, x_n, y_n]$. Since $a - f$ and $b - g$ belong to $\ker \mathcal{D}$, the ideal I is an integral ideal of $\mathcal{D}$, and we have that $D := \mathcal{D} \bmod I$ is well-defined, locally nilpotent and homogeneous on B.

It remains to show that $\text{rank}(D) = 2n$. If $Dv = 0$ for a variable $v \in B$, then by homogeneity, there exists a linear form $L = \sum(a_ix_i + b_iy_i)$ for scalars a_i, b_i such that $DL = 0$. But then $\sum(a_iDx_i + b_iDy_i) = 0$. So it suffices to show that the images $Dx_1, Dy_1, \ldots, Dx_n, Dy_n$ are linearly independent.

To this end, define a vector of univariate polynomials

$$\mathbf{t} = (t, t^2, t^3, t^4 - 1, t^5 - 1, t^6, \ldots, t^{2n})$$

noting that $f(\mathbf{t}) = -t$ and $g(\mathbf{t}) = -t^2$. Then for each i, we have:

$$\deg_t Dy_i(\mathbf{t}) = 2i + 3 \quad \text{and} \quad \deg_t Dx_i(\mathbf{t}) = 2i + 4$$

Since these degrees are all distinct for $1 \leq i \leq n$, it follows that these polynomials are linearly independent. □

Note that de Bondt's derivations are quasi-linear, in addition to being nice derivations.

In order to exhibit an example in odd dimension $2n + 1$ for $n \geq 3$, let $k^{[2n+1]} = B[z]$, and extend D to this ring. In particular, Dz should satisfy: (1) $Dz \in \ker D$, (2) $\deg Dz = 5$, and (3) Dz is not in the span of $Dx_1, \ldots, Dy_n$. For example, $h = x_2y_3 - x_3y_2 \in \ker D$, so we may take $Dz = h(fx_n - gy_n)$. Then $Dz \in \ker D$ and $\deg Dz = 5$. Moreover, $\deg_t Dz(\mathbf{t}) = 2n + 7$, so Dz is independent of the other images.

Remark 3.51 The examples of de Bondt given above are for $n \geq 6$ and have $\deg D = 4$. In [93], Cor. 3.4, de Bondt also gives examples with $n \geq 10$ and $\deg D = 3$. It is an open question whether there exist homogeneous $D \in \mathrm{LND}(k^{[n]})$ with $\deg D = 2$ and $\mathrm{rank}(D) = n$.

3.9.4 Rank-4 Example in Dimension 5

In the notation of de Bondt's examples, consider the case $n = 2$: Let $\mathcal{B} = k[a, b, x_1, y_1, x_2, y_2] = k^{[6]}$ and $R = k[a, b]$. In this case, replace the matrix N with:

$$N' = \begin{pmatrix} ab^2 & -b^4 \\ a^2 & -ab^2 \end{pmatrix}$$

This defines an R-linear $\mathcal{D} \in \mathrm{LND}_R(\mathcal{B})$, namely:

$$\mathcal{D} = \begin{pmatrix} N' & 0 \\ 0 & N' \end{pmatrix}_{4\times 4}$$

Note that we still have $f = x_1y_2 - x_2y_1 \in \ker \mathcal{D}$. Set $E = \mathcal{D} \bmod (a - f)$ on $B = \mathcal{B} \bmod (a - f) = k^{[5]}$. Then E is standard homogeneous of degree 4, and satisfies:

$$E^2b = E^2x_1 = E^2y_1 = E^2x_2 = E^2y_2 = 0$$

In addition, the rank of E is 4. To see this, it suffices to show that the images Ex_1, Ey_1, Ex_2, Ey_2 are linearly independent. As above, evaluate these polynomials at $\mathbf{t} = (1, t, t^2 - 1, t^3, t^4)$. Then:

$$Ex_1(\mathbf{t}) = t^4 - t^2 + 1 \ , \ Ey_1(\mathbf{t}) = t^7 - t^5 + t^3 \ , \ Ex_2(\mathbf{t}) = t^6 - t^4 \ , \ Ey_2(\mathbf{t}) = t^9 - t^7$$

Therefore, Ex_1, Ey_1, Ex_2, Ey_2 are linearly independent.

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