



Chapter 2

Tension and Compression in Bars

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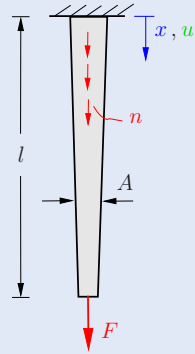
Tensile or compressive loading in bars

Assumptions:

- Length l of the bar is large compared to characteristic dimensions of the cross section $A(x)$.
- Axis of the bar (line connecting centroids of the cross sections) is a straight line.
- Common line of action (external loads F and $n(x)$ are aligned with the axis of the bar).
- Cross section $A(x)$ can only vary slightly.

Stress: Assuming a constant stress σ across the section A the following relation with the normal force N holds:

$$\sigma(x) = \frac{N(x)}{A(x)}.$$



Basic equations of a deformable bar:

<i>equilibrium condition</i>	$\frac{dN}{dx} = -n,$
<i>Hooke's law</i>	$\varepsilon = \frac{\sigma}{E} + \alpha_T \Delta T,$
<i>Kinematic relation</i>	$\varepsilon = \frac{du}{dx}$

E = Young's modulus,

α_T = coefficient of thermal expansion,

ΔT = temperature difference with respect to a reference state,

$u(x)$ = displacement of a point x within the bar.

The basic equations lead to a *single* differential equation for the displacements ($\{\cdot\}' := d\{\cdot\}/dx$):

$$(EAu')' = -n + (EA\alpha_T\Delta T)'$$

Elongation of a bar:
$$\Delta l = u(l) - u(0) = \int_0^l \varepsilon \, dx .$$

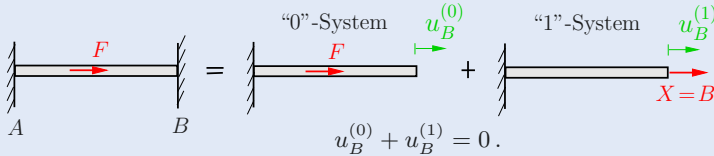
special cases:

$$\Delta l = \int_0^l \frac{N}{EA} dx \quad (\Delta T = 0) ,$$

$$\Delta l = \frac{Fl}{EA} \quad (N = F = \text{const}, EA = \text{const}, \Delta T = 0) ,$$

$$\Delta l = \alpha_T \Delta T l \quad (N = 0, EA = \text{const}, \alpha_T \Delta T = \text{const}) .$$

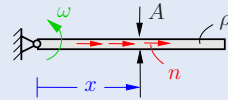
Superposition: The solution of a statically indeterminate problem can be achieved by *superposition* of solutions of associated statically determinate problems considering the compatibility conditions.



Rotating bar: A bar rotating with the angular velocity ω experiences an axial loading per unit length of

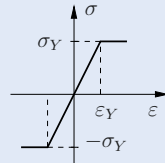
$$n = \rho A x \omega^2 .$$

Here ρ is the density and x represents the distance of the cross section A from the center of rotation.



Elastic-plastic bar: For an elastic-ideal-plastic material behavior, Hooke's law is valid only until a certain *yield limit* σ_Y :

$$\sigma = \begin{cases} E \varepsilon , & |\varepsilon| \leq \varepsilon_Y , \\ \sigma_Y \text{ sign}(\varepsilon) , & |\varepsilon| \geq \varepsilon_Y . \end{cases}$$

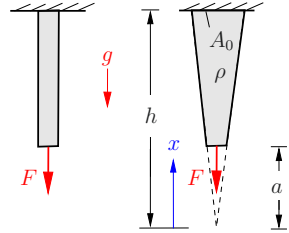


System of bars: The displacements are obtained by “disconnecting” and “reconnecting” of the bars from the nodes using a *displacement diagram*.

Note: In areas with rapidly changing cross sections (notches, holes) the above theory for bars is not applicable.

P2.1

Problem 2.1 Determine the stresses distribution $\sigma(x)$ in the homogeneous bar due to its weight. The bar has constant thickness and a linear varying width. Furthermore, identify the location and value of the smallest stress.



It is reasonable to introduce the x -coordinate at the intersection of the extended edges of the trapezoid. The x dependent cross section area follows then as

$$A(x) = A_0 x / l.$$

With the weight

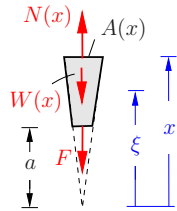
$$W(x) = \rho g V(x) = \rho g \int_a^x A(\xi) d\xi = \rho g A_0 \frac{x^2 - a^2}{2l}$$

of the lower part equilibrium provides

$$N(x) = F + W(x) = F + \rho g A_0 \frac{x^2 - a^2}{2h}.$$

This leads to the stress

$$\underline{\underline{\sigma(x) = \frac{N(x)}{A(x)} = \frac{Fh + \rho g \frac{A_0}{2} (x^2 - a^2)}{A_0 x}}}$$



The location x^* of the minimum is determined by condition $\sigma' = 0$:

$$\sigma' = -\frac{Fh}{A_0} \frac{1}{x^2} + \frac{\rho g}{2} \left(1 + \frac{a^2}{x^2} \right) = 0 \quad \rightsquigarrow \quad \underline{\underline{x^* = \sqrt{\frac{2Fh}{\rho g A_0} - a^2}}}$$

The value of the minimum stress is

$$\underline{\underline{\sigma_{\min} = \sigma(x^*) = \rho g \sqrt{\frac{2Fh}{\rho g A_0} - a^2} = \underline{\underline{\rho g x^*}}}}$$

Note:

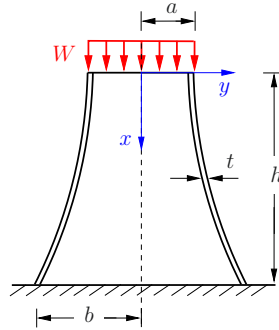
- For $\rho g = 0$ (“weight-less bar”) no minimum exists. The largest stress occurs at $x = a$.
- The minimum will be located within the bar, only if $a < x^* < h$ or $\rho g A_0 a^2 / (2h) < F < \rho g A_0 (h^2 + a^2) / (2h)$ holds.

Problem 2.2 The contour of a lighthouse with circular thin-walled cross section follows a hyperbolic equation

$$y^2 - \frac{b^2 - a^2}{h^2} x^2 = a^2.$$

Determine the stress distribution as a consequence of weight W of the lighthouse head (the weight of the structure can be neglected).

Given: $b = 2a$, $t \ll a$.

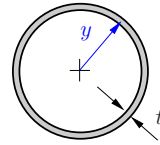


Solution As the weight W is the only acting external load, the normal force N is constant (compression):

$$N = -W.$$

The cross section area A is changing. It can be approximated by (thin-walled structure with $t \ll y$)

$$\begin{aligned} A(x) &= 2\pi y t = 2\pi t \sqrt{a^2 + \frac{b^2 - a^2}{h^2} x^2} \\ &= 2\pi t \sqrt{a^2 + 3 \frac{a^2}{h^2} x^2} \\ &= 2\pi a t \sqrt{1 + 3 \frac{x^2}{h^2}}. \end{aligned}$$



The stress follows now as

$$\sigma(x) = \frac{N}{A} = - \frac{W}{2\pi a t \sqrt{1 + 3 \frac{x^2}{h^2}}}.$$

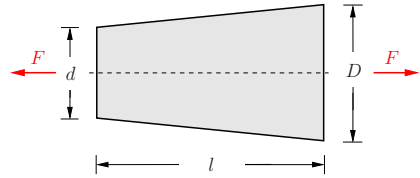
Especially at the top and bottom position we get

$$\sigma(x = 0) = - \frac{W}{2\pi a t} \quad \text{bzw.} \quad \sigma(x = h) = - \frac{W}{4\pi a t}.$$

Note: The stress at the top is twice as large as the stress at the bottom, which is a inefficient use of material. This situation changes if the weight of the thin-walled structure is included in the analysis.

P2.3

Problem 2.3 Determine the elongation Δl of the conical shaft (Young's modulus E) under the application of a tensile force F .



Solution The normal force $N = F$ is constant, while the cross section area A varies. With $\sigma = N/A$ the elongation is computed by

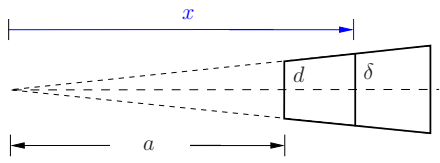
$$\Delta l = \int_0^l \varepsilon \, dx = \frac{1}{E} \int_0^l \sigma \, dx = \frac{1}{E} \int_0^l \frac{N \, dx}{A} = \frac{F}{E} \int_0^l \frac{dx}{A(x)}.$$

To describe the change of the cross section area $A(x)$ we start the x -axis at the peak of the frustum. Using the intercept theorem and the auxiliary variable a we obtain for the diameter

$$\delta(x) = d \frac{x}{a}$$

and for the area

$$A(x) = \frac{\pi}{4} \delta^2(x) = \frac{\pi}{4} d^2 \frac{x^2}{a^2}.$$



Introducing this in the relation for the elongation, then integration provides (integration limits!):

$$\Delta l = \frac{F}{E} \int_a^{a+l} \frac{dx}{\frac{\pi}{4} d^2 \frac{x^2}{a^2}} = \frac{4Fa^2}{\pi E d^2} \left(-\frac{1}{x} \right) \Big|_a^{a+l}.$$

With

$$\frac{a+l}{D} = \frac{a}{d} \quad \rightsquigarrow \quad a = \frac{d}{D} \frac{l}{1 - \frac{d}{D}}$$

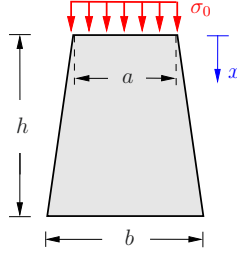
the elongation is

$$\underline{\underline{\Delta l = \frac{4Fl}{\pi E D d}}}.$$

Test: For $D = d$ (constant cross section) we obtain $\Delta l = \frac{4Fl}{\pi E d^2} = \frac{Fl}{EA}$.

Problem 2.4 A homogeneous frustum of a pyramid (Young's modulus E) with a square cross section is loaded on its top surface by a stress σ_0 .

Determine the displacement field $u(x)$ of a cross section at position x .



Solution The normal force $N = -\sigma_0 a^2$ is constant. From the kinematic relation $\varepsilon = du/dx$ and Hooke's law $\varepsilon = \sigma/E = N/EA$ we obtain a differential equation for the displacement u

$$EA(x) \frac{du}{dx} = -\sigma_0 a^2.$$

The area $A(x)$ follows from the intercept theorem:

$$A(x) = \left[a + (b - a) \frac{x}{h} \right]^2.$$

Thus we have

$$E \left(a + \frac{b - a}{h} x \right)^2 \frac{du}{dx} = -\sigma_0 a^2.$$

Separation of variables yields

$$du = -\frac{\sigma_0 a^2}{E} \frac{dx}{\left(\frac{b - a}{h} x + a \right)^2} \quad \rightsquigarrow \quad \int_{u(0)}^{u(x)} du = -\frac{\sigma_0 a^2}{E} \int_0^x \frac{d\xi}{\left(\frac{b - a}{h} \xi + a \right)^2}.$$

Using the substitution $z = a + (b - a) \xi/h$, $dz = (b - a) d\xi/h$ leads to

$$u(x) - u(0) = -\frac{\sigma_0 a^2}{E} \frac{h}{b - a} \left(-\frac{1}{z} \right) \Big|_a^{\frac{b-a}{h}x + a} = -\frac{\sigma_0 a^2}{E} \frac{h}{b - a} \left(\frac{1}{a} - \frac{1}{\frac{b-a}{h}x + a} \right).$$

The displacement $u(0)$ of the top surface follows from the boundary condition that the displacement has to vanish on the bottom edge $x = h$:

$$u(h) = 0 \quad \rightsquigarrow \quad u(0) = \frac{\sigma_0 a^2}{E} \frac{h}{b - a} \left(\frac{1}{a} - \frac{1}{b} \right) = \frac{\sigma_0 a h}{E b}.$$

From this relation the displacement follows

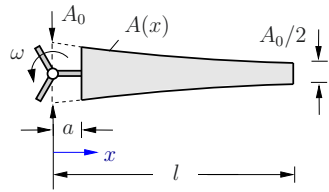
$$\underline{\underline{u(x) = \frac{\sigma_0 a^2}{E} \frac{h}{b - a} \left(-\frac{1}{b} + \frac{1}{\frac{b-a}{h}x + a} \right)}}.$$

P2.5

Problem 2.5 The cross section of a solid helicopter blade (density ρ , Young's modulus E) is described by the equation $A(x) = A_0 e^{-\alpha x/l}$.

Determine the stress distribution $\sigma(x)$, if the blade is rotating with a constant angular velocity ω .

Compute the elongation Δl under the assumption $a = 0$.



Solution First, the sketched geometry $A(l) = A_0/2$ yields

$$A_0 e^{-\alpha} = A_0/2 \quad \rightsquigarrow \quad e^{\alpha} = 2 \quad \rightsquigarrow \quad \alpha = \ln 2 = 0.693.$$

The rotation causes a distributed load per unit length

$$n = \rho \omega^2 x A(x) = \rho \omega^2 A_0 x e^{-\alpha x/l}.$$

The equilibrium condition $N' = -n$ provides the normal force by integration

$$N = - \int n dx = - \frac{\rho \omega^2 A_0 l^2}{\alpha^2} \left[-\frac{\alpha x}{l} e^{-\alpha x/l} - e^{-\alpha x/l} + C \right].$$

The integration constant C is determined by the boundary condition:

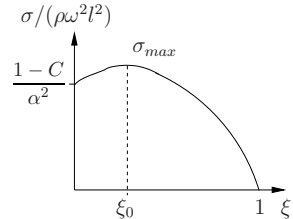
$$N(l) = 0 \quad \rightsquigarrow \quad C = (1 + \alpha) e^{-\alpha} = 0.847.$$

Introducing the dimensionless coordinate $\xi = x/l$ yields

$$N(\xi) = \frac{\rho \omega^2 A_0 l^2}{\alpha^2} [(1 + \alpha \xi) e^{-\alpha \xi} - C],$$

and for the stress distribution

$$\underline{\underline{\sigma(\xi) = \frac{N}{A} = \frac{\rho \omega^2 l^2}{\alpha^2} [1 + \alpha \xi - C e^{\alpha \xi}].}}$$

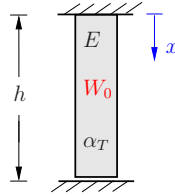


The elongation is calculated from

$$\begin{aligned} \underline{\underline{\Delta l}} &= \int_0^l \varepsilon dx = \frac{l}{E} \int_0^1 \sigma d\xi = \frac{\rho \omega^2 l^3}{\alpha^2 E} \left[\xi + \frac{\alpha \xi^2}{2} - \frac{C}{\alpha} e^{\alpha \xi} \right]_0^1 \\ &= \frac{\rho \omega^2 l^3}{E \alpha^2} \left[1 + \frac{\alpha}{2} - \frac{C}{\alpha} e^{\alpha} + \frac{C}{\alpha} \right] = 0.258 \frac{\rho \omega^2 l^3}{E}. \end{aligned}$$

Note: Due to the varying cross section the maximum stress occurs at the position $\xi_0 = -(\ln C)/\alpha = 0.24$ and attains the maximum value $\sigma_{\max} = -(\rho \omega^2 l^2 \ln C)/\alpha^2 = 0.347 \rho \omega^2 l^2$.

Problem 2.6 A massive bar (weight W_0 , cross section area A , thermal expansion coefficient α_T) is fixed at $x = 0$ and just touches the ground in a stress-free manner.



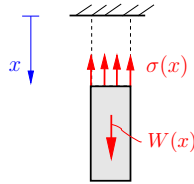
Determine the stress distribution $\sigma(x)$ in the bar after a uniform heating by ΔT . Which ΔT causes compression everywhere in the bar?

Solution We investigate the "two load cases", weight und heating. The weight causes a a normal force

$$N(x) = W(x) = W_0 \frac{h-x}{h} = W_0 \left(1 - \frac{x}{h}\right)$$

which is related to the stress distribution

$$\sigma_1(x) = \frac{N(x)}{A} = \frac{W_0}{A} \left(1 - \frac{x}{h}\right)$$



The *heating* produces an additional strain, which is blocked by the support on the bottom. The relation

$$\varepsilon = \frac{\sigma_2(x)}{E} + \alpha_T \Delta T = 0$$

yields

$$\sigma_2(x) = -E\alpha_T \Delta T.$$

Thus the total stress is computed by

$$\underline{\underline{\sigma(x) = \sigma_1 + \sigma_2 = \frac{W_0}{A} \left(1 - \frac{x}{h}\right) - E\alpha_T \Delta T.}}$$

Due to the blocked temperature strain, there exists a compressive stress at the end of the bar ($x = h$) at all times. As the stress distribution is linear, the stress will be compressive everywhere, if compression is present at the top edge. Thus the relation

$$\sigma(x = 0) < 0 \quad \text{bzw.} \quad \frac{W_0}{A} - E\alpha_T \Delta T < 0$$

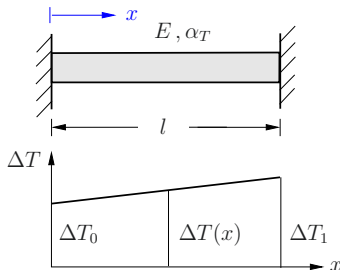
provides the necessary temperature difference

$$\underline{\underline{\Delta T > \frac{W_0}{EA\alpha_T} .}}$$

P2.7

Problem 2.7 An initially stress-free fixed bar (cross section area A) experiences a temperature increase varying linearly in x .

Determine the stress and strain distribution.



Solution The bar is supported in a statically indeterminate way. Thus we use equilibrium, kinematics and Hooke's law for the solution of the problem. With $n = 0$ and $\sigma = N/A$ these equations read

$$\sigma' = 0, \quad \varepsilon = u', \quad \varepsilon = \frac{\sigma}{E} + \alpha_T \Delta T(x)$$

with

$$\Delta T(x) = \Delta T_0 + (\Delta T_1 - \Delta T_0) \frac{x}{l}.$$

Combining the above relations renders the differential equation for the displacements

$$u'' = \alpha_T \Delta T' = \frac{\alpha_T}{l} (\Delta T_1 - \Delta T_0).$$

Integrating twice yields

$$u' = \frac{\alpha_T}{l} (\Delta T_1 - \Delta T_0) x + C_1,$$

$$u = \frac{\alpha_T}{l} (\Delta T_1 - \Delta T_0) \frac{x^2}{2} + C_1 x + C_2.$$

The two integration constants follow from the boundary conditions:

$$u(0) = 0 \rightsquigarrow C_2 = 0, \quad u(l) = 0 \rightsquigarrow C_1 = -\frac{\alpha_T}{2} (\Delta T_1 - \Delta T_0).$$

We obtain the displacement field

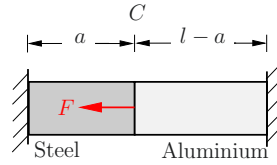
$$\underline{\underline{u(x) = \frac{\alpha_T l}{2} (\Delta T_1 - \Delta T_0) \left(\frac{x^2}{l^2} - \frac{x}{l} \right)}}$$

together with the (constant) stress

$$\underline{\underline{\sigma = E(u' - \alpha_T \Delta T) = -\frac{\alpha_T}{2} (\Delta T_1 + \Delta T_0) E.}}$$

Note: With constant heating $\Delta T_1 = \Delta T_0$ the displacement $u(x)$ vanishes. In this situation the stress is $\sigma = -\alpha_T \Delta T_0 E$.

Problem 2.8 A bar with a constant cross section A is fixed at both ends. The bar is made of two different materials, that are joint together at point C .



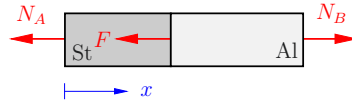
a) What are the reaction forces, if an external force F is applied at point C ?

b) Determine the normal force that is caused by a pure heating by ΔT ?

Given: $E_{St}/E_{Al} = 3$, $\alpha_{St}/\alpha_{Al} = 1/2$.

Solution We treat the system as two joint bars with constant normal forces.

to a)



equilibrium: $-N_A + N_B = F$,

kinematics: $\Delta l_{St} + \Delta l_{Al} = 0$,

Hooke's law: $\Delta l_{St} = \frac{N_A a}{E_{St} A}$, $\Delta l_{Al} = \frac{N_B (l-a)}{E_{Al} A}$.

The 4 equations for the 4 unknowns (N_A , N_B , Δl_{St} , Δl_{Al}) yield with the given numerical values

$$\underline{\underline{N_A = -F \frac{3(l-a)}{3l-2a}}} , \quad \underline{\underline{N_B = F \frac{a}{3l-2a}}} .$$

to b)

equilibrium: $N_A = N_B = N$,

kinematics: $\Delta l_{St} + \Delta l_{Al} = 0$,

Hooke's law: $\Delta l_{St} = \frac{N a}{E_{St} A} + \alpha_{St} \Delta T a$,

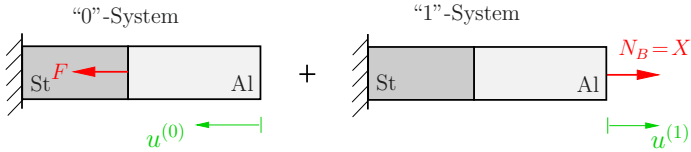
$$\Delta l_{Al} = \frac{N (l-a)}{E_{Al} A} + \alpha_{Al} \Delta T (l-a) .$$

Solving the system of equations for the normal force N yields with the given numerical values

$$\underline{\underline{N = -\frac{2l-a}{3l-2a} E_{St} \alpha_{St} A \Delta T}} .$$

P2.9 Problem 2.9 Solve Problem 2.8 by superposition.

Solution to a) We choose the reaction force N_B as statically redundant quantity.



Hooke's law provides

$$u^{(0)} = \frac{Fa}{E_{St}A}, \quad u^{(1)} = \frac{X(l-a)}{E_{Al}A} + \frac{Xa}{E_{St}A}.$$

As the right edge is fixed compatibility requires

$$u^{(0)} = u^{(1)}.$$

This condition yields

$$\underline{\underline{N_B}} = X = \frac{Fa}{a + (l-a)\frac{E_{St}A}{E_{Al}A}} = \underline{\underline{F \frac{a}{3l-2a}}}.$$

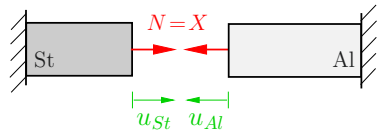
From equilibrium we have

$$\underline{\underline{N_A}} = N_B - F = -F \frac{3(l-a)}{3l-2a}.$$

to b) In the free body diagram we choose the normal force N as statically redundant quantity X . From Hooke's law

$$u_{St} = \frac{Xa}{E_{St}A} + \alpha_{St}\Delta T a,$$

$$u_{Al} = \frac{X(l-a)}{E_{Al}A} + \alpha_{Al}\Delta T(l-a)$$



and the compatibility

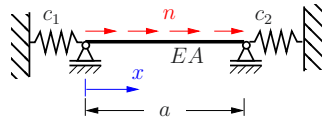
$$u_{St} + u_{Al} = 0$$

we obtain

$$\underline{\underline{N}} = X = -\frac{\alpha_{St}a + \alpha_{Al}(l-a)}{\frac{a}{E_{St}A} + \frac{(l-a)}{E_{Al}A}} = \underline{\underline{-\frac{2l-a}{3l-2a} E_{St} \alpha_{St} A \Delta T}}.$$

Problem 2.10 An elastically supported bar ($c_1 = 2c_2 = EA/2a$) is loaded by a constant axial load n .

Compute the distribution of the normal force $N(x)$ in the bar.

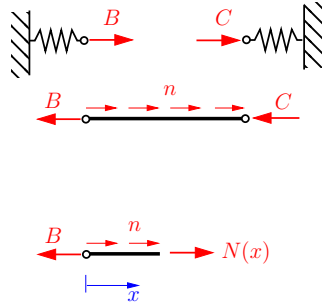


Solution Using the free body diagram with the forces B and C at the ends of the bar, the equilibrium conditions can be formulated

$$B + C = na, \quad N(x) = B - nx.$$

The elongation/shortening of the springs is given by

$$\Delta u_1 = \frac{B}{c_1}, \quad \Delta u_2 = \frac{C}{c_2}.$$



The elongation of the bar is computed from

$$\Delta u_{St} = \int_0^a \varepsilon dx = \int_0^a \frac{N}{EA} dx.$$

With $N = B - nx$ we obtain

$$\Delta u_{St} = \frac{Ba}{EA} - \frac{na^2}{2EA}.$$

Finally, the kinematic relation

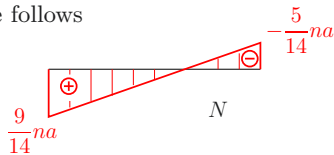
$$\Delta u_1 + \Delta u_{St} = \Delta u_2 \quad \rightsquigarrow \quad \frac{B}{c_1} + \frac{Ba}{EA} - \frac{na^2}{2EA} = \frac{C}{c_2}$$

with $C = -B + na$ and the given value for c_1 and c_2 yields

$$B \left(\frac{2a}{EA} + \frac{4a}{EA} + \frac{a}{EA} \right) = na \left(\frac{a}{2EA} + \frac{4a}{EA} \right) \quad \rightsquigarrow \quad B = \frac{9}{14} na$$

and the distribution of the normal force follows

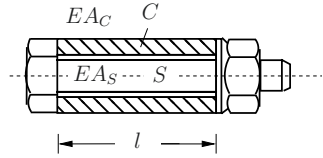
$$\underline{\underline{N(x) = \frac{9}{14} na - nx.}}$$



P2.11

Problem 2.11 Determine the compression Δl_C of a casing C of length l , if the nut of screw S (lead h) is turned by one revolution.

Given: $\frac{EA_C}{EA_S} = \frac{4}{3}$.



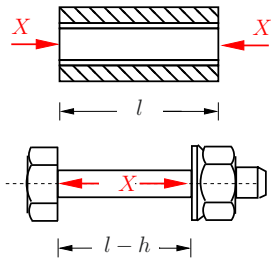
Solution After the revolution of the nut we cut the system of screw and casing and introduce the statically indeterminate force F between the two parts.

The casing experiences a compression

$$\Delta l_C = \frac{Xl}{EA_C}.$$

For the screw we obtain an elongation

$$\Delta l_S = \frac{Xl}{EA_S}.$$



The length changes have to be adjusted in such a way that casing and screw have the same length. Therefore compatibility can be written as

$$h = \Delta l_C + \Delta l_S.$$

Inserting the length changes yields the force

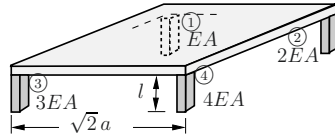
$$X = \frac{h}{l} \frac{1}{\frac{1}{EA_C} + \frac{1}{EA_S}}$$

and the compression of the casing

$$\underline{\underline{\Delta l_C}} = \frac{Xl}{EA_C} = h \frac{1}{1 + \frac{EA_C}{EA_S}} = h \frac{1}{1 + \frac{4}{3}} = \underline{\underline{\frac{3}{7}h}}.$$

Note: As the axial rigidity of the casing is larger than the one of the screw, the compression is only $3/7$ of the lead. If equal axial rigidities are present $EA_C = EA_S$, the length change of both parts will be equal, i. e. $\Delta l_C = \Delta l_S = h/2$.

Problem 2.12 A rigid quadratic plate (weight W , edge length $\sqrt{2} a$) is supported on 4 elastic posts. The posts are of equal length l , but possess different axial rigidities.



Determine the weight distribution on the 4 posts?

Determine the displacement f in the middle of the plate.

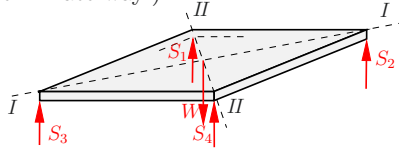
Solution The system is statically indeterminate of degree one (a table on 3 posts rests in a statically determinate way!).

Equilibrium yields

$$\uparrow: S_1 + S_2 + S_3 + S_4 = W,$$

$$\curvearrowright I: aS_4 = aS_1,$$

$$\curvearrowright II: aS_2 = aS_3.$$

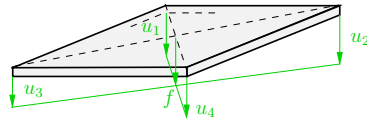


The displacement f in the middle is obtained from the average value of the displacements u_i ($=$ length change of the posts) at opposite corners (rigid plate). Accordingly the compatibility reads:

$$f = \frac{1}{2}(u_1 + u_4) = \frac{1}{2}(u_2 + u_3).$$

With Hooke's law

$$u_i = \frac{S_i l}{EA_i}$$



and $S_1 = S_4, S_2 = S_3$ we obtain as intermediate result

$$\frac{S_1 l}{EA} + \frac{S_1 l}{4EA} = \frac{S_2 l}{2EA} + \frac{S_2 l}{3EA} \quad \rightsquigarrow \quad \frac{5}{4} S_1 = \frac{5}{6} S_2.$$

Inserting this into the first equilibrium condition yields

$$S_1 + \frac{3}{2} S_1 + \frac{3}{2} S_1 + S_1 = G \quad \rightsquigarrow \quad \underline{\underline{S_1 = S_4 = \frac{1}{5} G}}, \quad \underline{\underline{S_2 = S_3 = \frac{3}{10} G}}.$$

from which the displacement follows:

$$\underline{\underline{f = \frac{1}{2} \left(\frac{S_1 l}{EA} + \frac{S_1 l}{4EA} \right) = \frac{1}{8} \frac{Gl}{EA}}}.$$

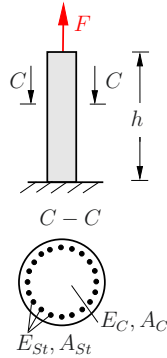
P2.13

Problem 2.13 A column of steel reinforced concrete is loaded by a tensile force F .

What are the stresses in the concrete and the steel as well as the height change Δh of the column, if we assume

- a perfect bonding between steel and concrete?
- the concrete is cracked and does not carry any load?

Given: $E_{St}/E_C = 6$, $A_{St}/A_C = 1/9$.



Solution to a) We consider the composite as a system of two "bars" of different materials, which experience under load F the same length change Δl . With this the basic equations of the system are:

equilibrium: $N_{St} + N_C = F$,

kinematics: $\Delta h_{St} = \Delta h_C = \Delta h$,

Hooke's law: $\Delta h_{St} = \frac{N_{St}h}{EA_{St}}$, $\Delta h_C = \frac{N_C h}{EA_C}$.

Solution of the system of equation yields –
with the stiffness ratio $EA_C/EA_{St} = 3/2$
– the normal forces

$$N_{St} = F \frac{1}{1 + \frac{EA_C}{EA_{St}}} = \frac{2}{5} F, \quad N_C = F \frac{\frac{EA_C}{EA_{St}}}{1 + \frac{EA_C}{EA_{St}}} = \frac{3}{5} F$$

and the height change

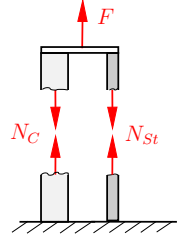
$$\underline{\underline{\Delta h}} = \frac{Fh}{EA_{St} + EA_C} \quad \text{i. e.} \quad \underline{\underline{\Delta h}} = \frac{Fh}{EA_{St}} \frac{1}{1 + \frac{EA_C}{EA_{St}}} = \frac{2}{5} \frac{Fl}{\underline{\underline{EA_{St}}}}$$

The stresses result from $A = A_C + A_{St}$ and $A_{St} = A/10$ and $A_C = 9A/10$

$$\underline{\underline{\sigma_{St}}} = \frac{N_{St}}{A_{St}} = 4 \frac{F}{\underline{\underline{A}}}, \quad \underline{\underline{\sigma_C}} = \frac{N_C}{A_C} = \frac{2}{3} \frac{F}{\underline{\underline{A}}}$$

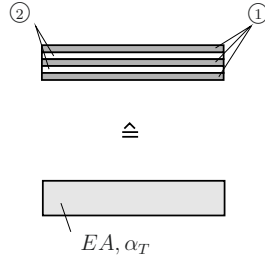
to b) If only the steel carries load, we will obtain with $N_{St} = F$

$$\underline{\underline{\sigma_{St}}} = \frac{F}{A_{St}} = 10 \frac{F}{\underline{\underline{A}}}, \quad \underline{\underline{\Delta h}} = \frac{Fh}{EA_{St}}$$



Problem 2.14 A laminated bar made of bonded layers of two different materials (respective axial rigidities EA_1 , EA_2 and coefficients of thermal expansion α_{T1} , α_{T2}) is to be replaced by a bar made of a homogeneous material.

Determine EA and α_T such that the homogeneous bar experiences the same elongation as the laminated bar under application of a force and a temperature change ?

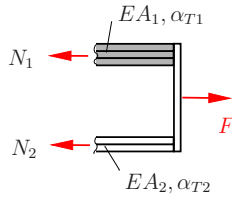


Solution For the laminated bar, subjected to a force F and a temperature increase ΔT , the basic equation yield

equilibrium: $N_1 + N_2 = F$,

kinematics: $\Delta l_1 = \Delta l_2 = \Delta l_{\text{lam}}$,

Hooke's law: $\Delta l_1 = \frac{N_1 l}{EA_1} + \alpha_{T1} \Delta T l$,
 $\Delta l_2 = \frac{N_2 l}{EA_2} + \alpha_{T2} \Delta T l$.



This yields

$$\Delta l_{\text{lam}} = \frac{Fl}{EA_1 + EA_2} + \frac{EA_1 \alpha_{T1} + EA_2 \alpha_{T2}}{EA_1 + EA_2} \Delta T l.$$

For a homogeneous bar under identical loading conditions, we have

$$\Delta l_{\text{hom}} = \frac{Fl}{EA} + \alpha_T \Delta T l.$$

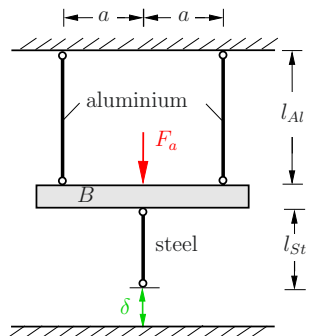
The length changes Δl_{lam} and Δl_{hom} agree for arbitrary F and ΔT only, if

$$\underline{\underline{EA = EA_1 + EA_2}}, \quad \underline{\underline{\alpha_T = \frac{EA_1 \alpha_{T1} + EA_2 \alpha_{T2}}{EA_1 + EA_2}}}.$$

P2.15

Problem 2.15 In the depicted support construction for the *rigid* body B the lower support bar is too short by the length δ . In order to assemble the structure a force F_a is applied, such that the end of the bar just touches the ground. After assembly the force F_a is removed. The diameters of all bars d_i are identical.

- a) Compute the required assembly force F_a .
 b) Determine the displacement v_B of the body and the forces in the bars after assembly.



Given: $l_{Al} = 1$ m, $d_{Al} = 2$ mm, $E_{Al} = 0.7 \cdot 10^5$ MPa, $l_{St} = 1.5$ m, $d_{St} = 2$ mm, $E_{St} = 2.1 \cdot 10^5$ MPa, $\delta = 5$ mm.

Solution to a) Each aluminium bar carries half of the assembly force (equilibrium) and elongates by the amount δ . This yields

$$S_{Al} = \frac{F_a}{2}, \quad \Delta l_{Al} = \frac{S_{Al} l_{Al}}{EA_{Al}} = \frac{F_a l_{Al}}{2EA_{Al}} = \delta,$$

$$\leadsto \underline{F_a} = 2 \frac{\delta}{l_{Al}} EA_{Al} = 2 \cdot \frac{5}{1000} \cdot 0,7 \cdot 10^5 \cdot \pi \cdot 1^2 = \underline{\underline{2200 \text{ N}}}.$$

to b) After removal of the force F_a new forces S_{Al} and S_{St} are present. This leads to the equilibrium condition

$$S_{St} = 2S_{Al},$$

Hooke's law

$$\Delta l_{Al} = \frac{S_{Al} l_{Al}}{EA_{Al}}, \quad \Delta l_{St} = \frac{S_{St} l_{St}}{EA_{St}}$$

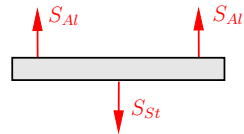
and the compatibility condition

$$\Delta l_{Al} + \Delta l_{St} = \delta.$$

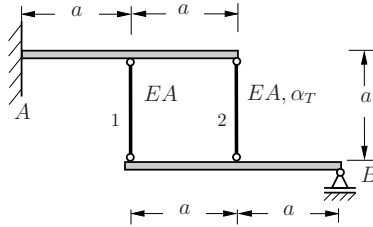
Solving the 4 equations yields

$$\underline{S_{Al}} = \frac{\delta}{l_{Al}} \frac{EA_{Al}}{1 + 2 \frac{l_{St}}{l_{Al}} \frac{EA_{Al}}{EA_{St}}} = \frac{5}{1000} \frac{0,7 \cdot 10^5 \cdot \pi \cdot 1^2}{1 + 2 \cdot \frac{3}{2} \cdot \frac{1}{3}} = \underline{\underline{550 \text{ N}}},$$

$$\underline{S_{St}} = 2S_{Al} = \underline{\underline{1100 \text{ N}}}, \quad \underline{v_K} = \Delta l_{Al} = \frac{S_{Al} l_{Al}}{EA_{Al}} = \underline{\underline{2,5 \text{ mm}}}.$$



Problem 2.16 Two rigid beams are connected by two elastic bars. The first beams is fixed at point A, while the second is simply supported at point B. Bar 2 is heated by a temperature ΔT .



Compute the forces in the two bars.

Solution We cut the system and use the following free body diagram to formulate the equilibrium conditions

$$\overset{\curvearrowright}{B} : 2aS_1 + aS_2 = 0,$$

Hooke's law

$$\Delta l_1 = \frac{S_1 a}{EA},$$

$$\Delta l_2 = \frac{S_2 a}{EA} + \alpha_T \Delta T \cdot a$$

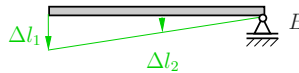
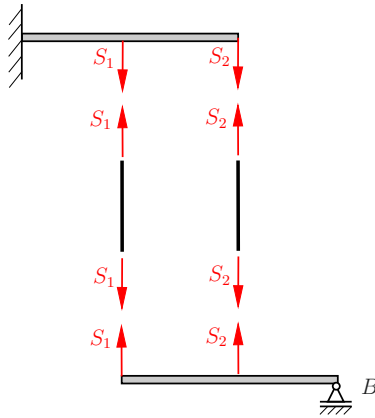
and the compatibility condition

$$\Delta l_1 = 2\Delta l_2.$$

Solving for the unknown forces in the bars yields

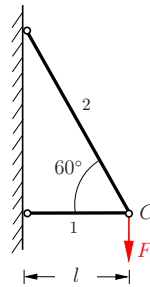
$$\underline{\underline{S_1 = \frac{2}{5} EA \alpha_T \Delta T}}, \quad \underline{\underline{S_2 = -\frac{4}{5} EA \alpha_T \Delta T}}.$$

Note: In the heated bar compressive forces are generated due to the constrained deformations.



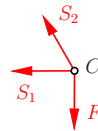
P2.17 Problem 2.17 In the depicted two bar system both bars have the same axial rigidity EA .

Determine the displacement of point C where the load is applied.



Solution From equilibrium we have

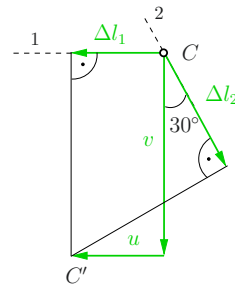
$$\begin{aligned} \uparrow: S_2 \sin 60^\circ &= F & \leadsto & S_2 = \frac{2}{3}\sqrt{3} F, \\ \rightarrow: -S_1 - S_2 \cos 60^\circ &= 0 & \leadsto & S_1 = -\frac{1}{3}\sqrt{3} F. \end{aligned}$$



Thus the elongation and shrinking of the bars follow as

$$\Delta l_2 = \frac{S_2 l_2}{EA} = \frac{\frac{2}{3}\sqrt{3} \frac{l}{\cos 60^\circ} F}{EA} = \frac{4\sqrt{3}}{3} \frac{Fl}{EA}, \quad \Delta l_1 = \frac{S_1 l_1}{EA} = -\frac{\sqrt{3}}{3} \frac{Fl}{EA}.$$

To determine the displacements of point C we construct the displacement diagram. In this diagram the length changes are introduced. As the length changes are small $\Delta l_i \ll l$ they are not drawn to the scale. In this example Δl_1 is a shrinkage (to the left) and Δl_2 an elongation. Considering that the bars can only rotate around the hinge points we introduce the right angles and read off the displacement diagram:

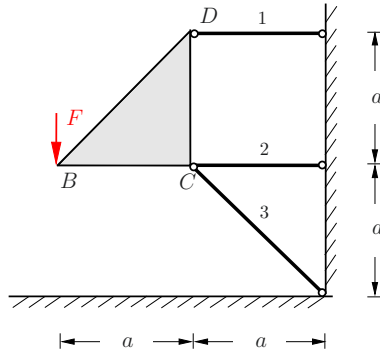


$$\underline{u} = |\Delta l_1| = \frac{\sqrt{3}}{3} \frac{Fl}{EA},$$

$$\underline{v} = \frac{\Delta l_2}{\cos 30^\circ} + \frac{u}{\tan 60^\circ} = \frac{4\sqrt{3}}{3} \frac{Fl}{EA} \frac{1}{\frac{1}{2}\sqrt{3}} + \frac{\sqrt{3}}{3} \frac{Fl}{EA} \frac{1}{\sqrt{3}} = 3 \frac{Fl}{EA}.$$

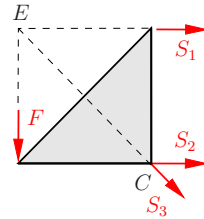
Problem 2.18 A rigid weightless triangle is supported by 3 bars with the axial rigidity EA . The triangle is loaded in point B by the force F .

- a) Determine the forces S_i in the 3 bars and their elongations Δl_i .
- b) Compute the displacement of point C .



Solution to a) The system is statically determinately supported. The forces in the bars follow immediately from the equilibrium conditions:

$$\begin{aligned} \curvearrowright C: aS_1 &= aF && \sim \underline{\underline{S_1 = F}}, \\ \curvearrowright E: aS_2 &= 0 && \sim \underline{\underline{S_2 = 0}}, \\ \uparrow: S_3 \sin 45^\circ + F &= 0 && \sim \underline{\underline{S_3 = -\sqrt{2} F}}. \end{aligned}$$

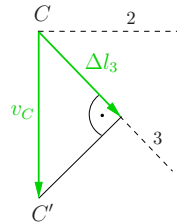


Related to these forces are the following elongations

$$\begin{aligned} \underline{\underline{\Delta l_1}} &= \frac{S_1 l_1}{EA} = \frac{Fa}{EA}, && \underline{\underline{\Delta l_2}} = 0, \\ \underline{\underline{\Delta l_3}} &= \frac{S_3 l_3}{EA} = -\frac{\sqrt{2} F \cdot \sqrt{2} a}{EA} = \underline{\underline{-2 \frac{Fa}{EA}}}. \end{aligned}$$

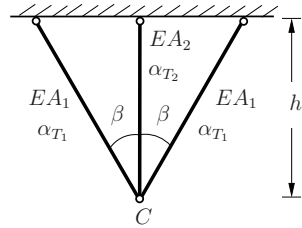
to b) The displacement of point C is sketched in the displacement diagram. As bar 2 experiences no force and thus no length change, the horizontal displacement vanishes. From the displacement diagram we obtain for the vertical displacement v_C :

$$\underline{\underline{v_C}} = \sqrt{2} |\Delta l_3| = \underline{\underline{2\sqrt{2} \frac{Fa}{EA}}}.$$



P2.19 Problem 2.19 In the depicted truss the members have the axial rigidities EA_1 , EA_2 and the coefficients of thermal expansion α_{T1} , α_{T2} .

Determine the axial forces in the trusses, if the system is heated by ΔT ?



Solution As the system is statically indeterminate, we have to use all basic equations. We start with the equilibrium

$$2S_1 \cos \beta + S_2 = 0$$

and continue with Hooke's law

$$\Delta l_1 = \frac{S_1 l_1}{EA_1} + l_1 \alpha_{T1} \Delta T,$$

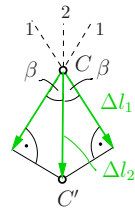
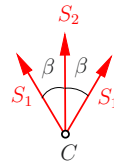
$$\Delta l_2 = \frac{S_2 l_2}{EA_2} + l_2 \alpha_{T2} \Delta T,$$

where

$$l_1 = \frac{h}{\cos \beta}, \quad l_2 = h.$$

The compatibility of the displacements is according to the displacement diagram:

$$\Delta l_1 = \Delta l_2 \cos \beta.$$



Solving the 4 equations for the two truss forces and the two elongations yields

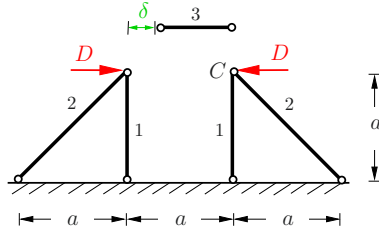
$$S_1 = EA_1 \frac{\alpha_{T2} \cos^2 \beta - \alpha_{T1}}{1 + 2 \cos^3 \beta \frac{EA_1}{EA_2}} \Delta T, \quad \underline{\underline{S_2 = -2 \cos \beta S_1.}}$$

Note: For $\cos \beta = \sqrt{\alpha_{T1}/\alpha_{T2}}$ we obtain $S_1 = S_2 = 0$: the trusses can then expand without causing forces! (special case $\alpha_{T1} = \alpha_{T2} \rightsquigarrow \beta = 0$)

Problem 2.20 Truss member 3 was produced too short to be assembled between two identical trusses.

- a) Determine the required assembly force D ?
- b) Calculate the normal force S_3 after the assembly ($D = 0$)?

Given: $EA_1 = EA_3 = EA$, $EA_2 = \sqrt{2}EA$.



Solution to a) The force D has to move point C by $\delta/2$ in horizontal direction during assembly. From equilibrium

$$\rightarrow: S_2 \cos 45^\circ = D,$$

$$\uparrow: S_1 = S_2 \cos 45^\circ,$$

kinematics (S_1 was positively introduced as compressive force!!) with the prescribed displacement

$$u_C = \Delta l_1 + \Delta l_2 \sqrt{2}, \quad u_C = \frac{\delta}{2},$$

and Hooke's law

$$\Delta l_1 = \frac{S_1 a}{EA}, \quad \Delta l_2 = \frac{S_2 a \sqrt{2}}{\sqrt{2} EA}$$

we obtain

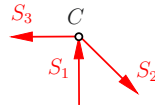
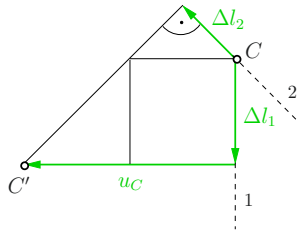
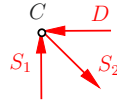
$$\underline{\underline{D = \frac{1}{6} \frac{\delta}{a} EA.}}$$

to b) Equilibrium, kinematics and Hooke's law are as in a), but D has to be replaced by S_3 . With the known compatibility condition

$$2u_C + \Delta l_3 = \delta \quad \text{and} \quad \Delta l_3 = \frac{S_3 a}{EA}$$

it follows

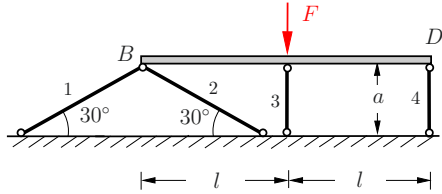
$$\underline{\underline{S_3 = \frac{1}{7} \frac{\delta}{a} EA.}}$$



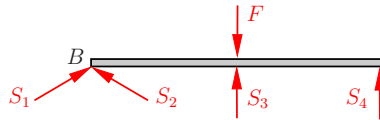
P2.21

Problem 2.21 A centric loaded rigid beam is supported by 4 elastic bars of equal axial rigidity EA .

Determine the forces in the bars?



Solution a) First, we solve the statically indeterminate system by applying all basic equations simultaneously. Using equilibrium



$$\rightarrow: S_1 = S_2,$$

$$\uparrow: (S_1 + S_2) \sin 30^\circ + S_3 + S_4 = F,$$

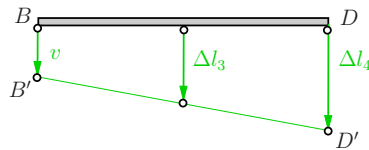
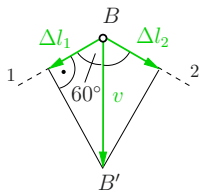
$$\curvearrowright_B: lS_3 + 2lS_4 = lF,$$

Hooke's laws

$$\Delta l_1 = \Delta l_2 = \frac{S_1 2a}{EA},$$

$$\Delta l_3 = \frac{S_3 a}{EA}, \quad \Delta l_4 = \frac{S_4 a}{EA}$$

and the geometry of the deformation



$$v = \frac{\Delta l_1}{\cos 60^\circ}$$

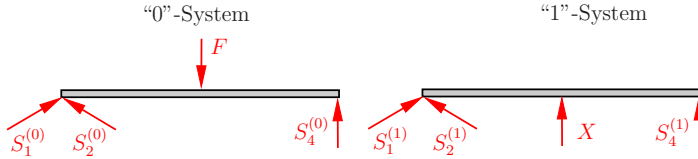
$$\Delta l_3 = \frac{1}{2}(v + \Delta l_4)$$

we obtain as solution

$$\underline{\underline{S_1 = S_2 = S_4 = \frac{2}{9} F}},$$

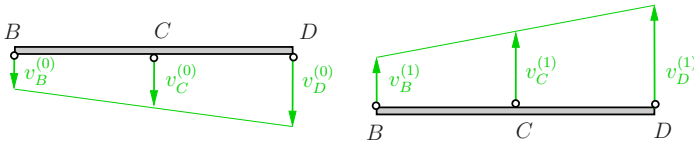
$$\underline{\underline{S_3 = \frac{5}{9} F}}.$$

b) Now, we solve the problem by superposition. The system is divided into two statically determinate basic systems:



Equilibrium yields

$$S_1^{(0)} = S_2^{(0)} = S_4^{(0)} = \frac{F}{2}, \quad S_1^{(1)} = S_2^{(1)} = S_4^{(1)} = \frac{X}{2}.$$



From geometry and Hooke's laws it follows

$$\begin{aligned} v_B^{(0)} &= \frac{\Delta l_1^{(0)}}{\cos 60^\circ} = \frac{F 2a}{EA}, & v_B^{(1)} &= \frac{X 2a}{EA}, \\ v_D^{(0)} &= \Delta l_4^{(0)} = \frac{Fa}{2EA}, & v_D^{(1)} &= \frac{Xa}{2EA}, \\ v_C^{(0)} &= \frac{1}{2} (v_B^{(0)} + v_D^{(0)}) = \frac{5 Fa}{4 EA}, & v_C^{(1)} &= \frac{5 Xa}{4 EA}, \\ & & \Delta l_3^{(1)} &= \frac{Xa}{EA}. \end{aligned}$$

The kinematic compatibility requires the total displacement of point C to coincide with the shortening of truss 3:

$$v_C^{(0)} - v_C^{(1)} = \Delta l_3^{(1)}.$$

Inserting the displacements yields

$$\underline{\underline{X = S_3 = \frac{5}{9} F}}$$

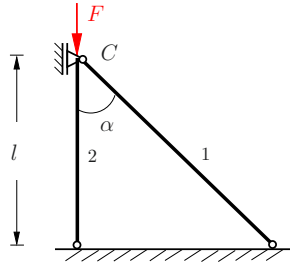
and

$$\underline{\underline{S_1 = S_1^{(0)} - S_1^{(1)} = \frac{2}{9} F}}, \quad \underline{\underline{S_4 = S_4^{(0)} - S_4^{(1)} = \frac{2}{9} F}}.$$

P2.22

Problem 2.22 The depicted truss system (axial rigidity EA) is loaded by the external force F and additionally pinned at point C .

- Determine the reaction force at point C .
- Calculate the vertical displacement of point C .



Solution to a) Using equilibrium

$$\downarrow: F + S_2 + S_1 \cos \alpha = 0,$$

$$\rightarrow: C + S_1 \sin \alpha = 0,$$

Hooke's laws

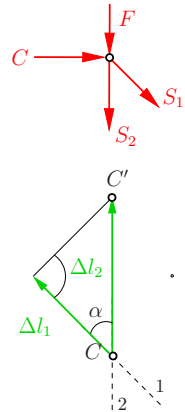
$$\Delta l_1 = \frac{S_1 l_1}{EA}, \quad \Delta l_2 = \frac{S_2 l_2}{EA},$$

and kinematics

$$\Delta l_1 = \Delta l_2 \cos \alpha$$

yields

$$\underline{\underline{C}} = \underline{\underline{\frac{\sin \alpha \cos^2 \alpha}{1 + \cos^3 \alpha} F}}, \quad \underline{\underline{S_1}} = \underline{\underline{-\frac{\cos^2 \alpha}{1 + \cos^3 \alpha} F}}, \quad \underline{\underline{S_2}} = \underline{\underline{-\frac{1}{1 + \cos^3 \alpha} F}}.$$



to b) Knowing S_2 the vertical displacement of point C follows as

$$\underline{\underline{v_C}} = \underline{\underline{\Delta l_2}} = \underline{\underline{\frac{S_2 l}{EA}}} = \underline{\underline{-\frac{1}{1 + \cos^3 \alpha} \frac{Fl}{EA}}}.$$

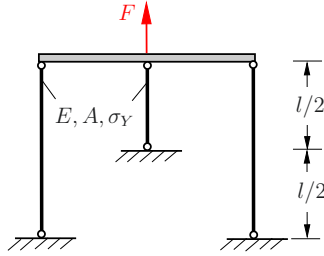
In contrast to the displacement diagram, in which tensile forces (elongations) are assumed, compressive force occur in the system. Due to shortening point C moves in downwards direction.

Test: $\alpha = \pi/2$ yields $S_1 = 0$ and $S_2 = -F$.

$\alpha = 0$ yields $S_1 = S_2 = -F/2$.

Problem 2.23 A rigid beam is supported by three bars of elastic-ideal-plastic material.

- a) At what force F_{max}^{el} and at which location in the bars is the yield stress σ_Y reached at first?
- b) At what force F_{max}^{pl} occurs plastic yielding in all bars of the system?



Solution to a) The system is statically indeterminate. Using symmetry equilibrium provides

$$2S_1 + S_2 = F$$

Kinematics is expressed by

$$\Delta l_1 = \Delta l_2.$$

Until plastic yielding Hooke's law can be used

$$\Delta l_1 = \frac{S_1 l}{EA}, \quad \Delta l_2 = \frac{S_2 l}{2EA}.$$

The solution provides forces and stresses in the bars

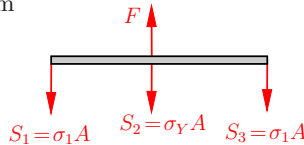
$$S_1 = \frac{F}{4}, \quad S_2 = \frac{F}{2} \quad \rightsquigarrow \quad \sigma_1 = \frac{F}{4A}, \quad \sigma_2 = \frac{F}{2A}.$$

As the stress in bar 2 is the highest, the yield limit is reached first there during load increase:

$$\sigma_2 = \sigma_Y \quad \rightsquigarrow \quad \underline{\underline{F_{max}^{el} = 2\sigma_Y A.}}$$

to b) For a load increase above F_{max}^{el} bar 1 and bar 3 still respond elastically, while bar two undergoes plastic deformation: $\sigma_2 = \sigma_Y$. Thus with $S_i = \sigma_i A$ it follows from equilibrium

$$2\sigma_1 A + \sigma_Y A = F \quad \rightsquigarrow \quad \sigma_1 = \frac{F}{2A} - \frac{\sigma_Y}{2}.$$

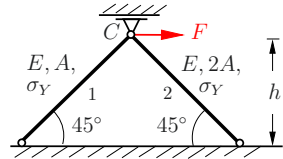


All bars are undego plastic deformation if

$$\sigma_1 = \sigma_Y \quad \rightsquigarrow \quad \frac{F}{2A} - \frac{\sigma_Y}{2} = \sigma_Y \quad \rightsquigarrow \quad \underline{\underline{F_{max}^{pl} = 3\sigma_Y A.}}$$

P2.24

Problem 2.24 In the depicted symmetric system all bars are made of the same elastic-ideal-plastic material, but have different cross sections.



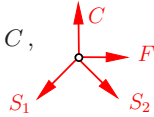
a) At what force F_{max}^{el} and at which location in the bars is the yield stress σ_Y reached at first? Determine the reaction force at C for this situation.

b) Determine the force F_{max}^{pl} when both bars deform plastically?

c) Calculate the displacement u_{max}^{el} of point C for case a)?

Solution to a) Until reaching the force F_{max}^{el} the system responds elastically. Therefore the equilibrium conditions are given by

$$\rightarrow: \frac{\sqrt{2}}{2} S_1 - \frac{\sqrt{2}}{2} S_2 = F, \quad \uparrow: \frac{\sqrt{2}}{2} S_1 + \frac{\sqrt{2}}{2} S_2 = C,$$

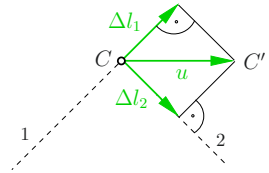


together with Hooke's law

$$\Delta l_1 = \frac{S_1 \sqrt{2} h}{EA}, \quad \Delta l_2 = \frac{S_2 \sqrt{2} h}{2EA}$$

and the kinematics (bar 2 will shorten)

$$\Delta l_1 = -\Delta l_2.$$



From the above relation we obtain

$$S_1 = \frac{\sqrt{2}}{3} F, \quad S_2 = -\frac{2\sqrt{2}}{3} F, \quad C = -\frac{F}{3}, \quad \Delta l_1 = -\Delta l_2 = \frac{2Fh}{3EA}$$

$$\leadsto \quad \sigma_1 = \frac{S_1}{A} = \frac{\sqrt{2}}{3} \frac{F}{A}, \quad \sigma_2 = \frac{S_2}{2A} = -\frac{\sqrt{2}}{3} \frac{F}{A}.$$

The absolute value of the stresses is identical in both bars. Yielding will occur if

$$\sigma_1 = |\sigma_2| = \sigma_Y \quad \leadsto \quad \underline{\underline{F_{max}^{el} = \frac{3}{2} \sqrt{2} \sigma_Y A}}, \quad \leadsto \quad \underline{\underline{C_{max}^{el} = -\frac{\sqrt{2}}{2} \sigma_Y A}}.$$

to b) As at F_{max}^{el} plastic yielding occurs in both bars, we have

$$\underline{\underline{F_{max}^{el} = F_{max}^{pl}}}.$$

to c) Until the yield limit is reached the displacement of C is given by

$$u = \sqrt{2} \Delta l_1 = \frac{2\sqrt{2}}{3} \frac{Fh}{EA}, \quad \leadsto \quad \underline{\underline{u_{max}^{el} = u(F_{max}^{el}) = 2 \frac{\sigma_Y}{E} h}}.$$



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