# Chapter 2
# Topological Spaces

A *topological space* $(X, \tau)$ is a set $X$ with a *topology* $\tau$, i.e., a collection of subsets of $X$ with the following properties:

1. $X \in \tau$, $\emptyset \in \tau$;
2. If $A, B \in \tau$, then $A \cap B \in \tau$;
3. For any collection $\{A_\alpha\}_\alpha$, if all $A_\alpha \in \tau$, then $\cup_\alpha A_\alpha \in \tau$.

The sets in $\tau$ are called *open sets*, and their complements are called *closed sets*. A *base* of the topology $\tau$ is a collection of open sets such that every open set is a union of sets in the base. The coarsest topology has two open sets, the empty set and $X$, and is called the *trivial topology* (or *indiscrete topology*). The finest topology contains all subsets as open sets, and is called the *discrete topology*.

In a metric space $(X, d)$ define the *open ball* as the set $B(x, r) = \{y \in X : d(x, y) < r\}$, where $x \in X$ (the *center* of the ball), and $r \in \mathbb{R}, r > 0$ (the *radius* of the ball). A subset of $X$ which is the union of (finitely or infinitely many) open balls, is called an *open set*. Equivalently, a subset $U$ of $X$ is called *open* if, given any point $x \in U$, there exists a real number $\epsilon > 0$ such that, for any point $y \in X$ with $d(x, y) < \epsilon$, $y \in U$.

Any metric space is a topological space, the topology (**metric topology**, *topology induced by the metric d*) being the set of all open sets. The metric topology is always $T_4$ (see below a list of topological spaces). A topological space which can arise in this way from a metric space, is called a **metrizable space**.

A *quasi-pseudo-metric topology* is a topology on $X$ induced similarly by a quasi-semimetric $d$ on $X$, using the set of open $d$-balls $B(x, r)$ as the base. In particular, *quasi-metric topology* and *pseudo-metric topology* are the terms used for the case of, respectively, quasi-metric and semimetric $d$. In general, those topologies are not $T_0$.

Given a topological space $(X, \tau)$, a *neighborhood* of a point $x \in X$ is a set containing an open set which in turn contains $x$. The *closure* of a subset of a topological space is the smallest closed set which contains it. An *open cover* of $X$ is a collection $\mathcal{L}$ of open sets, the union of which is $X$; its *subcover* is a cover $\mathcal{K}$



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such that every member of $\mathcal{K}$ is a member of $\mathcal{L}$; its *refinement* is a cover $\mathcal{K}$, where every member of $\mathcal{K}$ is a subset of some member of $\mathcal{L}$. A collection of subsets of $X$ is called *locally finite* if every point of $X$ has a neighborhood which meets only finitely many of these subsets.

A subset $A \subset X$ is called *dense* if $X = cl(A)$, i.e., it consists of $A$ and its *limit points*; cf. **closed subset of metric space** in Chap. 1. The *density* of a topological space is the least cardinality of its dense subset. A *local base* of a point $x \in X$ is a collection $\mathcal{U}$ of neighborhoods of $x$ such that every neighborhood of $x$ contains some member of $\mathcal{U}$.

A function from one topological space to another is called *continuous* if the preimage of every open set is open. Roughly, given $x \in X$, all points close to $x$ map to points close to $f(x)$. A function $f$ from one metric space $(X, d_X)$ to another metric space $(Y, d_Y)$ is *continuous* at the point $c \in X$ if, for any positive real number $\epsilon$, there exists a positive real number $\delta$ such that all $x \in X$ satisfying $d_X(x, c) < \delta$ will also satisfy $d_Y(f(x), f(c)) < \epsilon$; the function is continuous on an interval $I$ if it is continuous at any point of $I$.

The following classes of topological spaces (up to $T_4$) include any metric space.

- **$T_0$-space**

  A **$T_0$-space** (or *Kolmogorov space*) is a topological space in which every two distinct points are *topologically distinguishable*, i.e., have different neighborhoods.
- **$T_1$-space**

  A **$T_1$-space** (or *accessible space*) is a topological space in which every two distinct points are *separated*, i.e., each does not belong to other's closure. $T_1$-spaces are always $T_0$.
- **$T_2$-space**

  A **$T_2$-space** (or **Hausdorff space**) is a topological space in which every two distinct points are *separated by neighborhoods*, i.e., have disjoint neighborhoods. $T_2$-spaces are always $T_1$.

  A space is $T_2$ if and only if it is both $T_0$ and *pre-regular*, i.e., any two *topologically distinguishable* points are separated by neighborhoods.
- **Regular space**

  A **regular space** is a topological space in which every neighborhood of a point contains a closed neighborhood of the same point. A **$T_3$-space** (or *Vietoris space*, *regular Hausdorff space*) is a topological space which is $T_1$ and regular.

  Bing, Nagata, Smirnov showed in 1950–1951 that a topological space is metrizable if and only if it is regular, $T_0$ and has a countably locally finite base.

  A **completely regular space** (or *Tychonoff space*) is a **Hausdorff space** $(X, \tau)$ in which any closed set $A$ and any $x \notin A$ are *functionally separated*, i.e., there is a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.
- **Normal space**

  A **normal space** is a topological space in which, for any two disjoint closed sets $A$ and $B$, there exist two disjoint open sets $U$ and $V$ such that $A \subset U$, and

$B \subset V$. A $T_4$**-space** (or *Tietze space*, *normal Hausdorff space*) is a topological space which is $T_1$ and normal. Any metric space is a perfectly normal $T_4$-space.

A **completely** (or *hereditarily*) **normal space** is a topological space in which any two *separated* (i.e., disjoint from the other's closure) sets have disjoint neighborhoods. A $T_5$**-space** (or *completely normal Hausdorff space*) is a topological space which is completely normal and $T_1$. $T_5$-spaces are always $T_4$.

A **monotonically normal space** is a **completely normal space** in which any two *separated* subsets $A$ and $B$ are *strongly separated*, i.e., there exist open sets $U$ and $V$ with $A \subset U, B \subset V$ and $Cl(U) \cap Cl(V) = \emptyset$.

A **perfectly normal space** is a topological space $(X, \tau)$ in which any two disjoint closed subsets of $X$ are *functionally separated*. A $T_6$**-space** (or *perfectly normal Hausdorff space*) is a topological space which is $T_1$ and perfectly normal. $T_6$-spaces are always $T_5$.

- **Moore space**

  A **Moore space** is a **regular space** with a *development*.

  A *development* is a sequence $\{\mathcal{U}_n\}_n$ of open covers such that, for every $x \in X$ and every open set $A$ containing $x$, there exists $n$ such that $St(x, \mathcal{U}_n) = \cup\{U \in \mathcal{U}_n : x \in U\} \subset A$, i.e., $\{St(x, \mathcal{U}_n)\}_n$ is a *neighborhood base* at $x$.

- **Polish space**

  A **separable space** is a topological space which has a countable dense subset.

  A **Polish space** is a separable space which can be equipped with a complete metric. A *Lusin space* is a topological space such that some weaker topology makes it into a Polish space; every Polish space is Lusin. A *Souslin space* is a continuous image of a Polish space; every Lusin space is Suslin.

- **Lindelöf space**

  A **Lindelöf space** is a topological space in which every open cover has a countable subcover.

- **First-countable space**

  A topological space is called **first-countable** if every point has a countable local base. Every metric space is first-countable.

- **Second-countable space**

  A topological space is called **second-countable** if its topology has a countable base. Such space is **quasi-metrizable** and, if and only if it is a $T_3$-space, **metrizable**.

  Second-countable spaces are **first-countable**, **separable** and **Lindelöf**. The properties **second-countable**, **separable** and **Lindelöf** are equivalent for metric spaces.

  The Euclidean space $\mathbb{E}^n$ with its usual topology is second-countable.

- **Baire space**

  A **Baire space** is a topological space in which every intersection of countably many dense open sets is dense. Every complete metric space is a Baire space. Every locally compact $T_2$-space (hence, every $n$**-manifold**) is a Baire space.

- **Alexandrov space**

  An **Alexandrov space** is a topological space in which every intersection of arbitrarily many open sets is open.

A topological space is called a ***P*-space** if every $G_\delta$*-set* (i.e., the intersection of countably many open sets) is open.

A topological space $(X, \tau)$ is called a ***Q*-space** if every subset $A \subset X$ is a $G_\delta$-set.

- **Connected space**

A topological space $(X, \tau)$ is called **connected** if it is not the union of a pair of disjoint nonempty open sets. In this case the set $X$ is called a *connected set*.

A connected topological space $(X, \tau)$ is called *unicoherent* if the intersection $A \cap B$ is connected for any closed connected sets $A, B$ with $A \cup B = X$.

A topological space $(X, \tau)$ is called **locally connected** if every point $x \in X$ has a local base consisting of connected sets.

A topological space $(X, \tau)$ is called **path-connected** (or 0-*connected*) if for every points $x, y \in X$ there is a path $\gamma$ from $x$ to $y$, i.e., a continuous function $\gamma : [0, 1] \to X$ with $\gamma(x) = 0, \gamma(y) = 1$.

A topological space $(X, \tau)$ is called **simply connected** (or 1-*connected*) if it consists of one piece, and has no circle-shaped "holes" or "handles" or, equivalently, if every continuous curve of $X$ is *contractible*, i.e., can be reduced to one of its points by a *continuous deformation*.

A topological space $(X, \tau)$ is called **hyperconnected** (or *irreducible*) if $X$ cannot be written as the union of two proper closed sets.

- **Sober space**

A topological space $(X, \tau)$ is called **sober** if every **hyperconnected** closed subset of $X$ is the closure of exactly one point of $X$. Any sober space is a $T_0$-space.

Any $T_2$-space is a sober $T_1$-space but some sober $T_1$-spaces are not $T_2$.

- **Paracompact space**

A topological space is called **paracompact** if every open cover of it has an open locally finite refinement. Every **metrizable** space is paracompact.

- **Totally bounded space**

A topological space $(X, \tau)$ is called **totally bounded** (or *pre-compact*) if it can be covered by finitely many subsets of any fixed cardinality.

A metric space $(X, d)$ is a **totally bounded metric space** if, for every real number $r > 0$, there exist finitely many open balls of radius $r$, whose union is equal to $X$.

- **Compact space**

A topological space $(X, \tau)$ is called **compact** if every open cover of $X$ has a finite subcover.

Compact spaces are always **Lindelöf**, **totally bounded**, and **paracompact**. A metric space is compact if and only if it is **complete** and **totally bounded**. A subset of a Euclidean space $\mathbb{E}^n$ is compact if and only if it is closed and bounded.

There exist a number of topological properties which are equivalent to compactness in metric spaces, but are nonequivalent in general topological spaces. Thus, a metric space is compact if and only if it is a *sequentially compact space* (every sequence has a convergent subsequence), or a *countably compact space* (every countable open cover has a finite subcover), or a *pseudo-compact* *space* (every real-valued continuous function on the space is bounded), or a *weakly countably compact space* (i.e., every infinite subset has an accumulation point).

Sometimes, a compact **connected** $T_2$**-space** is called *continuum*; cf. **continuum** in Chap. 1.

- **Locally compact space**

A topological space is called **locally compact** if every point has a local base consisting of compact neighborhoods. The Euclidean spaces $\mathbb{E}^n$ and the spaces $\mathbb{Q}_p$ of *p-adic numbers* are locally compact.

A topological space $(X, \tau)$ is called a *k-space* if, for any compact set $Y \subset X$ and $A \subset X$, the set $A$ is closed whenever $A \cap Y$ is closed. The $k$-spaces are precisely quotient images of locally compact spaces.

- **Locally convex space**

A **topological vector space** is a real (complex) vector space $V$ which is a $T_2$**-space** with continuous vector addition and scalar multiplication. It is a **uniform space** (cf. Chap. 3).

A **locally convex space** is a topological vector space whose topology has a base, where each member is a *convex balanced absorbent* set. A subset $A$ of $V$ is called *convex* if, for all $x, y \in A$ and all $t \in [0, 1]$, the point $tx + (1-t)y \in A$, i.e., every point on the *line segment* connecting $x$ and $y$ belongs to $A$. A subset $A$ is *balanced* if it contains the line segment between $x$ and $-x$ for every $x \in A$; $A$ is *absorbent* if, for every $x \in V$, there exist $t > 0$ such that $tx \in A$.

The locally convex spaces are precisely vector spaces with topology induced by a family $\{||.||_\alpha\}$ of seminorms such that $x = 0$ if $||x||_\alpha = 0$ for every $\alpha$.

Any metric space $(V, ||x-y||)$ on a real (complex) vector space $V$ with a **norm metric** $||x-y||$ is a locally convex space; each point of $V$ has a local base consisting of convex sets. Every $L_p$ with $0 < p < 1$ is an example of a vector space which is not locally convex.

- ***n*-manifold**

Broadly, a *manifold* is a topological space locally homeomorphic to a **topological vector space** over the reals.

But usually, a **topological manifold** is a **second-countable** $T_2$**-space** that is locally homeomorphic to Euclidean space. An $n$**-manifold** is a topological manifold such that every point has a neighborhood homeomorphic to $\mathbb{E}^n$.

- **Fréchet space**

A **Fréchet space** is a **locally convex space** $(V, \tau)$ which is complete as a **uniform space** and whose topology is defined using a countable set of seminorms $||.||_1, \dots, ||.||_n, \dots$, i.e., a subset $U \subset V$ is *open in* $(V, \tau)$ if, for every $u \in U$, there exist $\epsilon > 0$ and $N \geq 1$ with $\{v \in V : ||u-v||_i < \epsilon \text{ if } i \leq N\} \subset U$.

A Fréchet space is precisely a locally convex **F-space** (cf. Chap. 5). Its topology can be induced by a **translation invariant metric** (Chap. 5) and it is a complete and **metrizable space** with respect to this topology. But this topology may be induced by many such metrics. Every **Banach space** is a Fréchet space.

- **Countably-normed space**

  A **countably-normed space** is a **locally convex space** $(V, \tau)$ whose topology is defined using a countable set of *compatible norms* $||.||_1, \dots, ||.||_n, \dots$. It means that, if a sequence $\{x_n\}_n$ of elements of $V$ that is fundamental in the norms $||.||_i$ and $||.||_j$ converges to zero in one of these norms, then it also converges in the other. A countably-normed space is a **metrizable space**, and its metric can be defined by

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{||x-y||_n}{1+||x-y||_n}.$$

- **Metrizable space**

  A topological space $(T, \tau)$ is called **metrizable** if it is homeomorphic to a metric space, i.e., $X$ admits a metric $d$ such that the set of open $d$-balls $\{B(x, r) : r > 0\}$ forms a neighborhood base at each point $x \in X$. If, moreover, $(X, d)$ is a complete metric space for one of such metrics $d$, then $(X, d)$ is a *completely metrizable* (or *topologically complete*) space.

  Metrizable spaces are always **paracompact** $T_2$**-spaces** (hence, **normal** and **completely regular**), and **first-countable**.

  A topological space is called **locally metrizable** if every point in it has a metrizable neighborhood.

  A topological space $(X, \tau)$ is called **submetrizable** if there exists a metrizable topology $\tau'$ on $X$ which is coarser than $\tau$.

  A topological space $(X, \tau)$ is called **proto-metrizable** if it is paracompact and has an *orthobase*, i.e., a base $\mathcal{B}$ such that, for $\mathcal{B}' \subset \mathcal{B}$, either $\cap\mathcal{B}'$ is open, or $\mathcal{B}'$ is a local base at the unique point in $\cap\mathcal{B}'$. It is not related to the **protometric** in Chap. 1.

  Some examples of other direct generalizations of metrizable spaces follow.

  A **sequential space** is a quotient image of a metrizable space.

  Morita's $M$**-space** is a topological space $(X, \tau)$ from which there exists a continuous map $f$ onto a metrizable topological space $(Y, \tau')$ such that $f$ is closed and $f^{-1}(y)$ is countably compact for each $y \in Y$.

  Ceder's $M_1$**-space** is a topological space $(X, \tau)$ having a $\sigma$-closure-preserving base (metrizable spaces have $\sigma$-locally finite bases).

  Okuyama's $\sigma$**-space** is a topological space $(X, \tau)$ having a $\sigma$-locally finite *net*, i.e., a collection $\mathcal{U}$ of subsets of $X$ such that, given of a point $x \in U$ with $U$ open, there exists $U' \in \mathcal{U}$ with $x \in U' \subset U$ (a base is a net consisting of open sets). Every compact subset of a $\sigma$-space is metrizable.

  Michael's **cosmic space** is a topological space $(X, \tau)$ having a countable net (equivalently, a Lindelöf $\sigma$-space). It is exactly a continuous image of a separable metric space. A $T_2$**-space** is called **analytic** if it is a continuous image of a complete separable metric space; it is called a **Lusin space** if, moreover, the image is one-to-one.

- **Quasi-metrizable space**

  A topological space $(X, \tau)$ is called a **quasi-metrizable space** if $X$ admits a quasi-metric $d$ such that the set of open $d$-balls $\{B(x,r) : r > 0\}$ forms a neighborhood base at each point $x \in X$.

  A more general $\gamma$**-space** is a topological space admitting a $\gamma$**-metric** $d$ (i.e., a function $d : X \times X \to \mathbb{R}_{\geq 0}$ with $d(x, z_n) \to 0$ whenever $d(x, y_n) \to 0$ and $d(y_n, z_n) \to 0$) such that the set of open *forward* $d$-balls $\{B(x,r) : r > 0\}$ forms a neighborhood base at each point $x \in X$.

  The *Sorgenfrey line* is the topological space $(\mathbb{R}, \tau)$ defined by the base $\{[a,b) : a, b \in \mathbb{R}, a < b\}$. It is not metrizable but it is a first-countable separable and paracompact $T_5$**-space**; neither it is second-countable, nor locally compact or locally connected. However, the Sorgenfrey line is quasi-metrizable by the **Sorgenfrey quasi-metric** (cf. Chap. 12) defined as $y - x$ if $y \geq x$, and 1, otherwise.
- **Symmetrizable space**

  A topological space $(X, \tau)$ is called **symmetrizable** (and $\tau$ is called the **distance topology**) if there is a **symmetric** $d$ on $X$ (i.e., a distance $d : X \times X \to \mathbb{R}_{\geq 0}$ with $d(x,y) = 0$ implying $x = y$) such that a subset $U \subset X$ is open if and only if, for each $x \in U$, there exists $\epsilon > 0$ with $B(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\} \subset U$.

  In other words, a subset $H \subset X$ is closed if and only if $d(x, H) = \inf_y\{d(x,y) : y \in H\} > 0$ for each $x \in X \backslash U$. A symmetrizable space is **metrizable** if and only if it is a Morita's $M$**-space**.

  In Topology, the term **semimetrizable space** refers to a topological space $(X, \tau)$ admitting a symmetric $d$ such that, for each $x \in X$, the family $\{B(x,\epsilon) : \epsilon > 0\}$ of balls forms a (not necessarily open) neighborhood base at $x$. In other words, a point $x$ is in the closure of a set $H$ if and only if $d(x, H) = 0$.

  A topological space is semimetrizable if and only if it is symmetrizable and **first-countable**. Also, a symmetrizable space is semimetrizable if and only if it is a *Fréchet–Urysohn space* (or *E-space*), i.e., for any subset $A$ and for any point $x$ of its closure, there is a sequence in $A$ converging to $x$.
- **Hyperspace**

  A **hyperspace** of a topological space $(X, \tau)$ is a topological space on the set $CL(X)$ of all nonempty closed (or, moreover, compact) subsets of $X$. The topology of a hyperspace of $X$ is called a *hypertopology*. Examples of such a *hit-and-miss topology* are the *Vietoris topology*, and the *Fell topology*. Examples of such a *weak hyperspace topology* are the *Hausdorff metric topology*, and the *Wijsman topology*.
- **Discrete topological space**

  A topological space $(X, \tau)$ is **discrete** if $\tau$ is the *discrete topology* (the finest topology on $X$), i.e., containing all subsets of $X$ as open sets. Equivalently, it does not contain any *limit point*, i.e., it consists only of *isolated points*.
- **Indiscrete topological space**

  A topological space $(X, \tau)$ is **indiscrete** if $\tau$ is the *indiscrete topology* (the coarsest topology on $X$), i.e., having only two open sets, $\emptyset$ and $X$.

It can be considered as the semimetric space $(X, d)$ with the **indiscrete semimetric**: $d(x, y) = 0$ for any $x, y \in X$.

- **Extended topology**

Consider a set $X$ and a map $cl : P(X) \to P(X)$, where $P(X)$ is the set of all subsets of $X$. The set $cl(A)$ (for $A \subset X$), its dual set $int(A) = X \backslash cl(X \backslash A)$ and the map $N : X \to P(X)$ with $N(x) = \{A \subset X : x \in int(A)\}$ are called the *closure*, *interior* and *neighborhood* map, respectively.

So, $x \in cl(A)$ is equivalent to $X \backslash A \in P(X) \backslash N(x)$. A subset $A \subset X$ is *closed* if $A = cl(A)$ and *open* if $A = int(A)$. Consider the following possible properties of $cl$; they are meant to hold for all $A, B \in P(X)$.

1. $cl(\emptyset) = \emptyset$;
2. $A \subseteq B$ implies $cl(A) \subseteq cl(B)$ (*isotony*);
3. $A \subseteq cl(A)$(*enlarging*);
4. $cl(A \cup B) = cl(A) \cup cl(B)$ (*linearity*, and, in fact, 4 implies 2);
5. $cl(cl(A)) = cl(A)$ (*idempotency*).

The pair $(X, cl)$ satisfying 1 is called an **extended topology** if 2 holds, a **Brissaud space** (Brissaud, 1974) if 3 holds, a **neighborhood space** (Hammer, 1964) if 2 and 3 hold, a **Smyth space** (Smyth, 1995) if 4 holds, a **pre-topology** (Čech, 1966) if 3 and 4 hold, and a **closure space** (Soltan, 1984) if 2, 3 and 5 hold.

$(X, cl)$ is the usual topology, in closure terms, if 1, 3, 4 and 5 hold.

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