Chapter 2
Ultraproducts of Admissible Models for Quantified Modal Logic

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Abstract Admissible models for quantified modal logic have a restriction on which sets of worlds are admissible as propositions. They give an actualist interpretation of quantifiers that leads to very general completeness results: for any propositional modal logic S there is a quantificational proof system QS that is complete for validity in models whose algebra of admissible propositions validates S. In this paper, we construct ultraproducts of admissible models and use them to derive compactness theorems that combine with completeness to yield *strong* completeness: any QS-consistent set of formulas is satisfiable in a model whose admissible propositions validate S. The Barcan Formula is analysed separately and shown to axiomatise certain logics that are strongly complete over admissible models in which the quantifiers are given their standard Kripkean interpretation.

Keywords Admissible semantics · Quantified modal logic · Ultraproduct · Actualist quantification · Compactness · Strong completeness · Kripkean interpretation · Barcan formula

2.1 Introduction

A theory of *admissible semantics* for quantified modal logics was set out by the author in [5]. Its aim is to address the problem of *incompleteness* of some such logics under their Kripkean possible-worlds semantics. This includes cases where completeness for validity in Kripke frames holds at the propositional level but fails to lift to the quantificational setting.

An example of this failure concerns the Gödel-Löb logic GL, the normal propositional modal logic with the axiom $\Box(\Box A \to A) \to \Box A$. It axiomatises the interpretation of $\Box$ as "it is provable in Peano arithmetic that". GL is a decidable logic that is complete for validity in its Kripke frames. These Kripke frames validating GL have a natural mathematical description as the transitive inverse well-founded ones. But the set of formulas that are valid in the Kripkean *quantificational* models over

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GL-frames is not recursively enumerable, so cannot be recursively axiomatised. What then are the prospects of developing a model theory that characterises logics defined proof-theoretically by adding standard axioms and inference rules for quantifiers to GL?

We answer this question by imposing a restriction on which sets of worlds count as propositions. Our models have a designated modal algebra *Prop* of sets of worlds, called the *admissible propositions*. Every formula is interpreted as an admissible proposition. For propositional modal languages such structures are called *general frames* and provide a complete semantics for any logic. In models for languages with quantification of individual variables, each world w of a general frame is assigned a subset Dw of some fixed universe U of possible individuals. Dw is the domain of individuals that exist, or are actual, in w.

In Kripkean models, a universal quantifier $\forall x$ is interpreted at w by taking the variable x to range over the domain Dw. This is the *actualist* interpretation of quantification, validating the Actual Instantiation scheme

AI: $\forall y(\forall x\varphi \to \varphi(y/x))$, where y is free for x in φ,

but not the Universal Instantiation scheme

UI: $\forall x\varphi \to \varphi(y/x)$, where y is free for x in φ

(because the value of y may not be actual in a particular world).

In an admissible model we take $\forall x\varphi$ to have the same meaning as the conjunction of the assertions "if a exists then $\varphi(a/x)$" for all $a \in U$. The conjunction operation is interpreted as the *meet*, or greatest lower bound, operation in the set (*Prop*, $\subseteq$) of admissible propositions under the partial ordering $\subseteq$ of entailment (= set inclusion). In a Kripkean model, the meet of a set $\mathscr{X}$ of propositions is just its set-theoretic intersection $\bigcap \mathscr{X}$. But in an admissible model, the meet $\bigsqcap \mathscr{X}$ of $\mathscr{X}$ is the largest *admissible* subset of $\bigcap \mathscr{X}$. This can be understood as the weakest admissible proposition that entails every member of $\mathscr{X}$, and may have $\bigsqcap \mathscr{X} \subsetneq \bigcap \mathscr{X}$.

Using these ideas we have shown that for every propositional modal logic S there is a naturally axiomatised quantified logic QS (with axioms including AI and all instances of S-theorems), which is *complete* for validity in models whose underlying general frame of admissible propositions validates S. Completeness here means that every QS-consistent formula is satsifiable in a model of the kind just described. It is noteworthy that such models need not validate the commuting quantifiers axiom

$$\text{CQ}: \quad \forall x\forall y\varphi \to \forall y\forall x\varphi,$$

which is valid in Kripkean models (see [6]).

In this paper, we take up the question of *strong* completeness, meaning that every consistent *set* of formulas is satisfiable in a model of the required kind. We introduce a definition of the ultraproduct $\mathscr{M}_\mu$ of a family $\{\mathscr{M}_i : i \in I\}$ of admissible models with respect to an ultrafilter μ on the index set I. We show that Łoś' Theorem, the so-called "fundamental theorem of ultraproducts", continues to hold for our admissible interpretation of the quantifier $\forall$. This theorem states that a formula is satisfiable in $\mathscr{M}_\mu$ iff it is satisfiable in “almost all” of the models $\mathscr{M}_i$. Armed with Łoś’ Theorem it is then a matter of using standard arguments to derive a compactness theorem for admissible model theory and combine it with completeness to infer strong completeness for QS.

We then take up the question of the Barcan Formula BF: $\forall x\Box\varphi \to \Box\forall x\varphi$, and its converse CBF. In Kripkean models validity of BF is often identified with the condition of *contracting domains*: wRu implies $Dw \supseteq Du$. But admissible models can have contracting domains without validating BF. Perhaps surprisingly, every logic of the form QS is characterised by models with contracting domains. Imposition of this contracting domains condition on admissible models does not force the general validity of any non-theorems of QS. It is only in *Kripkean* models with contracting domains that validity of BF is guaranteed.

We apply our ultraproducts method to prove strong completeness of QS over contracting-domains models; of QS + CBF over models with *constant domains* (wRu implies $Dw = Du$); and of QS + CBF + CQ + BF over Kripkean constant-domain models. The proof for the last case works for arbitrarily large languages, overcoming a countability restriction on the original proof of completeness. The whole analysis reveals that the real role of BF in admissible model theory is to enable us to build models that give the quantifier $\forall$ its standard Kripkean interpretation.

Finally, we examine the universal instantiation axiom UI, which corresponds to the condition that a model has *one universal domain*: $D_w = U$ for all worlds w. The axioms CBF and CQ are derivable from UI. We show that QS + UI is strongly complete for validity in one-universal-domain admissible models whose underlying general frame validates S, and that QS + UI + BF is strongly complete over Kripkean models of this kind.

2.2 Admissible Models

Here, we set out the basic syntax of quantified modal logic, and its admissible semantics. Let $\{x_0, \dots, x_n, \dots\}$ be a fixed denumerable set of individual variables. The letters x, y will be used for arbitrary variables. Let $\mathscr{L}$ be a *signature*: a set of individual constants c, predicate symbols P, and function symbols F. An *$\mathscr{L}$-term* is any individual variable, any constant c, or inductively any expression $F\tau_1 \cdots \tau_n$ where F is an n-ary function symbol from $\mathscr{L}$, and $\tau_1, \dots, \tau_n$ are $\mathscr{L}$-terms.

An *atomic $\mathscr{L}$-formula* is any expression $P\tau_1 \cdots \tau_n$ where P is an n-ary predicate symbol from $\mathscr{L}$, and $\tau_1, \dots, \tau_n$ are $\mathscr{L}$-terms. The set of *$\mathscr{L}$-formulas* is generated from the atomic ones and a constant formula $\bot$ (Falsum) in the usual way, using the connectives $\wedge$ (conjunction), $\neg$ (negation), the modality $\Box$ and universal quantifiers $\forall x$ for each variable x.

A *model structure* is a system $\mathscr{S} = (W, R, Prop, U, D)$ such that:

- W is a non-empty set (of "worlds"), and R is a binary relation on W.
- *Prop* is a non-empty subset of the powerset $\wp W$ of W that is closed under binary intersections $X \cap Y$ and complements $-X$, hence under binary unions $X \cup Y$ and Boolean implications $X \Rightarrow Y = (-X) \cup Y$. Hence $\emptyset, W \in Prop$.
- *Prop* is closed under the operation $[R]$ defined by

$$[R]X = \{w \in W : \forall v \in W(wRv \text{ implies } v \in X)\}.$$

- U is a non-empty set, called the *universe* of $\mathscr{S}$; and
- D is a function assigning to each element w of W a subset Dw of U, called the *domain of* w.

A subset of W is *admissible* if it belongs to *Prop*. Members of *Prop* are also called the *admissible propositions* of $\mathscr{S}$.

The triple $(W, R, Prop)$ is sometimes called a *general frame*. Such a structure is used to provide semantics for propositional modal logic, in a manner that will be described in Sect. 2.6 below. When extracted from a model structure $\mathscr{S}$ as above it may be called the *underlying* general frame of $\mathscr{S}$.

An operation $\bigsqcap$ on collections of subsets of W is defined by putting, for each $\mathscr{X} \subseteq \wp W$,

$$\bigsqcap \mathscr{X} = \bigcup\{Y \in Prop : Y \subseteq \bigcap \mathscr{X}\}.$$

Thus $\bigsqcap \mathscr{X}$ is the union of all admissible subsets of $\bigcap \mathscr{X}$, hence $\bigsqcap \mathscr{X} \subseteq \bigcap \mathscr{X}$. It is not required that $\mathscr{X} \subseteq Prop$ in this definition: $\bigsqcap \mathscr{X}$ is defined for arbitrary $\mathscr{X} \subseteq \wp W$ and need not be admissible in general, even when $\mathscr{X} \subseteq Prop$. If we do have $\mathscr{X} \subseteq Prop$ and $\bigsqcap \mathscr{X} \in Prop$, then $\bigsqcap \mathscr{X}$ is the *greatest lower bound* of $\mathscr{X}$ in the partially ordered set $(Prop, \subseteq)$, i.e. the largest admissible set included in every member of $\mathscr{X}$. If $\bigcap \mathscr{X}$ is admissible, then $\bigsqcap \mathscr{X} = \bigcap \mathscr{X}$.

For each $a \in U$ we define $Ea = \{w \in W : a \in Dw\}$, representing the proposition "a exists". Sets of the form Ea may be referred to as "existence sets" or "existence propositions". They are not required to be admissible.

A *premodel* $\mathscr{M} = (\mathscr{S}, |-|^{\mathscr{M}})$ for signature $\mathscr{L}$, based on a model structure $\mathscr{S}$, is given by an interpretation function $|-|^{\mathscr{M}}$ on $\mathscr{L}$ that assigns:

- to each individual constant $\mathsf{c} \in \mathscr{L}$ an element $|\mathsf{c}|^{\mathscr{M}}$ of the universe U.
- to each n-ary function symbol $F \in \mathscr{L}$ an n-ary function $|F|^{\mathscr{M}}$ on the universe U, i.e. $|F|^{\mathscr{M}} : U^n \to U$.
- to each n-ary predicate symbol $P \in \mathscr{L}$ a function $|P|^{\mathscr{M}} : U^n \to \wp W$.

Intuitively, $|P|^{\mathscr{M}}(a_1, \ldots, a_n)$ represents the proposition that the predicate P holds of the n-tuple $(a_1, \ldots, a_n)$. A *variable-assignment* in a premodel is a function from the set ω of natural numbers into U. Thus, the set of variable-assignments is the set U^ω of all functions $f : \omega \to U$. The idea here is that f assigns the value fn to the variable x_n. Such an f then assigns to each $\mathscr{L}$-term τ a value $|\tau|^{\mathscr{M}} f \in U$, so overall $\mathcal{M}$ interprets τ as a function $|\tau|^{\mathcal{M}} : U^{\omega} \to U$. The inductive definition of $|\tau|^{\mathcal{M}} f$ is:

- $|x|^{\mathcal{M}} f = fn$, if x is the variable x_n.
- $|\mathsf{c}|^{\mathcal{M}} f = |\mathsf{c}|^{\mathcal{M}}$.
- $|F\tau_1 \cdots \tau_n|^{\mathcal{M}} f = |F|^{\mathcal{M}}(|\tau_1|^{\mathcal{M}} f, \dots, |\tau_n|^{\mathcal{M}} f)$.

We write fx for fn when x is x_n, so we get $|x|^{\mathcal{M}} f = fx$. The notation $f[a/x]$ will be used for the function that *updates* f by assigning the value a to x and otherwise acting identically to f. Thus, $f[a/x]x = a$ and $f[a/x]y = fy$ if $y \neq x$.

A premodel gives an interpretation $|\varphi|^{\mathcal{M}} : U^{\omega} \to \wp W$ to each $\mathcal{L}$-formula. This interpretation is a *propositional function*, i.e. a function whose values are propositions (not necessarily admissible ones). For each assignment f, $|\varphi|^{\mathcal{M}} f$ is to be the *truth set* of all worlds at which φ is true under f. This is defined by induction on the formation of φ:

- $|P\tau_1 \cdots \tau_n|^{\mathcal{M}} f = |P|^{\mathcal{M}}(|\tau_1|^{\mathcal{M}} f, \dots, |\tau_n|^{\mathcal{M}} f)$.
- $|\bot|^{\mathcal{M}} f = \emptyset$.
- $|\varphi \wedge \psi|^{\mathcal{M}} f = |\varphi|^{\mathcal{M}} f \cap |\psi|^{\mathcal{M}} f$.
- $|\neg\varphi|^{\mathcal{M}} f = W - |\varphi|^{\mathcal{M}} f$.
- $|\Box\varphi|^{\mathcal{M}} f = [R]|\varphi|^{\mathcal{M}} f$.
- $|\forall x\varphi|^{\mathcal{M}} f = \bigsqcap_{a\in U} \big(Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\big)$.

Writing $\mathcal{M}, w, f \models \varphi$ to mean that $w \in |\varphi|^{\mathcal{M}} f$, we get the following standard clauses for this truth/satisfaction relation $\models$.

- $\mathcal{M}, w, f \models P\tau_1 \cdots \tau_n$ iff $w \in |P|^{\mathcal{M}}(|\tau_1|^{\mathcal{M}} f, \dots, |\tau_n|^{\mathcal{M}} f)$.
- $\mathcal{M}, w, f \not\models \bot$.
- $\mathcal{M}, w, f \models \varphi \wedge \psi$ iff $\mathcal{M}, w, f \models \varphi$ and $\mathcal{M}, w, f \models \psi$.
- $\mathcal{M}, w, f \models \neg\varphi$ iff not $\mathcal{M}, w, f \models \varphi$.
- $\mathcal{M}, w, f \models \Box\varphi$ iff for all $v \in W(wRv$ implies $\mathcal{M}, v, f \models \varphi)$.

For the universal quantifier, the condition for $\mathcal{M}, w, f \models \forall x\varphi$ is that

$$\text{there is an } X \in \mathit{Prop} \text{ such that } w \in X \text{ and } X \subseteq \bigcap_{a\in U} \big(Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\big). \quad (2.1)$$

Informally, this asserts that there is an admissible proposition X that is true at w and entails the assertions “if a exists then $\varphi(a/x)$” for all $a \in U$.

From (2.1) we see that

$$\mathcal{M}, w, f \models \forall x\varphi \;\; \textit{only if} \;\; \text{for all } a \in Dw,\ \mathcal{M}, w, f[a/x] \models \varphi. \quad (2.2)$$

The converse need not hold [5, Example 1.6.6]. If it does hold, then $\mathcal{M}$ will be called *Kripkean*, because this means that $\forall$ gets the varying-domain semantics of [7]:

$$\mathcal{M}, w, f \models \forall x\varphi \;\; \text{iff} \;\; \text{for all } a \in Dw,\ \mathcal{M}, w, f[a/x] \models \varphi.$$

Thus a Kripkean premodel is one that always has

$$|\forall x\varphi|^{\mathscr{M}} f = \bigcap_{a \in U} \big(Ea \Rightarrow |\varphi|^{\mathscr{M}} f[a/x]\big).$$

A formula φ is *valid in* premodel $\mathscr{M}$, written $\mathscr{M} \models \varphi$, if $|\varphi|^{\mathscr{M}} f = W$ for all f, i.e. if $\mathscr{M}, w, f \models \varphi$ for all $w \in W$ and $f \in U^{\omega}$.

An *admissible model*, or just *model*, for $\mathscr{L}$ is, by definition, a premodel in which every $\mathscr{L}$-formula φ is *admissible* in the sense that the function $|\varphi|^{\mathscr{M}}$ has the form $U^{\omega} \to Prop$, i.e. $|\varphi|^{\mathscr{M}} f \in Prop$ for all $f \in U^{\omega}$.

Informally, a model interprets a sentence $\forall x\varphi$ as the weakest admissible proposition that entails the assertions "if a exists then $\varphi(a/x)$" for all $a \in U$.

2.3 Ultraproducts of Premodels

Let μ be an ultrafilter on a set I. Recall that this means that μ is a collection of subsets of I such that $I \in \mu$; the complement $I - J$ of a subset $J \subseteq I$ belongs to μ iff $J \notin \mu$; and an intersection $J \cap K$ belongs to μ iff $J \in \mu$ and $K \in \mu$. Such a μ is closed under supersets: if $J \in \mu$ and $J \subseteq K$, then $K \in \mu$.

Each $J \in \mu$ is a "large" subset of I. We think of J as containing *almost all* members of I.

For any I-indexed collection $\{X_i : i \in I\}$ of sets, let $\prod_I X_i$ be the Cartesian product set whose points are the functions $f : I \to \bigcup_I X_i$ having $f(i) \in X_i$ for all $i \in I$. Define an equivalence relation $=_{\mu}$ on $\prod_I X_i$ by putting

$$f =_{\mu} g \quad \text{iff} \quad \{i \in I : f(i) = g(i)\} \in \mu.$$

The relation $f =_{\mu} g$ can be thought of as asserting that f and g agree at almost all $i \in I$. Let f_{μ} be the equivalence class $\{g \in \prod_I X_i : f =_{\mu} g\}$, and put $\prod_{\mu} X_i = \{f_{\mu} : f \in \prod_I X_i\}$. Then $\prod_{\mu} X_i$ is called the *ultraproduct* of the sets X_i with respect to μ. Many properties can be specified as holding of $\prod_{\mu} X_i$ iff they hold correspondingly of almost all of the X_i's.

Let $\{\mathscr{S}_i : i \in I\}$ be an I-indexed collection of model structures, with $\mathscr{S}_i = (W_i, R_i, Prop_i, U_i, D_i)$. We define the ultraproduct of the $\mathscr{S}_i$'s with respect to μ as a structure

$$\mathscr{S}_{\mu} = (W_{\mu}, R_{\mu}, Prop_{\mu}, U_{\mu}, D_{\mu}),$$

(which could also be denoted $\prod_{\mu} \mathscr{S}_i$). Here W_{μ} is the ultraproduct $\prod_{\mu} W_i$ of the W_i's and U_{μ} is the ultraproduct $\prod_{\mu} U_i$ of the U_i's. The binary relation R_{μ} on W_{μ} is well defined by putting, for all $f, g \in \prod_I W_i$,

$$f_{\mu} R_{\mu} g_{\mu} \quad \text{iff} \quad \{i \in I : f(i) R_i g(i)\} \in \mu.$$

The domain function $D_\mu : W_\mu \to \wp U_\mu$ is defined by putting

$$D_\mu f_\mu = \{g_\mu \in U_\mu : \{i : g(i) \in D_i f(i)\} \in \mu\}$$

for all $f \in \prod_I W_i$. This definition[1] can be seen as an example of the general procedure of lifting an operation to an ultraproduct by lifting it to the direct product and then transferring it to the $=_\mu$-equivalence classes. For this and other purposes it is convenient to lift the set membership relation to a relation $\in_\mu$ between any functions h, k with domain I by putting

$$h \in_\mu k \quad \text{iff} \quad \{i \in I : h(i) \in k(i)\} \in \mu.$$

Now the domain functions D_i induce the function $D_I : \prod_I W_i \to \prod_I \wp U_i$ where, for any $f \in \prod_I W_i$, the function $D_I f \in \prod_I \wp U_i$ is defined by putting, for each $i \in I$,

$$(D_I f)(i) = D_i f(i) \subseteq U_i.$$

Then the definition of D_μ becomes that

$$D_\mu f_\mu = \{g_\mu \in U_\mu : g \in_\mu D_I f\}.$$

We will write E_i for the existence operator in $\mathscr{S}_i$, so that $f(i) \in E_i g(i))$ iff $g(i) \in D_i g(i)$. The existence operator in $\mathscr{S}_\mu$ will be denoted E_μ, so that $f_\mu \in E_\mu g_\mu$ iff $g_\mu \in D_\mu f_\mu$. Thus

$$f_\mu \in E_\mu g_\mu \quad \text{iff} \quad \{i \in I : f(i) \in E_i g(i)\} \in \mu. \tag{2.3}$$

It remains to construct $Prop_\mu$ as a modal algebra of subsets of W_μ, closed under the Boolean set operations and the unary modal operator $[R_\mu]$ induced on $\wp W_\mu$ by the relation R_μ. This construction was carried out for general modal frames in [3], reproduced in [4]. It constructs $Prop_\mu$, not as the ultraproduct $\prod_\mu Prop_i$ of the modal algebras $Prop_i$, but as an algebra of subsets of W_μ that is isomorphic to $\prod_\mu Prop_i$. For each element σ of the Cartesian product $\prod_I Prop_i$, define a subset $S(\sigma)$ of W_μ by putting

$$S(\sigma) = \{f_\mu \in W_\mu : f \in_\mu \sigma\}.$$

Then, we put $Prop_\mu = \{S(\sigma) : \sigma \in \prod_I Prop_i\}$.

Now it can be shown that $S(\sigma)$ is well defined and that in general $\sigma =_\mu \sigma'$ iff $S(\sigma) = S(\sigma')$. Thus the map $\sigma_\mu \mapsto S(\sigma)$ is a bijection between $\prod_\mu Prop_i$ and $Prop_\mu$. Moreover, we have

[1] As with many operations on ultraproducts, it needs to be checked that D_μ is well defined, i.e. that $f_\mu = f'_\mu$ implies $D_\mu(f_\mu) = D_\mu(f'_\mu)$. Such checking is left to the reader in routine cases.

$$\begin{aligned} S(\sigma) \cap S(\sigma') &= S(\sigma \cap \sigma') \\ W_\mu - S(\sigma) &= S(-\sigma) \\ [R_\mu]S(\sigma) &= S([R_I]\sigma), \end{aligned} \tag{2.4}$$

where $\sigma \cap \sigma'$, $-\sigma$ and $[R_I]\sigma$ are the members of $\prod_I Prop_i$ defined pointwise by the corresponding operations on the algebras $Prop_i$, i.e. $(\sigma \cap \sigma')(i) = \sigma(i) \cap \sigma'(i)$, $(-\sigma)(i) = W_i - \sigma(i)$ and $([R_I]\sigma)(i) = [R_i]\sigma_i$.

It follows from (2.4) that $Prop_\mu$ is closed under $\cap$, $-$ and $[R_\mu]$. Full details of this construction can be found in [4, Sect. 1.7]. That completes the description of the ultraproduct of the the $\mathscr{S}_i$'s with respect to μ.

2.4 Łoś' Theorem

Given a collection $\{\mathscr{M}_i \in I\}$ of premodels for $\mathscr{L}$, with $\mathscr{M}_i = (\mathscr{S}_i, |-|^{\mathscr{M}_i})$, and an ultrafilter μ on I, we define a premodel $\mathscr{M}_\mu = (\mathscr{S}_\mu, |-|^{\mathscr{M}_\mu})$ on the ultraproduct $\mathscr{S}_\mu$ of the $\mathscr{S}_i$'s with respect to μ. We use *tuple* notation for functions here: a function f with domain I may be written as the tuple $\langle f(i) : i \in I \rangle$, and then $\langle f(i) : i \in I \rangle_\mu$ denotes f_μ.

The interpretation function $|-|^{\mathscr{M}_i}$ is defined as follows.

- For each individual constant $\mathsf{c} \in \mathscr{L}$, $|\mathsf{c}|^{\mathscr{M}_\mu} = \langle |\mathsf{c}|^{\mathscr{M}_i} : i \in I \rangle_\mu \in U_\mu$.
- For each n-ary function symbol $F \in \mathscr{L}$, the function $|F|^{\mathscr{M}_\mu} : {U_\mu}^n \to U_\mu$ is defined by putting, for all $f_1, \ldots, f_n \in \prod_I U_i$,

$$|F|^{\mathscr{M}_\mu}(f_{1\mu}, \ldots, f_{n\mu}) = \langle |F|^{\mathscr{M}_i}(f_1(i), \ldots, f_n(i)) : i \in I \rangle_\mu .$$

- For each n-ary predicate symbol $P \in \mathscr{L}$, the function $|P|^{\mathscr{M}_\mu} : {U_\mu}^n \to \wp W_\mu$ is defined by putting, for all $f_1, \ldots, f_n \in \prod_I U_i$ and all $g \in \prod_I W_i$,

$$g_\mu \in |P|^{\mathscr{M}_\mu}(f_{1\mu}, \ldots, f_{n\mu}) \quad \text{iff} \quad \{i \in I : g(i) \in |P|^{\mathscr{M}_i}(f_1(i), \ldots, f_n(i))\} \in \mu. \tag{2.5}$$

The fundamental property of ultraproducts of models of (non-modal) first-order logic, due to Łoś, is in essence that a formula is satisfiable in an ultraproduct iff it is satisfiable in almost all of the component models. We now formulate the corresponding result for the admissible semantics of our premodels.

A sequence $f = \langle f_0, \ldots, f_n, \ldots \rangle \in (\prod_I U_i)^\omega$ of elements f_n of the Cartesian product $\prod_I U_i$ determines, for each $i \in I$, the sequence

$$f \cdot i = \langle f_0(i), \ldots, f_n(i), \ldots \rangle \in U_i^\omega \tag{2.6}$$

of elements of U_i. We write f_μ for the sequence $\langle f_{0\mu}, \ldots, f_{n\mu}, \ldots \rangle \in {U_\mu}^\omega$. Then it is straightforward to check that for any $g \in \prod_I U_i$ and any variable x we have

$$f_\mu[g_\mu/x] = f[g/x]_\mu, \tag{2.7}$$

and for each $i \in I$,

$$f[g/x] \cdot i = (f \cdot i)[g(i)/x]. \tag{2.8}$$

An induction on term formation shows that for any $\mathscr{L}$-term τ, the function $|\tau|^{\mathscr{M}_\mu} : {U_\mu}^\omega \to U_\mu$ has

$$|\tau|^{\mathscr{M}_\mu} f_\mu = \langle |\tau|^{\mathscr{M}_i} f \cdot i : i \in I \rangle_\mu. \tag{2.9}$$

The proof is essentially as in the classical case of ultraproducts of non-modal first-order logic [2, Sect. 4.1].

To formulate the fundamental theorem we define, for each formula φ and each $h \in \prod_I W_i$, the "truth set"

$$[\![h, \varphi, f]\!] = \{i \in I : h(i) \in |\varphi|^{\mathscr{M}_i} f \cdot i\} = \{i \in I : \mathscr{M}_i, h(i), f \cdot i \models \varphi\}.$$

Theorem 1 (Łoś' Theorem) *Let φ be any $\mathscr{L}$-formula. Then for all $f \in (\prod_I U_i)^\omega$ and $h \in \prod_I W_i$,*

$$h_\mu \in |\varphi|^{\mathscr{M}_\mu} f_\mu \quad \textit{iff} \quad [\![h, \varphi, f]\!] \in \mu.$$

In other words,

$$\mathscr{M}_\mu, h_\mu, f_\mu \models \varphi \quad \textit{iff} \quad \{i \in I : \mathscr{M}_i, h(i), f \cdot i \models \varphi\} \in \mu.$$

Proof This proceeds by induction on the formation of φ. For the case that φ is the atomic formula $P\tau_1 \cdots \tau_n$, the definition of $|P|^{\mathscr{M}_\mu}$ in (2.5) combines with (2.9) to show that $h_\mu \in |\varphi|^{\mathscr{M}_\mu} f_\mu$ iff the set

$$\{i \in I : h(i) \in |P|^{\mathscr{M}_i}(|\tau_1|^{\mathscr{M}_i} f \cdot i, \ldots, |\tau_n|^{\mathscr{M}_i} f \cdot i)\}$$

belongs to μ. But this set is $[\![h, P\tau_1 \cdots \tau_n, f]\!]$.

The case that φ is $\bot$ and the inductive steps for the connectives $\neg$, $\wedge$ and $\Box$ are as for propositional modal logic in [4, Sect. 1.7] (see also (2.13) below).

The really new case here is to show that the theorem holds for a formula $\forall x\varphi$ under the induction hypothesis that it holds for φ. Assume first that $[\![h, \forall x\varphi, f]\!] \in \mu$. To prove that $h_\mu \in |\forall x\varphi|^{\mathscr{M}_\mu} f_\mu$ we prove, in accordance with (2.1), that there exists some admissible set $S(\sigma) \in Prop_\mu$ such that $h_\mu \in S(\sigma)$ and

$$S(\sigma) \subseteq \bigcap_{g_\mu \in U_\mu} \big(E_\mu g_\mu \Rightarrow |\varphi|^{\mathscr{M}_\mu} f_\mu[g_\mu/x]\big). \tag{2.10}$$

Now if $i \in [\![h, \forall x\varphi, f]\!]$, then $h(i) \in |\forall x\varphi|^{\mathscr{M}_i} f \cdot i$, so applying (2.1) in $\mathscr{M}_i$, there is some admissible set $\sigma(i) \in Prop_i$ such that $h(i) \in \sigma(i)$ and

$$\sigma(i) \subseteq \bigcap_{d \in U_i} \big(E_i d \Rightarrow |\varphi|^{\mathscr{M}_i} f \cdot i[d/x]\big). \tag{2.11}$$

For $i \notin [\![h, \forall x\varphi, f]\!]$, put $\sigma(i) = \emptyset$. We have now defined a function $\sigma \in \prod_I Prop_i$ with $[\![h, \forall x\varphi, f]\!] \subseteq \{i : h(i) \in \sigma(i)\}$. Hence $\{i : h(i) \in \sigma(i)\} \in \mu$, so $h \in_\mu \sigma$ and therefore $h_\mu \in S(\sigma) \in Prop_\mu$. It remains to prove (2.10).

Take any $k_\mu \in S(\sigma)$, where $k \in \prod_I W_i$ and $k \in_\mu \sigma$. Let $g_\mu \in U_\mu$. If $k_\mu \in E_\mu g_\mu$, then the intersection

$$J = [\![h, \forall x\varphi, f]\!] \cap \{i : k(i) \in \sigma(i)\} \cap \{i : k(i) \in E_i g(i)\}$$

belongs to μ, since each of the three sets involved belongs to μ [cf. (2.3)]. But if $i \in J$, then (2.11) holds, and so as $k(i)$ belongs to $\sigma(i)$ and to $E_i g(i)$ we infer that it belongs to $|\varphi|^{\mathscr{M}_i} f \cdot i[g(i)/x]$, which is equal to $|\varphi|^{\mathscr{M}_i} f[g/x] \cdot i$ by (2.8). This shows that

$$J \subseteq \{i : k(i) \in |\varphi|^{\mathscr{M}_i} f[g/x] \cdot i\} = [\![k, \varphi, f[g/x]]\!].$$

Therefore, $[\![k, \varphi, f[g/x]]\!] \in \mu$, and so by the induction hypothesis on φ, k_μ belongs to $|\varphi|^{\mathscr{M}_\mu} f[g/x]_\mu$, which is equal to $|\varphi|^{\mathscr{M}_\mu} f_\mu[g_\mu/x]$ by (2.7). Altogether this proves that $k_\mu \in E_\mu g_\mu \Rightarrow |\varphi|^{\mathscr{M}_\mu} f_\mu[g_\mu/x]$, which completes the proof of (2.10), and hence the proof that $[\![h, \forall x\varphi, f]\!] \in \mu$ implies $h_\mu \in |\forall x\varphi|^{\mathscr{M}_\mu} f_\mu$.

For the converse, suppose that $[\![h, \forall x\varphi, f]\!] \notin \mu$. Then to show that $h_\mu \notin |\forall x\varphi|^{\mathscr{M}_\mu} f_\mu$ we take an arbitrary $S(\sigma) \in Prop_\mu$ such that $h_\mu \in S(\sigma)$, and will show that (2.10) fails. As μ is an ultrafilter we have $(I - [\![h, \forall x\varphi, f]\!]) \in \mu$, so as $h \in_\mu \sigma$ we get that the set

$$J' = (I - [\![h, \forall x\varphi, f]\!]) \cap \{i : h(i) \in \sigma(i)\}$$

belongs to μ. Now if $i \in J'$ we have $h(i) \notin |\forall x\varphi|^{\mathscr{M}_i} f \cdot i$ and $h(i) \in \sigma(i) \in Prop_i$, so (2.11) must fail. Hence there must be some $k(i) \in \sigma(i)$ and some $g(i) \in U_i$ with $k(i) \in E_i g(i) - |\varphi|^{\mathscr{M}_i} f \cdot i[g(i)/x]$. For $i \notin J'$ choose $k(i) \in W_i$ and $g(i) \in U_i$ arbitrarily. This defines $k \in \prod_I W_i$ and $g \in \prod_I U_i$.

Since $J' \subseteq \{i : k(i) \in E_i g(i)\}$ we get $k_\mu \in E_\mu g_\mu$. Whenever $i \in J'$ we have

$$k(i) \notin |\varphi|^{\mathscr{M}_i} f \cdot i[g(i)/x] = |\varphi|^{\mathscr{M}_i} f[g/x] \cdot i,$$

so $J' \cap [\![k, \varphi, f[g/x]]\!] = \emptyset$, and hence $[\![k, \varphi, f[g/x]]\!] \notin \mu$. The induction hypothesis on φ then gives

$$k_\mu \notin |\varphi|^{\mathscr{M}_\mu} f[g/x]_\mu = |\varphi|^{\mathscr{M}_\mu} f_\mu[g_\mu/x].$$

Altogether, this proves that $k_\mu \notin E_\mu g_\mu \Rightarrow |\varphi|^{\mathcal{M}_\mu} f_\mu[g_\mu/x]$, which shows that (2.10) fails, proving that $h_\mu \notin |\forall x\varphi|^{\mathcal{M}_\mu} f_\mu$, and hence completing the proof that the theorem holds for $\forall x\varphi$. □

Theorem 2 *$\mathcal{M}_\mu$ is a model if almost all of the $\mathcal{M}_i$'s are models.*

Proof Let $M = \{i \in I : \mathcal{M}_i \text{ is a model}\}$. Suppose that $M \in \mu$. To prove that $\mathcal{M}_\mu$ is a model we have to show that for any formula φ and any $f \in \prod_I U_i$, the set $|\varphi|^{\mathcal{M}_\mu} f_\mu$ is admissible in $\mathcal{M}_\mu$, i.e. belongs to $Prop_\mu$.

Define $\sigma \in \prod_I Prop_i$ by putting $\sigma(i) = |\varphi|^{\mathcal{M}_i} f \cdot i$ when $i \in M$, and $\sigma(i) = \emptyset$ otherwise. Note that when $i \in M$, φ is admissible in the model $\mathcal{M}_i$, so indeed $|\varphi|^{\mathcal{M}_i} f \cdot i \in Prop_i$. We will show that $|\varphi|^{\mathcal{M}_\mu} f_\mu = S(\sigma)$, giving the desired result that $|\varphi|^{\mathcal{M}_\mu} f_\mu \in Prop_\mu$.

Take any $h \in \prod_I W_i$. Then

$$[\![h, \varphi, f]\!] \cap M = \{i : h(i) \in \sigma(i)\} \cap M,$$

for if $i \in M$ then $|\varphi|^{\mathcal{M}_i} f \cdot i = \sigma(i)$, so $h(i) \in |\varphi|^{\mathcal{M}_i} f \cdot i$ iff $h(i) \in \sigma(i)$. Since $M \in \mu$ and μ is a filter, it follows that $[\![h, \varphi, f]\!] \in \mu$ iff $\{i : h(i) \in \sigma(i)\} \in \mu$. By Łoś' Theorem 1 and the definition of $S(\sigma)$, this says that $h_\mu \in |\varphi|^{\mathcal{M}_\mu} f_\mu$ iff $h_\mu \in S(\sigma)$, which gives the desired result. □

Theorem 3 *$\mathcal{M}_\mu$ is Kripkean if almost all of the $\mathcal{M}_i$'s are Kripkean.*

Proof Let $K = \{i \in I : \mathcal{M}_i \text{ is Kripkean}\}$. Suppose that $K \in \mu$. To prove that $\mathcal{M}_\mu$ is Kripkean we have to show that in general

$$|\forall x\varphi|^{\mathcal{M}_\mu} f_\mu = \bigcap_{g_\mu \in U_\mu} \big(E_\mu g_\mu \Rightarrow |\varphi|^{\mathcal{M}_\mu} f_\mu[g_\mu/x]\big). \tag{2.12}$$

Now from (2.2), which holds in any premodel, it follows that

$$|\forall x\varphi|^{\mathcal{M}_\mu} f_\mu \subseteq E_\mu g_\mu \Rightarrow |\varphi|^{\mathcal{M}_\mu} f_\mu[g_\mu/x]$$

for any $g_\mu \in U_\mu$. So the left to right inclusion of (2.12) holds. For the converse, suppose that $h_\mu \notin |\forall x\varphi|^{\mathcal{M}_\mu} f_\mu$. Then $[\![h, \forall x\varphi, f]\!] \notin \mu$ by Łoś' Theorem. Hence the set

$$J = (I - [\![h, \forall x\varphi, f]\!]) \cap K$$

belongs to μ. But if $i \in J$ then $h(i) \notin |\forall x\varphi|^{\mathcal{M}_i} f \cdot i$ and $\mathcal{M}_i$ is Kripkean, so there exists some element $g(i)$ of U_i with $h(i) \in E_i g(i) - |\varphi|^{\mathcal{M}_i} f \cdot i[g(i)/x]$. For $i \notin J$ choose $g(i) \in U_i$ arbitrarily. This defines $g \in \prod_I U_i$.

Since $J \subseteq \{i : h(i) \in E_i g(i)\}$ we get $h_\mu \in E_\mu g_\mu$. Whenever $i \in J$ we have

$$h(i) \notin |\varphi|^{\mathcal{M}_i} f \cdot i[g(i)/x] = |\varphi|^{\mathcal{M}_i} f[g/x] \cdot i,$$

so $J \cap [\![h, \varphi, f[g/x]]\!] = \emptyset$, and hence $[\![h, \varphi, f[g/x]]\!] \notin \mu$. Łoś' Theorem then gives

$$h_\mu \notin |\varphi|^{\mathscr{M}_\mu} f[g/x]_\mu = |\varphi|^{\mathscr{M}_\mu} f_\mu[g_\mu/x].$$

Altogether this proves that $h_\mu \notin E_\mu g_\mu \Rightarrow |\varphi|^{\mathscr{M}_\mu} f_\mu[g_\mu/x]$, showing that h_μ does not belong to the intersection on the right of (2.12), which completes the proof of (2.12). □

2.5 Compactness

We say that a formula φ is *satisfiable in* $\mathscr{M}$ if $|\varphi|^{\mathscr{M}} f \neq \emptyset$ for some $f \in U^\omega$, i.e. if $\mathscr{M}, w, f \models \varphi$ for some f and some $w \in W$. φ is *valid in* $\mathscr{M}$ if $\neg\varphi$ is not satisfiable in $\mathscr{M}$, which means that $\mathscr{M}, w, f \models \varphi$ for all $w \in W$ and $f \in U^\omega$.

If Γ is a set of formulas, we write $\mathscr{M}, w, f \models \Gamma$ to mean that for all $\varphi \in \Gamma$, $\mathscr{M}, w, f \models \varphi$. If this holds for some w and f then Γ is *satisfiable in* $\mathscr{M}$.

We say that a class $\mathbb{M}$ of premodels is *closed under ultraproducts* if, for all indexed subsets $\{\mathscr{M}_i : i \in I\}$ of $\mathbb{M}$ and all ultrafilters μ on I, the ultraproduct $\mathscr{M}_\mu$ belongs to $\mathbb{M}$.

Theorem 4 (Compactness) *Let $\mathscr{L}$ be any signature and $\mathbb{M}$ any class of premodels for $\mathscr{L}$ that is closed under utraproducts. For any set Γ of $\mathscr{L}$-formulas, if each finite subset of Γ is satisfiable in some member of $\mathbb{M}$, then Γ itself is satisfiable in some member of $\mathbb{M}$.*

Proof This follows the pattern of the standard ultraproducts proof of compactness for first-order logic.

Let $I = \{i \subseteq \Gamma : i \text{ is finite}\}$, and for each $i \in I$ put $J_i = \{i' \in I : i \subseteq i'\}$. Then the collection $\{J_i : i \in I\}$ has the finite intersection property, since for $i_1, \ldots, i_n \in I$, the intersection $J_{i_1} \cap \cdots \cap J_{i_n}$ contains $i_1 \cup \cdots \cup i_n$. It follows that there is an ultrafilter μ on I such that $J_i \in \mu$ for all $i \in I$.

For each $i \in I$ there is by hypothesis a premodel $\mathscr{M}_i \in \mathbb{M}$ with set of worlds W_i and universe U_i such that $\mathscr{M}_i, w_i, f_i \models i$ for some $w_i \in W_i$ and some $f_i \in {U_i}^\omega$. Define a sequence $f = \langle f_0, \ldots, f_n, \ldots \rangle \in (\prod_I U_i)^\omega$ by putting $f_n(i) = f_i(n)$ for all $n < \omega$ and $i \in I$. Then for each $i \in I$, the sequence $f \cdot i \in U_i^\omega$ given by (2.6) is just f_i.

Now if $\varphi \in \Gamma$, consider $\{\varphi\} \in I$. For $i \in J_{\{\varphi\}}$, we have $\mathscr{M}_i, w_i, f_i \models \varphi$ as $\varphi \in i$. Hence

$$J_{\{\varphi\}} \subseteq \{i \in I : \mathscr{M}_i, w_i, f \cdot i \models \varphi\} = [\![h, \varphi, f]\!]$$

where $h(i) = w_i$ for all $i \in I$. Thus $[\![h, \varphi, f]\!] \in \mu$, and so $\mathscr{M}_\mu, h_\mu, f_\mu \models \varphi$ by Łoś' Theorem 1.

This shows that $\mathscr{M}_\mu, h_\mu, f_\mu \models \Gamma$, so Γ is satisfiable in the premodel $\mathscr{M}_\mu$, which belongs to the ultraproducts-closed class $\mathbb{M}$. □

Corollary 1 *For any set* Γ *of* $\mathscr{L}$*-formulas, if each finite subset of* Γ *is satisfiable in some (Kripkean)* $\mathscr{L}$*-model, then* Γ *is satisfiable in some (Kripkean)* $\mathscr{L}$*-model.*

Proof This follows from the theorem first by taking $\mathbb{M}$ to be the class of all $\mathscr{L}$-models, which is closed under ultraproducts by Theorem 2, and then by taking $\mathbb{M}$ to be the class of all Kripkean $\mathscr{L}$-models, which is closed under ultraproducts by Theorems 2 and 3. □

2.6 Propositional Logic

The formulas for propositional modal logic are generated from a denumerable list $\{p_n : n < \omega\}$ of propositional variables and the constant $\bot$ by using the connectives $\wedge$, $\neg$, and $\Box$. This language can be interpreted by models on a *general frame* $\mathscr{G} = (W, R, Prop)$, comprising a binary relation R on W and a set $Prop \subseteq \wp W$ closed under intersection $\cap$ complementation $-$ and the operation $[R]$, as in Sect. 2.2.

A model $\mathscr{M}$ on a general frame $\mathscr{G}$ is given by a variable assignment $|-|^{\mathscr{M}}$ such that $|p|^{\mathscr{M}} \in Prop$ for every propositional variable p. This assignment is then extended to define a truth set $|A|^{\mathscr{M}}$ for each propositional formula A, by induction on formula formation, as follows:

$$\begin{aligned} |\bot|^{\mathscr{M}} &= \emptyset. \\ |A \wedge B|^{\mathscr{M}} &= |A|^{\mathscr{M}} \cap |B|^{\mathscr{M}}. \\ |\neg A|^{\mathscr{M}} &= W - |A|^{\mathscr{M}}. \\ |\Box A|^{\mathscr{M}} &= [R]|A|^{\mathscr{M}}. \end{aligned}$$

The closure conditions on *Prop* then ensure that every formula is interpreted in $\mathscr{M}$ as an admissible proposition: $|A|^{\mathscr{M}} \in Prop$ for all propositional modal A. A formula A is *valid in the frame* $\mathscr{G}$, symbolised $\mathscr{G} \models A$, when $|A|^{\mathscr{M}} = W$ for all models $\mathscr{M}$ on $\mathscr{G}$. Thus $\mathscr{G} \models A$ when A is true at every point in every model on $\mathscr{G}$. A set S of propositional formulas is *valid in* $\mathscr{G}$, symbolised $\mathscr{G} \models$ S, when every member of S is valid in $\mathscr{G}$.

Let $\{\mathscr{G}_i : i \in I\}$ be a collection of general frames, with $\mathscr{G}_i = (W_i, R_i, Prop_i)$. If μ is an ultrafilter on the index set I, then we take the ultraproduct of the $\mathscr{G}_i$'s with respect to μ to be the structure

$$\mathscr{G}_\mu = (W_\mu, R_\mu, Prop_\mu)$$

who components were defined in Sect. 2.3. For any propositional modal formula A it can be shown [4, Corollary 1.7.13] that

$$\mathscr{G}_\mu \models A \quad \text{iff} \quad \{i \in I : \mathscr{G}_i \models A\} \in \mu. \tag{2.13}$$

In particular, if $\mathscr{G}_i \models A$ for all $i \in I$, then $\mathscr{G}_\mu \models A$. This implies

Theorem 5 *For any set* S *of propositional modal formulas, the class* $\{\mathscr{G} : \mathscr{G} \models \mathrm{S}\}$ *of all general frames validating* S *is closed under ultraproducts.* □

A *propositional modal logic* is a set S of propositional modal formulas that includes all such formulas that are Boolean tautologies or instances of the scheme

$$\mathrm{K}: \quad \Box(A \to B) \to (\Box A \to \Box B),$$

and is closed under the rules of Modus Ponens (from A and $A \to B$ infer B) and Necessitation (from A infer $\Box A$).

For each general frame $\mathscr{G}$, the set $\mathrm{S}_{\mathscr{G}} = \{A : \mathscr{G} \models A\}$ of all propositional formulas valid in $\mathscr{G}$ is a propositional modal logic that is closed under the rule of uniform substitution for propositional variables. Conversely, there is a canonical frame construction showing that if a logic S is closed under uniform substitution, then it is equal to $S_{\mathscr{G}}$ for some general frame $\mathscr{G}$ (see [1, Sect. 5.5]).

2.7 Quantified Logics

For a given signature $\mathscr{L}$, a *quantified modal logic* is defined to be any set L of $\mathscr{L}$-formulas that includes all Boolean tautologies and instances of the axiom schemes listed in Fig. 2.1, and is closed under the inference rules of that Figure. A member φ of L is called an L-*theorem*, which we indicate by writing $\vdash_{\mathrm{L}} \varphi$.

Fig. 2.1 Axioms and rules for quantified modal logics

Axiom schemes:

K: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
AI: $\forall y(\forall x\varphi \to \varphi(y/x))$, where y is free for x in φ.
UD: $\forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$
VQ: $\varphi \to \forall x\varphi$, where x is not free in φ.

Inference rules:

MP:	$\dfrac{\varphi,\ \varphi \to \psi}{\psi}$	*Modus Ponens*
N:	$\dfrac{\varphi}{\Box\varphi}$	*Necessitation*
UG:	$\dfrac{\varphi}{\forall x\varphi}$	*Universal Generalisation*
TI:	$\dfrac{\varphi}{\varphi(\tau/x)}$, if τ is free for x in φ.	*Term Instantiation*
GC:	$\dfrac{\varphi(\mathsf{c}/x)}{\varphi}$, if c is not in φ.	*Generalisation on Constants*

A set Γ of $\mathscr{L}$-formulas is said to be L-consistent if there is no finite subset Γ_0 of Γ with $\vdash_L \neg \bigwedge \Gamma_0$, where $\bigwedge \Gamma_0$ is the conjunction of the members of Γ_0. In particular, a single formula φ is L-consistent iff $\{\varphi\}$ is L-consistent, which means that $\neg\varphi \notin L$.

If S is any set of propositional modal formulas, we use the name QS for the *smallest* quantified modal logic that contains every $\mathscr{L}$-formula that is a substitution-instance of a member of S. In other words, QS is the intersection of all such quantified logics. If S is itself the smallest propositional modal logic that includes some set S_{ax} of propositional modal formulas, then $QS = QS_{ax}$ (see [5, Theorem 1.2.5], which also characterises QS-theorems in terms of derivability from substitution-instances of members of S by the axioms and rules of Fig. 2.1). Theorem 1.10.2 of [5] established the following characterisation:

> If S is any set of propositional modal formulas, then QS is characterised by validity in all models for $\mathscr{L}$ whose underlying general frame validates S.

The proof of this involves a canonical model construction that requires $\mathscr{L}$ to contain a denumerable infinity of individual constants. From now on we assume that all signatures have this property when required. It is a harmless assumption, since any logic can be conservatively extended by the addition of such constants.

The above characterisation of QS has two parts:

- *Soundness* If $\vdash_{QS} \varphi$, then φ is valid in all models for $\mathscr{L}$ whose underlying general frame validates the propositional logic S.
- *Completeness* If φ is valid in all models whose underlying general frame validates S, then $\vdash_{QS} \varphi$.

Since a formula φ is QS-consistent iff $\nvdash_{QS} \neg\varphi$, it is readily seen that completeness is equivalent to the statement

- Any QS-consistent formula φ is satisfiable in a model whose underlying general frame validates S.

Now a finite set of formulas is QS-consistent iff its conjunction is, and is satisfied at a point of a model iff its conjunction is. From this we see that completeness implies that

- Any *finite* QS-consistent set of formulas is satisfiable in a model whose underlying general frame validates S.

Strong completeness is the assertion that satisfiability holds for infinite consistent sets as well as finite ones. Here we can derive this stronger conclusion by combining completeness with an ultraproducts-based compactness argument.

Theorem 6 (Strong Completeness of QS) *If* S *is any set of propositional modal formulas, then for any signature* $\mathscr{L}$, *any* QS*-consistent set of* $\mathscr{L}$*-formulas is satisfiable in a model whose underlying general frame validates* S.

Proof Given S and $\mathscr{L}$, let $\mathbb{M}$ be the class of all models for $\mathscr{L}$ whose underlying general frame validates S. Now the property of being a model for $\mathscr{L}$ is preserved by ultraproducts (Theorem 2), as is the property of being a general frame that validates S (Theorem 5). Hence $\mathbb{M}$ is closed under ultraproducts.

Now if Γ is any QS-consistent set, then each finite subset of Γ is QS-consistent, and so is satisfiable in a member of $\mathbb{M}$ by the completeness of QS as stated above. Hence Theorem 4 implies that Γ is satisfiable in a member of $\mathbb{M}$, as required. □

2.8 Strong Completeness with the Barcan Formulas

The Barcan Formula is the axiom scheme,

$$\text{BF:}\quad \forall x\Box\varphi \to \Box\forall x\varphi,$$

while the Converse Barcan Formula is

$$\text{CBF:}\quad \Box\forall x\varphi \to \forall x\Box\varphi.$$

We write L + BF and L + CBF for the least extensions of a quantified modal logic L that include BF and CBF, respectively.

Now validity of BF is often associated with the condition that a model structure has *contracting domains*: for all $w, u \in W$, wRu implies $Dw \supseteq Du$. Validity of CBF is often associated with *expanding domains*: wRu implies $Dw \subseteq Du$. However, these connections really only apply to models whose underlying frame is *full* in the sense that every set of worlds is admissible, i.e. $Prop = \wp W$. Full models are not adequate to characterise logics QS in general. *Admissible* models based on general frames are adequate, but in such models the relationship between contracting and expanding domains and the schemes BF and CBF is more complex. For instance, there exists admissible models that have contracting domains but do not validate BF. In fact there are such models that falsify BF and have *constant domains*: wRu implies $Dw = Du$. Admissible models rejecting BF even include ones with a single domain, having $D_w = U$ for all $w \in W$.

On the other hand, CBF is valid in all admissible models with expanding domains, and any logic of the form QS + CBF is characterised by models with expanding domains. But, perhaps surprisingly, these same logics are also characterised by models with *constant* domains. The class of expanding domain structures includes the constant domain ones, and these constant ones are sufficient to characterise QS + CBF, even when BF is not amongst its theorems.

What underlies these observations about QS + CBF is the perhaps more surprising fact that every logic of the form QS is characterised by models with *contracting* domains. In admissible models, imposition of this contracting domains condition does not force the general validity of any non-theorems of QS. Addition of the expanding domains condition to such models then compels the contracting domains to be constant. The work of Chap. 2 of [5] yields the following completeness results:

If S is any set of propositional modal formulas, then any finite QS-consistent set of formulas is satisfiable in a model whose underlying general frame validates S and has contracting domains.
Moreover, any finite QS + CBF-consistent set of formulas is satisfiable in a model whose underlying general frame validates S and has constant domains.

We now apply our ultraproduct construction to strengthen these facts to strong completeness results.

Theorem 7 *An ultraproduct* $\mathscr{M}_\mu = \prod_\mu \mathscr{M}_i$ *has contracting/expanding/constant domains if almost all of the* $\mathscr{M}_i$*'s have likewise.*

Proof Let $J = \{i \in I : \mathscr{M}_i$ has contracting domains$\}$ and suppose $J \in \mu$. We prove that $\mathscr{M}_\mu$ has contracting domains.

Let $f_\mu R_\mu g_\mu$. If $h_\mu \in D_\mu g_\mu$ then we have that the sets $\{i \in I : f(i) R_i g(i)\}$ and $\{i : h(i) \in D_i g(i)\}$ both belong to μ, and so the intersection

$$J \cap \{i \in I : f(i) R_i g(i)\} \cap \{i : h(i) \in D_i g(i)\}$$

belongs to μ. But the set $\{i : h(i) \in D_i f(i)\}$ includes this intersection, so it belongs to μ as well, showing that $h_\mu \in D_\mu f_\mu$. Hence $D_\mu f_\mu \supseteq D_\mu g_\mu$ as required.

The cases of expanding and constant domains, respectively, are similar. □

This theorem combines with the argument of Theorem 6, taking $\mathbb{M}$ to be the class of all contracting domains models whose underlying general frame validates S, and then restricting it those models with constant domains. In both cases, Theorem 7 implies that we get a class of models that is closed under ultraproducts. Given the above Completeness results we infer:

Theorem 8 (Contracting and Constant Domains Strong Completeness) *If* S *is any set of propositional modal formulas, then for any signature* $\mathscr{L}$*, any* QS*-consistent set of* $\mathscr{L}$*-formulas is satisfiable in a model whose underlying general frame validates* S *and has contracting domains. Moreover, any* QS + CBF*-consistent set of formulas is satisfiable in a model whose underlying general frame validates* S *and has constant domains.* □

Turning now to the Barcan Formula, we have already noted that it need not be valid in a contracting-domains model. In general it is only in *Kripkean* models with contracting domains that validity of BF is guaranteed. The real role of BF in admissible model theory is to enable us to build models that give the quantifier ∀ its standard Kripkean interpretation. In that context we also need to use the commuting quantifiers axiom

$$\text{CQ}: \quad \forall x \forall y \varphi \to \forall y \forall x \varphi$$

which is valid in Kripkean models, but not in general. In [5, Sect. 2.6] a canonical model construction was given that provides a completeness result for certain logics containing BF and which depends on the background signature being countable. The upshot is this:

> If S is any set of propositional modal formulas, then for any countable signature $\mathscr{L}$, any finite QS + CBF + CQ + BF-consistent set of $\mathscr{L}$-formulas is satisfiable in a Kripkean constant-domains $\mathscr{L}$-model whose underlying general frame validates S.

We now lift this result to a strong completeness theorem, overcoming the countability restriction.

Theorem 9 (Strong Completeness for QS + CBF + CQ + BF) *If* S *is any set of propositional modal formulas, then for any signature $\mathscr{L}$, any* QS + CBF + CQ + BF*-consistent set of $\mathscr{L}$-formulas is satisfiable in a Kripkean constant-domains $\mathscr{L}$-model whose underlying general frame validates* S.

Proof Let $\mathbb{M}$ be the class of all Kripkean constant-domains $\mathscr{L}$-models whose underlying general frame validates S. Then $\mathbb{M}$ is closed under ultraproducts, by Theorems 2, 3, 5 and 7.

Let L be the logic QS + CBF + CQ + BF as a set of $\mathscr{L}$-formulas, and let Γ be an L-consistent set of $\mathscr{L}$-formulas. Put $I = \{i \subseteq \Gamma : i \text{ is finite}\}$. For each $i \in I$, let $\mathscr{L}_i$ be a countable subset of $\mathscr{L}$ that firstly includes all the (finitely many) members of $\mathscr{L}$ that occur in i; secondly has infinitely many constants, including some particular constant c_0; and thirdly for each positive integer n includes some particular n-ary function symbol F_n if $\mathscr{L}$ has n-ary function symbols. Then i is a set of $\mathscr{L}_i$-formulas.

Define L_i to be the logic QS + CBF + CQ + BF in the language $\mathscr{L}_i$. Then $\mathrm{L}_i \subseteq \mathrm{L}$, so if $\neg(\bigwedge i) \in \mathrm{L}_i$ we would have $\neg(\bigwedge i) \in \mathrm{L}$, contradicting the L-consistency of Γ. Hence $\neg(\bigwedge i) \notin \mathrm{L}_i$, showing i is L_i-consistent. Since the signature $\mathscr{L}_i$ is countable, the above completeness result for QS + CBF + CQ + BF implies that i is satisfiable in some Kripkean constant-domains $\mathscr{L}_i$-model $\mathscr{M}$ whose underlying general frame validates S.

Let $\mathscr{S} = (W, R, Prop, U, D)$ be the model structure of $\mathscr{M}$. We now expand $\mathscr{M}$ to an $\mathscr{L}$-premodel $\mathscr{M}'$ on $\mathscr{S}$ by declaring $\mathscr{M}'$ to be identical to $\mathscr{M}$ on $\mathscr{L}_i$, and for symbols ζ in $\mathscr{L} - \mathscr{L}_i$ putting $|\zeta|^{\mathscr{M}'} = |\mathsf{c}_0|^{\mathscr{M}}$ if ζ is a constant; $|\zeta|^{\mathscr{M}'} = |F_n|^{\mathscr{M}}$ if ζ is an n-ary function symbol; and if ζ is an n-ary predicate symbol, letting $|\zeta|^{\mathscr{M}'}$ be the n-ary function on U with constant value $\emptyset$.

For each $\mathscr{L}$-term τ, let τ' be the $\mathscr{L}_i$-term resulting from replacing any constant of τ not in $\mathscr{L}_i$ by c_0, and any n-ary function symbol of τ not in $\mathscr{L}_i$ by F_n. A routine induction on term-formation shows that in general $|\tau|^{\mathscr{M}'} = |\tau'|^{\mathscr{M}}$.

Then for each $\mathscr{L}$-formula φ, let φ' be the $\mathscr{L}_i$-formula resulting from replacing each atomic formula $P\tau_1 \cdots \tau_n$ within φ by $P\tau_1' \cdots \tau_n'$ if $P \in \mathscr{L}_i$, and by $\bot$ if $P \notin \mathscr{L}_i$. An induction on formula formation then shows that in general $|\varphi|^{\mathscr{M}'} = |\varphi'|^{\mathscr{M}}$.

It follows that $\mathscr{M}'$ is an $\mathscr{L}$-model: for any $\mathscr{L}$-formula φ and $f \in U^\omega$, $|\varphi|^{\mathscr{M}'} f = |\varphi'|^{\mathscr{M}} f \in Prop$ as $\mathscr{M}$ is an $\mathscr{L}_i$-model. So every $\mathscr{L}$-formula is admissible in $\mathscr{M}'$.

It also follows that $\mathscr{M}'$ is Kripkean. To see this, take any $\mathscr{L}$-formula φ, variable x, and $f \in U^\omega$, and let $\mathscr{X} = \{Ea \Rightarrow |\varphi|^{\mathscr{M}'} f[a/x] : a \in U\}$. So $|\forall x\varphi|^{\mathscr{M}'} f = \bigsqcap \mathscr{X}$. But

$$\bigcap \mathscr{X} = \bigcap_{a \in U} \big(Ea \Rightarrow |\varphi'|^{\mathscr{M}} f[a/x] : a \in U\big) = |\forall x(\varphi')|^{\mathscr{M}} f \in Prop,$$

because $\mathcal{M}$ is Kripkean and an $\mathcal{L}_i$-model. Thus $\bigcap \mathcal{X} \in Prop$, which implies that $\bigcap \mathcal{X} = \prod \mathcal{X}$. So $|\forall x \varphi|^{\mathcal{M}'} = \bigcap \mathcal{X}$, making $\mathcal{M}'$ a Kripkean model.

Now the underlying structure $\mathcal{S}$ of $\mathcal{M}'$ has constant domains and its general frame validates S. So $\mathcal{M}'$ belongs to the class $\mathbb{M}$ defined at the start of this proof. But each $\varphi \in i$ is an $\mathcal{L}_i$-formula so has $\varphi' = \varphi$, and hence $|\varphi|^{\mathcal{M}'} = |\varphi|^{\mathcal{M}}$. Since i is satisfiable in $\mathcal{M}$ it follows that it is satisfiable in $\mathcal{M}'$.

We have now established that any finite subset i of Γ is satisfiable in an $\mathcal{L}$-model belonging to the ultraproducts-closed class $\mathbb{M}$. Hence Theorem 4 implies that Γ is satisfiable in a member of $\mathbb{M}$, as required. □

2.9 One Universal Domain

A model structure $\mathcal{S}$ has *one universal domain* if $D_w = U$ for all w in $\mathcal{S}$. If this holds, then $Ea = W$ for all $a \in U$, and so $(Ea \Rightarrow X) = X$ in general. This implies that in any model $\mathcal{M}$ on $\mathcal{S}$ we have

$$|\forall x \varphi|^{\mathcal{M}} f = \prod_{a \in U} |\varphi|^{\mathcal{M}} f[a/x].$$

A model with one universal domain validates the universal instantiation axiom

$$\text{UI:} \quad \forall x \varphi \to \varphi(\tau/x), \quad \text{where } \tau \text{ is free for } x \text{ in } \varphi.$$

It was shown in [5, Sect. 2.4] that any quantifed modal logic of the form QS + UI is complete for validity in one-universal-domain admissible models whose underlying general frame validates S.

Now it is readily seen that the property of having one universal domain is preserved by an ultraproduct $\mathcal{M}_\mu = \prod_\mu \mathcal{M}_i$. For if the set

$$J = \{i \in I : \mathcal{M}_i \text{ has one universal domain}\}$$

belongs to μ, then for any $f_u \in W_\mu$ and any $g_\mu \in U_\mu$, the set $\{i \in I : g(i) \in D_i(f(i))\}$ includes J and so belongs to μ. It follows that $g_\mu \in D_\mu f_\mu$. Hence $D_\mu f_\mu = U_\mu$, implying $\mathcal{M}_\mu$ has one universal domain.

Applying this observation to our earlier arguments, we can conclude that any logic of the form QS + UI is *strongly* complete for validity in one-universal-domain admissible models whose underlying general frame validates S.

A logic containing UI also contains the schemes CBF and CQ, but need not contain BF. For instance, BF is not derivable in QS4 + UI. Section 2.7 of [5] showed that any quantifed modal logic of the form QS + UI + BF in a countable signature is complete for validity in *Kripkean* one-universal-domain admissible models whose underlying general frame validates S. Here, our ultraproduct analysis allows us to strengthen this to conclude that, in arbitrary signatures, QS + UI + BF is *strongly* complete for validity in such models.

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