Chapter 2
Nonlinear Discrete Systems

In this chapter, a theory for nonlinear discrete systems is reviewed. The local and global theory of stability and bifurcation for nonlinear discrete systems is presented. The stability switching and bifurcation on specific eigenvectors of the linearized system at fixed points under a specific period are discussed. The higher-order singularity and stability for nonlinear discrete systems on the specific eigenvectors are also presented.

2.1 Definitions

Definition 2.1 For $\Omega_\alpha \subseteq \mathscr{R}^n$ and $\Lambda \subseteq \mathscr{R}^m$ with $\alpha \in \mathbb{Z}$, consider a vector function $\mathbf{f}_\alpha : \Omega_\alpha \times \Lambda \to \Omega_\alpha$ which is C^r ($r \geq 1$)-continuous, and there is a discrete (or difference) equation in a form of

$$\mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha) \text{ for } \mathbf{x}_k,\ \mathbf{x}_{k+1} \in \Omega_\alpha, \quad k \in \mathbb{Z} \ \text{ and } \ \mathbf{p}_\alpha \in \Lambda \tag{2.1}$$

with an initial condition of $\mathbf{x}_k = \mathbf{x}_0$, the solution of Eq. (2.1) is given by

$$\begin{aligned} &\mathbf{x}_k = \underbrace{\mathbf{f}_\alpha(\mathbf{f}_\alpha(\ldots(\mathbf{f}_\alpha}_{k}(\mathbf{x}_0, \mathbf{p}_\alpha)))) \\ &\text{for } \mathbf{x}_k \in \Omega_\alpha, \quad k \in \mathbb{Z} \ \text{ and } \ \mathbf{p} \in \Lambda. \end{aligned} \tag{2.2}$$

(i) The difference equation with the initial condition is called a *discrete dynamical system*.
(ii) The vector function $\mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ is called a *discrete vector field* on Ω_α.
(iii) The solution $\mathbf{x}_k$ for each $k \in \mathbb{Z}$ is called a *flow* of discrete system.
(iv) The solution $\mathbf{x}_k$ for all $k \in \mathbb{Z}$ on domain Ω_α is called the trajectory, phase curve, or orbit of the discrete dynamical system, which is defined as

$$\Gamma = \{\mathbf{x}_k | \mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha) \ \text{ for } k \in \mathbb{Z} \ \text{ and } \mathbf{p}_\alpha \in \Lambda\} \subseteq \cup_\alpha \Omega_\alpha. \tag{2.3}$$

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(v) The discrete dynamical system is called *a uniform discrete system* if

$$\mathbf{x}_{k+1} = \mathbf{f}_{\alpha}(\mathbf{x}_k, \mathbf{p}_{\alpha}) = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \quad \text{for } k \in \mathbb{Z} \text{ and } \mathbf{x}_k \in \mathbf{\Omega}_{\alpha}. \tag{2.4}$$

Otherwise, this discrete dynamical system is called a *non-uniform discrete system*.

Definition 2.2 For the discrete dynamical system in Eq. (2.1), the relation between state $\mathbf{x}_k$ and state $\mathbf{x}_{k+1}$ ($k \in \mathbb{Z}$) is called a discrete map if

$$P_{\alpha} : \mathbf{x}_k \xrightarrow{\mathbf{f}_{\alpha}} \mathbf{x}_{k+1} \quad \text{and} \quad \mathbf{x}_{k+1} = P_{\alpha}\mathbf{x}_k \tag{2.5}$$

with the following properties:

$$P_{(k;l)} : \mathbf{x}_k \xrightarrow{\mathbf{f}_{\alpha_1}, \mathbf{f}_{\alpha_2}, \ldots, \mathbf{f}_{\alpha_l}} \mathbf{x}_{k+l} \text{ and } \mathbf{x}_{k+l} = P_{\alpha_l} \circ P_{\alpha_{l-1}} \circ \cdots \circ P_{\alpha_1}\mathbf{x}_k \tag{2.6}$$

where

$$P_{(k;l)} = P_{\alpha_l} \circ P_{\alpha_{l-1}} \circ \cdots \circ P_{\alpha_1}. \tag{2.7}$$

If $P_{\alpha_l} = P_{\alpha_{l-1}} = \cdots = P_{\alpha_1} = P_{\alpha}$, then

$$P_{(\alpha;l)} \equiv P_{\alpha}^{(l)} = P_{\alpha} \circ P_{\alpha} \circ \cdots \circ P_{\alpha} \tag{2.8}$$

with

$$P_{\alpha}^{(n)} = P_{\alpha} \circ P_{\alpha}^{(n-1)} \text{ and } P_{\alpha}^{(0)} = \mathbf{I}. \tag{2.9}$$

The total map with l-different sub-maps is shown in Fig. 2.1. The map P_{α_k} with the relation function $\mathbf{f}_{\alpha_k}(\alpha_k \in \mathbb{Z})$ is given by Eq. (2.5). The total map $P_{(k;l)}$ is given in Eq. (2.7). The domains $\Omega_{\alpha_k}(\alpha_k \in \mathbb{Z})$ can fully overlap each other or can be completely separated without any intersection.

Definition 2.3 For a vector function in $\mathbf{f}_{\alpha} \in \mathscr{R}^n$, $\mathbf{f}_{\alpha} : \mathscr{R}^n \to \mathscr{R}^n$. The operator norm of $\mathbf{f}_{\alpha}$ is defined by

$$\|\mathbf{f}_{\alpha}\| = \sum_{i=1}^{n} \max_{\|\mathbf{x}_k\| \le 1, \mathbf{p}_{\alpha}} |f_{\alpha(i)}(\mathbf{x}_k, \mathbf{p}_{\alpha})|. \tag{2.10}$$

For an $n \times n$ matrix $\mathbf{f}_{\alpha}(\mathbf{x}_k, \mathbf{p}_{\alpha}) = \mathbf{A}_{\alpha}\mathbf{x}_k$ and $\mathbf{A}_{\alpha} = (a_{ij})_{n \times n}$, the corresponding norm is defined by

$$\|\mathbf{A}_{\alpha}\| = \sum_{i,j=1}^{n} |a_{ij}|. \tag{2.11}$$

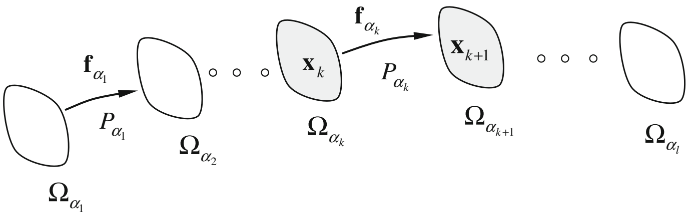

Fig. 2.1 Maps and vector functions on each sub-domain for discrete dynamical system

Definition 2.4 For $\Omega_\alpha \subseteq \mathscr{R}^n$ and $\Lambda \subseteq \mathscr{R}^m$ with $\alpha \in \mathbb{Z}$, the vector function $\mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ with $\mathbf{f}_\alpha : \Omega_\alpha \times \Lambda \to \mathscr{R}^n$ is differentiable at $\mathbf{x}_k \in \Omega_\alpha$ if

$$\left.\frac{\partial \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)}{\partial \mathbf{x}_k}\right|_{(\mathbf{x}_k, \mathbf{p})} = \lim_{\Delta \mathbf{x}_k \to \mathbf{0}} \frac{\mathbf{f}_\alpha(\mathbf{x}_k + \Delta \mathbf{x}_k, \mathbf{p}_\alpha) - \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)}{\Delta \mathbf{x}_k}. \tag{2.12}$$

$\partial \mathbf{f}_\alpha / \partial \mathbf{x}_k$ is called the spatial derivative of $\mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ at $\mathbf{x}_k$, and the derivative is given by the Jacobian matrix

$$\frac{\partial \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)}{\partial \mathbf{x}_k} = \left[\frac{\partial f_{\alpha(i)}}{\partial x_{k(j)}}\right]_{n \times n}. \tag{2.13}$$

Definition 2.5 For $\Omega_\alpha \subseteq \mathscr{R}^n$ and $\Lambda \subseteq \mathscr{R}^m$, consider a vector function $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ with $\mathbf{f} : \Omega_\alpha \times \Lambda \to \mathscr{R}^n$ where $\mathbf{x}_k \in \Omega_\alpha$ and $\mathbf{p} \in \Lambda$ with $k \in \mathbb{Z}$. The vector function $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is said to satisfy the Lipschitz condition if

$$\|\mathbf{f}(\mathbf{y}_k, \mathbf{p}) - \mathbf{f}(\mathbf{x}_k, \mathbf{p})\| \le L \|\mathbf{y}_k - \mathbf{x}_k\| \tag{2.14}$$

with $\mathbf{x}_k, \mathbf{y}_k \in \Omega_\alpha$ and L a constant. The constant L is called the Lipschitz constant.

2.2 Fixed Points and Stability

Definition 2.6 Consider a discrete, dynamical system $\mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ in Eq. (2.4).

(i) A point $\mathbf{x}_k^* \in \Omega_\alpha$ is called a fixed point or period-1 solution of a discrete nonlinear system $\mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ under a map P_α if for $\mathbf{x}_{k+1} = \mathbf{x}_k = \mathbf{x}_k^*$

$$\mathbf{x}_k^* = \mathbf{f}_\alpha(\mathbf{x}_k^*, \mathbf{p}). \tag{2.15}$$

The linearized system of the nonlinear discrete system $\mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ in Eq. (2.4) at the fixed point $\mathbf{x}_k^*$ is given by

$$\mathbf{y}_{k+1} = DP_{\alpha}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k = D\mathbf{f}_{\alpha}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k \tag{2.16}$$

where

$$\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^* \quad \text{and} \quad \mathbf{y}_{k+1} = \mathbf{x}_{k+1} - \mathbf{x}_{k+1}^*. \tag{2.17}$$

(ii) A set of points $\mathbf{x}_j^* \in \Omega_{\alpha_j}$ $(\alpha_j \in \mathbb{Z})$ is called the fixed point set or period-1 point set of the total map $P_{(k;l)}$ with l-different sub-maps in nonlinear discrete system of Eq. (2.5) if

$$\begin{aligned} &\mathbf{x}_{k+j+1}^* = \mathbf{f}_{\alpha_{j'}}(\mathbf{x}_{k+j}^*, \mathbf{p}_{\alpha_{j'}}) \quad \text{for } j \in \mathbb{Z}_+ \text{ and } j' = \mathrm{mod}(j,l) + 1;\\ &\mathbf{x}_{k+\mathrm{mod}(j,l)}^* = \mathbf{x}_k^*. \end{aligned} \tag{2.18}$$

The linearized equation of the total map $P_{(k;l)}$ gives

$$\begin{aligned} &\mathbf{y}_{k+j+1} = DP_{\alpha_{j'}}(\mathbf{x}_{k+j}^*, \mathbf{p}_{\alpha_{j'}})\mathbf{y}_{k+j} = D\mathbf{f}_{\alpha_{j'}}(\mathbf{x}_{k+j}^*, \mathbf{p}_{\alpha_{j'}})\mathbf{y}_{k+j} \text{ with}\\ &\mathbf{y}_{k+j+1} = \mathbf{x}_{k+j+1} - \mathbf{x}_{k+j+1}^* \quad \text{and} \quad \mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_{k+j}^* \text{ for}\\ &j \in \mathbb{Z}_+ \quad \text{and} \quad j' = \mathrm{mod}(j,l) + 1. \end{aligned} \tag{2.19}$$

The resultant equation for each individual map is

$$\mathbf{y}_{k+j+1} = DP_{(k,l)}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j} \quad \text{for } j \in \mathbb{Z}_+ \tag{2.20}$$

where

$$\begin{aligned} DP_{(k,n)}(\mathbf{x}_k^*, \mathbf{p}) &= \prod\nolimits_{j=l}^{1} DP_{\alpha_j}(\mathbf{x}_{k+j-1}^*, \mathbf{p})\\ &= DP_{\alpha_l}(\mathbf{x}_{k+l-1}^*, \mathbf{p}_{\alpha_n}) \cdot \cdots \cdot DP_{\alpha_2}(\mathbf{x}_{k+1}^*, \mathbf{p}_{\alpha_2}) \cdot DP_{\alpha_1}(\mathbf{x}_k^*, \mathbf{p}_{\alpha_1})\\ &= D\mathbf{f}_{(\alpha_l)}(\mathbf{x}_{k+l-1}^*, \mathbf{p}_{\alpha_n}) \cdot \cdots \cdot D\mathbf{f}_{(\alpha_2)}(\mathbf{x}_{k+1}^*, \mathbf{p}_{\alpha_2}) \cdot D\mathbf{f}_{(\alpha_1)}(\mathbf{x}_k^*, \mathbf{p}_{\alpha_1}). \end{aligned} \tag{2.21}$$

The fixed point $\mathbf{x}_k^*$ lies in the intersected set of two domains Ω_k and Ω_{k+1}, as shown in Fig. 2.2. In the vicinity of the fixed point $\mathbf{x}_k^*$, the incremental relations in the two domains Ω_k and Ω_{k+1} are different. In other words, setting $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$ and $\mathbf{y}_{k+1} = \mathbf{x}_{k+1} - \mathbf{x}_{k+1}^*$, the corresponding linearization is generated as in Eq. (2.16). Similarly, the fixed point of the total map with n-different sub-maps requires the intersection set of two domains Ω_k and Ω_{k+n}, and there are a set of equations to obtain the fixed points from Eq. (2.18). The other values of fixed points lie in different domains, i.e., $\mathbf{x}_j^* \in \Omega_j$ $(j = k+1, k+2, \ldots, k+n-1)$, as shown in Fig. 2.3.

The corresponding linearized equations are given in Eq. (2.19). From Eq. (2.20), the local characteristics of the total map can be discussed as a single map. Thus, the

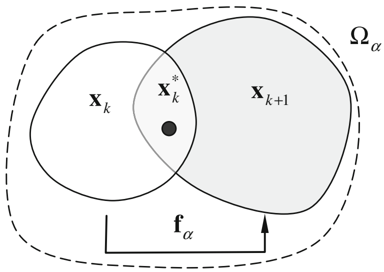

Fig. 2.2 A fixed point between domains Ω_k and Ω_{k+1} for a discrete dynamical system

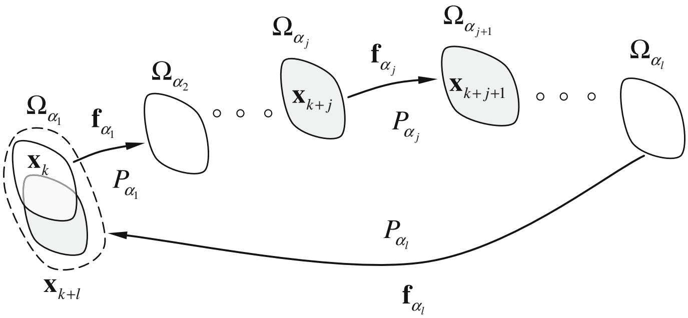

Fig. 2.3 Fixed points with l-maps for discrete dynamical system

dynamical characteristics for the fixed point of the single map will be discussed comprehensively, and the fixed points for resultant map are applicable. The results can be extended to any period-m flows with $P^{(m)}$.

Definition 2.7 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The linearized system of the discrete nonlinear system in the neighborhood of $\mathbf{x}_k^*$ is $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ ($\mathbf{y}_l = \mathbf{x}_l - \mathbf{x}_k^*$ and $l = k, k+1$) in Eq. (2.16). The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n_1 real eigenvalues $|\lambda_j| < 1$ ($j \in N_1$), n_2 real eigenvalues $|\lambda_j| > 1$ ($j \in N_2$), n_3 real eigenvalues $\lambda_j = 1$ ($j \in N_3$), and n_4 real eigenvalues $\lambda_j = -1$ ($j \in N_4$). $N = \{1, 2, \ldots, n\}$ and $N_i = \{i_1, i_2, \ldots, i_{n_i}\} \cup \varnothing$ ($i = 1, 2, 3, 4$) with $i_m \in N$ ($m = 1, 2, \ldots, n_i$) and $\Sigma_{i=1}^4 n_i = n$. $N_i \subseteq N \cup \varnothing$, $\cup_{i=1}^4 N_i = N$, $N_i \cap N_p = \varnothing$ ($p \neq i$). $N_i = \varnothing$ if $n_i = 0$. The corresponding eigenvectors for contraction, expansion, invariance, and flip oscillation are $\{\mathbf{v}_j\}$ ($j \in N_i$) ($i = 1, 2, 3, 4$), respectively. The stable, unstable, invariant, and flip subspaces of $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ in Eq. (2.16) are linear subspace spanned by $\{\mathbf{v}_j\}$ ($j \in N_i$) ($i = 1, 2, 3, 4$), respectively, i.e.,

$$
\begin{aligned}
\mathscr{E}^{\mathrm{s}} &= \operatorname{span}\left\{ \mathbf{v}_j \middle| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ |\lambda_j| < 1, j \in N_1 \subseteq N \cup \varnothing \end{array} \right\}; \\
\mathscr{E}^{\mathrm{u}} &= \operatorname{span}\left\{ \mathbf{v}_j \middle| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ |\lambda_j| > 1, j \in N_2 \subseteq N \cup \varnothing \end{array} \right\}; \\
\mathscr{E}^{\mathrm{i}} &= \operatorname{span}\left\{ \mathbf{v}_j \middle| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j = 1, j \in N_3 \subseteq N \cup \varnothing \end{array} \right\}; \\
\mathscr{E}^{\mathrm{f}} &= \operatorname{span}\left\{ \mathbf{v}_j \middle| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j = -1, j \in N_4 \subseteq N \cup \varnothing \end{array} \right\}
\end{aligned}
\tag{2.22}
$$

where

$$
\begin{aligned}
&\mathscr{E}^{\mathrm{s}} = \mathscr{E}^{\mathrm{s}}_{\mathrm{m}} \cup \mathscr{E}^{\mathrm{s}}_{\mathrm{o}} \cup \mathscr{E}^{\mathrm{s}}_{\mathrm{z}} \text{ with} \\
&\mathscr{E}^{\mathrm{s}}_{\mathrm{m}} = \operatorname{span}\left\{ \mathbf{v}_j \middle| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ 0 < \lambda_j < 1, j \in N_1^{\mathrm{m}} \subseteq N \cup \varnothing \end{array} \right\}; \\
&\mathscr{E}^{\mathrm{s}}_{\mathrm{o}} = \operatorname{span}\left\{ \mathbf{v}_j \middle| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ -1 < \lambda_j < 0, j \in N_1^{\mathrm{o}} \subseteq N \cup \varnothing \end{array} \right\}; \\
&\mathscr{E}^{\mathrm{s}}_{\mathrm{z}} = \operatorname{span}\left\{ \mathbf{v}_j \middle| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j = 0, j \in N_1^{\mathrm{z}} \subseteq N \cup \varnothing \end{array} \right\};
\end{aligned}
\tag{2.23}
$$

$$
\begin{aligned}
&\mathscr{E}^{\mathrm{u}} = \mathscr{E}^{\mathrm{u}}_{\mathrm{m}} \cup \mathscr{E}^{\mathrm{u}}_{\mathrm{o}} \text{ with} \\
&\mathscr{E}^{\mathrm{u}}_{\mathrm{m}} = \operatorname{span}\left\{ \mathbf{v}_j \middle| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j > 1, j \in N_2^{\mathrm{m}} \subseteq N \cup \varnothing \end{array} \right\}; \\
&\mathscr{E}^{\mathrm{u}}_{\mathrm{o}} = \operatorname{span}\left\{ \mathbf{v}_j \middle| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ -1 > \lambda_j, j \in N_2^{\mathrm{o}} \subseteq N \cup \varnothing \end{array} \right\}.
\end{aligned}
\tag{2.24}
$$

Herein, subscripts “m” and “o” represent the monotonic and oscillatory evolutions.

Definition 2.8 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The linearized system of the discrete nonlinear system in the neighborhood of $\mathbf{x}_k^*$ is $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ ($\mathbf{y}_l = \mathbf{x}_l - \mathbf{x}_k^*$ and $l = k, k+1$) in Eq. (2.16). The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ has complex eigenvalues $\alpha_j \pm \mathbf{i}\beta_j$ with eigenvectors $\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j$ ($j \in \{1, 2, \ldots, n\}$), and the base of vector is

$$
\mathbf{B} = \{\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{u}_j, \mathbf{v}_j, \ldots, \mathbf{u}_n, \mathbf{v}_n\}. \tag{2.25}
$$

The stable, unstable, center subspaces of $\mathbf{y}_{k+1} = D\mathbf{f}_k(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ in Eq. (2.16) are linear subspaces spanned by $\{\mathbf{u}_j, \mathbf{v}_j\}(j \in N_i,\ \ i = 1, 2, 3)$, respectively. Set $N = \{1, 2, \ldots, n\}$ plus $N_i = \{i_1, i_2, \ldots, i_{n_i}\} \cup \varnothing \subseteq N \cup \varnothing$ with $i_m \in N$ ($m = 1, 2, \ldots, n_i$) and $\Sigma_{i=1}^4 n_i = n$. $\cup_{i=1}^4 N_i = N$ with $N_i \cap N_p = \varnothing (p \neq i)$. $N_i = \varnothing$ if $n_i = 0$. The stable, unstable, center subspaces of $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ in Eq. (2.16) are defined by

$$\begin{aligned}
\mathscr{E}^{\mathrm{s}} &= \mathrm{span}\left\{ (\mathbf{u}_j, \mathbf{v}_j) \middle| \begin{array}{l} r_j = \sqrt{\alpha_j^2 + \beta_j^2} < 1, \\ (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - (\alpha_j \pm \mathbf{i}\beta_j)\mathbf{I})(\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j) = \mathbf{0}, \\ j \in N_1 \subseteq \{1, 2, \ldots, n\} \cup \varnothing \end{array} \right\}; \\
\mathscr{E}^{\mathrm{u}} &= \mathrm{span}\left\{ (\mathbf{u}_j, \mathbf{v}_j) \middle| \begin{array}{l} r_j = \sqrt{\alpha_j^2 + \beta_j^2} > 1, \\ (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - (\alpha_j \pm \mathbf{i}\beta_j)\mathbf{I})(\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j) = \mathbf{0}, \\ j \in N_2 \subseteq \{1, 2, \ldots, n\} \cup \varnothing \end{array} \right\}; \\
\mathscr{E}^{\mathrm{c}} &= \mathrm{span}\left\{ (\mathbf{u}_j, \mathbf{v}_j) \middle| \begin{array}{l} r_j = \sqrt{\alpha_j^2 + \beta_j^2} = 1, \\ (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - (\alpha_j \pm \mathbf{i}\beta_j)\mathbf{I})(\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j) = \mathbf{0}, \\ j \in N_3 \subseteq \{1, 2, \ldots, n\} \cup \varnothing \end{array} \right\}.
\end{aligned} \tag{2.26}$$

Definition 2.9 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The linearized system of the discrete nonlinear system in the neighborhood of $\mathbf{x}_k^*$ is $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ ($\mathbf{y}_l = \mathbf{x}_l - \mathbf{x}_k^*$ and $l = k, k+1$) in Eq. (2.16). The fixed point or period-1 point is *hyperbolic* if no any eigenvalues of $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ are on the unit circle (i.e., $|\lambda_i| \neq 1$ for $i = 1, 2, \ldots, n$).

Theorem 2.1 *Consider a discrete, nonlinear dynamical system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *in* Eq. (2.4) *with a fixed point* $\mathbf{x}_k^*$. *The linearized system of the discrete nonlinear system in the neighborhood of* $\mathbf{x}_k^*$ *is* $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ ($\mathbf{y}_j = \mathbf{x}_j - \mathbf{x}_k^*$ *and* $j = k, k+1$) *in* Eq. (2.16). *The eigenspace of* $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ (*i.e.,* $\mathscr{E} \subseteq \mathscr{R}^n$) *in the linearized dynamical system is expressed by direct sum of three subspaces*

$$\mathscr{E} = \mathscr{E}^{\mathrm{s}} \oplus \mathscr{E}^{\mathrm{u}} \oplus \mathscr{E}^{\mathrm{c}} \tag{2.27}$$

where $\mathscr{E}^{\mathrm{s}}, \mathscr{E}^{\mathrm{u}}$ and $\mathscr{E}^{\mathrm{c}}$ *are the stable, unstable, and center subspaces, respectively.*

Proof The proof can be referred to Luo (2011). □

Definition 2.10 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. Suppose there is a neighborhood of the equilibrium $\mathbf{x}_k^*$ as $U_k(\mathbf{x}_k^*) \subset \Omega_k$, and in the neighborhood,

$$\lim_{||\mathbf{y}_k|| \to 0} \frac{||\mathbf{f}(\mathbf{x}_k^* + \mathbf{y}_k, \mathbf{p}) - D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k||}{||\mathbf{y}_k||} = 0, \tag{2.28}$$

and

$$\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k. \tag{2.29}$$

(i) A C^r invariant manifold

$$\mathscr{S}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*) = \{\mathbf{x}_k \in U_k(\mathbf{x}_k^*) | \lim_{j\to+\infty} \mathbf{x}_{k+j} = \mathbf{x}_k^* \quad \text{and} \\ \mathbf{x}_{k+j} \in U_k(\mathbf{x}_k^*) \text{ with } j \in \mathbb{Z}_+\} \tag{2.30}$$

is called the local stable manifold of $\mathbf{x}_k^*$, and the corresponding global stable manifold is defined as

$$\mathscr{S}(\mathbf{x}_k, \mathbf{x}_k^*) = \cup_{j\in\mathbb{Z}_-} \mathbf{f}(\mathscr{S}_{\text{loc}}(\mathbf{x}_{k+j}, \mathbf{x}_{k+j}^*)) = \cup_{j\in\mathbb{Z}_-} \mathbf{f}^{(j)}(\mathscr{S}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)). \tag{2.31}$$

(ii) A C^r invariant manifold $\mathscr{U}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$

$$\mathscr{U}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*) = \{\mathbf{x}_k \in U_k(\mathbf{x}_k^*) | \lim_{j\to-\infty} \mathbf{x}_{k+j} = \mathbf{x}_k^* \text{ and} \\ \mathbf{x}_{k+j} \in U_k(\mathbf{x}_k^*) \text{ with } j \in \mathbb{Z}_-\} \tag{2.32}$$

is called the local unstable manifold of $\mathbf{x}^*$, and the corresponding global unstable manifold is defined as

$$\mathscr{U}(\mathbf{x}_k, \mathbf{x}_k^*) = \cup_{j\in\mathbb{Z}_+} \mathbf{f}(\mathscr{U}_{\text{loc}}(\mathbf{x}_{k+j}, \mathbf{x}_{k+j}^*)) = \cup_{j\in\mathbb{Z}_+} \mathbf{f}^{(j)}(\mathscr{U}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)). \tag{2.33}$$

(iii) A C^{r-1} invariant manifold $\mathscr{C}_{\text{loc}}(\mathbf{x}, \mathbf{x}^*)$ is called the center manifold of $\mathbf{x}^*$ if $\mathscr{C}_{\text{loc}}(\mathbf{x}, \mathbf{x}^*)$ possesses the same dimension of $\mathscr{E}^{\text{c}}$ for $\mathbf{x}^* \in \mathscr{C}_{\text{loc}}(\mathbf{x}, \mathbf{x}^*)$, and the tangential space of $\mathscr{C}_{\text{loc}}(\mathbf{x}, \mathbf{x}^*)$ is identical to $\mathscr{E}^{\text{c}}$.

As in continuous dynamical systems, the stable and unstable manifolds are unique, but the center manifold is not unique. If the nonlinear vector field $\mathbf{f}$ is C^∞-continuous, then a C^r center manifold can be found for any $r < \infty$.

Theorem 2.2 *Consider a discrete, nonlinear dynamical system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *in* Eq. (2.4) *with a hyperbolic fixed point* $\mathbf{x}_k^*$. *The corresponding solution is* $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ *with* $j \in \mathbb{Z}$. *Suppose there is a neighborhood of the hyperbolic fixed point* $\mathbf{x}_k^*$ *(i.e.,* $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$*), and* $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *is* C^r $(r \geq 1)$*-continuous in* $U_k(\mathbf{x}_k^*)$. *The linearized system is* $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ *in* $U_k(\mathbf{x}_k^*)$. *If the homeomorphism between the local invariant subspace* $E(\mathbf{x}_k^*) \subset U_k(\mathbf{x}_k^*)$ *and the eigenspace* $\mathscr{E}$ *of the linearized system exists with the condition in* Eq. (2.28)*, the local invariant subspace is decomposed by*

$$E(\mathbf{x}_k, \mathbf{x}_k^*) = \mathscr{S}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*) \oplus \mathscr{U}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*). \tag{2.34}$$

(a) *The local stable invariant manifold* $\mathscr{S}_{\text{loc}}(\mathbf{x}, \mathbf{x}^*)$ *possesses the following properties*:

   (i) *for* $\mathbf{x}_k^* \in \mathscr{S}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$, $\mathscr{S}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$ *possesses the same dimension of* $\mathscr{E}^{\text{s}}$ *and the tangential space of* $\mathscr{S}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$ *is identical to* $\mathscr{E}^{\text{s}}$;

(ii) *for* $\mathbf{x}_k \in \mathscr{S}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$, $\mathbf{x}_{k+j} \in \mathscr{S}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$ *and* $\lim_{j\to\infty} \mathbf{x}_{k+j} = \mathbf{x}_k^*$ *for all* $j \in \mathbb{Z}_+$;

(iii) *For* $\mathbf{x}_k \notin \mathscr{S}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$, $||\mathbf{x}_{k+j} - \mathbf{x}_k^*|| \geq \delta$ *for* $\delta > 0$ *with* $j, j_1 \in \mathbb{Z}_+$ *and* $j \geq j_1 \geq 0$.

(b) *The local unstable invariant manifold* $\mathscr{U}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$ *possesses the following properties*:

(i) *for* $\mathbf{x}_k^* \in \mathscr{U}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$, $\mathscr{U}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$ *possesses the same dimension of* $\mathscr{E}^{\text{u}}$ *and the tangential space of* $\mathscr{U}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$ *is identical to* $\mathscr{E}^{\text{u}}$;

(ii) *for* $\mathbf{x}_k \in \mathscr{U}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$, $\mathbf{x}_{k+j} \in \mathscr{U}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$ *and* $\lim_{j\to-\infty} \mathbf{x}_{k+j} = \mathbf{x}_k^*$ *for all* $j \in \mathbb{Z}_-$;

(iii) *for* $\mathbf{x}_k \notin \mathscr{U}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k{}^*)$, $||\mathbf{x}_{k+j} - \mathbf{x}_k^*|| \geq \delta$ *for* $\delta > 0$ *with* $j_1, j \in \mathbb{Z}_-$ *and* $j \leq j_1 \leq 0$.

Proof See Nitecki (1971). □

Theorem 2.3 *Consider a discrete, nonlinear dynamical system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *in* Eq. (2.4) *with a fixed point* $\mathbf{x}_k^*$. *The corresponding solution is* $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ *with* $j \in \mathbb{Z}$. *Suppose there is a neighborhood of the fixed point* $\mathbf{x}_k^*$ (*i.e.,* $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), *and* $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *is* C^r ($r \geq 1$)*-continuous in* $U_k(\mathbf{x}_k^*)$. *The linearized system is* $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ ($\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*$) *in* $U_k(\mathbf{x}_k^*)$. *If the homeomorphism between the local invariant subspace* $E(\mathbf{x}_k^*) \subset U_k(\mathbf{x}_k^*)$ *and the eigenspace* $\mathscr{E}$ *of the linearized system exists with the condition in* Eq. (2.28), *in addition to the local stable and unstable invariant manifolds, there is a* C^{r-1} *center manifold* $\mathscr{C}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$. *The center manifold possesses the same dimension of* $\mathscr{E}^{\text{c}}$ *for* $\mathbf{x}^* \in \mathscr{C}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*)$, *and the tangential space of* $\mathscr{C}_{\text{loc}}(\mathbf{x}, \mathbf{x}^*)$ *is identical to* $\mathscr{E}^{\text{c}}$. *Thus, the local invariant subspace is decomposed by*

$$E(\mathbf{x}_k, \mathbf{x}_k^*) = \mathscr{S}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*) \oplus \mathscr{U}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*) \oplus \mathscr{C}_{\text{loc}}(\mathbf{x}_k, \mathbf{x}_k^*). \tag{2.35}$$

Proof See Guckenhiemer and Holmes (1990). □

Definition 2.11 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) on domain $\Omega_\alpha \in \mathscr{R}^n$. Suppose there is a metric space (Ω_α, ρ), then the map P under the vector function $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is called the contraction map if

$$\rho(\mathbf{x}_{k+1}^{(1)}, \mathbf{x}_{k+1}^{(2)}) = \rho(\mathbf{f}(\mathbf{x}_k^{(1)}, \mathbf{p}), \mathbf{f}(\mathbf{x}_k^{(2)}, \mathbf{p})) \leq \lambda \rho(\mathbf{x}_k^{(1)}, \mathbf{x}_k^{(2)}) \tag{2.36}$$

for $\lambda \in (0, 1)$ and $\mathbf{x}_k^{(1)}, \mathbf{x}_k^{(2)} \in \Omega_\alpha$ with $\rho(\mathbf{x}_k^{(1)}, \mathbf{x}_k^{(2)}) = ||\mathbf{x}_k^{(1)} - \mathbf{x}_k^{(2)}||$.

Theorem 2.4 *Consider a discrete, nonlinear dynamical system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *in* Eq. (2.4) *on domain* $\Omega_\alpha \in \mathscr{R}^n$. *Suppose there is a metric space* (Ω_α, ρ), *if the map* P *under the vector function* $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *is the contraction map, then there is a unique fixed point* $\mathbf{x}_k^*$ *which is globally stable.*

Proof The proof can be referred to Luo (2011). □

Definition 2.12 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$. The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. Consider a real eigenvalue λ_i of matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ $(i \in N = \{1, 2, \ldots, n\})$ and there is a corresponding eigenvector $\mathbf{v}_i$. On the invariant eigenvector $\mathbf{v}_k^{(i)} = \mathbf{v}_i$, consider $\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i = \lambda_i c_k^{(i)}\mathbf{v}_i$, and thus, $c_{k+1}^{(i)} = \lambda_i c_k^{(i)}$.

(i) $\mathbf{x}_k^{(i)}$ on the direction $\mathbf{v}_i$ is stable if

$$\lim_{k\to\infty} |c_k^{(i)}| = \lim_{k\to\infty} |(\lambda_i)^k| \times |c_0^{(i)}| = 0 \quad \text{for } |\lambda_i| < 1. \tag{2.37}$$

(ii) $\mathbf{x}_k^{(i)}$ on the direction $\mathbf{v}_i$ is unstable if

$$\lim_{k\to\infty} |c_k^{(i)}| = \lim_{k\to\infty} |(\lambda_i)^k| \times |c_0^{(i)}| = \infty \quad \text{for } |\lambda_i| > 1. \tag{2.38}$$

(iii) $\mathbf{x}_k^{(i)}$ on the direction $\mathbf{v}_i$ is invariant if

$$\lim_{k\to\infty} c_k^{(i)} = \lim_{k\to\infty} (\lambda_i)^k c_0^{(i)} = c_0^{(i)} \quad \text{for } \lambda_i = 1. \tag{2.39}$$

(iv) $\mathbf{x}_k^{(i)}$ on the direction $\mathbf{v}_i$ is flipped if

$$\left.\begin{aligned} \lim_{2k\to\infty} c_k^{(i)} &= \lim_{2k\to\infty} (\lambda_i)^{2k} \times c_0^{(i)} = c_0^{(i)} \\ \lim_{2k+1\to\infty} c_k^{(i)} &= \lim_{2k+1\to\infty} (\lambda_i)^{2k+1} \times c_0^{(i)} = -c_0^{(i)} \end{aligned}\right\} \text{for } \lambda_i = -1. \tag{2.40}$$

(v) $\mathbf{x}_k^{(i)}$ on the direction $\mathbf{v}_i$ is degenerate if

$$c_k^{(i)} = (\lambda_i)^k c_0^{(i)} = 0 \quad \text{for } \lambda_i = 0. \tag{2.41}$$

Definition 2.13 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$. Consider a pair of complex eigenvalues $\alpha_i \pm \mathbf{i}\beta_i$ of matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ $(i \in N = \{1, 2, \ldots, n\}, \mathbf{i} = \sqrt{-1})$ and there is a corresponding eigenvector $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. On the invariant plane of $(\mathbf{u}_k^{(i)}, \mathbf{v}_k^{(i)}) = (\mathbf{u}_i, \mathbf{v}_i)$, consider $\mathbf{x}_k^{(i)} = \mathbf{x}_{k+}^{(i)} + \mathbf{x}_{k-}^{(i)}$ with

$$\mathbf{x}_k^{(i)} = c_k^{(i)}\mathbf{u}_i + d_k^{(i)}\mathbf{v}_i, \mathbf{x}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{u}_i + d_{k+1}^{(i)}\mathbf{v}_i. \tag{2.42}$$

Thus, $\mathbf{c}_k^{(i)} = (c_k^{(i)}, d_k^{(i)})^{\mathrm{T}}$ with

$$\mathbf{c}_{k+1}^{(i)} = \mathbf{E}_i\mathbf{c}_k^{(i)} = r_i\mathbf{R}_i\mathbf{c}_k^{(i)} \tag{2.43}$$

where

$$\begin{gathered}\mathbf{E}_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \quad \text{and} \quad \mathbf{R}_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix}, \\ r_i = \sqrt{\alpha_i^2 + \beta_i^2},\ \cos\theta_i = \alpha_i/r_i \quad \text{and} \quad \sin\theta_i = \beta_i/r_i;\end{gathered} \tag{2.44}$$

and

$$\mathbf{E}_i^k = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}^k \quad \text{and} \quad \mathbf{R}_i^k = \begin{bmatrix} \cos k\theta_i & \sin k\theta_i \\ -\sin k\theta_i & \cos k\theta_i \end{bmatrix}. \tag{2.45}$$

(i) $\mathbf{x}_k^{(i)}$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally stable if

$$\lim_{k\to\infty} ||\mathbf{c}_k^{(i)}|| = \lim_{k\to\infty} r_i^k||\mathbf{R}_i^k|| \times ||\mathbf{c}_0^{(i)}|| = 0 \quad \text{for } r_i = |\lambda_i| < 1. \tag{2.46}$$

(ii) $\mathbf{x}_k^{(i)}$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally unstable if

$$\lim_{k\to\infty} ||\mathbf{c}_k^{(i)}|| = \lim_{k\to\infty} r_i^k||\mathbf{R}_i^k|| \times ||\mathbf{c}_0^{(i)}|| = \infty \quad \text{for } r_i = |\lambda_i| > 1. \tag{2.47}$$

(iii) $\mathbf{x}_k^{(i)}$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is on the invariant circles if

$$||\mathbf{c}_k^{(i)}|| = r_i^k||\mathbf{R}_i^k|| \times ||\mathbf{c}_0^{(i)}|| = ||\mathbf{c}_0^{(i)}|| \quad \text{for } r_i = |\lambda_i| = 1. \tag{2.48}$$

(iv) $\mathbf{x}_k^{(i)}$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is degenerate in the direction of $\mathbf{u}_i$ if $\beta_i = 0$.

Definition 2.14 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n eigenvalues λ_i $(i = 1, 2, \ldots, n)$.

(i) The fixed point $\mathbf{x}_k^*$ is called a hyperbolic point if $|\lambda_i| \neq 1$ $(i = 1, 2, \ldots, n)$.
(ii) The fixed point $\mathbf{x}_k^*$ is called a sink if $|\lambda_i| < 1$ $(i = 1, 2, \ldots, n)$.

(iii) The fixed point $\mathbf{x}_k^*$ is called a source if $|\lambda_i| > 1$ $(i = 1, 2, \ldots, n)$.
(iv) The fixed point $\mathbf{x}_k^*$ is called a center if $|\lambda_i| = 1$ $(i = 1, 2, \ldots, n)$ with distinct eigenvalues.

Definition 2.15 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n eigenvalues λ_i $(i = 1, 2, \ldots, n)$.

(i) The fixed point $\mathbf{x}_k^*$ is called a stable node if $|\lambda_i| < 1 (i = 1, 2, \ldots, n)$.
(ii) The fixed point $\mathbf{x}_k^*$ is called an unstable node if $|\lambda_i| > 1$ $(i = 1, 2, \ldots, n)$.
(iii) The fixed point $\mathbf{x}_k^*$ is called an $(l_1 : l_2)$-saddle if at least one $|\lambda_i| > 1$ $(i \in L_1 \subset \{1, 2, \ldots, n\})$ and the other $|\lambda_j| < 1$ $(j \in L_2 \subset \{1, 2, \ldots, n\})$ with $L_1 \cup L_2 = \{1, 2, \ldots, n\}$ and $L_1 \cap L_2 = \emptyset$. $l_1 = span(L_1)$ and $l_2 = span(L_2)$.
(iv) The fixed point $\mathbf{x}_k^*$ is called an lth-order degenerate case if $\lambda_i = 0$ $(i \in L \subseteq \{1, 2, \ldots, n\})$.

Definition 2.16 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is $C^r (r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n-pairs of complex eigenvalues λ_i $(i = 1, 2, \ldots, n)$.

(i) The fixed point $\mathbf{x}_k^*$ is called a spiral sink if $|\lambda_i| < 1$ $(i = 1, 2, \ldots, n)$ and $\mathrm{Im}\lambda_j \neq 0$ $(j \in \{1, 2, \ldots, n\})$.
(ii) The fixed point $\mathbf{x}_k^*$ is called a spiral source if $|\lambda_i| > 1$ $(i = 1, 2, \ldots, n)$ with $\mathrm{Im}\lambda_j \neq 0$ $(j \in \{1, 2, \ldots, n\})$.
(iii) The fixed point $\mathbf{x}_k^*$ is called a center if $|\lambda_i| = 1$ with distinct $\mathrm{Im}\lambda_i \neq 0$ $(i = 1, 2, \ldots, n)$.

The generalized stability and bifurcation of flows in linearized, nonlinear dynamical systems in Eq. (2.4) will be discussed as follows.

Definition 2.17 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+s} = \mathbf{f}(\mathbf{x}_{k+s-1}, \mathbf{p})$ with $s \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+s+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+s}$ $(\mathbf{y}_{k+s} = \mathbf{x}_{k+s} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n eigenvalues λ_i $(i = 1, 2, \ldots, n)$. Set $N = \{1, 2, \ldots, m, m + 1, \ldots, (n + m)/2\}$, $N_j = \{j_1, j_2, \ldots, j_{n_j}\} \cup \emptyset$ with $j_p \in N$ $(p = 1, 2, \ldots, n_j; j = 1, 2, \ldots, 7)$, $\Sigma_{j=1}^4 n_j = m$ and $2\Sigma_{j=5}^7 n_j = n - m$. $\cup_{j=1}^7 N_j = N$ with $N_j \cap N_l = \emptyset (l \neq j)$. $N_j = \emptyset$ if $n_j = 0$. $N_\alpha = N_\alpha^{\mathrm{m}} \cup N_\alpha^{\mathrm{o}}$ $(\alpha = 1, 2)$ and $N_\alpha^{\mathrm{m}} \cap N_\alpha^{\mathrm{o}} = \emptyset$ with $n_\alpha^{\mathrm{m}} + n_\alpha^{\mathrm{o}} = n_\alpha$

where superscripts "m" and "o" represent monotonic and oscillatory evolutions. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n_1-stable, n_2-unstable, n_3-invariant, and n_4-flip real eigenvectors plus n_5-stable,n_6-unstable, and n_7-center pairs of complex eigenvectors. Without repeated complex eigenvalues of $|\lambda_i| = 1$ $(i \in N_3 \cup N_4 \cup N_7)$, an iterative response of $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4]|n_5 : n_6 : n_7)$ flow in the neighborhood of the fixed point $\mathbf{x}_k^*$. With repeated complex eigenvalues of $|\lambda_i| = 1$ $(i \in N_3 \cup N_4 \cup N_7)$, an iterative response of $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4]|n_5 : n_6 : [n_7, l; \kappa_7])$ flow in the neighborhood of the fixed point $\mathbf{x}_k^*$, where $\kappa_j \in \{\varnothing, m_j\}$ $(j = 3, 4, 7)$. The meanings of notations in the aforementioned structures are defined as follows:

(i) $[n_1^{\mathrm{m}}, n_1^{\mathrm{o}}]$ represents n_1-sinks with n_1^{m}-monotonic convergence and n_1^{o}-oscillatory convergence among n_1-directions of $\mathbf{v}_i$ if $|\lambda_i| < 1$ $(i \in N_1$ and $1 \le n_1 \le n)$ with distinct or repeated eigenvalues.
(ii) $[n_2^{\mathrm{m}}, n_2^{\mathrm{o}}]$ represents n_2-sources with n_2^{m}-monotonic divergence and n_2^{o}-oscillatory divergence among n_2-directions of $\mathbf{v}_i$ if $|\lambda_i| > 1$ $(i \in N_2$ and $1 \le n_2 \le n)$ with distinct or repeated eigenvalues.
(iii) $n_3 = 1$ represents an invariant center on 1-direction of $\mathbf{v}_i$ if $\lambda_i = 1$ $(i \in N_3$ and $n_3 = 1)$.
(iv) $n_4 = 1$ represents a flip center on 1-direction of $\mathbf{v}_i$ if $\lambda_i = -1$ $(i \in N_4$ and $n_4 = 1)$.
(v) n_5 represents n_5-spiral sinks on n_5-pairs of $(\mathbf{u}_i, \mathbf{v}_i)$ if $|\lambda_i| < 1$ and $\mathrm{Im}\lambda_i \neq 0$ $(i \in N_5$ and $1 \le n_5 \le n)$ with distinct or repeated eigenvalues.
(vi) n_6 represents n_6-spiral sources on n_6-directions of $(\mathbf{u}_i, \mathbf{v}_i)$ if $|\lambda_i| > 1$ and $\mathrm{Im}\lambda_i \neq 0$ $(i \in N_6$ and $1 \le n_6 \le n)$ with distinct or repeated eigenvalues.
(vii) n_7 represents n_7-invariant centers on n_7-pairs of $(\mathbf{u}_i, \mathbf{v}_i)$ if $|\lambda_i| = 1$ and $\mathrm{Im}\lambda_i \neq 0$ $(i \in N_7$ and $1 \le n_7 \le n)$ with distinct eigenvalues.
(viii) $\varnothing$ represents none if $n_j = 0$ $(j \in \{1, 2, \ldots, 7\})$.
(ix) $[n_3; \kappa_3]$ represents $(n_3 - \kappa_3)$-invariant centers on $(n_3 - \kappa_3)$-directions of $\mathbf{v}_{i_3}$ $(i_3 \in N_3)$ and κ_3-sources in κ_3-directions of $\mathbf{v}_{j_3}(j_3 \in N_3$ and $j_3 \neq i_3)$ if $\lambda_i = 1$ $(i \in N_3$ and $n_3 \le n)$ with the $(\kappa_3 + 1)$th-order nilpotent matrix $\mathbf{N}_3^{\kappa_3+1} = \mathbf{0}$ $(0 < \kappa_3 \le n_3 - 1)$.
(x) $[n_3; \varnothing]$ represents n_3-invariant centers on n_3-directions of $\mathbf{v}_i$ if $\lambda_i = 1$ $(i \in N_3$ and $1 < n_3 \le n)$ with a nilpotent matrix $\mathbf{N}_3 = \mathbf{0}$.
(xi) $[n_4; \kappa_4]$ represents $(n_4 - \kappa_4)$-flip oscillatory centers on $(n_4 - \kappa_4)$-directions of $\mathbf{v}_{i_4}$ $(i_4 \in N_4)$ and κ_4-sources in κ_4-directions of $\mathbf{v}_{j_4}$ $(j_4 \in N_4$ and $j_4 \neq i_4)$ if $\lambda_i = -1$ $(i \in N_4$ and $n_4 \le n)$ with the $(\kappa_4 + 1)$th-order nilpotent matrix $\mathbf{N}_4^{\kappa_4+1} = \mathbf{0}$ $(0 < \kappa_4 \le n_4 - 1)$.
(xii) $[n_4; \varnothing]$ represents n_4 flip oscillatory centers on n_4-directions of $\mathbf{v}_i$ if $\lambda_i = -1$ $(i \in N_4$ and $1 < n_4 \le n)$ with a nilpotent matrix $\mathbf{N}_4 = \mathbf{0}$.
(xiii) $[n_7, l; \kappa_7]$ represents $(n_7 - \kappa_7)$-invariant centers on $(n_7 - \kappa_7)$-pairs of $(\mathbf{u}_{i_7}, \mathbf{v}_{i_7})$ $(i_7 \in N_7)$ and κ_7-sources on κ_7-pairs of $(\mathbf{u}_{j_7}, \mathbf{v}_{j_7})$ $(j_7 \in N_7$ and $j_7 \neq i_7)$ if

$|\lambda_i| = 1$ and $\text{Im}\lambda_i \neq 0$ ($i \in N_7$ and $n_7 \leq n$) for $(l+1)$-pairs of repeated eigenvalues with the (κ_7+1)th-order nilpotent matrix $\mathbf{N}_7^{\kappa_7+1} = \mathbf{0}$ $(0 < \kappa_7 \leq l)$.

(xiv) $[n_7, l; \emptyset]$ represents n_7-invariant centers on n_7-pairs of $(\mathbf{u}_i, \mathbf{v}_i)$ if $|\lambda_i| = 1$ and $\text{Im}\lambda_i \neq 0$ ($i \in N_7$ and $1 \leq n_7 \leq n$) for $(l+1)$-pairs of repeated eigenvalues with a nilpotent matrix $\mathbf{N}_7 = \mathbf{0}$.

2.3 Stability Switching Theory

To extend the idea of Definitions 2.11 and 2.12, a new function will be defined to determine the stability and the stability state switching.

Definition 2.18 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is $C^r(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ ($\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*$) in $U_k(\mathbf{x}_k^*)$ and there are n linearly independent vectors $\mathbf{v}_i$ ($i = 1, 2, \ldots, n$). For a perturbation of fixed point $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$, let $\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$,

$$s_k^{(i)} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k = \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*) \tag{2.49}$$

where $s_k^{(i)} = c_k^{(i)}||\mathbf{v}_i||^2$. Define the following functions

$$G_i(\mathbf{x}_k, \mathbf{p}) = \mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_k, \mathbf{p}) - \mathbf{x}_k^*] \tag{2.50}$$

and

$$\begin{aligned} G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k, \mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}} \cdot D_{s_k^{(i)}}\mathbf{f}(\mathbf{x}_k(s_k^{(i)}), \mathbf{p}) = \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k(s_k^{(i)}), \mathbf{p})\partial_{c_k^{(i)}}\mathbf{x}_k\partial_{s_k^{(i)}}c_k^{(i)} \\ &= \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{x}}\mathbf{f}(\mathbf{x}_k(s_k^{(i)}), \mathbf{p})\mathbf{v}_i||\mathbf{v}_i||^{-2} \end{aligned} \tag{2.51}$$

$$G_{s_k^{(i)}}^{(m)}(\mathbf{x}_k, \mathbf{p}) = \mathbf{v}_i^{\mathrm{T}} \cdot D_{s_k^{(i)}}^{(m)}\mathbf{f}(\mathbf{x}_k(s_k^{(i)}), \mathbf{p}) = \mathbf{v}_i^{\mathrm{T}} \cdot D_{s_k^{(i)}}(D_{s_k^{(i)}}^{(m-1)}\mathbf{f}(\mathbf{x}_k(s_k^{(i)}), \mathbf{p})) \tag{2.52}$$

where $D_{s_k^{(i)}}(\cdot) = \partial(\cdot)/\partial s_k^{(i)}$ and $D_{s_k^{(i)}}^{(m)}(\cdot) = D_{s_k^{(i)}}(D_{s_k^{(i)}}^{(m-1)}(\cdot))$.

Definition 2.19 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ ($\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*$) in

$U_k(\mathbf{x}_k^*)$ and there are n linearly independent vectors $\mathbf{v}_i$ $(i = 1, 2, \ldots, n)$. For a perturbation of fixed point $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$, let $\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$.

(i) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is stable if

$$|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \tag{2.53}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called a sink (or stable node) on the direction $\mathbf{v}_i$.

(ii) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is unstable if

$$|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \tag{2.54}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called a source (or unstable node) on the direction $\mathbf{v}_i$.

(iii) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is invariant if

$$\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*) = \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*) \tag{2.55}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called to be degenerate on the direction $\mathbf{v}_i$.

(iv) $\mathbf{x}_{k+j}^{(i)}$ $(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is symmetrically flipped if

$$\text{(v) } \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*) = -\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*) \tag{2.56}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called to be degenerate on the direction $\mathbf{v}_i$.

The stability of fixed points for a specific eigenvector is presented in Fig. 2.4. The solid curve is $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_k, \mathbf{p})$. The circular symbol is fixed point. The shaded regions are stable. The horizontal solid line is for a degenerate case. The vertical solid line is for a line with infinite slope. The monotonically stable node (sink) is presented in Fig. 2.4a. From the fixed point $\mathbf{x}_k^*$, let $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$ and $\mathbf{y}_{k+1} = \mathbf{x}_{k+1} - \mathbf{x}_k^*$. $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1}$ and $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1} = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k$ are represented by dashed and dotted lines, respectively. The iterative responses approach the fixed point. However, the monotonically unstable (source) is presented in Fig. 2.4b. The iterative responses go away from the fixed point. Similarly, the oscillatory stable node (sink) after iteration with a flip $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1}$ is presented in Fig. 2.4c. The dashed and dotted lines are used for two lines $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1} = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k$ and $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1}$, respectively. In a similar fashion, the oscillatory unstable node (source) is presented in Fig. 2.4d. This illustration can be easily observed from the stability of fixed points. In Fig. 2.4e, f, the oscillatory stable and unstable nodes are presented as usual through the two-time iterations.

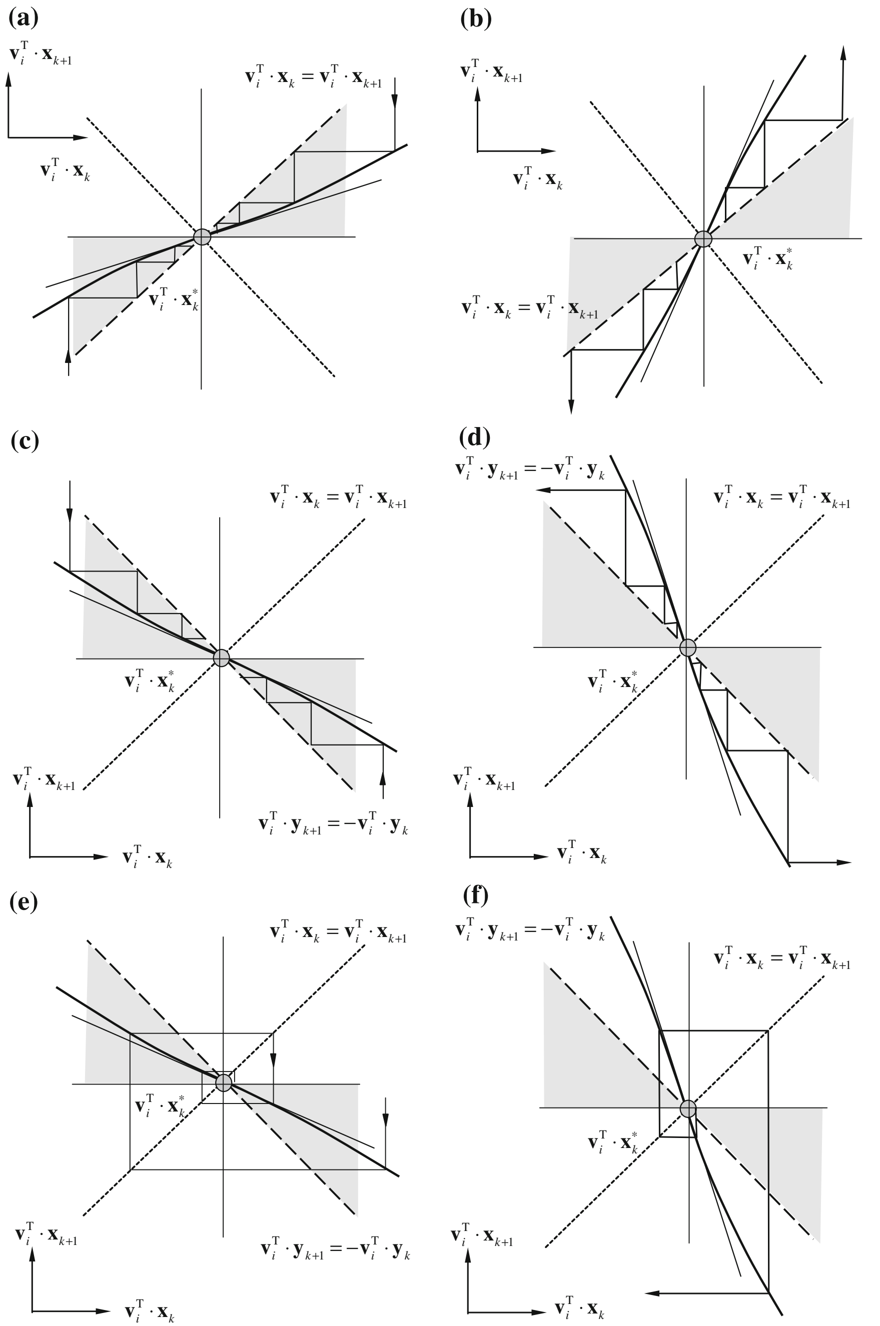

Fig. 2.4 Stability of fixed points: **a** monotonically stable node (sink); **b** monotonically unstable node (source); **c** oscillatory stable node (sink) and **d** oscillatory unstable node (source); **e** oscillatory stable node (sink) and **f** oscillatory unstable node (sink). Shaded areas are stable zones. ($\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$ and $\mathbf{y}_{k+1} = \mathbf{x}_{k+1} - \mathbf{x}_k^*$)

Theorem 2.5 *Consider a discrete, nonlinear dynamical system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ *in* Eq. (2.4) *with a fixed point* $\mathbf{x}_k^*$. *The corresponding solution is given by* $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ *with* $j \in \mathbb{Z}$. *Suppose there is a neighborhood of the fixed point* $\mathbf{x}_k^*$ *(i.e.,* $U_k(\mathbf{x}_k^*) \subset \Omega$*), and* $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *is* C^r $(r \geq 1)$*-continuous in* $U_k(\mathbf{x}_k^*)$ *with* Eq. (2.28). *The linearized system is* $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ *in* $U_k(\mathbf{x}_k^*)$ *and there are n linearly independent vectors* $\mathbf{v}_i$ $(i = 1, 2, \ldots, n)$. *For a perturbation of fixed point* $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$, *let* $\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ *and* $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$.

(i) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is stable if and only if*

$$G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i \in (-1, 1) \tag{2.57}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(ii) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is unstable if and only if*

$$G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i \in (1, \infty) \text{ and } (-\infty, -1) \tag{2.58}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(iii) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is invariant if and only if*

$$G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1 \quad \text{and} \quad G^{(m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \; m_i = 2, 3, \ldots \tag{2.59}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(iv) $\mathbf{x}_{k+j}^{(i)}$ $(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_k$ *is symmetrically flip if and only if*

$$G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1 \quad \text{and} \quad G^{(m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \, m_i = 2, 3, \ldots \tag{2.60}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

Proof The proof can be referred to Luo (2012). □

The monotonic stability of fixed points with higher-order singularity for a specific eigenvector is presented in Fig. 2.5. The solid curve is $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_k, \mathbf{p})$. The circular symbol is fixed pointed. The shaded regions are stable. The horizontal solid line is also for the degenerate case. The vertical solid line is for a line with infinite slope. The monotonically stable node (sink) of the $(2m_i + 1)$th-order is sketched in Fig. 2.5a. The dashed and dotted lines are for $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1}$ and $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1} = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k$, respectively. The nonlinear curve lies in the stable zone, and the iterative responses approach the fixed point. However, the monotonically

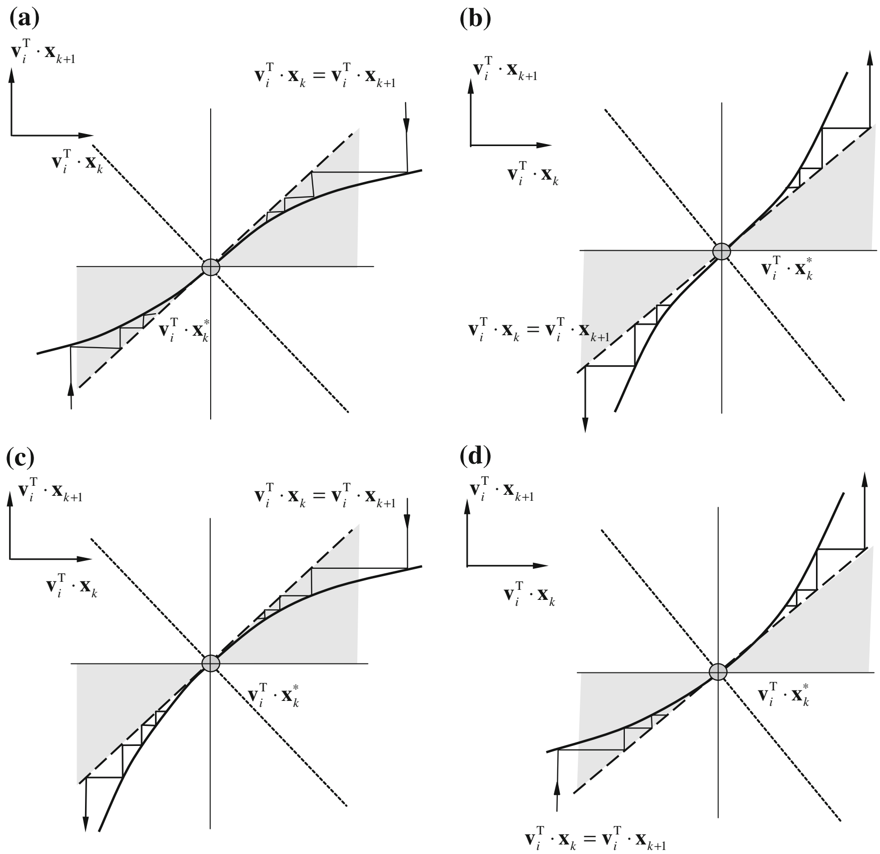

Fig. 2.5 Monotonic stability of fixed points with higher-order singularity: **a** monotonically stable node (sink) of $(2m_i + 1)$th-order, **b** monotonically unstable node (source) of $(2m_i + 1)$th-order, **c** monotonically lower saddle of $(2m_i)$th-order, and **d** monotonically upper saddle of $(2m_i)$th-order. *Shaded areas* are stable zones. ($\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$ and $\mathbf{y}_{k+1} = \mathbf{x}_{k+1} - \mathbf{x}_k^*$)

unstable (source) of the $(2m_i + 1)$th-order is presented in Fig. 2.5b. The nonlinear curve lies in the unstable zone, and the iterative responses go away from the fixed point. The monotonically lower saddle of the $(2m_i)$th-order is presented in Fig. 2.5c. The nonlinear curve is tangential to the line of $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1}$ with the $(2m_i)$th-order, and the upper branch is in the stable zone and the lower branch is in the unstable zone. Similarly, the monotonically upper saddle of the $(2m_i)$th-order is presented in Fig. 2.5d. The oscillatory stability of fixed points with higher-order singularity for a specific eigenvector after iteration with a flip $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1}$ is presented in Fig. 2.6. The oscillatory stable node (sink) of the $(2m_i + 1)$th-order is sketched in Fig. 2.6a. The dashed and dotted lines are for $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1} = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k$

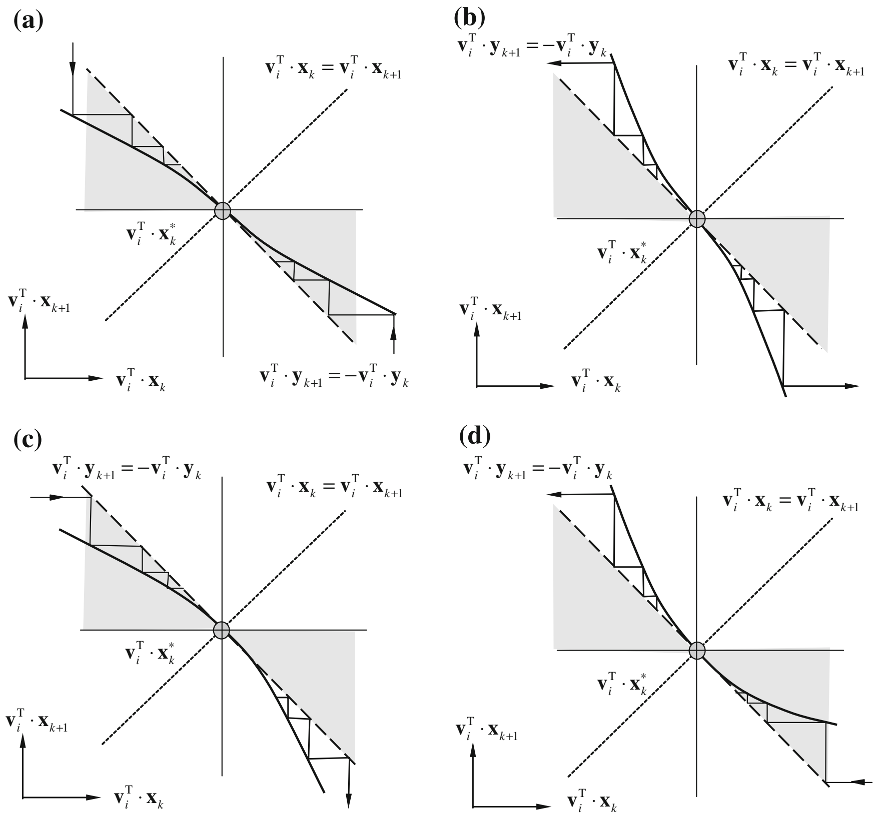

Fig. 2.6 Oscillatory stability of fixed points with higher-order singularity after iteration with a flip $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1}$: **a** oscillatory stable node (sink) of $(2m_i + 1)$th-order, **b** oscillatory unstable node (source) of $(2m_i + 1)$th-order, **c** oscillatory lower saddle of $(2m_i)$th-order, and **d** oscillatory upper saddle of $(2m_i)$th-order. *Shaded areas* are stable zones. ($\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$ and $\mathbf{y}_{k+1} = \mathbf{x}_{k+1} - \mathbf{x}_k^*$)

and $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1}$, respectively. The nonlinear curve lies in the stable zone, and the iterative responses approach the fixed point. However, the oscillatory unstable (source) of the $(2m_i + 1)$th-order is presented in Fig. 2.6b. The nonlinear curve lies in the unstable zone, and the iterative responses go away from the fixed point. The oscillatory lower saddle of the $(2m_i)$th-order is presented in Fig. 2.6c. The nonlinear curve is tangential to and below the line of $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1} = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k$ with the $(2m_i)$th-order, and the upper branch is in the stable zone and the lower branch is in the unstable zone. Finally, the oscillatory upper saddle of the $(2m_i)$th-order is presented in Fig. 2.6d. For clear illustrations, the oscillatory stability of fixed points with higher-order singularity for the two-time iterations is presented in Fig. 2.7.

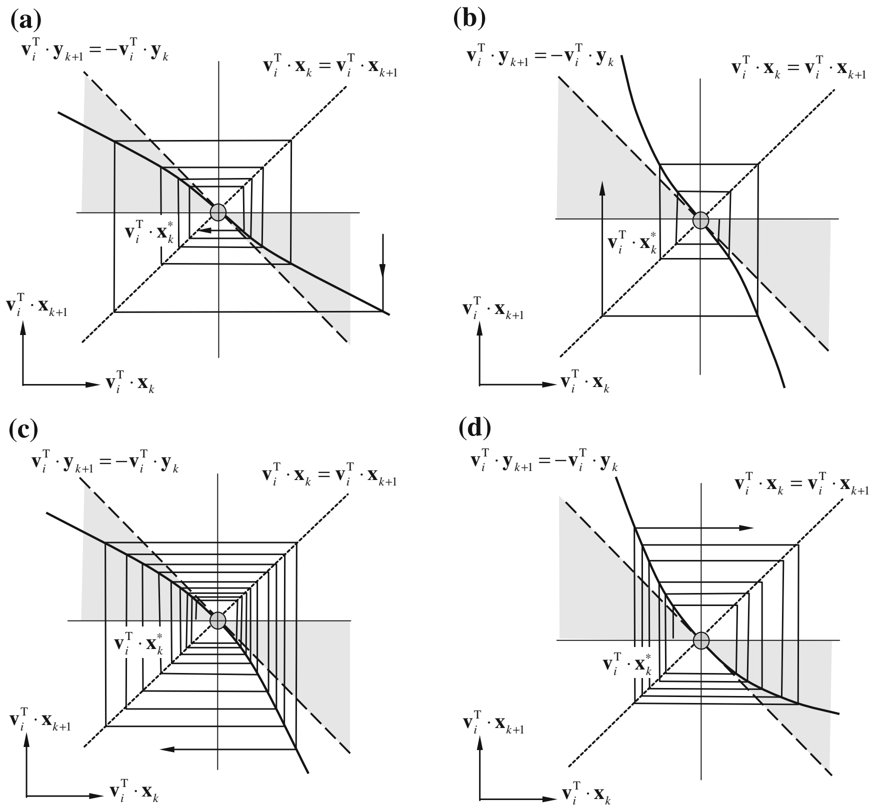

Fig. 2.7 Oscillatory stability of fixed points with higher-order singularity for the two-time iterations: **a** oscillatory stable node (sink) of $(2m_i + 1)$th-order, **b** oscillatory unstable node (source) of $(2m_i + 1)$th-order, **c** oscillatory lower saddle of $(2m_i)$th-order, and **d** oscillatory upper saddle of $(2m_i)$th-order. *Shaded areas* are stable zones. ($\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$ and $\mathbf{y}_{k+1} = \mathbf{x}_{k+1} - \mathbf{x}_k^*$)

Definition 2.20 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^{2n}$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ ($\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*$) in $U_k(\mathbf{x}_k^*)$ and there are n linearly independent vectors $\mathbf{v}_i$ $(i = 1, 2, \ldots, n)$. For a perturbation of fixed point $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$, let $\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$.

(i) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is monotonically stable of the $(2m_i + 1)$th-order if

$$\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \text{for } r_i = 2, 3, \ldots, 2m_i,\\
&G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) \neq 0,\\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|
\end{aligned} \tag{2.61}$$

for $\mathbf{x}_k \in U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called a monotonic sink (or stable node) of the $(2m_i + 1)$th-order on the direction $\mathbf{v}_i$.

(ii) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i + 1)$th-order if

$$\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \text{for } r_i = 2, 3, \ldots, 2m_i;\\
&G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) \neq 0;\\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|
\end{aligned} \tag{2.62}$$

for $\mathbf{x}_k \in U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called a monotonic source (or unstable node) of the $(2m_i + 1)$th-order on the direction $\mathbf{v}_i$.

(iii) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i)$th-order, lower saddle if

$$\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \text{for } r_i = 2, 3, \ldots, 2m_i - 1;\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) \neq 0,\\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \quad \text{for } s_k^{(i)} > 0,\\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \quad \text{for } s_k^{(i)} < 0
\end{aligned} \tag{2.63}$$

for $\mathbf{x}_k \in U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called a monotonic, lower saddle of the $(2m_i)$th-order on the direction $\mathbf{v}_i$.

(iv) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i)$th-order, upper saddle if

$$\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \text{for } r_i = 2, 3, \ldots, 2m_i - 1;\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) \neq 0,
\end{aligned}$$

$$
\begin{aligned}
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \quad \text{for } s_k^{(i)} > 0, \\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \quad \text{for } s_k^{(i)} < 0
\end{aligned} \tag{2.64}
$$

for $\mathbf{x}_k \in U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called a monotonic, upper saddle of the $(2m_i)$th-order on the direction $\mathbf{v}_i$.

(v) $\mathbf{x}_{k+j}$ ($j \in \mathbb{Z}$) at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is oscillatory stable of the $(2m_i + 1)$th-order if

$$
\begin{aligned}
&G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1, \\
&G_{s_k^{(i)}}^{(r_i)}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \text{for } r_i = 2, 3, \ldots, 2m_i; \\
&G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{x}_k^*, \mathbf{p}) \neq 0; \\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|
\end{aligned} \tag{2.65}
$$

for $\mathbf{x}_k \in U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called an oscillatory sink (or stable node) of the $(2m_i + 1)$th-order on the direction $\mathbf{v}_i$.

(vi) $\mathbf{x}_{k+j}$ ($j \in \mathbb{Z}$) at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is oscillatory unstable of the $(2m_i + 1)$th-order if

$$
\begin{aligned}
&G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1; \\
&G_{s_k^{(i)}}^{(r_i)}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \text{for } r_i = 2, 3, \ldots, 2m_i; \\
&G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{x}_k^*, \mathbf{p}) \neq 0, \\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|
\end{aligned} \tag{2.66}
$$

for $\mathbf{x}_k \in U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called an oscillatory source (or unstable node) of the $(2m_i + 1)$th-order on the direction $\mathbf{v}_i$.

(vii) $\mathbf{x}_{k+j}$ ($j \in \mathbb{Z}$) at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is oscillatory unstable of the $(2m_i)$th-order, lower saddle if

$$
\begin{aligned}
&G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1, \\
&G_{s_k^{(i)}}^{(r_i)}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \text{for } r_i = 2, 3, \ldots, 2m_i - 1; \\
&G_{s_k^{(i)}}^{(2m_i)}(\mathbf{x}_k^*, \mathbf{p}) \neq 0, \\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \quad \text{for } s_k^{(i)} > 0, \\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \quad \text{for } s_k^{(i)} < 0
\end{aligned} \tag{2.67}
$$

for $\mathbf{x}_k \in U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called an oscillatory lower saddle of the $(2m_i)$th-order on the direction $\mathbf{v}_i$.

(viii) $\mathbf{x}_{k+j}$ ($j \in \mathbb{Z}$) at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is oscillatory unstable of the $(2m_i)$th-order, upper saddle if

$$\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \text{for } r_i = 2, 3, \ldots, 2m_i - 1;\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) \neq 0,\\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \quad \text{for } s_k^{(i)} > 0,\\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \quad \text{for } s_k^{(i)} < 0
\end{aligned} \tag{2.68}$$

for $\mathbf{x}_k \in U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called an oscillatory, upper saddle of the $(2m_i)$th-order on the direction $\mathbf{v}_i$.

Theorem 2.6 *Consider a discrete, nonlinear dynamical system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ *in* Eq. (2.4) *with a fixed point* $\mathbf{x}_k^*$. *The corresponding solution is given by* $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ *with* $j \in \mathbb{Z}$. *Suppose there is a neighborhood of the fixed point* $\mathbf{x}_k^*$ *(i.e.,* $U_k(\mathbf{x}_k^*) \subset \Omega$*), and* $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *is* C^r ($r \geq 1$)*-continuous in* $U_k(\mathbf{x}_k^*)$ *with* Eq. (2.28). *The linearized system is* $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ ($\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*$) *in* $U_k(\mathbf{x}_k^*)$ *and there are n linearly independent vectors* $\mathbf{v}_i$ ($i = 1, 2, \ldots, n$). *For a perturbation of fixed point* $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$, *let* $\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ *and* $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$.

(i) $\mathbf{x}_{k+j}$ ($j \in \mathbb{Z}$) *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is monotonically stable of the* $(2m_i + 1)$th-*order if and only if*

$$\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \;\; \textit{for}\; r_i = 2, 3, \ldots, 2m_i,\\
&G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) < 0
\end{aligned} \tag{2.69}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(ii) $\mathbf{x}_{k+j}$ ($j \in \mathbb{Z}$) *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is monotonically unstable of the* $(2m_i + 1)$th-*order if and only if*

$$\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \;\; \textit{for}\; r_i = 2, 3, \ldots, 2m_i,\\
&G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) > 0
\end{aligned} \tag{2.70}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(iii) $\mathbf{x}_{k+j}$ ($j \in \mathbb{Z}$) *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is monotonically unstable of the* $(2m_i)$th-*order, lower saddle if and only if*

$$\begin{aligned} &G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1, \\ &G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \textit{for } r_i = 2, 3, \ldots, 2m_i - 1, \\ &G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) < 0 \quad \textit{stable for } s_k^{(i)} > 0; \\ &G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) < 0 \quad \textit{unstable for } s_k^{(i)} < 0 \end{aligned} \tag{2.71}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(iv) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is monotonically unstable of the* $(2m_i)$th-*order if and only if*

$$\begin{aligned} &G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1, \\ &G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \textit{for } r_i = 2, 3, \ldots, 2m_i - 1, \\ &G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) > 0 \quad \textit{unstable for } s_k^{(i)} > 0; \\ &G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) > 0 \quad \textit{stable for } s_k^{(i)} < 0 \end{aligned} \tag{2.72}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(v) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is oscillatory stable of the* $(2m_i + 1)$th-*order if and only if*

$$\begin{aligned} &G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1, \\ &G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \textit{for } r_i = 2, 3, \ldots, 2m_i, \\ &G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) > 0 \end{aligned} \tag{2.73}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(vi) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is oscillatory unstable of the* $(2m_i + 1)$th-*order if and only if*

$$\begin{aligned} &G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1, \\ &G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \textit{for } r_i = 2, 3, \ldots, 2m_i, \\ &G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) < 0 \end{aligned} \tag{2.74}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(vii) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is oscillatory unstable of the* $(2m_i)$th-*order, upper saddle if and only if*

$$
\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \textit{for}\ r_i = 2, 3, \ldots, 2m_i - 1,\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) > 0 \quad \textit{stable for}\ s_k^{(i)} > 0;\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) > 0 \quad \textit{unstable for}\ s_k^{(i)} < 0
\end{aligned}
\tag{2.75}
$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(viii) $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is oscillatory unstable of the* $(2m_i)$th-*order, lower saddle if and only if*

$$
\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \textit{for}\ r_i = 2, 3, \ldots, 2m_i - 1,\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) < 0 \quad \textit{stable for}\ s_k^{(i)} < 0;\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) < 0 \quad \textit{unstable for}\ s_k^{(i)} > 0
\end{aligned}
\tag{2.76}
$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

Proof The proof can be referred to Luo (2012). □

Definition 2.21 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is $C^r(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. Consider a pair of complex eigenvalues $\alpha_i \pm \mathbf{i}\beta_i$ $(i \in N = \{1, 2, \ldots, n\}$, $\mathbf{i} = \sqrt{-1})$ of matrix $D\mathbf{f}(\mathbf{x}^*, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. On the invariant plane of $(\mathbf{u}_i, \mathbf{v}_i)$, consider $\mathbf{r}_k^{(i)} = \mathbf{y}_k^{(i)} = \mathbf{y}_{k+}^{(i)} + \mathbf{y}_{k-}^{(i)}$ with

$$
\begin{aligned}
\mathbf{r}_k^{(i)} &= c_k^{(i)}\mathbf{u}_i + d_k^{(i)}\mathbf{v}_i,\\
\mathbf{r}_{k+1}^{(i)} &= c_{k+1}^{(i)}\mathbf{u}_i + d_{k+1}^{(i)}\mathbf{v}_i
\end{aligned}
\tag{2.77}
$$

and

$$
\begin{aligned}
&c_k^{(i)} = \frac{1}{\Delta}[\Delta_2(\mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{y}_k) - \Delta_{12}(\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k)],\\
&d_k^{(i)} = \frac{1}{\Delta}[\Delta_1(\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k) - \Delta_{12}(\mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{y}_k)];\\
&\Delta_1 = ||\mathbf{u}_i||^2,\ \Delta_2 = ||\mathbf{v}_i||^2,\ \Delta_{12} = \mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{v}_i;\\
&\Delta = \Delta_1\Delta_2 - \Delta_{12}^2.
\end{aligned}
\tag{2.78}
$$

Consider a polar coordinate of (r_k, θ_k) defined by

$$c_k^{(i)} = r_k^{(i)} \cos\theta_k^{(i)}, \quad \text{and} \quad d_k^{(i)} = r_k^{(i)} \sin\theta_k^{(i)};$$
$$r_k^{(i)} = \sqrt{(c_k^{(i)})^2 + (d_k^{(i)})^2}, \quad \text{and} \quad \theta_k^{(i)} = \arctan(d_k^{(i)}/c_k^{(i)}). \tag{2.79}$$

Thus,

$$\begin{aligned} c_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_2 G_{c_k^{(i)}}(\mathbf{x}_k, \mathbf{p}) - \Delta_{12} G_{d_k^{(i)}}(\mathbf{x}_k, \mathbf{p})] \\ d_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_1 G_{d_k^{(i)}}(\mathbf{x}_k, \mathbf{p}) - \Delta_{12} G_{c_k^{(i)}}(\mathbf{x}_k, \mathbf{p})] \end{aligned} \tag{2.80}$$

where

$$\begin{aligned} G_{c_k^{(i)}}(\mathbf{x}_k, \mathbf{p}) &= \mathbf{u}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_k, \mathbf{p}) - \mathbf{x}_k^*] = \sum_{m_i=1}^{\infty} \frac{1}{m_i!} G_{c_k^{(i)}}^{(m_i)}(\theta_k^{(i)})(r_k^{(i)})^{m_i}, \\ G_{d_k^{(i)}}(\mathbf{x}_k, \mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_k, \mathbf{p}) - \mathbf{x}_k^*] = \sum_{m_i=1}^{\infty} \frac{1}{m_i!} G_{d_k^{(i)}}^{(m_i)}(\theta_k^{(i)})(r_k^{(i)})^{m_i}; \end{aligned} \tag{2.81}$$

$$\begin{aligned} G_{c_k^{(i)}}^{(m_i)}(\theta_k^{(i)}) &= \mathbf{u}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}^{(m_i)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})[\mathbf{u}_i \cos\theta_k^{(i)} + \mathbf{v}_i \sin\theta_k^{(i)}]^{m_i}\Big|_{(\mathbf{x}_k^*, \mathbf{p})}, \\ G_{d_k^{(i)}}^{(m_i)}(\theta_k^{(i)}) &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}^{(m_i)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})[\mathbf{u}_i \cos\theta_k^{(i)} + \mathbf{v}_i \sin\theta_k^{(i)}]^{m_i}\Big|_{(\mathbf{x}_k^*, \mathbf{p})}. \end{aligned} \tag{2.82}$$

Thus,

$$\begin{aligned} r_{k+1}^{(i)} &= \sqrt{(c_{k+1}^{(i)})^2 + (d_{k+1}^{(i)})^2} = \sqrt{\textstyle\sum_{m=2}^{\infty} (r_k^{(i)})^{m_i} G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)})} \\ &= \sqrt{G_{r_{k+1}^{(i)}}^{(2)}}\, r_k^{(i)} \sqrt{1 + (G_{r_{k+1}^{(i)}}^{(2)})^{-1} \textstyle\sum_{m=3}^{\infty} (r_k^{(i)})^{m_i - 2} G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)})} \\ \theta_{k+1}^{(i)} &= \arctan(d_{k+1}^{(i)}/c_{k+1}^{(i)}) \end{aligned} \tag{2.83}$$

where

$$\begin{aligned} &G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)}) \\ &= \sum_{r_i=1}^{\infty} \sum_{s_i=1}^{\infty} \frac{1}{r_i!} \frac{1}{s_i!} [G_{c_{k+1}^{(i)}}^{(r_i)}(\theta_k^{(i)}) G_{c_{k+1}^{(i)}}^{(s_i)}(\theta_k^{(i)}) + G_{d_{k+1}^{(i)}}^{(r_i)}(\theta_k^{(i)}) G_{d_{k+1}^{(i)}}^{(s_i)}(\theta_k^{(i)})] \delta_{m_i}^{(r_i+s_i)} \\ &= \frac{1}{m_i!} \sum_{r_i=1}^{m_i} C_{m_i}^{r_i} G_{c_{k+1}^{(i)}}^{(r_i)}(\theta_k^{(i)}) G_{c_{k+1}^{(i)}}^{(m_i-r_i)}(\theta_k^{(i)}) + G_{d_{k+1}^{(i)}}^{(r_i)}(\theta_k^{(i)}) G_{d_{k+1}^{(i)}}^{(m_i-r_i)}(\theta_k^{(i)}) \end{aligned} \tag{2.84}$$

and

$$\begin{aligned}G^{(m_i)}_{c^{(i)}_{k+1}}(\theta^{(i)}_k) &= \frac{1}{\Delta}[\Delta_2 G^{(m_i)}_{c^{(i)}_k}(\theta^{(i)}_k) - \Delta_{12} G^{(m_i)}_{d^{(i)}_k}(\theta^{(i)}_k)],\\ G^{(m_i)}_{d^{(i)}_{k+1}}(\theta^{(i)}_k) &= \frac{1}{\Delta}[\Delta_1 G^{(m_i)}_{d^{(i)}_k}(\theta^{(i)}_k) - \Delta_{12} G^{(m_i)}_{c^{(i)}_k}(\theta^{(i)}_k)].\end{aligned} \tag{2.85}$$

From the foregoing definition, consider the first-order terms of G-function

$$\begin{aligned}G^{(1)}_{c^{(i)}_k}(\mathbf{x}_k,\mathbf{p}) &= G^{(1)}_{c^{(i)}_k 1}(\mathbf{x}_k,\mathbf{p}) + G^{(1)}_{c^{(i)}_k 2}(\mathbf{x}_k,\mathbf{p}),\\ G^{(1)}_{d^{(i)}_k}(\mathbf{x}_k,\mathbf{p}) &= G^{(1)}_{d^{(i)}_k 1}(\mathbf{x}_k,\mathbf{p}) + G^{(1)}_{d^{(i)}_k 2}(\mathbf{x}_k,\mathbf{p})\end{aligned} \tag{2.86}$$

where

$$\begin{aligned}G^{(1)}_{c^{(i)}_k 1}(\mathbf{x}_k,\mathbf{p}) &= \mathbf{u}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\partial_{c^{(i)}_k}\mathbf{x}_k = \mathbf{u}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\mathbf{u}_i\\ &= \mathbf{u}_i^{\mathrm{T}} \cdot (-\beta_i \mathbf{v}_i + \alpha_i \mathbf{u}_i) = \alpha_i\Delta_1 - \beta_i\Delta_{12},\\ G^{(1)}_{c^{(i)}_k 2}(\mathbf{x}_k,\mathbf{p}) &= \mathbf{u}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\partial_{d^{(i)}_k}\mathbf{x}_k = \mathbf{u}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\mathbf{v}_i\\ &= \mathbf{u}_i^{\mathrm{T}} \cdot (\beta_i \mathbf{u}_i + \alpha_i \mathbf{v}_i) = \alpha_i\Delta_{12} + \beta_i\Delta_1;\end{aligned} \tag{2.87}$$

and

$$\begin{aligned}G^{(1)}_{d^{(i)}_k 1}(\mathbf{x}_k,\mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\partial_{c^{(i)}_k}\mathbf{x}_k = \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\mathbf{u}_i\\ &= \mathbf{v}_i^{\mathrm{T}} \cdot (-\beta_i \mathbf{v}_i + \alpha_i \mathbf{u}_i) = -\beta_i\Delta_2 + \alpha_i\Delta_{12},\\ G^{(1)}_{d^{(i)}_k 2}(\mathbf{x},\mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\partial_{d^{(i)}_k}\mathbf{x}_k = \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\mathbf{v}_i\\ &= \mathbf{v}_i^{\mathrm{T}} \cdot (\beta_i \mathbf{u}_i + \alpha_i \mathbf{v}_i) = \alpha_i\Delta_2 + \beta_i\Delta_{12}.\end{aligned} \tag{2.88}$$

Substitution of Eqs. (2.86)–(2.88) into Eq. (2.82) gives

$$\begin{aligned}G^{(1)}_{c^{(i)}_k}(\theta^{(i)}_k) &= G^{(1)}_{c^{(i)}_k 1}(\mathbf{x}_k,\mathbf{p})\cos\theta^{(i)}_k + G^{(1)}_{c^{(i)}_k 2}(\mathbf{x}_k,\mathbf{p})\sin\theta^{(i)}_k\\ &= (\alpha_i\Delta_1 - \beta_i\Delta_{12})\cos\theta^{(i)}_k + (\alpha_i\Delta_{12} + \beta_i\Delta_1)\sin\theta^{(i)}_k,\\ G^{(1)}_{d^{(i)}_k}(\theta^{(i)}_k) &= G^{(1)}_{d^{(i)}_k 1}(\mathbf{x}_k,\mathbf{p})\cos\theta^{(i)}_k + G^{(1)}_{d^{(i)}_k 2}(\mathbf{x}_k,\mathbf{p})\sin\theta^{(i)}_k\\ &= (-\beta_i\Delta_2 + \alpha_i\Delta_{12})\cos\theta^{(i)}_k + (\alpha_i\Delta_2 + \beta_i\Delta_{12})\sin\theta^{(i)}_k.\end{aligned} \tag{2.89}$$

From Eq. (2.85), we have

$$\begin{aligned} G^{(1)}_{c^{(i)}_{k+1}}(\theta^{(i)}_k) &= \frac{1}{\Delta}[\Delta_2 G^{(1)}_{c^{(i)}_k}(\theta^{(i)}_k) - \Delta_{12} G^{(1)}_{d^{(i)}_k}(\theta^{(i)}_k)] \\ &= \alpha_i \cos\theta^{(i)}_k + \beta_i \sin\theta^{(i)}_k, \\ G^{(1)}_{d^{(i)}_{k+1}}(\theta^{(i)}_k) &= \frac{1}{\Delta}[\Delta_1 G^{(1)}_{d^{(i)}_k}(\theta^{(i)}_k) - \Delta_{12} G^{(1)}_{c^{(i)}_k}(\theta^{(i)}_k)] \\ &= \alpha_i \sin\theta^{(i)}_k - \beta_i \cos\theta^{(i)}_k. \end{aligned} \tag{2.90}$$

Thus,

$$\begin{aligned} G^{(2)}_{r^{(i)}_{k+1}}(\theta^{(i)}_k) &= [G^{(1)}_{c^{(i)}_{k+1}}(\theta^{(i)}_k) G^{(1)}_{c^{(i)}_{k+1}}(\theta^{(i)}_k) + G^{(1)}_{d^{(i)}_{k+1}}(\theta^{(i)}_k) G^{(1)}_{d^{(i)}_{k+1}}(\theta^{(i)}_k)] \\ &= \alpha_i^2 + \beta_i^2. \end{aligned} \tag{2.91}$$

Furthermore, Eq. (2.83) gives

$$r^{(i)}_{k+1} = \rho_i r^{(i)}_k + o(r^{(i)}_k) \quad \text{and} \quad \theta^{(i)}_{k+1} = \theta^{(i)}_k - \vartheta_i + o(r^{(i)}_k). \tag{2.92}$$

where

$$\vartheta_i = \arctan(\beta_i/\alpha_i) \quad \text{and} \quad \rho_i = \sqrt{\alpha_i^2 + \beta_i^2}. \tag{2.93}$$

As $r^{(i)}_k \ll 1$ and $r^{(i)}_k \to 0$, we have

$$r^{(i)}_{k+1} = \rho_i r^{(i)}_k \quad \text{and} \quad \theta^{(i)}_{k+1} = \vartheta_i - \theta^{(i)}_k. \tag{2.94}$$

With an initial condition of $r^{(i)}_k = r^0_k$ and $\theta^{(i)}_k = \theta^{(i)}_k$, the corresponding solution of Eq. (2.94) is

$$r^{(i)}_{k+j} = (\rho_i)^j r^0_k \quad \text{and} \quad \theta^{(i)}_{k+j} = j\vartheta_i - \theta^{(i)}_k. \tag{2.95}$$

From Eqs. (2.80), (2.81), and (2.90), we have

$$\begin{aligned} c^{(i)}_{k+1} &= \alpha_i r^{(i)}_k \cos\theta^{(i)}_k + \beta_i r^{(i)}_k \sin\theta^{(i)}_k = \alpha_i c^{(i)}_k + \beta_i d^{(i)}_k, \\ d^{(i)}_{k+1} &= \alpha_i r^{(i)}_k \sin\theta^{(i)}_k - \beta_i r^{(i)}_k \cos\theta^{(i)}_k = -\beta_i c^{(i)}_k + \alpha_i d^{(i)}_k. \end{aligned} \tag{2.96}$$

That is,

$$\begin{Bmatrix} c^{(i)}_{k+1} \\ d^{(i)}_{k+1} \end{Bmatrix} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \begin{Bmatrix} c^{(i)}_k \\ d^{(i)}_k \end{Bmatrix} = \rho_i \begin{bmatrix} \cos\vartheta_i & \sin\vartheta_i \\ -\sin\vartheta_i & \cos\vartheta_i \end{bmatrix} \begin{Bmatrix} c^{(i)}_k \\ d^{(i)}_k \end{Bmatrix}. \tag{2.97}$$

From the foregoing equation, we have

$$\begin{Bmatrix} c_{k+j}^{(i)} \\ d_{k+j}^{(i)} \end{Bmatrix} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}^j \begin{Bmatrix} c_k^{(i)} \\ d_k^{(i)} \end{Bmatrix} = (\rho_i)^j \begin{bmatrix} \cos j\vartheta_i & \sin j\vartheta_i \\ -\sin j\vartheta_i & \cos j\vartheta_i \end{bmatrix} \begin{Bmatrix} c_k^{(i)} \\ d_k^{(i)} \end{Bmatrix}. \tag{2.98}$$

Definition 2.22 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ ($\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*$) in $U_k(\mathbf{x}_k^*)$. Consider a pair of complex eigenvalues $\alpha_i \pm \mathbf{i}\beta_i$ ($i \in N = \{1, 2, \ldots, n\}$, $\mathbf{i} = \sqrt{-1}$) of matrix $D\mathbf{f}(\mathbf{x}^*, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. On the invariant plane of $(\mathbf{u}_i, \mathbf{v}_i)$, consider $\mathbf{r}_k^{(i)} = \mathbf{y}_k^{(i)} = \mathbf{y}_{k+}^{(i)} + \mathbf{y}_{k-}^{(i)}$ with Eqs. (2.73) and (2.75). For any arbitrarily small $\varepsilon > 0$, the stability of the fixed point $\mathbf{x}_k^*$ on the invariant plane of $(\mathbf{u}_i, \mathbf{v}_i)$ can be determined.

(i) $\mathbf{x}_k^{(i)}$ at the fixed point $\mathbf{x}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally stable if

$$r_{k+1}^{(i)} - r_k^{(i)} < 0. \tag{2.99}$$

(ii) $\mathbf{x}_k^{(i)}$ at the fixed point $\mathbf{x}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally unstable if

$$r_{k+1}^{(i)} - r_k^{(i)} > 0. \tag{2.100}$$

(iii) $\mathbf{x}_k^{(i)}$ at the fixed point $\mathbf{x}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally stable with the m_ith-order singularity if for $\theta_k^{(i)} \in [0, 2\pi]$

$$\begin{gathered} \rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1, \\ G_{r_{k+1}^{(i)}}^{(s_k^{(i)})}(\theta_k) = 0 \quad \text{for } s_k^{(i)} = 1, 2, \ldots, m_i - 1, \\ r_{k+1}^{(i)} - r_k^{(i)} < 0. \end{gathered} \tag{2.101}$$

(iv) $\mathbf{x}_k^{(i)}$ at the fixed point $\mathbf{x}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally unstable with the m_ith-order singularity if for $\theta_k^{(i)} \in [0, 2\pi]$

$$\begin{gathered} \rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1, \\ G_{r_{k+1}^{(i)}}^{(s_k^{(i)})}(\theta_k) = 0 \quad \text{for } s_k^{(i)} = 1, 2, \ldots, m_i - 1, \\ r_{k+1}^{(i)} - r_k^{(i)} > 0. \end{gathered} \tag{2.102}$$

(v) $\mathbf{x}_k^{(i)}$ at the fixed point $\mathbf{x}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is circular if for $\theta_k^{(i)} \in [0, 2\pi]$

$$r_{k+1}^{(i)} - r_k^{(i)} = 0. \tag{2.103}$$

(vi) $\mathbf{x}_k^{(i)}$ at the fixed point $\mathbf{x}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is degenerate in the direction of $\mathbf{u}_i$ if

$$\beta_i = 0 \text{ and } \theta_{k+1}^{(i)} - \theta_k^{(i)} = 0. \tag{2.104}$$

Theorem 2.7 *Consider a discrete, nonlinear dynamical system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ *in* Eq. (2.4) *with a fixed point* $\mathbf{x}_k^*$. *The corresponding solution is given by* $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ *with* $j \in \mathbb{Z}$. *Suppose there is a neighborhood of the fixed point* $\mathbf{x}_k^*$ *(i.e.,* $U_k(\mathbf{x}_k^*) \subset \Omega$*), and* $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *is* C^r $(r \geq 1)$*-continuous in* $U_k(\mathbf{x}_k^*)$ *with* Eq. (2.28). *The linearized system is* $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ *in* $U_k(\mathbf{x}_k^*)$. *Consider a pair of complex eigenvalues* $\alpha_i \pm \mathbf{i}\beta_i$ $(i \in N = \{1, 2, \ldots, n\}$, $\mathbf{i} = \sqrt{-1})$ *of matrix* $D\mathbf{f}(\mathbf{x}^*, \mathbf{p})$ *with a pair of eigenvectors* $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. *On the invariant plane of* $(\mathbf{u}_i, \mathbf{v}_i)$*, consider* $\mathbf{r}_k^{(i)} = \mathbf{y}_k^{(i)} = \mathbf{y}_{k+}^{(i)} + \mathbf{y}_{k-}^{(i)}$ *with Eqs.* (2.73) *and* (2.75). *For any arbitrarily small* $\varepsilon > 0$*, the stability of the equilibrium* $\mathbf{x}_k^*$ *on the invariant plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *can be determined.*

(i) $\mathbf{x}_k^{(i)}$ *at the fixed point* $\mathbf{x}_k^*$ *on the plane of* $(\mathbf{u}_k, \mathbf{v}_k)$ *is spirally stable if and only if*

$$\rho_i < 1. \tag{2.105}$$

(ii) $\mathbf{x}_k^{(i)}$ *at the fixed point* $\mathbf{x}_k^*$ *on the plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *is spirally unstable if and only if*

$$\rho_i > 1. \tag{2.106}$$

(iii) $\mathbf{x}_k^{(i)}$ *at the fixed point* $\mathbf{x}_k^*$ *on the plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *is stable with the* m_ith-*order singularity if and only if for* $\theta_k^{(i)} \in [0, 2\pi]$

$$\begin{aligned}
&\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1, \\
&G_{r_k^{(i)}}^{(s_k^{(i)})}(\theta_k^{(i)}) = 0 \ \ \textit{for } s_k = 1, 2, \ldots, m_i - 1, \\
&G_{r_k^{(i)}}^{(m_i)}(\theta_k^{(i)}) < 0.
\end{aligned} \tag{2.107}$$

(iv) $\mathbf{x}_k^{(i)}$ *at the fixed point* $\mathbf{x}_k^*$ *on the plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *is spirally unstable with the* m_ith-*order singularity if and only if for* $\theta_k^{(i)} \in [0, 2\pi]$

$$\begin{aligned}
&\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1, \\
&G^{(s_k^{(i)})}_{r_k^{(i)}}(\theta_k^{(i)}) = 0 \ \ for\ s_k^{(i)} = 0, 1, 2, \ldots, m_i - 1, \\
&G^{(m_i)}_{r_k^{(i)}}(\theta_k^{(i)}) > 0.
\end{aligned} \tag{2.108}$$

(v) $\mathbf{x}_k^{(i)}$ *at the fixed point* $\mathbf{x}_k^*$ *on the plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *is circular if and only if for* $\theta_k^{(i)} \in [0, 2\pi]$

$$\begin{aligned}
&\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1, \\
&G^{(s_k^{(i)})}_{r_k^{(i)}}(\theta_k^{(i)}) = 0 \ \ for\ s_k^{(i)} = 0, 1, 2, \ldots.
\end{aligned} \tag{2.109}$$

Proof The proof can be referred to Luo (2011). □

2.4 Bifurcation Theory

Definition 2.23 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r ($r \geq 1$)-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ ($\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*$) in $U_k(\mathbf{x}_k^*)$ and there are n linearly independent vectors $\mathbf{v}_i$ ($i = 1, 2, \ldots, n$). For a perturbation of fixed point $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$, let $\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$.

$$s_k^{(i)} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k = \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*) \tag{2.110}$$

where $s_k^{(i)} = c_k^{(i)} \|\mathbf{v}_i\|^2$.

$$s_{k+1}^{(i)} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1} = \mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_k, \mathbf{p}) - \mathbf{x}_k^*]. \tag{2.111}$$

In the vicinity of point $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$, $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ can be expanded for $(0 < \theta < 1)$ as

$$\begin{aligned}
\mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_k, \mathbf{p}) - \mathbf{x}_{k(0)}^*] = {} & a_i(s_k^{(i)} - s_{k(0)}^{(i)*}) + \mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) \\
& + \sum_{q=2}^{m} \frac{1}{q!} \sum_{r=0}^{q} C_q^r \mathbf{a}_i^{(q-r,r)} (s_k^{(i)} - s_{k(0)}^{(i)*})^{q-r} (\mathbf{p} - \mathbf{p}_0)^r \\
& + \frac{1}{(m+1)!} [(s_k^{(i)} - s_{k(0)}^{(i)*})\partial_{s_k^{(i)}} + (\mathbf{p} - \mathbf{p}_0)\partial_{\mathbf{p}}]^{m+1} \\
& \times (\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_{k(0)}^* + \theta \Delta \mathbf{x}_k, \mathbf{p}_0 + \theta \Delta \mathbf{p}))
\end{aligned} \tag{2.112}$$

where

$$
\begin{aligned}
a_i &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)}, \ \mathbf{b}_i^{\mathrm{T}} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)}, \\
\mathbf{a}_i^{(r,s)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(r)} \partial_{\mathbf{p}}^{(s)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)}.
\end{aligned} \tag{2.113}
$$

If $a_i = 1$ and $\mathbf{p} = \mathbf{p}_0$, the stability of the fixed point $\mathbf{x}_k^*$ on an eigenvector $\mathbf{v}_i$ changes from stable to unstable state (or from unstable to stable state). The bifurcation manifold on the direction of $\mathbf{v}_i$ is determined by

$$
\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) + \sum_{q=2}^{m} \frac{1}{q!} \sum_{r=0}^{q} C_q^r \mathbf{a}_i^{(q-r,r)} (s_k^{(i)} - s_{k(0)}^{(i)*})^{q-r} (\mathbf{p} - \mathbf{p}_0)^r = 0. \tag{2.114}
$$

In the neighborhood of $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$, when other components of fixed point $\mathbf{x}_k^*$ on the eigenvector of $\mathbf{v}_j$ for all $j \neq i$, $(i, j \in N)$ do not change their stability states, Eq. (2.114) possesses l-branch solutions of equilibrium $s_k^{(i)*}$ $(0 < l \leq m)$ with l_1-stable and l_2-unstable solutions $(l_1, l_2 \in \{0, 1, 2, \ldots, l\})$. Such l-branch solutions are called the bifurcation solutions of fixed point $\mathbf{x}_k^*$ on the eigenvector of $\mathbf{v}_i$ in the neighborhood of $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$. Such a bifurcation at point $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$ is called the hyperbolic bifurcation of mth-order on the eigenvector of $\mathbf{v}_i$. Consider two special cases herein.

(i) If

$$
\mathbf{a}_i^{(1,1)} = \mathbf{0} \ \text{ and } \ \mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) + \frac{1}{2!} a_i^{(2,0)} (s_k^{(i)*} - s_{k0}^{(i)*})^2 = 0 \tag{2.115}
$$

where

$$
\begin{aligned}
a_i^{(2,0)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(2)} \partial_{\mathbf{p}}^{(0)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(2)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} \\
&= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}}^{(2)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})(\mathbf{v}_k \mathbf{v}_k)\big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = G_{s_k^{(i)}}^{(2)} (\mathbf{x}_{k(0)}^*, \mathbf{p}_0) \neq 0, \\
\mathbf{b}_i^{\mathrm{T}} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} \neq \mathbf{0},
\end{aligned} \tag{2.116}
$$

$$
a_i^{(2,0)} \times [\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0)] < 0, \tag{2.117}
$$

such a bifurcation at point $(\mathbf{x}_0^*, \mathbf{p}_0)$ is called the *saddle–node* bifurcation on the eigenvector of $\mathbf{v}_i$.

(ii) If

$$
\begin{aligned}
&\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) = 0 \ \text{ and} \\
&\mathbf{a}_i^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0)(s_k^{(i)*} - s_{k(0)}^{(i)*}) + \frac{1}{2!} a_i^{(2,0)} (s_k^{(i)*} - s_{k(0)}^{(i)*})^2 = 0
\end{aligned} \tag{2.118}
$$

where

$$\begin{aligned} a_i^{(2,0)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(2)} \partial_{\mathbf{p}}^{(0)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(2)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_0^*, \mathbf{p}_0)} \\ &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}^{(2)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})(\mathbf{v}_i \mathbf{v}_i)\big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = G_{s_k^{(i)}}^{(2)}(\mathbf{x}_{k(0)}^*, \mathbf{p}_0) \neq 0, \\ \mathbf{a}_i^{(1,1)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(1)} \partial_{\mathbf{p}}^{(1)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}} \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} \\ &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k} \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \mathbf{v}_i\big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} \neq \mathbf{0}, \end{aligned} \tag{2.119}$$

$$a_i^{(2,0)} \times [\mathbf{a}_i^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0)] < 0, \tag{2.120}$$

such a bifurcation at point $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$ is called the *transcritical* bifurcation on the eigenvector of $\mathbf{v}_i$.

Definition 2.24 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r ($r \geq 1$)-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ ($\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*$) in $U_k(\mathbf{x}_k^*)$ and there are n linearly independent vectors $\mathbf{v}_i$ ($i = 1, 2, \ldots, n$). For a perturbation of fixed point $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$, let $\mathbf{y}_k^{(i)} = c_k^{(i)} \mathbf{v}_i$ and $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)} \mathbf{v}_i$. Equations (2.110), (2.111), and (2.113) hold. In the vicinity of point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$, $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ can be expended for $(0 < \theta < 1)$ as

$$\begin{aligned} \mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_k, \mathbf{p}) - \mathbf{x}_{k+1(0)}^*] = {} & a_i(s_k^{(i)} - s_{k(0)}^{(i)*}) + \mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) \\ & + \sum_{q=2}^{m} \frac{1}{q!} \sum_{r=0}^{q} C_q^r \mathbf{a}_i^{(q-r,r)} (s_k^{(i)} - s_{k(0)}^{(i)*})^{q-r} (\mathbf{p} - \mathbf{p}_0)^r \\ & + \frac{1}{(m+1)!} [(s_k^{(i)} - s_{k(0)}^{(i)*}) \partial_{s_k^{(i)}} + (\mathbf{p} - \mathbf{p}_0) \partial_{\mathbf{p}}]^{m+1} \\ & \times (\mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_{k0}^* + \theta \Delta \mathbf{x}_k, \mathbf{p}_0 + \theta \Delta \mathbf{p})) \end{aligned} \tag{2.121}$$

and

$$\begin{aligned} \mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_{k+1}, \mathbf{p}) - \mathbf{x}_{k(0)}^*] = {} & a_i(s_{k+1}^{(i)} - s_{k+1(0)}^*) + \mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) \\ & + \sum_{q=2}^{m} \frac{1}{q!} \sum_{r=0}^{q} C_q^r \mathbf{a}_i^{(q-r,r)} (s_{k+1}^{(i)} - s_{k+1(0)}^{(i)*})^{q-r} (\mathbf{p} - \mathbf{p}_0)^r \\ & + \frac{1}{(m+1)!} [(s_{k+1}^{(i)} - s_{k+1(0)}^*) \partial_{s_{k+1}^{(i)}} + (\mathbf{p} - \mathbf{p}_0) \partial_{\mathbf{p}}]^{m+1} \\ & \times (\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_{k+1(0)}^* + \theta \Delta \mathbf{x}_{k+1}, \mathbf{p}_0 + \theta \Delta \mathbf{p})). \end{aligned} \tag{2.122}$$

If $a_i = -1$ and $\mathbf{p} = \mathbf{p}_0$, the stability of current equilibrium $\mathbf{x}_k^*$ on an eigenvector $\mathbf{v}_i$ changes from stable to unstable state (or from unstable to stable state). The bifurcation manifold in the direction of $\mathbf{v}_i$ is determined by

$$\begin{aligned}
&\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) + a_i(s_k^{(i)*} - s_{k(0)}^{(i)*}) \\
&+ \sum_{q=2}^{m} \frac{1}{q!} \sum_{r=0}^{q} C_q^r \mathbf{a}_i^{(q-r,r)} (s_k^{(i)} - s_{k(0)}^{(i)*})^{q-r} (\mathbf{p} - \mathbf{p}_0)^r = (s_{k+1}^{(i)*} - s_{k+1(0)}^{(i)*}); \\
&\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) + a_i(s_{k+1}^{(i)*} - s_{k+1(0)}^{(i)*}) \\
&+ \sum_{q=2}^{m} \frac{1}{q!} \sum_{r=0}^{q} C_q^r \mathbf{a}_i^{(q-r,r)} (s_{k+1}^{(i)} - s_{k+1(0)}^{(i)*})^{q-r} (\mathbf{p} - \mathbf{p}_0)^r = (s_k^{(i)*} - s_{k(0)}^{(i)*}).
\end{aligned} \tag{2.123}$$

In the neighborhood of $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$, when other components of fixed point $\mathbf{x}_{k(0)}^*$ on the eigenvector of $\mathbf{v}_j$ for all $j \neq i$, $(j, i \in N)$ do not change their stability states, Eq. (2.123) possesses l-branch solutions of equilibrium $s_k^{(i)*} (0 < l \leq m)$ with l_1-stable and l_2-unstable solutions ($l_1, l_2 \in \{0, 1, 2, \ldots, l\}$). Such l-branch solutions are called the bifurcation solutions of fixed point $\mathbf{x}_k^*$ on the eigenvector of $\mathbf{v}_i$ in the neighborhood of $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$. Such a bifurcation at point $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$ is called the *hyperbolic bifurcation* of mth-order with doubling iterations on the eigenvector of $\mathbf{v}_i$. Consider a special case. If

$$\begin{aligned}
&\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) = 0,\, a_i = -1,\, a_i^{(2,0)} = 0, \quad \mathbf{a}_i^{(2,1)} = 0, \quad \mathbf{a}_i^{(1,2)} = 0, \\
&[\mathbf{a}^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0) + a_i](s_k^{(i)*} - s_{k(0)}^{(i)*}) + \frac{1}{3!} a_i^{(3,0)} (s_k^* - s_{k(0)}^*)^3 = (s_{k+1}^{(i)*} - s_{k+1(0)}^{(i)*}), \\
&[\mathbf{a}^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0) + a_i](s_{k+1}^{(i)*} - s_{k+1(0)}^{(i)*}) + \frac{1}{3!} a_i^{(3,0)} (s_{k+1}^* - s_{k+1(0)}^*)^3 = s_k^{(i)*} - s_{k(0)}^{(i)*})
\end{aligned} \tag{2.124}$$

where

$$\begin{aligned}
a_i^{(3,0)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(3)} \partial_{\mathbf{p}}^{(0)} \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(3)} \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} \\
&= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}^{(3)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})(\mathbf{v}_i \mathbf{v}_i \mathbf{v}_i) \big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = G_i^{(3)}(\mathbf{x}_{k(0)}^*, \mathbf{p}_0) \neq 0, \\
\mathbf{a}_i^{(1,1)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(1)} \partial_{\mathbf{p}}^{(1)} \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}} \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} \\
&= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k} \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \mathbf{v}_i \big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} \neq \mathbf{0},
\end{aligned} \tag{2.125}$$

$$a_i^{(3,0)} \times [\mathbf{a}_i^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0)] < 0, \tag{2.126}$$

such a bifurcation at point $(\mathbf{x}_{k(0)}^{*}, \mathbf{p}_0)$ is called the *pitchfork* bifurcation (or period-doubling bifurcation) on the eigenvector of $\mathbf{v}_i$.

For the saddle–node bifurcation of the first kind, the $(2m)$th-order singularity of the fixed point at the bifurcation point exists as a saddle of the $(2m)$th-order. For the transcritical bifurcation, the $(2m)$th-order singularity of the fixed point at the bifurcation point exists as a saddle of the $(2m)$th-order. However, for the stable pitchfork bifurcation (or saddle–node bifurcation of the second kind, or period-doubling bifurcation), the $(2m+1)$th-order singularity of the fixed point at the bifurcation point exists as an oscillatory sink of the $(2m+1)$th-order. For the unstable pitchfork bifurcation (or the unstable saddle–node bifurcation of the second kind, or unstable period-doubling bifurcation), the $(2m+1)$th-order singularity of the fixed point at the bifurcation point exists as an oscillatory source of the $(2m+1)$th-order.

Definition 2.25 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. Consider a pair of complex eigenvalues $\alpha_i \pm \mathbf{i}\beta_i$ $(i \in N = \{1, 2, \ldots, n\}$, $\mathbf{i} = \sqrt{-1})$ of matrix $D\mathbf{f}(\mathbf{x}^*, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. On the invariant plane of $(\mathbf{u}_i, \mathbf{v}_i)$, consider $\mathbf{r}_k^{(i)} = \mathbf{y}_k^{(i)} = \mathbf{y}_{k+}^{(i)} + \mathbf{y}_{k-}^{(i)}$ with

$$\mathbf{r}_k^{(i)} = c_k^{(i)}\mathbf{u}_i + d_k^{(i)}\mathbf{v}_i \quad \text{and} \quad \mathbf{r}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{u}_i + d_{k+1}^{(i)}\mathbf{v}_i. \tag{2.127}$$

and

$$\begin{aligned} c_k^{(i)} &= \frac{1}{\Delta}[\Delta_2(\mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{y}_k) - \Delta_{12}(\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k)], \\ d_k^{(i)} &= \frac{1}{\Delta}[\Delta_1(\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k) - \Delta_{12}(\mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{y}_k)]; \\ \Delta_1 &= \|\mathbf{u}_i\|^2,\ \Delta_2 = \|\mathbf{v}_i\|^2,\ \Delta_{12} = \mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{v}_i; \\ \Delta &= \Delta_1\Delta_2 - \Delta_{12}^2. \end{aligned} \tag{2.128}$$

Consider a polar coordinate of (r_k, θ_k) defined by

$$\begin{aligned} c_k^{(i)} &= r_k^{(i)}\cos\theta_k^{(i)}, \quad \text{and} \quad d_k^{(i)} = r_k^{(i)}\sin\theta_k^{(i)}; \\ r_k^{(i)} &= \sqrt{(c_k^{(i)})^2 + (d_k^{(i)})^2}, \quad \text{and} \quad \theta_k^{(i)} = \arctan(d_k^{(i)}/c_k^{(i)}). \end{aligned} \tag{2.129}$$

Thus,

$$\begin{aligned} c_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_2 G_{c_k^{(i)}}(\mathbf{x}_k, \mathbf{p}) - \Delta_{12} G_{d_k^{(i)}}(\mathbf{x}_k, \mathbf{p})], \\ d_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_1 G_{d_k^{(i)}}(\mathbf{x}_k, \mathbf{p}) - \Delta_{12} G_{c_k^{(i)}}(\mathbf{x}_k, \mathbf{p})] \end{aligned} \tag{2.130}$$

where

$$\begin{aligned}
G_{c_k^{(i)}}(\mathbf{x}_k,\mathbf{p}) &= \mathbf{u}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_k,\mathbf{p}) - \mathbf{x}_{k(0)}^*] \\
&= \mathbf{a}_i^{\mathrm{T}} \cdot (\mathbf{p}-\mathbf{p}_0) + a_{i11}(c_k^{(i)} - c_{k(0)}^{(i)*}) + a_{i12}(d_k^{(i)} - d_{k(0)}^{(i)*}) \\
&\quad + \sum_{q=2}^{m_i} \frac{1}{q!} \sum_{r=0}^{q} C_{m_i}^{r_i} \mathbf{G}_{c_k^{(i)}}^{(m_i-r_i,r_i)}(\mathbf{x}_k^*,\mathbf{p}_0)(\mathbf{p}-\mathbf{p}_0)^{r_i}(r_k^{(i)})^{m_i-r_i} \\
&\quad + \frac{1}{(m_i+1)!}[(c_k^{(i)} - c_{k(0)}^{(i)*})\partial_{c_k^{(i)}} + (d_k^{(i)} - d_{k(0)}^{(i)*})\partial_{d_k^{(i)}} + (\mathbf{p}-\mathbf{p}_0)\partial_{\mathbf{p}}]^{m_i+1} \\
&\quad \times (\mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_{k0}^* + \theta\Delta\mathbf{x}_k, \mathbf{p}_0 + \theta\Delta\mathbf{p})),
\end{aligned}$$

$$\begin{aligned}
G_{d_k^{(i)}}(\mathbf{x}_k,\mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_k,\mathbf{p}) - \mathbf{x}_{k(0)}^*] \\
&= \mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p}-\mathbf{p}_0) + a_{i21}(c_k^{(i)} - c_{k(0)}^{(i)*}) + a_{i22}(d_k^{(i)} - d_{k(0)}^{(i)*}) \\
&\quad + \sum_{q=2}^{m_i} \frac{1}{q!} \sum_{r=0}^{q} C_{m_i}^{r_i} \mathbf{G}_{d_k^{(i)}}^{(m_i-r_i,r_i)}(\mathbf{x}_k^*,\mathbf{p}_0)(\mathbf{p}-\mathbf{p}_0)^{r_i} r_k^{m_i-r_i} \\
&\quad + \frac{1}{(m_i+1)!}[(c_k^{(i)} - c_{k(0)}^{(i)*})\partial_{c_k^{(i)}} + (d_k^{(i)} - d_{k(0)}^{(i)*})\partial_{d_k^{(i)}} + (\mathbf{p}-\mathbf{p}_0)\partial_{\mathbf{p}}]^{m_i+1} \\
&\quad \times (\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_{k(0)}^* + \theta\Delta\mathbf{x}, \mathbf{p}_0 + \theta\Delta\mathbf{p}));
\end{aligned} \tag{2.131}$$

and

$$\begin{aligned}
&\mathbf{G}_{c_k^{(i)}}^{(s,r)}(\mathbf{x}_{k(0)}^*,\mathbf{p}_0) \\
&= \mathbf{u}_i^{\mathrm{T}} \cdot [\partial_{\mathbf{x}_k}()\mathbf{u}_i \cos\theta_k^{(i)} + \partial_{\mathbf{x}_k}()\mathbf{v}_i \sin\theta_k^{(i)}]^s \partial_{\mathbf{p}}^{(r)} \mathbf{f}(\mathbf{x}_k,\mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*,\mathbf{p}_0)}, \\
&\mathbf{G}_{d_k^{(i)}}^{(s,r)}(\mathbf{x}_{k(0)}^*,\mathbf{p}_0) \\
&= \mathbf{v}_i^{\mathrm{T}} \cdot [\partial_{\mathbf{x}_k}()\mathbf{u}_i \cos\theta_k^{(i)} + \partial_{\mathbf{x}_k}()\mathbf{v}_i \sin\theta_k^{(i)}]^s \partial_{\mathbf{p}}^{(r)} \mathbf{f}(\mathbf{x}_k,\mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*,\mathbf{p}_0)};
\end{aligned} \tag{2.132}$$

$$\begin{aligned}
&\mathbf{a}_i^{\mathrm{T}} = \mathbf{u}_i^{\mathrm{T}} \cdot \partial_{\mathbf{p}}\mathbf{f}(\mathbf{x}_k,\mathbf{p}),\ \mathbf{b}_i^{\mathrm{T}} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{p}}\mathbf{f}(\mathbf{x}_k,\mathbf{p}); \\
&a_{i11} = \mathbf{u}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\mathbf{u}_i,\ a_{i12} = \mathbf{u}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\mathbf{u}_i; \\
&a_{i21} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\mathbf{u}_i,\ a_{i22} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\mathbf{v}_i.
\end{aligned} \tag{2.133}$$

Suppose

$$\mathbf{a}_i = \mathbf{0} \quad \text{and} \quad \mathbf{b}_i = \mathbf{0} \tag{2.134}$$

then

$$
\begin{aligned}
r_{k+1}^{(i)} &= \sqrt{(c_{k+1}^{(i)})^2 + (d_{k+1}^{(i)})^2} = \sqrt{\sum_{m=2}^{\infty} (r_k^{(i)})^m G_{r_{k+1}^{(i)}}^{(m)}} \\
&= \sqrt{G_{r_{k+1}^{(i)}}^{(2,0)}}\, r_k^{(i)} \sqrt{1 + \lambda^{(i)} + \sum_{m=3}^{\infty} \lambda_m^{(i)} (r_k^{(i)})^{m-2}} \\
\theta_{k+1}^{(i)} &= \arctan(d_{k+1}^{(i)}/c_{k+1}^{(i)})
\end{aligned}
\tag{2.135}
$$

where

$$
\begin{aligned}
G_{r_{k+1}^{(i)}}^{(2)} &= G_{r_{k+1}^{(i)}}^{(2,0)} + G_{r_{k+1}^{(i)}}^{(1,1)} \quad \text{and} \quad \lambda^{(i)} = G_{r_{k+1}^{(i)}}^{(1,1)}/G_{r_{k+1}^{(i)}}^{(2,0)} \text{ with} \\
G_{r_{k+1}^{(i)}}^{(2,0)} &= [G_{c_{k+1}^{(i)}}^{(1,0)}(\theta_k^{(i)}, \mathbf{p}_0)]^2 + [G_{d_{k+1}^{(i)}}^{(1,0)}(\theta_k^{(i)}, \mathbf{p}_0)]^2, \\
G_{r_{k+1}^{(i)}}^{(1,1)} &= \sum_{r=1}^{M}\sum_{s=1}^{M} [\mathbf{G}_{c_{k+1}^{(i)}}^{(1,r)}(\theta_k^{(i)}, \mathbf{p}_0) \cdot \mathbf{G}_{c_{k+1}^{(i)}}^{(1,s)}(\theta_k^{(i)}, \mathbf{p}_0)(\mathbf{p} - \mathbf{p}_0)^{r+s} \\
&\quad + [\mathbf{G}_{d_{k+1}^{(i)}}^{(1,r)}(\theta_k^{(i)}, \mathbf{p}_0) \cdot \mathbf{G}_{d_{k+1}^{(i)}}^{(1,s)}(\theta_k^{(i)}, \mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0)^{r+s}];
\end{aligned}
\tag{2.136}
$$

and

$$
\begin{aligned}
\lambda_m^{(i)} &= G_{r_{k+1}^{(i)}}^{(m)}/G_{r_{k+1}^{(i)}}^{(2,0)} \text{ with} \\
G_{r_{k+1}^{(i)}}^{(m)} &= \sum_{m_i=0}^{M}\sum_{m_j=0}^{M} \frac{1}{m_i!}\frac{1}{m_j!} [\mathbf{G}_{c_{k+1}^{(i)}}^{(m_i-r_i,r_i)}(\theta_k^{(i)}, \mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0)^{m_i-r_i}] \\
&\quad \times \mathbf{G}_{c_{k+1}^{(i)}}^{(m_j-s_j,s_j)}(\theta_k^{(i)}, \mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0)^{m_j-s_j} \\
&\quad + \mathbf{G}_{d_{k+1}^{(i)}}^{(m_i-r_i,r_i)}(\theta_k^{(i)}, \mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0)^{m_i-r_i}] \\
&\quad \times \mathbf{G}_{d_{k+1}^{(i)}}^{(m_j-s_j,s_j)}(\theta_k^{(i)}, \mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0)^{m_j-s_j}]\delta_m^{(r_i+s_j)} \\
&= \frac{1}{m!}\sum_{r=1}^{m-1} C_m^r \sum_{s=1}^{M} \frac{1}{s!}\frac{1}{(2M-m)!} [\mathbf{G}_{c_{k+1}^{(i)}}^{(r,s)}(\theta_k^{(i)}, \mathbf{p}_0) \cdot \mathbf{G}_{c_{k+1}^{(i)}}^{(m-r,2M-m-r)}(\theta_k^{(i)}, \mathbf{p}_0) \\
&\quad + \mathbf{G}_{d_{k+1}^{(i)}}^{(r,s)}(\theta_k^{(i)}, \mathbf{p}_0) \cdot \mathbf{G}_{d_{k+1}^{(i)}}^{(m-r,2M-m-r)}(\theta_k^{(i)}, \mathbf{p}_0)] \cdot (\mathbf{p} - \mathbf{p}_0)^{M-m},
\end{aligned}
\tag{2.137}
$$

$$
\begin{aligned}
\mathbf{G}_{c_{k+1}^{(i)}}^{(m-r,r)}(\theta_k, \mathbf{p}_0) &= \frac{1}{\Delta}[\Delta_2 \mathbf{G}_{c_k^{(i)}}^{(m-r,r)}(\mathbf{x}_{k(0)}^*, \mathbf{p}_0) - \Delta_{12}\mathbf{G}_{d_k^{(i)}}^{(m-r,r)}(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)], \\
\mathbf{G}_{d_{k+1}^{(i)}}^{(m-r,r)}(\theta_k, \mathbf{p}_0) &= \frac{1}{\Delta}[\Delta_1 \mathbf{G}_{d_k^{(i)}}^{(m-r,r)}(\mathbf{x}_{k(0)}^*, \mathbf{p}_0) - \Delta_{12}\mathbf{G}_{c_k^{(i)}}^{(m-r,r)}(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)].
\end{aligned}
\tag{2.138}
$$

If $G^{(2,0)}_{r^{(i)}_{k+1}} = 1$ and $\mathbf{p} = \mathbf{p}_0$, the stability of current fixed point $\mathbf{x}^*_k$ on an eigenvector plane of $(\mathbf{u}_i, \mathbf{v}_i)$ changes from stable to unstable state (or from unstable to stable state). The bifurcation manifold in the direction of $\mathbf{v}_i$ is determined by

$$\lambda^{(i)} + \sum\nolimits_{m=3}^{\infty} \lambda^{(i)}_m (r^{(i)}_k)^{m-2} = 0. \tag{2.139}$$

Such a bifurcation at the fixed point $(\mathbf{x}^*_{k(0)}, \mathbf{p}_0)$ is called the generalized Neimark bifurcation on the eigenvector plane of $(\mathbf{u}_i, \mathbf{v}_i)$.

For a special case, if

$$\lambda^{(i)} + \lambda^{(i)}_4 (r^{(i)}_k)^2 = 0, \quad \text{for} \quad \lambda^{(i)} \times \lambda^{(i)}_4 < 0 \quad \text{and} \quad \lambda^{(i)}_3 = 0 \tag{2.140}$$

such a bifurcation at point $(\mathbf{x}^*_0, \mathbf{p}_0)$ is called the Neimark bifurcation on the eigenvector plane of $(\mathbf{u}_i, \mathbf{v}_i)$.

For the repeating eigenvalues of $DP(\mathbf{x}^*_k, \mathbf{p})$, the bifurcation of fixed point $\mathbf{x}^*_k$ can be similarly discussed in the foregoing Theorems 2.5 and 2.6. Herein, such a procedure will not be repeated. From the foregoing analysis of the Neimark bifurcation, the Neimark bifurcation points possess the higher-order singularity of the flow in discrete dynamical system in the radial direction. For the stable Neimark bifurcation, the mth-order singularity of the flow at the bifurcation point exists as a sink of the mth-order in the radial direction. For the unstable Neimark bifurcation, the mth-order singularity of the flow at the bifurcation point exists as a source of the mth-order in the radial direction.

Consider a 2D map

$$P : \mathbf{x}_k \to \mathbf{x}_{k+1} \quad \text{with } \mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \tag{2.141}$$

where $\mathbf{x}_k = (x_k, y_k)^{\mathrm{T}}$ and $\mathbf{f} = (f_1, f_2)^{\mathrm{T}}$ with a parameter vector $\mathbf{p}$. The period-n fixed point for Eq. (2.141) is $(\mathbf{x}^*_k, \mathbf{p})$, i.e., $P^{(n)}\mathbf{x}^*_k = \mathbf{x}^*_{k+n}$ where $P^{(n)} = P \circ P^{(n-1)}$ and $P^{(0)} = 1$, and its stability and bifurcation conditions are given as follows.

(i) period-doubling (flip or pitchfork) bifurcation

$$\mathrm{tr}(DP^{(n)}) + \det(DP^{(n)}) + 1 = 0, \tag{2.142}$$

(ii) saddle–node bifurcation

$$\det(DP^{(n)}) + 1 = \mathrm{tr}(DP^{(n)}), \tag{2.143}$$

(iii) Neimark bifurcation

$$\det(DP^{(n)}) = 1. \tag{2.144}$$

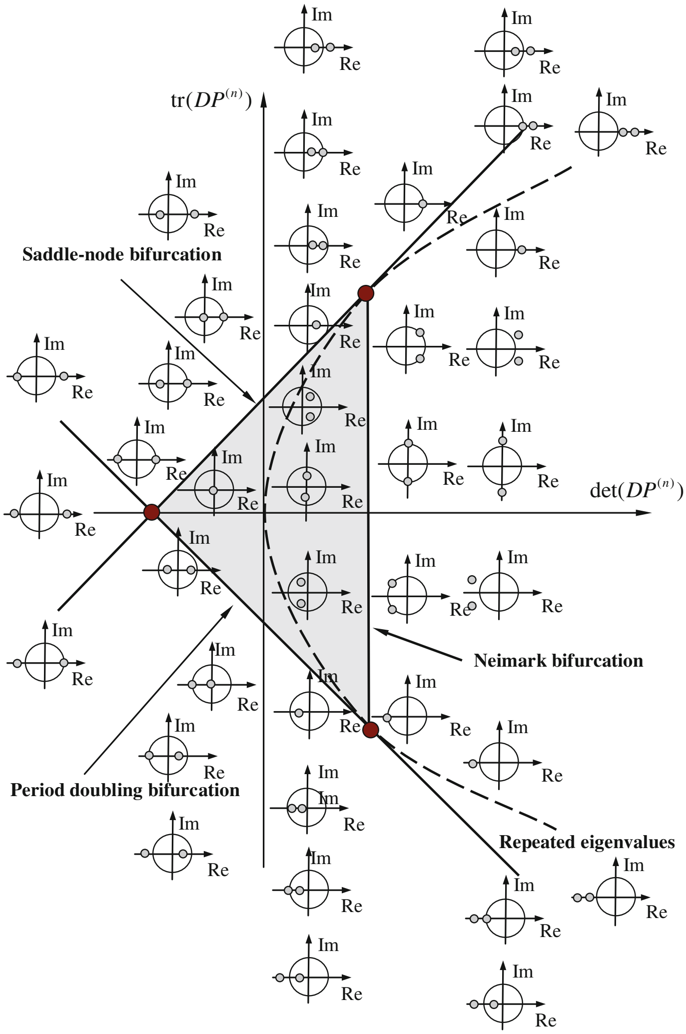

Fig. 2.8 Stability and bifurcation diagrams through the complex plane of eigenvalues for 2D discrete dynamical systems

The bifurcation and stability conditions for the solution of period-n for Eq. (2.141) are summarized in Fig. 2.8 with $\det(DP^{(n)}) = \det(DP^{(n)}(\mathbf{x}^*_{k(0)}, \mathbf{p}_0))$ and $\mathrm{tr}(DP^{(n)}) = \mathrm{tr}(DP^{(n)}(\mathbf{x}^*_{k(0)}, \mathbf{p}_0))$. The thick dashed lines are bifurcation lines. The stability of the fixed point is given by the eigenvalues in complex plane. The stability of the fixed point for higher-dimensional systems can be identified by using a naming of stability for linear dynamical systems in Luo (2011, 2012). The saddle–node bifurcation possesses stable saddle–node bifurcation (critical) and unstable saddle–node bifurcation (degenerate).

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