Chapter 2
Nonlinear Discrete Systems

In this chapter, a theory for nonlinear discrete systems is reviewed. The local and global theory of stability and bifurcation for nonlinear discrete systems is presented. The stability switching and bifurcation on specific eigenvectors of the linearized system at fixed points under a specific period are discussed. The higher-order singularity and stability for nonlinear discrete systems on the specific eigenvectors are also presented.

2.1 Definitions

Definition 2.1 For $\Omega_x \subseteq \mathbb{R}^n$ and $\Lambda \subseteq \mathbb{R}^m$ with $x \in \mathbb{Z}$, consider a vector function $f_z : \Omega_x \times \Lambda \rightarrow \Omega_x$ which is $C^r$ ($r \geq 1$)-continuous, and there is a discrete (or difference) equation in a form of

$$x_{k+1} = f_z(x_k, p_z) \text{ for } x_k, x_{k+1} \in \Omega_x, \quad k \in \mathbb{Z} \text{ and } p_z \in \Lambda$$

(2.1)

with an initial condition of $x_k = x_0$, the solution of Eq. (2.1) is given by

$$x_k = f_z(f_z(\ldots(f_z(x_0, p_z))))$$

(2.2)

for $x_k \in \Omega_x, \quad k \in \mathbb{Z} \text{ and } p_z \in \Lambda$.

(i) The difference equation with the initial condition is called a discrete dynamical system.

(ii) The vector function $f_z(x_k, p_z)$ is called a discrete vector field on $\Omega_x$.

(iii) The solution $x_k$ for each $k \in \mathbb{Z}$ is called a flow of discrete system.

(iv) The solution $x_k$ for all $k \in \mathbb{Z}$ on domain $\Omega_x$ is called the trajectory, phase curve, or orbit of the discrete dynamical system, which is defined as

$$\Gamma = \{ x_k | x_{k+1} = f_z(x_k, p_z) \text{ for } k \in \mathbb{Z} \text{ and } p_z \in \Lambda \} \subseteq \cup_z \Omega_x.$$  

(2.3)
The discrete dynamical system is called a uniform discrete system if
\[ x_{k+1} = f_x(x_k, p_x) = f(x_k, p) \quad \text{for} \ k \in \mathbb{Z} \quad \text{and} \ x_k \in \Omega_x. \] (2.4)

Otherwise, this discrete dynamical system is called a non-uniform discrete system.

**Definition 2.2** For the discrete dynamical system in Eq. (2.1), the relation between state \( x_k \) and state \( x_{k+1} \) \((k \in \mathbb{Z})\) is called a discrete map if
\[ P_x : x_k \xrightarrow{f_x} x_{k+1} \quad \text{and} \quad x_{k+1} = P_x x_k \] (2.5)
with the following properties:
\[ P_{(k;l)} : x_k \xrightarrow{f_x, f_x, \ldots, f_y} x_{k+l} \quad \text{and} \quad x_{k+l} = P_{x_l} \circ P_{x_{l-1}} \circ \cdots \circ P_{x_1} x_k \] (2.6)
where
\[ P_{(k;l)} = P_{x_l} \circ P_{x_{l-1}} \circ \cdots \circ P_{x_1}. \] (2.7)

If \( P_{x_l} = P_{x_{l-1}} = \cdots = P_{x_1} = P_x \), then
\[ P_{(x;l)} \equiv P_{x}^{(l)} = P_x \circ P_x \circ \cdots \circ P_x \] (2.8)
with
\[ P_{x}^{(n)} = P_x \circ P_{x}^{(n-1)} \quad \text{and} \quad P_{x}^{(0)} = I. \] (2.9)

The total map with \( l \)-different sub-maps is shown in Fig. 2.1. The map \( P_{x_k} \) with the relation function \( f_{x_k}(x_k \in \mathbb{Z}) \) is given by Eq. (2.5). The total map \( P_{(k;l)} \) is given in Eq. (2.7). The domains \( \Omega_{x_k}(x_k \in \mathbb{Z}) \) can fully overlap each other or can be completely separated without any intersection.

**Definition 2.3** For a vector function in \( f_x \in \mathcal{R}^n \), \( f_x : \mathcal{R}^n \to \mathcal{R}^n \). The operator norm of \( f_x \) is defined by
\[ \| f_x \| = \sum_{i=1}^{n} \max_{|x_i| \leq 1, p} |f_x(i)(x_k, p_x)|. \] (2.10)

For an \( n \times n \) matrix \( f_x(x_k, p_x) = A_{x} x_k \) and \( A_{x} = (a_{ij})_{n \times n} \), the corresponding norm is defined by
\[ \| A_{x} \| = \sum_{i,j=1}^{n} |a_{ij}|. \] (2.11)
Definition 2.4 For $\Omega_z \subseteq \mathbb{R}^m$ and $\Lambda \subseteq \mathbb{R}^m$ with $\alpha \in \mathbb{Z}$, the vector function $f_x(x_k, p_x)$ with $f_x : \Omega_x \times \Lambda \rightarrow \mathbb{R}^n$ is differentiable at $x_k \in \Omega_x$ if

$$\left. \frac{\partial f_x(x_k, p_x)}{\partial x_k} \right|_{(x_k, p)} = \lim_{\Delta x_k \to 0} \frac{f_x(x_k + \Delta x_k, p_x) - f_x(x_k, p_x)}{\Delta x_k}. \quad (2.12)$$

$\partial f_x/\partial x_k$ is called the spatial derivative of $f_x(x_k, p_x)$ at $x_k$, and the derivative is given by the Jacobian matrix

$$\frac{\partial f_x(x_k, p_x)}{\partial x_k} = \left[ \frac{\partial f_x(i)}{\partial x_k(j)} \right]_{n \times n}. \quad (2.13)$$

Definition 2.5 For $\Omega_z \subseteq \mathbb{R}^m$ and $\Lambda \subseteq \mathbb{R}^m$, consider a vector function $f(x_k, p)$ with $f : \Omega_z \times \Lambda \rightarrow \mathbb{R}^n$ where $x_k \in \Omega_x$ and $p \in \Lambda$ with $k \in \mathbb{Z}$. The vector function $f(x_k, p)$ is said to satisfy the Lipschitz condition if

$$\|f(y_k, p) - f(x_k, p)\| \leq L\|y_k - x_k\| \quad (2.14)$$

with $x_k, y_k \in \Omega_z$ and $L$ a constant. The constant $L$ is called the Lipschitz constant.

2.2 Fixed Points and Stability

Definition 2.6 Consider a discrete, dynamical system $x_{k+1} = f_x(x_k, p_x)$ in Eq. (2.4).

(i) A point $x^*_k \in \Omega_z$ is called a fixed point or period-1 solution of a discrete nonlinear system $x_{k+1} = f_x(x_k, p_x)$ under a map $P_x$ if for $x_{k+1} = x_k = x^*_k$

$$x^*_k = f_x(x^*_k, p). \quad (2.15)$$

The linearized system of the nonlinear discrete system $x_{k+1} = f_x(x_k, p_x)$ in Eq. (2.4) at the fixed point $x^*_k$ is given by
\[ y_{k+1} = DP_x(x_k^+, p)y_k = DF_x(x_k^+, p)y_k \]  

(2.16)

where

\[ y_k = x_k - x_k^+ \quad \text{and} \quad y_{k+1} = x_{k+1} - x_{k+1}^+. \]  

(2.17)

(ii) A set of points \( x_j^+ \in \Omega_j \) \( (x_j \in \mathbb{Z}) \) is called the fixed point set or period-1 point set of the total map \( P_{(k,l)} \) with \( l \)-different sub-maps in nonlinear discrete system of Eq. (2.5) if

\[ x_{k+j+1}^+ = f_{z_j}(x_{k+j}, p_{z_j}) \quad \text{for} \quad j \in \mathbb{Z}_+ \quad \text{and} \quad j' = \text{mod}(j, l) + 1; \]

\[ x_{k+\text{mod}(j,l)}^+ = x_k^+. \]  

(2.18)

The linearized equation of the total map \( P_{(k,l)} \) gives

\[ y_{k+j+1} = DP_{z_j}(x_{k+j}^+, p_{z_j})y_{k+j} = DF_{z_j}(x_{k+j}^+, p_{z_j})y_{k+j} \quad \text{with} \]

\[ y_{k+j+1} = x_{k+j+1} - x_{k+j+1}^+ \quad \text{and} \quad y_{k+j} = x_{k+j} - x_{k+j}^+ \quad \text{for} \]

\( j \in \mathbb{Z}_+ \quad \text{and} \quad j' = \text{mod}(j, l) + 1. \)  

(2.19)

The resultant equation for each individual map is

\[ y_{k+j+1} = DP_{(k,l)}(x_k^+, p)y_{k+j} \quad \text{for} \quad j \in \mathbb{Z}_+ \]  

(2.20)

where

\[
DP_{(k,n)}(x_k^+, p) = \prod_{j=1}^{l} DP_{z_j}(x_{k+j-1}^+, p) \\
= DP_{z_1}(x_{k+1-1}^+, p_{z_1}) \cdots \cdot DP_{z_l}(x_{k+l-1}^+, p_{z_l}) \cdot DP_{z_1}(x_k^+, p_{z_1}) \\
= DF_{(z_1)}(x_{k+1-1}^+, p_{z_1}) \cdots \cdot DF_{(z_l)}(x_{k+l-1}^+, p_{z_l}) \cdot DF_{(z_1)}(x_k^+, p_{z_1}).
\]  

(2.21)

The fixed point \( x_k^+ \) lies in the intersected set of two domains \( \Omega_k \) and \( \Omega_{k+1} \), as shown in Fig. 2.2. In the vicinity of the fixed point \( x_k^+ \), the incremental relations in the two domains \( \Omega_k \) and \( \Omega_{k+1} \) are different. In other words, setting \( y_k = x_k - x_k^+ \) and \( y_{k+1} = x_{k+1} - x_{k+1}^+ \), the corresponding linearization is generated as in Eq. (2.16). Similarly, the fixed point of the total map with \( n \)-different sub-maps requires the intersection set of two domains \( \Omega_k \) and \( \Omega_{k+n} \), and there are a set of equations to obtain the fixed points from Eq. (2.18). The other values of fixed points lie in different domains, i.e., \( x_j^+ \in \Omega_j (j = k+1, k+2, \ldots, k+n-1) \), as shown in Fig. 2.3.

The corresponding linearized equations are given in Eq. (2.19). From Eq. (2.20), the local characteristics of the total map can be discussed as a single map. Thus, the
dynamical characteristics for the fixed point of the single map will be discussed comprehensively, and the fixed points for resultant map are applicable. The results can be extended to any period-$m$ flows with $P^{(m)}$.

**Definition 2.7** Consider a discrete, nonlinear dynamical system $x_{k+1} = f(x_k, p)$ in Eq. (2.4) with a fixed point $x_k^*$. The linearized system of the discrete nonlinear system in the neighborhood of $x_k^*$ is $y_{k+1} = Df(x_k^*, p)y_k$ ($y_l = x_l - x_k^*$ and $l = k, k+1$) in Eq. (2.16). The matrix $Df(x_k^*, p)$ possesses $n_1$ real eigenvalues $|\lambda_j| < 1$ ($j \in N_1$), $n_2$ real eigenvalues $|\lambda_j| > 1$ ($j \in N_2$), $n_3$ real eigenvalues $\lambda_j = 1$ ($j \in N_3$), and $n_4$ real eigenvalues $\lambda_j = -1$ ($j \in N_4$). $N = \{1, 2, \ldots, n\}$ and $N_i = \{i_1, i_2, \ldots, i_{n_i}\} \cup \emptyset$ ($i = 1, 2, 3, 4$) with $i_m \in N$ ($m = 1, 2, \ldots, n_i$) and $\Sigma_{i=1}^{4} n_i = n$. $\Omega_i \subseteq N \cup \emptyset$, $\cup_{i=1}^{4} N_i = N$, $N_i \cap N_p = \emptyset$ ($p \neq i$). $\Omega_i = \emptyset$ if $n_i = 0$. The corresponding eigenvectors for contraction, expansion, invariance, and flip oscillation are $\{v_j\}$ ($j \in N_i$) ($i = 1, 2, 3, 4$), respectively. The stable, unstable, invariant, and flip subspaces of $y_{k+1} = Df(x_k^*, p)y_k$ in Eq. (2.16) are linear subspace spanned by $\{v_j\}$ ($j \in N_i$) ($i = 1, 2, 3, 4$), respectively, i.e.,
Herein, subscripts with eigenvectors \( l_f \) spanned by are used to denote the stable, unstable, center subspaces of the eigenvalues \( \lambda_j \) of the matrix \( f \) in the neighborhood of the fixed point \( x^*_k \). The linearized system of the discrete nonlinear system in the neighborhood of the fixed point \( x^*_k \) is given by \( \frac{d}{dt} x(t) = f(x(t), p) \). The matrix \( Df(x^*_k, p) \) has complex eigenvalues \( \alpha_j \pm i\beta_j \) with eigenvectors \( u_j = e^{\alpha_j t} v_j \). The base of vector is

\[
B = \{ u_1, v_1, \ldots, u_j, v_j, \ldots, u_n, v_n \}. 
\]
\[ \delta^s = \text{span} \left\{ (u, v) \mid r_j = \sqrt{x_j^2 + \beta_j^2} < 1, \quad (Df(x_k^*, p) - (x_j \pm i\beta_j)I)(u_j \pm iv_j) = 0, \quad j \in N_1 \subseteq \{1, 2, \ldots, n\} \cup \emptyset \right\} \]

\[ \delta^u = \text{span} \left\{ (u, v) \mid r_j = \sqrt{x_j^2 + \beta_j^2} > 1, \quad (Df(x_k^*, p) - (x_j \pm i\beta_j)I)(u_j \pm iv_j) = 0, \quad j \in N_2 \subseteq \{1, 2, \ldots, n\} \cup \emptyset \right\} \]

\[ \delta^c = \text{span} \left\{ (u, v) \mid r_j = \sqrt{x_j^2 + \beta_j^2} = 1, \quad (Df(x_k^*, p) - (x_j \pm i\beta_j)I)(u_j \pm iv_j) = 0, \quad j \in N_3 \subseteq \{1, 2, \ldots, n\} \cup \emptyset \right\} \]

**Definition 2.9** Consider a discrete, nonlinear dynamical system \( x_{k+1} = f(x_k, p) \) in Eq. (2.4) with a fixed point \( x_k^* \). The linearized system of the discrete nonlinear system in the neighborhood of \( x_k^* \) is \( y_{k+1} = Df(x_k^*, p)y_k \) (\( y_j = x_j - x_k^* \) and \( j = k, k + 1 \)) in Eq. (2.16). The fixed point or period-1 point is **hyperbolic** if no any eigenvalues of \( Df(x_k^*, p) \) are on the unit circle (i.e., \( |\lambda_j| \neq 1 \) for \( i = 1, 2, \ldots, n \)).

**Theorem 2.1** Consider a discrete, nonlinear dynamical system \( x_{k+1} = f(x_k, p) \) in Eq. (2.4) with a fixed point \( x_k^* \). The linearized system of the discrete nonlinear system in the neighborhood of \( x_k^* \) is \( y_{k+1} = Df(x_k^*, p)y_k \) (\( y_j = x_j - x_k^* \) and \( j = k, k + 1 \)) in Eq. (2.16). The eigenspace of \( Df(x_k^*, p) \) (i.e., \( \delta \subseteq \mathbb{R}^n \)) in the linearized dynamical system is expressed by direct sum of three subspaces

\[ \delta = \delta^s \oplus \delta^u \oplus \delta^c \]  

(2.27)

where \( \delta^s, \delta^u \) and \( \delta^c \) are the stable, unstable, and center subspaces, respectively.

**Proof** The proof can be referred to Luo (2011).

**Definition 2.10** Consider a discrete, nonlinear dynamical system \( x_{k+1} = f(x_k, p) \) in Eq. (2.4) with a fixed point \( x_k^* \). Suppose there is a neighborhood of the equilibrium \( x_k^* \) as \( U_k(x_k^*) \subseteq \Omega_k \), and in the neighborhood,

\[ \lim_{||y_k|| \to 0} \frac{||f(x_k^* + y_k, p) - Df(x_k^*, p)y_k||}{||y_k||} = 0, \]  

(2.28)

and

\[ y_{k+1} = Df(x_k^*, p)y_k. \]  

(2.29)
(i) A $C^r$ invariant manifold

$$
\mathcal{S}_{\text{loc}}(x_k, x_k^*) = \{ x_k \in U_k(x_k^*) \mid \lim_{j \to +\infty} x_{k+j} = x_k^* \} \quad \text{and} \quad x_{k+j} \in U_k(x_k^*) \text{ with } j \in \mathbb{Z}_+ \}
$$

is called the local stable invariant manifold of $x_k^*$, and the corresponding global stable manifold is defined as

$$
\mathcal{S}(x_k, x_k^*) = \bigcup_{j \in \mathbb{Z}} f(\mathcal{S}_{\text{loc}}(x_{k+j}, x_{k+j}^*)) = \bigcup_{j \in \mathbb{Z}} f^{(j)}(\mathcal{S}_{\text{loc}}(x_k, x_k^*)). \tag{2.31}
$$

(ii) A $C^r$ invariant manifold $\mathcal{U}_{\text{loc}}(x_k, x_k^*)$

$$
\mathcal{U}_{\text{loc}}(x_k, x_k^*) = \{ x_k \in U_k(x_k^*) \mid \lim_{j \to -\infty} x_{k+j} = x_k^* \} \quad \text{and} \quad x_{k+j} \in U_k(x_k^*) \text{ with } j \in \mathbb{Z}_- \}
$$

is called the local unstable manifold of $x_k^*$, and the corresponding global unstable manifold is defined as

$$
\mathcal{U}(x_k, x_k^*) = \bigcup_{j \in \mathbb{Z}} f(\mathcal{U}_{\text{loc}}(x_{k+j}, x_{k+j}^*)) = \bigcup_{j \in \mathbb{Z}} f^{(j)}(\mathcal{U}_{\text{loc}}(x_k, x_k^*)). \tag{2.33}
$$

(iii) A $C^{r-1}$ invariant manifold $\mathcal{C}_{\text{loc}}(x, x^*)$ is called the center manifold of $x^*$ if $\mathcal{C}_{\text{loc}}(x, x^*)$ possesses the same dimension of $\mathcal{C}^c$ for $x^* \in \mathcal{C}_{\text{loc}}(x, x^*)$, and the tangential space of $\mathcal{C}_{\text{loc}}(x, x^*)$ is identical to $\mathcal{C}^c$.

As in continuous dynamical systems, the stable and unstable manifolds are unique, but the center manifold is not unique. If the nonlinear vector field $f$ is $C^\infty$-continuous, then a $C^r$ center manifold can be found for any $r < \infty$.

**Theorem 2.2** Consider a discrete, nonlinear dynamical system $x_{k+1} = f(x_k, p)$ in Eq. (2.4) with a hyperbolic fixed point $x_k^*$. The corresponding solution is $x_{k+j} = f(x_{k+j-1}, p)$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the hyperbolic fixed point $x_k^*$ (i.e., $U_k(x_k^*) \subset \Omega_x$), and $f(x_k, p)$ is $C^r$ ($r \geq 1$)-continuous in $U_k(x_k^*)$. The linearized system is $y_{k+j+1} = Df(x_k^*, p)y_{k+j}$ ($y_{k+j} = x_{k+j} - x_k^*$) in $U_k(x_k^*)$. If the homeomorphism between the local invariant subspace $E(x_k^*) \subset U_k(x_k^*)$ and the eigenspace $\mathcal{C}$ of the linearized system exists with the condition in Eq. (2.28), the local invariant subspace is decomposed by

$$
E(x_k, x_k^*) = \mathcal{S}_{\text{loc}}(x_k, x_k^*) \oplus \mathcal{U}_{\text{loc}}(x_k, x_k^*). \tag{2.34}
$$

(a) The local stable invariant manifold $\mathcal{S}_{\text{loc}}(x, x^*)$ possesses the following properties:

(i) for $x_k^* \in \mathcal{S}_{\text{loc}}(x_k, x_k^*)$, $\mathcal{S}_{\text{loc}}(x_k, x_k^*)$ possesses the same dimension of $\mathcal{C}^s$ and the tangential space of $\mathcal{S}_{\text{loc}}(x_k, x_k^*)$ is identical to $\mathcal{C}^s$;
2.2 Fixed Points and Stability

(ii) for \(x_k \in S_{loc}(x_k, x^*_k)\), \(x_{k+j} \in S_{loc}(x_k, x^*_k)\) and \(\lim_{j \to \infty} x_{k+j} = x^*_k\) for all \(j \in \mathbb{Z}_+\).

(iii) For \(x_k \not\in S_{loc}(x_k, x^*_k)\), \(||x_{k+j} - x^*_k|| \geq \delta\) for \(\delta > 0\) with \(j, j_1 \in \mathbb{Z}_+\) and \(j \geq j_1 \geq 0\).

(b) The local unstable invariant manifold \(U_{loc}(x_k, x^*_k)\) possesses the following properties:

(i) for \(x^*_k \in U_{loc}(x_k, x^*_k)\), \(U_{loc}(x_k, x^*_k)\) possesses the same dimension of \(\mathcal{E}^u\) and the tangential space of \(U_{loc}(x_k, x^*_k)\) is identical to \(\mathcal{E}^u\);

(ii) for \(x_k \in U_{loc}(x_k, x^*_k)\), \(x_{k+j} \in U_{loc}(x_k, x^*_k)\) and \(\lim_{j \to -\infty} x_{k+j} = x^*_k\) for all \(j \in \mathbb{Z}_-\);

(iii) for \(x_k \not\in U_{loc}(x_k, x^*_k)\), \(||x_{k+j} - x^*_k|| \geq \delta\) for \(\delta > 0\) with \(j, j_1 \in \mathbb{Z}_-\) and \(j \leq j_1 \leq 0\).

Proof See Nitecki (1971). □

Theorem 2.3 Consider a discrete, nonlinear dynamical system \(x_{k+1} = f(x_k, p)\) in Eq. (2.4) with a fixed point \(x^*_k\). The corresponding solution is \(x_{k+j} = f(x_{k+j-1}, p)\) with \(j \in \mathbb{Z}\). Suppose there is a neighborhood of the fixed point \(x^*_k\) (i.e., \(U_k(x^*_k) \subset \Omega_x\)), and \(f(x_k, p)\) is \(C^r\) \((r \geq 1)\)-continuous in \(U_k(x^*_k)\). The linearized system is \(y_{k+1} = Df(x^*_k, p)y_k + (y_{k+1} - x^*_k)\) in \(U_k(x^*_k)\). If the homeomorphism between the local invariant subspace \(E(x^*_k) \subset U_k(x^*_k)\) and the eigenspace \(\mathcal{E}\) of the linearized system exists with the condition in Eq. (2.28), in addition to the local stable and unstable invariant manifolds, there is a \(C^{r-1}\) center manifold \(E_{loc}(x_k, x^*_k)\). The center manifold possesses the same dimension of \(\mathcal{E}^c\) for \(x^*_k \in E_{loc}(x_k, x^*_k)\), and the tangential space of \(E_{loc}(x, x^*)\) is identical to \(\mathcal{E}^c\). Thus, the local invariant subspace is decomposed by

\[
E(x_k, x^*_k) = S_{loc}(x_k, x^*_k) \oplus U_{loc}(x_k, x^*_k) \oplus E_{loc}(x_k, x^*_k).
\]

(2.35)


Definition 2.11 Consider a discrete, nonlinear dynamical system \(x_{k+1} = f(x_k, p)\) in Eq. (2.4) on domain \(\Omega_x \subset \mathbb{R}^n\). Suppose there is a metric space \((\Omega_x, \rho)\), then the map \(P\) under the vector function \(f(x_k, p)\) is called the contraction map if

\[
\rho(x^{(1)}_{k+1}, x^{(2)}_{k+1}) = \rho(f(x^{(1)}_k, p), f(x^{(2)}_k, p)) \leq \lambda \rho(x^{(1)}_k, x^{(2)}_k)
\]

(2.36)

for \(\lambda \in (0, 1)\) and \(x^{(1)}_k, x^{(2)}_k \in \Omega_x\) with \(\rho(x^{(1)}_k, x^{(2)}_k) = ||x^{(1)}_k - x^{(2)}_k||\).

Theorem 2.4 Consider a discrete, nonlinear dynamical system \(x_{k+1} = f(x_k, p)\) in Eq. (2.4) on domain \(\Omega_x \subset \mathbb{R}^n\). Suppose there is a metric space \((\Omega_x, \rho)\), if the map \(P\) under the vector function \(f(x_k, p)\) is the contraction map, then there is a unique fixed point \(x^*_k\) which is globally stable.

Proof The proof can be referred to Luo (2011). □
Definition 2.12 Consider a discrete, nonlinear dynamical system \( x_{k+1} = f(x_k, p) \) in Eq. (2.4) with a fixed point \( x_k \). The corresponding solution is given by \( x_{k+j} = f(x_{k+j-1}, p) \) with \( j \in \mathbb{Z} \). Suppose there is a neighborhood of the fixed point \( x_k \) (i.e., \( U_k(x_k) \subset \Omega_x \)), and \( f(x_k, p) \) is \( C^r \) \((r \geq 1)\)-continuous in \( U_k(x_k) \). The linearized system is \( y_{k+j+1} = Df(x_k, p)y_{k+j} \) \((y_{k+j} = x_{k+j} - x_k)\) in \( U_k(x_k) \). Consider a real eigenvalue \( \lambda_i \) of matrix \( Df(x_k, p) \) \((i \in \mathbb{N} = \{1, 2, \ldots, n\})\) and there is a corresponding eigenvector \( v_i \). On the invariant eigenvector \( v_i \), consider \( y_k^{(i)} = c_k^{(i)}v_i \) and \( y_{k+1}^{(i)} = c_{k+1}^{(i)}v_i = \lambda_ic_k^{(i)}v_i \), and thus, \( c_{k+1}^{(i)} = \lambda_ic_k^{(i)} \).

(i) \( x_k^{(i)} \) on the direction \( v_i \) is stable if

\[
\lim_{k \to \infty} |c_k^{(i)}| = \lim_{k \to \infty} |(\lambda_i)^k| |c_0^{(i)}| = 0 \quad \text{for} \quad |\lambda_i| < 1. \tag{2.37}
\]

(ii) \( x_k^{(i)} \) on the direction \( v_i \) is unstable if

\[
\lim_{k \to \infty} |c_k^{(i)}| = \lim_{k \to \infty} |(\lambda_i)^k| |c_0^{(i)}| = \infty \quad \text{for} \quad |\lambda_i| > 1. \tag{2.38}
\]

(iii) \( x_k^{(i)} \) on the direction \( v_i \) is invariant if

\[
\lim_{k \to \infty} c_k^{(i)} = \lim_{k \to \infty} (\lambda_i)^kc_0^{(i)} = c_0^{(i)} \quad \text{for} \quad \lambda_i = 1. \tag{2.39}
\]

(iv) \( x_k^{(i)} \) on the direction \( v_i \) is flipped if

\[
\lim_{2k \to \infty} c_k^{(i)} = \lim_{2k \to \infty} (\lambda_i)^{2k}c_0^{(i)} = c_0^{(i)} \quad \text{for} \quad \lambda_i = 1.
\]
\[
\lim_{2k+1 \to \infty} c_k^{(i)} = \lim_{2k+1 \to \infty} (\lambda_i)^{2k+1}c_0^{(i)} = -c_0^{(i)} \quad \text{for} \quad \lambda_i = -1. \tag{2.40}
\]

(v) \( x_k^{(i)} \) on the direction \( v_i \) is degenerate if

\[
c_k^{(i)} = (\lambda_i)^kc_0^{(i)} = 0 \quad \text{for} \quad \lambda_i = 0. \tag{2.41}
\]

Definition 2.13 Consider a discrete, nonlinear dynamical system \( x_{k+1} = f(x_k, p) \) in Eq. (2.4) with a fixed point \( x_k \). The corresponding solution is given by \( x_{k+j} = f(x_{k+j-1}, p) \) with \( j \in \mathbb{Z} \). Suppose there is a neighborhood of the fixed point \( x_k \) (i.e., \( U_k(x_k) \subset \Omega_x \)), and \( f(x_k, p) \) is \( C^r \) \((r \geq 1)\)-continuous in \( U_k(x_k) \). Consider a pair of complex eigenvalues \( \alpha_i \pm i\beta_i \) of matrix \( Df(x_k, p) \) \((i \in \mathbb{N} = \{1, 2, \ldots, n\}, i = \sqrt{-1})\) and there is a corresponding eigenvector \( u_i \pm iv_i \). On the invariant plane of \((u_k^{(i)}, v_k^{(i)}) = (u_i, v_i)\), consider \( x_k^{(i)} = x_{k+}^{(i)} + x_{k-}^{(i)} \) with
Thus, \( \mathbf{c}_{k}^{(i)} = (c_{k}^{(i)}, d_{k}^{(i)})^{T} \) with
\[
\mathbf{c}_{k+1}^{(i)} = \mathbf{E}_{i} \mathbf{c}_{k}^{(i)} = r_{i} \mathbf{R}_{i} \mathbf{c}_{k}^{(i)}
\] (2.43)

where
\[
\mathbf{E}_{i} = \begin{bmatrix} \alpha_{i} & \beta_{i} \\ -\beta_{i} & \alpha_{i} \end{bmatrix} \quad \text{and} \quad \mathbf{R}_{i} = \begin{bmatrix} \cos \theta_{i} & \sin \theta_{i} \\ -\sin \theta_{i} & \cos \theta_{i} \end{bmatrix},
\]
\[
r_{i} = \sqrt{\alpha_{i}^{2} + \beta_{i}^{2}}, \quad \cos \theta_{i} = \frac{\alpha_{i}}{r_{i}} \quad \text{and} \quad \sin \theta_{i} = \frac{\beta_{i}}{r_{i}};
\]
and
\[
\mathbf{E}_{i}^{k} = \begin{bmatrix} \alpha_{i} & \beta_{i} \\ -\beta_{i} & \alpha_{i} \end{bmatrix}^{k} \quad \text{and} \quad \mathbf{R}_{i}^{k} = \begin{bmatrix} \cos k\theta_{i} & \sin k\theta_{i} \\ -\sin k\theta_{i} & \cos k\theta_{i} \end{bmatrix}.
\] (2.45)

(i) \( \mathbf{x}_{k}^{(i)} \) on the plane of \((\mathbf{u}_{i}, \mathbf{v}_{i})\) is spirally stable if
\[
\lim_{k \to \infty} \| \mathbf{c}_{k}^{(i)} \| = \lim_{k \to \infty} r_{i}^{k} \| \mathbf{R}_{i}^{k} \| \times \| \mathbf{c}_{0}^{(i)} \| = 0 \quad \text{for} \quad r_{i} = |\lambda_{i}| < 1.
\] (2.46)

(ii) \( \mathbf{x}_{k}^{(i)} \) on the plane of \((\mathbf{u}_{i}, \mathbf{v}_{i})\) is spirally unstable if
\[
\lim_{k \to \infty} \| \mathbf{c}_{k}^{(i)} \| = \lim_{k \to \infty} r_{i}^{k} \| \mathbf{R}_{i}^{k} \| \times \| \mathbf{c}_{0}^{(i)} \| = \infty \quad \text{for} \quad r_{i} = |\lambda_{i}| > 1.
\] (2.47)

(iii) \( \mathbf{x}_{k}^{(i)} \) on the plane of \((\mathbf{u}_{i}, \mathbf{v}_{i})\) is on the invariant circles if
\[
\| \mathbf{c}_{k}^{(i)} \| = r_{i}^{k} \| \mathbf{R}_{i}^{k} \| \times \| \mathbf{c}_{0}^{(i)} \| = \| \mathbf{c}_{0}^{(i)} \| \quad \text{for} \quad r_{i} = |\lambda_{i}| = 1.
\] (2.48)

(iv) \( \mathbf{x}_{k}^{(i)} \) on the plane of \((\mathbf{u}_{i}, \mathbf{v}_{i})\) is degenerate in the direction of \(\mathbf{u}_{i}\) if \(\beta_{i} = 0\).

**Definition 2.14** Consider a discrete, nonlinear dynamical system \( \mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_{k}, p) \) in Eq. (2.4) with a fixed point \( \mathbf{x}^{*} \). The corresponding solution is given by \( \mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, p) \) with \( j \in \mathbb{Z} \). Suppose there is a neighborhood of the fixed point \( \mathbf{x}^{*} \) (i.e., \( U_{k}(\mathbf{x}^{*}) \subset \Omega_{k} \)), and \( \mathbf{f}(\mathbf{x}_{k}, p) \) is \( C^{r} \) \((r \geq 1)\)-continuous in \( U_{k}(\mathbf{x}^{*}) \) with Eq. (2.28). The linearized system is \( \mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}^{*}, p)\mathbf{y}_{k+j} \) \((\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}^{*})\) in \( U_{k}(\mathbf{x}^{*}) \). The matrix \( D\mathbf{f}(\mathbf{x}^{*}, p) \) possesses \( n \) eigenvalues \( \lambda_{i} \) \((i = 1, 2, \ldots, n)\).

(i) The fixed point \( \mathbf{x}^{*} \) is called a hyperbolic point if \( |\lambda_{i}| \neq 1 \) \((i = 1, 2, \ldots, n)\).

(ii) The fixed point \( \mathbf{x}^{*} \) is called a sink if \( |\lambda_{i}| < 1 \) \((i = 1, 2, \ldots, n)\).
(iii) The fixed point \( x_k^* \) is called a source if \( |\lambda_i| > 1 \) (\( i = 1, 2, \ldots, n \)).

(iv) The fixed point \( x_k^* \) is called a center if \( |\lambda_i| = 1 \) (\( i = 1, 2, \ldots, n \)) with distinct eigenvalues.

**Definition 2.15** Consider a discrete, nonlinear dynamical system \( x_{k+1} = f(x_k, p) \) in Eq. (2.4) with a fixed point \( x_k^* \). The corresponding solution is given by \( x_{k+j} = f(x_{k+j-1}, p) \) with \( j \in \mathbb{Z} \). Suppose there is a neighborhood of the fixed point \( x_k^* \) (i.e., \( U_k(x_k^*) \subset \Omega_{x_k^*} \)), and \( f(x_k, p) \) is \( C^r \) \((r \geq 1)\)-continuous in \( U_k(x_k^*) \) with Eq. (2.28). The linearized system is \( y_{k+j+1} = Df(x_k^*, p)y_{k+j} \) (\( y_{k+j} = x_{k+j} - x_k^* \)) in \( U_k(x_k^*) \). The matrix \( Df(x_k^*, p) \) possesses \( n \) eigenvalues \( \lambda_i \) (\( i = 1, 2, \ldots, n \)).

(i) The fixed point \( x_k^* \) is called a stable node if \( |\lambda_i| < 1 \) (\( i = 1, 2, \ldots, n \)).

(ii) The fixed point \( x_k^* \) is called an unstable node if \( |\lambda_i| > 1 \) (\( i = 1, 2, \ldots, n \)).

(iii) The fixed point \( x_k^* \) is called an \( (l_1 : l_2) \)-saddle if at least one \( |\lambda_i| > 1 \) (\( i \in L_1 \subset \{1, 2, \ldots, n\} \)) and the other \( |\lambda_j| < 1 \) (\( j \in L_2 \subset \{1, 2, \ldots, n\} \)) with \( L_1 \cup L_2 = \{1, 2, \ldots, n\} \) and \( L_1 \cap L_2 = \emptyset \). \( l_1 = \text{span}(L_1) \) and \( l_2 = \text{span}(L_2) \).

(iv) The fixed point \( x_k^* \) is called an \( l \)-th order degenerate case if \( \lambda_i = 0 \) (\( i \in L \subset \{1, 2, \ldots, n\} \)).

**Definition 2.16** Consider a discrete, nonlinear dynamical system \( x_{k+1} = f(x_k, p) \) in Eq. (2.4) with a fixed point \( x_k^* \). The corresponding solution is given by \( x_{k+j} = f(x_{k+j-1}, p) \) with \( j \in \mathbb{Z} \). Suppose there is a neighborhood of the fixed point \( x_k^* \) (i.e., \( U_k(x_k^*) \subset \Omega_{x_k^*} \)), and \( f(x_k, p) \) is \( C^r \) \((r \geq 1)\)-continuous in \( U_k(x_k^*) \) with Eq. (2.28). The linearized system is \( y_{k+j+1} = Df(x_k^*, p)y_{k+j} \) (\( y_{k+j} = x_{k+j} - x_k^* \)) in \( U_k(x_k^*) \). The matrix \( Df(x_k^*, p) \) possesses \( n \)-pairs of complex eigenvalues \( \lambda_i \) (\( i = 1, 2, \ldots, n \)).

(i) The fixed point \( x_k^* \) is called a spiral sink if \( |\lambda_i| < 1 \) (\( i = 1, 2, \ldots, n \)) and \( \text{Im} \lambda_j \neq 0 \) (\( j \in \{1, 2, \ldots, n\} \)).

(ii) The fixed point \( x_k^* \) is called a spiral source if \( |\lambda_i| > 1 \) (\( i = 1, 2, \ldots, n \)) with \( \text{Im} \lambda_j \neq 0 \) (\( j \in \{1, 2, \ldots, n\} \)).

(iii) The fixed point \( x_k^* \) is called a center if \( |\lambda_i| = 1 \) with distinct \( \text{Im} \lambda_i \neq 0 \) (\( i = 1, 2, \ldots, n \)).

The generalized stability and bifurcation of flows in linearized, nonlinear dynamical systems in Eq. (2.4) will be discussed as follows.

**Definition 2.17** Consider a discrete, nonlinear dynamical system \( x_{k+1} = f(x_k, p) \) in Eq. (2.4) with a fixed point \( x_k^* \). The corresponding solution is given by \( x_{k+s} = f(x_{k+s-1}, p) \) with \( s \in \mathbb{Z} \). Suppose there is a neighborhood of the fixed point \( x_k^* \) (i.e., \( U_k(x_k^*) \subset \Omega_{x_k^*} \)), and \( f(x_k, p) \) is \( C^r \) \((r \geq 1)\)-continuous in \( U_k(x_k^*) \) with Eq. (2.28). The linearized system is \( y_{k+s+1} = Df(x_k^*, p)y_{k+s} \) (\( y_{k+s} = x_{k+s} - x_k^* \)) in \( U_k(x_k^*) \). The matrix \( Df(x_k^*, p) \) possesses \( n \) eigenvalues \( \lambda_i \) (\( i = 1, 2, \ldots, n \)). Set \( N = \{1, 2, \ldots, m, m+1, \ldots, (n+m)/2\} \), \( N_j = \{j_1, j_2, \ldots, j_m\} \cup \emptyset \) with \( j_p \in N \) (\( p = 1, 2, \ldots, n \)), \( j_1 < j_2 < \cdots < j_m < j_1 = 1, 2, \ldots, 7 \), \( \sum_{j=1}^m n_j = m \) and \( 2\sum_{j=1}^m n_j = n - m \). \( \cup_{j_1} j_n = N \) with \( N_j \cap N_l = \emptyset \) (\( l \neq j \)). \( N_j = \emptyset \) if \( n_j = 0 \). \( N_1 = N_2^m \cup N_2^n \) (\( z = 1, 2 \)) and \( N_1^m \cap N_2^n = \emptyset \) with \( n_1^m + n_2^m = n_2^m \).
where superscripts “m” and “o” represent monotonic and oscillatory evolutions. The matrix $Df(x^{*}, p)$ possesses $n_1$-stable, $n_2$-unstable, $n_3$-invariant, and $n_4$-flip real eigenvectors plus $n_5$-stable, $n_6$-unstable, and $n_7$-center pairs of complex eigenvectors. Without repeated complex eigenvalues of $|\lambda_i| = 1$ ($i \in N_3 \cup N_4 \cup N_7$), an iterative response of $x_{k+1} = f(x_k, p)$ is an $((n_1^m, n_1^o) : [n_2^m, n_2^o] : [n_3; \kappa_3] : [n_4; \kappa_4] : n_5 : n_6 : n_7)$ flow in the neighborhood of the fixed point $x^{*}_i$. With repeated complex eigenvalues of $|\lambda_i| = 1$ ($i \in N_3 \cup N_4 \cup N_7$), an iterative response of $x_{k+1} = f(x_k, p)$ is an $((n_1^m, n_1^o) : [n_2^m, n_2^o] : [n_3; \kappa_3] : [n_4; \kappa_4] : n_5 : n_6 : n_7)$ flow in the neighborhood of the fixed point $x^{*}_i$, where $\kappa_j \in \{\mathcal{O}, m_j\}$ ($j = 3, 4, 7$). The meanings of notations in the aforementioned structures are defined as follows:

(i) $[n_1^m, n_1^o]$ represents $n_1$-sinks with $n_1^m$-monotonic convergence and $n_1^o$-oscillatory convergence among $n_1$-directions of $v_i$ if $|\lambda_i| < 1$ ($i \in N_1$ and $1 \leq n_1 \leq n$) with distinct or repeated eigenvalues.

(ii) $[n_2^m, n_2^o]$ represents $n_2$-sources with $n_2^m$-monotonic divergence and $n_2^o$-oscillatory divergence among $n_2$-directions of $v_i$ if $|\lambda_i| > 1$ ($i \in N_2$ and $1 \leq n_2 \leq n$) with distinct or repeated eigenvalues.

(iii) $n_3 = 1$ represents an invariant center on 1-direction of $v_i$ if $\lambda_i = 1$ ($i \in N_3$ and $n_3 = 1$).

(iv) $n_4 = 1$ represents a flip center on 1-direction of $v_i$ if $\lambda_i = -1$ ($i \in N_4$ and $n_4 = 1$).

(v) $n_5$ represents $n_5$-spiral sinks on $n_5$-pairs of $(u_i, v_i)$ if $|\lambda_i| < 1$ and $\text{Im} \lambda_i \neq 0$ ($i \in N_5$ and $1 \leq n_5 \leq n$) with distinct or repeated eigenvalues.

(vi) $n_6$ represents $n_6$-spiral sources on $n_6$-directions of $(u_i, v_i)$ if $|\lambda_i| > 1$ and $\text{Im} \lambda_i \neq 0$ ($i \in N_6$ and $1 \leq n_6 \leq n$) with distinct or repeated eigenvalues.

(vii) $n_7$ represents $n_7$-invariant centers on $n_7$-pairs of $(u_i, v_i)$ if $|\lambda_i| = 1$ and $\text{Im} \lambda_i \neq 0$ ($i \in N_7$ and $1 \leq n_7 \leq n$) with distinct eigenvalues.

(viii) $\mathcal{O}$ represents none if $n_j = 0$ ($j \in \{1, 2, \ldots, 7\}$).

(ix) $[n_3; \kappa_3]$ represents $(n_3 - \kappa_3)$-invariant centers on $(n_3 - \kappa_3)$-directions of $v_i$ ($i \in N_3$) and $\kappa_3$-sources in $\kappa_3$-directions of $v_j$ ($j_3 \in N_3$ and $j_3 \neq i_3$) if $\lambda_i = 1$ ($i \in N_3$ and $n_3 \leq n$) with the $(\kappa_3 + 1)$th-order nilpotent matrix $N_3^{\kappa_3+1} = 0$ ($0 < \kappa_3 \leq n_3 - 1$).

(x) $[n_3; \mathcal{O}]$ represents $n_3$-invariant centers on $n_3$-directions of $v_i$ if $\lambda_i = 1$ ($i \in N_3$ and $1 < n_3 \leq n$) with a nilpotent matrix $N_3 = 0$.

(xi) $[n_4; \kappa_4]$ represents $(n_4 - \kappa_4)$-flip oscillatory centers on $(n_4 - \kappa_4)$-directions of $v_i$ ($i_4 \in N_4$) and $\kappa_4$-sources in $\kappa_4$-directions of $v_j$ ($j_4 \in N_4$ and $j_4 \neq i_4$) if $\lambda_i = -1$ ($i \in N_4$ and $n_4 \leq n$) with the $(\kappa_4 + 1)$th-order nilpotent matrix $N_4^{\kappa_4+1} = 0$ ($0 < \kappa_4 \leq n_4 - 1$).

(xii) $[n_4; \mathcal{O}]$ represents $n_4$-flip oscillatory centers on $n_4$-directions of $v_i$ if $\lambda_i = -1$ ($i \in N_4$ and $1 < n_4 \leq n$) with a nilpotent matrix $N_4 = 0$.

(xiii) $[n_7; l; \kappa_7]$ represents $(n_7 - \kappa_7)$-invariant centers on $(n_7 - \kappa_7)$-pairs of $(u_i, v_i)$ ($i_7 \in N_7$) and $\kappa_7$-sources on $\kappa_7$-pairs of $(u_j, v_j)$ ($j_7 \in N_7$ and $j_7 \neq i_7$) if


\[ |\lambda_i| = 1 \text{ and } \text{Im}\lambda_i \neq 0 \ (i \in N_7 \text{ and } n_7 \leq n) \text{ for } (l + 1)\text{-pairs of repeated eigenvalues with the } (k_7 + 1)\text{-order nilpotent matrix } N_7^{k_7+1} = 0 \ (0 < k_7 \leq l), \]

(xiv) \[ [n_7, l; \Theta] \text{ represents } n_7\text{-invariant centers on } n_7\text{-pairs of } (u_i, v_i) \text{ if } |\lambda_i| = 1 \text{ and } \text{Im}\lambda_i \neq 0 \ (i \in N_7 \text{ and } 1 \leq n_7 \leq n) \text{ for } (l + 1)\text{-pairs of repeated eigenvalues with a nilpotent matrix } N_7 = 0. \]

## 2.3 Stability Switching Theory

To extend the idea of Definitions 2.11 and 2.12, a new function will be defined to determine the stability and the stability state switching.

**Definition 2.18** Consider a discrete, nonlinear dynamical system \( x_{k+1} = f(x_k, p) \in \mathbb{R}^n \) in Eq. (2.4) with a fixed point \( x^*_k \). The corresponding solution is given by \( x_{k+j} = f(x_{k+j-1}, p) \) with \( j \in \mathbb{Z} \). Suppose there is a neighborhood of the fixed point \( x^*_k \) (i.e., \( U_k(x^*_k) \subset \Omega \)), and \( f(x_k, p) \) is \( C'(r \geq 1)\text{-continuous in } U_k(x^*_k) \) with Eq. (2.28). The linearized system is \( y_{k+j+1} = Df(x^*_k, p)y_{k+j} \) \( (y_{k+j} = x_{k+j} - x^*_k) \) in \( U_k(x^*_k) \) and there are \( n \) linearly independent vectors \( v_i \ (i = 1, 2, \ldots, n) \). For a perturbation of fixed point \( y_k = x_k - x^*_k \), let \( y^{(i)}_k = c^{(i)}_k v_i \) and \( y^{(i)}_{k+1} = c^{(i)}_{k+1} v_i \),

\[
s^{(i)}_k = v_i^T \cdot y_k = v_i^T \cdot (x_k - x^*_k) \tag{2.49}
\]

where \( s^{(i)}_k = c^{(i)}_k ||v_i||^2 \). Define the following functions

\[
G_i(x_k, p) = v_i^T \cdot (f(x_k, p) - x^*_k) \tag{2.50}
\]

and

\[
G^{(1)}_{s^{(i)}_k}(x_k, p) = v_i^T \cdot D_{s^{(i)}_k}f(x_k(s^{(i)}_k), p) = v_i^T \cdot D_{x_i}f(x_k(s^{(i)}_k), p)\partial_{s^{(i)}_k}x_k\partial_{s^{(i)}_k}c^{(i)}_k 
\]

\[
= v_i^T \cdot D_xf(x_k(s^{(i)}_k), p)v_i||v_i||^{-2} \tag{2.51}
\]

\[
G^{(m)}_{s^{(i)}_k}(x_k, p) = v_i^T \cdot D_{s^{(i)}_k}^{(m)}f(x_k(s^{(i)}_k), p) = v_i^T \cdot D_{x_i}^{(m)}(D_{s^{(i)}_k}^{(m-1)}f(x_k(s^{(i)}_k), p)) \tag{2.52}
\]

where \( D_{s^{(i)}_k}(\cdot) = \partial(\cdot)/\partial s^{(i)}_k \) and \( D_{s^{(i)}_k}^{(m)}(\cdot) = D_{s^{(i)}_k}^{(m-1)}(\cdot) \).

**Definition 2.19** Consider a discrete, nonlinear dynamical system \( x_{k+1} = f(x_k, p) \in \mathbb{R}^n \) in Eq. (2.4) with a fixed point \( x^*_k \). The corresponding solution is given by \( x_{k+j} = f(x_{k+j-1}, p) \) with \( j \in \mathbb{Z} \). Suppose there is a neighborhood of the fixed point \( x^*_k \) (i.e., \( U_k(x^*_k) \subset \Omega \)), and \( f(x_k, p) \) is \( C' \ (r \geq 1)\text{-continuous in } U_k(x^*_k) \) with Eq. (2.28). The linearized system is \( y_{k+j+1} = Df(x^*_k, p)y_{k+j} \) \( (y_{k+j} = x_{k+j} - x^*_k) \) in
2.3 Stability Switching Theory

The solid curve is

\[ \text{The dashed and dotted lines are used for two lines} \]

\[ v^{(i)} \]

\[ \text{presented in Fig. 2.4d. This illustration can be easily observed from} \]

the stability of

\[ (\text{sink}) \]

\[ \text{for} \]

\[ k \]

\[ \text{is presented in Fig. 2.4b. The dashed and dotted lines, respectively.} \]

\[ \text{The iterative responses approach the} \]

vertical solid line is for a line with in

\[ \text{shaded regions are stable. The horizontal solid line is for a degenerate case.} \]

\[ \text{The circular symbol is fixed point. The monotonically unstable (source) is} \]

\[ \text{for} \]

\[ 1 \]

\[ \text{and there are} n \text{ linearly independent vectors} v_i \ (i = 1, 2, \ldots, n). \]

\[ \text{For a perturbation of fixed point} y_k = x_k - x_k^* \], let \[ y_k^{(i)} = \epsilon_k^{(i)} v_i \]

and \[ y_{k+1} = y_k^{(i)} + v_i. \]

(i) \[ x_{k+j} \ (j \in \mathbb{Z}) \] at fixed point \( x_k^* \) on the direction \( v_i \) is stable if

\[ |v_i^T \cdot (x_{k+1} - x_k^*)| < |v_i^T \cdot (x_k - x_k^*)| \]  \hspace{1cm} (253)

for \( x_k \in U(x_k^*) \subseteq \Omega_x \). The fixed point \( x_k^* \) is called a sink (or stable node) on the direction \( v_i \).

(ii) \[ x_{k+j} \ (j \in \mathbb{Z}) \] at fixed point \( x_k^* \) on the direction \( v_i \) is unstable if

\[ |v_i^T \cdot (x_{k+1} - x_k^*)| > |v_i^T \cdot (x_k - x_k^*)| \]  \hspace{1cm} (254)

for \( x_k \in U(x_k^*) \subseteq \Omega_x \). The fixed point \( x_k^* \) is called a source (or unstable node) on the direction \( v_i \).

(iii) \[ x_{k+j} \ (j \in \mathbb{Z}) \] at fixed point \( x_k^* \) on the direction \( v_i \) is invariant if

\[ v_i^T \cdot (x_{k+1} - x_k^*) = v_i^T \cdot (x_k - x_k^*) \]  \hspace{1cm} (255)

for \( x_k \in U(x_k^*) \subseteq \Omega_x \). The fixed point \( x_k^* \) is called to be degenerate on the direction \( v_i \).

(iv) \[ x_{k+j}^{(i)} \ (j \in \mathbb{Z}) \] at fixed point \( x_k^* \) on the direction \( v_i \) is symmetrically flipped if

\[ (v) \ v_i^T \cdot (x_{k+1} - x_k^*) = -v_i^T \cdot (x_k - x_k^*) \]  \hspace{1cm} (256)

for \( x_k \in U(x_k^*) \subseteq \Omega_x \). The fixed point \( x_k^* \) is called to be degenerate on the direction \( v_i \).

The stability of fixed points for a specific eigenvector is presented in Fig. 2.4.

The shaded regions are stable. The horizontal solid line is for a degenerate case. The vertical solid line is for a line with infinite slope. The monotonically stable node (sink) is presented in Fig. 2.4a. From the fixed point \( x_k^* \), let \( y_k = x_k - x_k^* \) and

\[ y_{k+1} = x_{k+1} - x_k^*. \]

\[ v_i^T \cdot x_k = v_i^T \cdot x_{k+1} \] and \[ v_i^T \cdot y_{k+1} = -v_i^T \cdot y_k \]

are represented by dashed and dotted lines, respectively. The iterative responses approach the fixed point. However, the monotonically unstable (source) is presented in Fig. 2.4b. The iterative responses go away from the fixed point. Similarly, the oscillatory stable node (sink) after iteration with a flip \( v_i^T \cdot y_k = -v_i^T \cdot y_{k+1} \) is presented in Fig. 2.4c. The dashed and dotted lines are used for two lines \[ v_i^T \cdot y_k = v_i^T \cdot x_k \]

and \[ v_i^T \cdot y_{k+1} = -v_i^T \cdot y_k \]

respectively. In a similar fashion, the oscillatory unstable node (source) is presented in Fig. 2.4d. This illustration can be easily observed from the stability of fixed points. In Fig. 2.4e, f, the oscillatory stable and unstable nodes are presented as usual through the two-time iterations.
Fig. 2.4 Stability of fixed points: (a) monotonically stable node (sink); (b) monotonically unstable node (source); (c) oscillatory stable node (sink) and (d) oscillatory unstable node (source); (e) oscillatory stable node (sink) and (f) oscillatory unstable node (sink). Shaded areas are stable zones. ($y_k = x_k - x_k^*$ and $y_{k+1} = x_{k+1} - x_k^*$)
Theorem 2.5 Consider a discrete, nonlinear dynamical system \( x_{k+1} = f(x_k, p) \) in Eq. (2.4) with a fixed point \( x_k^* \). The corresponding solution is given by \( x_{k+j} = f(x_{k+j-1}, p) \) with \( j \in \mathbb{Z} \). Suppose there is a neighborhood of the fixed point \( x_k^* \) (i.e., \( U_k(x_k^*) \subset \Omega \)), and \( f(x_k, p) \) is \( C^r \) \((r \geq 1)\)-continuous in \( U_k(x_k^*) \) with Eq. (2.28). The linearized system is \( y_{k+j} = Df(x_k, p)y_{k+j} \) \((y_{k+j} = x_{k+j} - x_k^*)\) in \( U_k(x_k^*) \) and there are \( n \) linearly independent vectors \( v_i \) \((i = 1, 2, \ldots, n)\). For a perturbation of fixed point \( y_k = x_k - x_k^* \), let \( y_k^{(i)} = c_k^{(i)} v_i \) and \( y_{k+1}^{(i)} = c_{k+1}^{(i)} v_i \).

1. \( x_{k+j} \) \((j \in \mathbb{Z})\) at fixed point \( x_k^* \) on the direction \( v \) is stable if and only if
   \[
   G_{s_k}^{(1)}(x_k^*, p) = \lambda_i \in (-1, 1) \tag{2.57}
   \]
   for \( x_k \in U(x_k^*) \subset \Omega_2 \).
2. \( x_{k+j} \) \((j \in \mathbb{Z})\) at fixed point \( x_k^* \) on the direction \( v \) is unstable if and only if
   \[
   G_{s_k}^{(1)}(x_k^*, p) = \lambda_i \in (1, \infty) \text{ and } (-\infty, -1) \tag{2.58}
   \]
   for \( x_k \in U(x_k^*) \subset \Omega_2 \).
3. \( x_{k+j} \) \((j \in \mathbb{Z})\) at fixed point \( x_k^* \) on the direction \( v \) is invariant if and only if
   \[
   G_{s_k}^{(1)}(x_k^*, p) = \lambda_i = 1 \quad \text{and} \quad G_{s_k}^{(m)}(x_k^*, p) = 0 \text{ for } m_i = 2, 3, \ldots \tag{2.59}
   \]
   for \( x_k \in U(x_k^*) \subset \Omega_2 \).
4. \( x_{k+j} \) \((j \in \mathbb{Z})\) at fixed point \( x_k^* \) on the direction \( v \) is symmetrically flip if and only if
   \[
   G_{s_k}^{(1)}(x_k^*, p) = \lambda_i = -1 \quad \text{and} \quad G_{s_k}^{(m)}(x_k^*, p) = 0 \text{ for } m_i = 2, 3, \ldots \tag{2.60}
   \]
   for \( x_k \in U(x_k^*) \subset \Omega_2 \).

Proof The proof can be referred to Luo (2012). \( \square \)

The monotonic stability of fixed points with higher-order singularity for a specific eigenvector is presented in Fig. 2.5. The solid curve is \( v_i^T \cdot x_{k+1} = v_i^T \cdot f(x_k, p) \). The circular symbol is fixed pointed. The shaded regions are stable. The horizontal solid line is also for the degenerate case. The vertical solid line is for a line with infinite slope. The monotonically stable node (sink) of the \((2m_i + 1)\)th-order is sketched in Fig. 2.5a. The dashed and dotted lines are for \( v_i^T \cdot x_k = v_i^T \cdot x_{k+1} \) and \( v_i^T \cdot y_{k+1} = -v_i^T \cdot y_k \), respectively. The nonlinear curve lies in the stable zone, and the iterative responses approach the fixed point. However, the monotonically
unstable (source) of the \( (2m_i+1) \)-th order is presented in Fig. 2.5b. The nonlinear curve lies in the unstable zone, and the iterative responses go away from the fixed point. The monotonically lower saddle of \( (2m_i) \)-th order is presented in Fig. 2.5c. The nonlinear curve is tangential to the line of \( y_k = x_k - x_k^* \) and \( y_{k+1} = x_{k+1} - x_k^* \) with the \( (2m_i) \)-th-order, and the upper branch is in the stable zone and the lower branch is in the unstable zone. Similarly, the monotonically upper saddle of \( (2m_i) \)-th order is presented in Fig. 2.5d. The oscillatory stability of fixed points with higher-order singularity for a specific eigenvector after iteration with a flip \( v_i^T \cdot y_k = -v_i^T \cdot y_{k+1} \) is presented in Fig. 2.6. The oscillatory stable node (sink) of \( (2m_i+1) \)-th order is sketched in Fig. 2.6a. The dashed and dotted lines are for \( v_i^T \cdot y_{k+1} = -v_i^T \cdot y_k \)

---

**Fig. 2.5** Monotonic stability of fixed points with higher-order singularity: a monotonically stable node (sink) of \( (2m_i+1) \)-th order, b monotonically unstable node (source) of \( (2m_i+1) \)-th order, c monotonically lower saddle of \( (2m_i) \)-th order, and d monotonically upper saddle of \( (2m_i) \)-th order. *Shaded areas* are stable zones. \( y_k = x_k - x_k^* \) and \( y_{k+1} = x_{k+1} - x_k^* \)
The nonlinear curve lies in the stable zone, and the iterative responses approach the fixed point. However, the oscillatory unstable (source) of the $(2m_i + 1)$th-order is presented in Fig. 2.6b. The nonlinear curve lies in the unstable zone, and the iterative responses go away from the fixed point. The oscillatory lower saddle of the $(2m_i)$th-order is presented in Fig. 2.6c. The nonlinear curve is tangential to and below the line of $v^T_x \cdot y_k = v^T_x \cdot x_{k+1}$ with the $(2m_i)$th-order, and the upper branch is in the stable zone and the lower branch is in the unstable zone. Finally, the oscillatory upper saddle of the $(2m_i)$th-order is presented in Fig. 2.6d. For clear illustrations, the oscillatory stability of fixed points with higher-order singularity for the two-time iterations is presented in Fig. 2.7.
**Definition 2.20** Consider a discrete, nonlinear dynamical system $x_{k+1} = f(x_k, p)$ in Eq. (2.4) with a fixed point $x_k^*$. The corresponding solution is given by $x_{k+j} = f(x_{k+j-1}, p)$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $x_k^*$ (i.e., $U_k(x_k^*) \subset \Omega$), and $f(x_k, p)$ is $C^r$ ($r \geq 1$)-continuous in $U_k(x_k^*)$ with Eq. (2.28). The linearized system is $y_{k+j+1} = Df(x_k^*, p)y_{k+j}$ ($y_{k+j} = x_{k+j} - x_k^*$) in $U_k(x_k^*)$ and there are $n$ linearly independent vectors $v_i$ ($i = 1, 2, \ldots, n$). For a perturbation of fixed point $y_k = x_k - x_k^*$, let $y_k^{(i)} = c_k^{(i)} v_i$ and $y_{k+1}^{(i)} = c_{k+1}^{(i)} v_i$.

(i) $x_{k+j}$ ($j \in \mathbb{Z}$) at fixed point $x_k^*$ on the direction $v_i$ is monotonically stable of the $(2m_i + 1)$th-order if
\[ G^{(1)}_{s^j}(x^*_k, p) = \lambda_i = 1, \]
\[ G^{(r)}_{s^j}(x^*_k, p) = 0 \quad \text{for} \quad r_i = 2, 3, \ldots, 2m_i, \]
\[ C^{(2m_i+1)}_{s^j}(x^*_k, p) \neq 0, \]
\[ |v^T_i \cdot (x_{k+1} - x^*_k)| < |v^T_i \cdot (x_k - x^*_k)| \]

for \( x_k \in U_k(x^*_k) \subset \Omega_2 \). The fixed point \( x^*_k \) is called a monotonic sink (or stable node) of the \((2m_i + 1)\)th-order on the direction \( v_i \).

(ii) \( x_{k+j} \ (j \in \mathbb{Z}) \) at fixed point \( x^*_k \) on the direction \( v_i \) is monotonically unstable of the \((2m_i + 1)\)th-order if

\[ G^{(1)}_{s^j}(x^*_k, p) = \lambda_i = 1, \]
\[ G^{(r)}_{s^j}(x^*_k, p) = 0 \quad \text{for} \quad r_i = 2, 3, \ldots, 2m_i; \]
\[ C^{(2m_i+1)}_{s^j}(x^*_k, p) \neq 0; \]
\[ |v^T_i \cdot (x_{k+1} - x^*_k)| > |v^T_i \cdot (x_k - x^*_k)| \]

for \( x_k \in U_k(x^*_k) \subset \Omega_2 \). The fixed point \( x^*_k \) is called a monotonic source (or unstable node) of the \((2m_i + 1)\)th-order on the direction \( v_i \).

(iii) \( x_{k+j} \ (j \in \mathbb{Z}) \) at fixed point \( x^*_k \) on the direction \( v_i \) is monotonically unstable of the \((2m_i)\)th-order, lower saddle if

\[ G^{(1)}_{s^j}(x^*_k, p) = \lambda_i = 1, \]
\[ G^{(r)}_{s^j}(x^*_k, p) = 0 \quad \text{for} \quad r_i = 2, 3, \ldots, 2m_i - 1; \]
\[ C^{(2m_i)}_{s^j}(x^*_k, p) \neq 0, \]
\[ |v^T_i \cdot (x_{k+1} - x^*_k)| < |v^T_i \cdot (x_k - x^*_k)| \quad \text{for} \quad s^{(i)}_k > 0, \]
\[ |v^T_i \cdot (x_{k+1} - x^*_k)| > |v^T_i \cdot (x_k - x^*_k)| \quad \text{for} \quad s^{(i)}_k < 0 \]

for \( x_k \in U_k(x^*_k) \subset \Omega_2 \). The fixed point \( x^*_k \) is called a monotonic, lower saddle of the \((2m_i)\)th-order on the direction \( v_i \).

(iv) \( x_{k+j} \ (j \in \mathbb{Z}) \) at fixed point \( x^*_k \) on the direction \( v_i \) is monotonically unstable of the \((2m_i)\)th-order, upper saddle if

\[ G^{(1)}_{s^j}(x^*_k, p) = \lambda_i = 1, \]
\[ G^{(r)}_{s^j}(x^*_k, p) = 0 \quad \text{for} \quad r_i = 2, 3, \ldots, 2m_i - 1; \]
\[ C^{(2m_i)}_{s^j}(x^*_k, p) \neq 0, \]
for $x_k \in U_k(x_k^*) \subset \Omega_2$. The fixed point $x_k^*$ is called a monotonic, upper saddle of the $(2m_i)$th-order on the direction $v_i$.

(v) $x_{k+j}$ $(j \in \mathbb{Z})$ at fixed point $x_k^*$ on the direction $v_i$ is oscillatory stable of the $(2m_i + 1)$th-order if

$$
G^{(1)}_{s_k^j}(x_k^*, p) = \lambda_i = -1,
G^{(r)}_{s_k^j}(x_k^*, p) = 0 \text{ for } r_i = 2, 3, \ldots, 2m_i;
G^{(2m_i + 1)}_{s_k^j}(x_k^*, p) \neq 0;
|v_i^T \cdot (x_{k+1} - x_k^*)| < |v_i^T \cdot (x_k - x_k^*)|
$$

for $x_k \in U_k(x_k^*) \subset \Omega_2$. The fixed point $x_k^*$ is called an oscillatory sink (or stable node) of the $(2m_i + 1)$th-order on the direction $v_i$.

(vi) $x_{k+j}$ $(j \in \mathbb{Z})$ at fixed point $x_k^*$ on the direction $v_i$ is oscillatory unstable of the $(2m_i + 1)$th-order if

$$
G^{(1)}_{s_k^j}(x_k^*, p) = \lambda_i = -1;
G^{(r)}_{s_k^j}(x_k^*, p) = 0 \text{ for } r_i = 2, 3, \ldots, 2m_i;
G^{(2m_i + 1)}_{s_k^j}(x_k^*, p) \neq 0,
|v_i^T \cdot (x_{k+1} - x_k^*)| > |v_i^T \cdot (x_k - x_k^*)|
$$

for $x_k \in U_k(x_k^*) \subset \Omega_2$. The fixed point $x_k^*$ is called an oscillatory source (or unstable node) of the $(2m_i + 1)$th-order on the direction $v_i$.

(vii) $x_{k+j}$ $(j \in \mathbb{Z})$ at fixed point $x_k^*$ on the direction $v_i$ is oscillatory unstable of the $(2m_i)$th-order, lower saddle if

$$
G^{(1)}_{s_k^j}(x_k^*, p) = \lambda_i = -1,
G^{(r)}_{s_k^j}(x_k^*, p) = 0 \text{ for } r_i = 2, 3, \ldots, 2m_i - 1;
G^{(2m_i)}_{s_k^j}(x_k^*, p) \neq 0,
|v_i^T \cdot (x_{k+1} - x_k^*)| > |v_i^T \cdot (x_k - x_k^*)| \text{ for } s_k^{(i)} > 0,
|v_i^T \cdot (x_{k+1} - x_k^*)| < |v_i^T \cdot (x_k - x_k^*)| \text{ for } s_k^{(i)} < 0
$$

for $x_k \in U_k(x_k^*) \subset \Omega_2$. The fixed point $x_k^*$ is called an oscillatory lower saddle of the $(2m_i)$th-order on the direction $v_i$. 

\[
|v_i^T \cdot (x_{k+1} - x_k^*)| > |v_i^T \cdot (x_k - x_k^*)| \quad \text{for } s_k^{(i)} > 0, \\
|v_i^T \cdot (x_{k+1} - x_k^*)| < |v_i^T \cdot (x_k - x_k^*)| \quad \text{for } s_k^{(i)} < 0
\]
there are \( n \) linearly independent vectors \( \mathbf{x} \).

The linearized system is

\[ \begin{align*}
G_{s_j}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) &= \lambda_i = -1, \\
G_{s_j}^{(r_i)}(\mathbf{x}_k^*, \mathbf{p}) &= 0 \quad \text{for } r_i = 2, 3, \ldots, 2m_i - 1; \\
G_{s_j}^{(2m_i)}(\mathbf{x}_k^*, \mathbf{p}) &\neq 0, \\
|\mathbf{v}_i^T \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| &< |\mathbf{v}_i^T \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \quad \text{for } s_k^{(i)} > 0, \\
|\mathbf{v}_i^T \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| &> |\mathbf{v}_i^T \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \quad \text{for } s_k^{(i)} < 0
\end{align*} \]

for \( \mathbf{x}_k \in U_k(\mathbf{x}_k^*) \subset \Omega_2 \). The fixed point \( \mathbf{x}_k^* \) is called an oscillatory, upper saddle of the \((2m_i)\)th-order on the direction \( \mathbf{v}_i \).

**Theorem 2.6** Consider a discrete, nonlinear dynamical system \( \mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathbb{R}^n \) in Eq. (2.4) with a fixed point \( \mathbf{x}_k^* \). The corresponding solution is given by \( \mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p}) \) with \( j \in \mathbb{Z} \). Suppose there is a neighborhood of the fixed point \( \mathbf{x}_k^* \) (i.e., \( U_k(\mathbf{x}_k^*) \subset \Omega \)), and \( \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \) is \( C^r \) \((r \geq 1)\)-continuous in \( U_k(\mathbf{x}_k^*) \) with Eq. (2.28). The linearized system is \( \mathbf{y}_{k+j} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j} \) \((\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)\) in \( U_k(\mathbf{x}_k^*) \) and there are \( n \) linearly independent vectors \( \mathbf{v}_i \) \((i = 1, 2, \ldots, n)\). For a perturbation of fixed point \( \mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^* \), let \( \mathbf{y}_k^{(i)} = c_k^{(i)} \mathbf{v}_i \) and \( \mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)} \mathbf{v}_i \).

(i) \( \mathbf{x}_{k+j} \) \((j \in \mathbb{Z})\) at fixed point \( \mathbf{x}_k^* \) on the direction \( \mathbf{v}_i \) is monotonically stable of the \((2m_i + 1)\)th-order if and only if

\[ \begin{align*}
G_{s_k}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) &= \lambda_i = 1, \\
G_{s_k}^{(r_i)}(\mathbf{x}_k^*, \mathbf{p}) &= 0 \quad \text{for } r_i = 2, 3, \ldots, 2m_i, \\
G_{s_k}^{(2m_i+1)}(\mathbf{x}_k^*, \mathbf{p}) &< 0
\end{align*} \]

for \( \mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_2 \).

(ii) \( \mathbf{x}_{k+j} \) \((j \in \mathbb{Z})\) at fixed point \( \mathbf{x}_k^* \) on the direction \( \mathbf{v}_i \) is monotonically unstable of the \((2m_i + 1)\)th-order if and only if

\[ \begin{align*}
G_{s_k}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) &= \lambda_i = 1, \\
G_{s_k}^{(r_i)}(\mathbf{x}_k^*, \mathbf{p}) &= 0 \quad \text{for } r_i = 2, 3, \ldots, 2m_i, \\
G_{s_k}^{(2m_i+1)}(\mathbf{x}_k^*, \mathbf{p}) &> 0
\end{align*} \]

for \( \mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_2 \).

(iii) \( \mathbf{x}_{k+j} \) \((j \in \mathbb{Z})\) at fixed point \( \mathbf{x}_k^* \) on the direction \( \mathbf{v}_i \) is monotonically unstable of the \((2m_i)\)th-order, lower saddle if and only if
\[ G^{(1)}_{s_k}(x_k^*, p) = \lambda_i = 1, \]
\[ G^{(r)}_{s_k}(x_k^*, p) = 0 \text{ for } r_i = 2, 3, \ldots, 2m_i - 1, \] (2.71)
\[ G^{(2m)}_{s_k}(x_k^*, p) < 0 \text{ stable for } s_k^{(i)} > 0; \]
\[ G^{(2m)}_{s_k}(x_k^*, p) < 0 \text{ unstable for } s_k^{(i)} < 0 \]

for \( x_k \in U(x_k^*) \subset \Omega_2 \).

(iv) \( x_{k+i} \ (j \in \mathbb{Z}) \) at fixed point \( x_k^* \) on the direction \( v_i \) is monotonically unstable of the \((2m_i)\)th-order if and only if
\[ G^{(1)}_{s_k}(x_k^*, p) = \lambda_i = 1, \]
\[ G^{(r)}_{s_k}(x_k^*, p) = 0 \text{ for } r_i = 2, 3, \ldots, 2m_i - 1, \] (2.72)
\[ G^{(2m)}_{s_k}(x_k^*, p) > 0 \text{ unstable for } s_k^{(i)} > 0; \]
\[ G^{(2m)}_{s_k}(x_k^*, p) > 0 \text{ stable for } s_k^{(i)} < 0 \]

for \( x_k \in U(x_k^*) \subset \Omega_2 \).

(v) \( x_{k+i} \ (j \in \mathbb{Z}) \) at fixed point \( x_k^* \) on the direction \( v_i \) is oscillatory stable of the \((2m_i + 1)\)th-order if and only if
\[ G^{(1)}_{s_k}(x_k^*, p) = \lambda_i = -1, \]
\[ G^{(r)}_{s_k}(x_k^*, p) = 0 \text{ for } r_i = 2, 3, \ldots, 2m_i, \] (2.73)
\[ G^{(2m+1)}_{s_k}(x_k^*, p) > 0 \]

for \( x_k \in U(x_k^*) \subset \Omega_2 \).

(vi) \( x_{k+i} \ (j \in \mathbb{Z}) \) at fixed point \( x_k^* \) on the direction \( v_i \) is oscillatory unstable of the \((2m_i + 1)\)th-order if and only if
\[ G^{(1)}_{s_k}(x_k^*, p) = \lambda_i = -1, \]
\[ G^{(r)}_{s_k}(x_k^*, p) = 0 \text{ for } r_i = 2, 3, \ldots, 2m_i, \] (2.74)
\[ G^{(2m+1)}_{s_k}(x_k^*, p) < 0 \]

for \( x_k \in U(x_k^*) \subset \Omega_2 \).

(vii) \( x_{k+i} \ (j \in \mathbb{Z}) \) at fixed point \( x_k^* \) on the direction \( v_i \) is oscillatory unstable of the \((2m_i)\)th-order, upper saddle if and only if
2.3 Stability Switching Theory

\[ G_{s_k}^{(1)}(x_k^*, p) = \lambda_i = -1, \]

\[ G_{s_k}^{(r)}(x_k^*, p) = 0 \quad \text{for} \quad r_i = 2, 3, \ldots, 2m_i - 1, \]

\[ G_{s_k}^{(2m_i)}(x_k^*, p) > 0 \quad \text{stable for} \quad s_k^{(i)} > 0; \]

\[ G_{s_k}^{(2m_i)}(x_k^*, p) > 0 \quad \text{unstable for} \quad s_k^{(i)} < 0 \]

for \( x_k \in U(x_k^*) \subset \Omega_s. \)

(viii) \( x_{k+j} \) (\( j \in \mathbb{Z} \)) at fixed point \( x_k^* \) on the direction \( v_i \) is oscillatory unstable of the \( (2m_i)\)th-order, lower saddle if and only if

\[ G_{s_k}^{(1)}(x_k^*, p) = \lambda_i = -1, \]

\[ G_{s_k}^{(r)}(x_k^*, p) = 0 \quad \text{for} \quad r_i = 2, 3, \ldots, 2m_i - 1, \]

\[ G_{s_k}^{(2m_i)}(x_k^*, p) < 0 \quad \text{stable for} \quad s_k^{(i)} < 0; \]

\[ G_{s_k}^{(2m_i)}(x_k^*, p) < 0 \quad \text{unstable for} \quad s_k^{(i)} > 0 \]

for \( x_k \in U(x_k^*) \subset \Omega_s. \)

**Proof** The proof can be referred to Luo (2012). \( \square \)

**Definition 2.21** Consider a discrete, nonlinear dynamical system \( x_{k+1} = f(x_k, p) \) \( \in \mathbb{R}^n \) in Eq. (2.4) with a fixed point \( x_k^* \). The corresponding solution is given by \( x_{k+j} = f(x_{k+j-1}, p) \) with \( j \in \mathbb{Z} \). Suppose there is a neighborhood of the fixed point \( x_k^* \) (i.e., \( U_k(x_k^*) \subset \Omega \)), and \( f(x_k, p) \) is \( C^r (r \geq 1) \)-continuous in \( U_k(x_k^*) \) with Eq. (2.28). The linearized system is \( y_{k+j} = Df(x_k^*, p)y_{k+j} \) (\( y_{k+j} = x_{k+j} - x_k^* \)) in \( U_k(x_k^*) \). Consider a pair of complex eigenvalues \( \lambda_i \pm i\beta_i \) (\( i \in \mathbb{N} = \{1, 2, \ldots, n\} \), \( i = \sqrt{-1} \)) of matrix \( Df(x_k^*, p) \) with a pair of eigenvectors \( u_i \pm iv_i \). On the invariant plane of \( (u_i, v_i) \), consider \( r_k^{(i)} = y_k^{(i)} = y_k^{(i)} + y_k^{(i)} \) with

\[
    r_k^{(i)} = c_k^{(i)} u_i + d_k^{(i)} v_i, \\
    r_{k+1}^{(i)} = c_{k+1}^{(i)} u_i + d_{k+1}^{(i)} v_i
\]

and

\[
    c_k^{(i)} = \frac{1}{\Delta} \left[ \Delta_2 (u_i^T \cdot y_k) - \Delta_12 (v_i^T \cdot y_k) \right], \\
    d_k^{(i)} = \frac{1}{\Delta} \left[ \Delta_1 (v_i^T \cdot y_k) - \Delta_12 (u_i^T \cdot y_k) \right];
\]

\[
    \Delta_1 = ||u_i||^2, \Delta_2 = ||v_i||^2, \Delta_12 = u_i^T \cdot v_i; \\
    \Delta = \Delta_1 \Delta_2 - \Delta_12^2.
\]
Consider a polar coordinate of \((r_k, \theta_k)\) defined by
\[
C_k^{(i)} = r_k^{(i)} \cos \theta_k^{(i)}, \quad \text{and} \quad d_k^{(i)} = r_k^{(i)} \sin \theta_k^{(i)};
\]
\[
r_k^{(i)} = \sqrt{(c_k^{(i)})^2 + (d_k^{(i)})^2}, \quad \text{and} \quad \theta_k^{(i)} = \arctan(d_k^{(i)}/c_k^{(i)}).
\]
(2.79)
Thus,
\[
c_{k+1}^{(i)} = \frac{1}{\Delta} \left[ \Delta_2 C_k^{(i)}(x_k, p) - \Delta_1 G_{d_k^{(i)}}(x_k, p) \right]
\]
\[
d_{k+1}^{(i)} = \frac{1}{\Delta} \left[ \Delta_1 G_{d_k^{(i)}}(x_k, p) - \Delta_2 G_{c_k^{(i)}}(x_k, p) \right]
\]
(2.80)
where
\[
G_{c_k^{(i)}}(x_k, p) = u^T \cdot \left[ f(x_k, p) - x_k^* \right] = \sum_{m_l=1}^\infty \frac{1}{m_l!} G_{c_k^{(i)}}^{(m_l)}(\theta_k^{(i)})(r_k^{(i)})^{m_l},
\]
\[
G_{d_k^{(i)}}(x_k, p) = v^T \cdot \left[ f(x_k, p) - x_k^* \right] = \sum_{m_l=1}^\infty \frac{1}{m_l!} G_{d_k^{(i)}}^{(m_l)}(\theta_k^{(i)})(r_k^{(i)})^{m_l};
\]
(2.81)
\[
G_{c_k^{(i)}}^{(m_l)}(\theta_k^{(i)}) = u^T \cdot \partial x_k^{(m_l)} f(x_k, p) [ u_i \cos \theta_k^{(i)} + v_i \sin \theta_k^{(i)} ] \right|_{(x_k^*, p)},
\]
\[
G_{d_k^{(i)}}^{(m_l)}(\theta_k^{(i)}) = v^T \cdot \partial x_k^{(m_l)} f(x_k, p) [ u_i \cos \theta_k^{(i)} + v_i \sin \theta_k^{(i)} ] \right|_{(x_k^*, p)}.
\]
(2.82)
Thus,
\[
r_{k+1}^{(i)} = \sqrt{(c_{k+1}^{(i)})^2 + (d_{k+1}^{(i)})^2} = \sqrt{\sum_{m_l=2}^\infty (r_k^{(i)})^{m_l} G_{c_k^{(i)}}^{(m_l)}(\theta_k^{(i)})}
\]
\[
= \sqrt{G_{r_k^{(i)}}^{(2)}(r_k^{(i)})} \sqrt{1 + (G_{r_k^{(i)}}^{(2)})^{-1} \sum_{m_l=3}^\infty (r_k^{(i)})^{m_l-2} G_{c_k^{(i)}}^{(m_l)}(\theta_k^{(i)})}
\]
\[
\theta_{k+1}^{(i)} = \arctan(d_{k+1}^{(i)}/c_{k+1}^{(i)})
\]
(2.83)
where
\[
G_{r_k^{(i)}}^{(m_l)}(\theta_k^{(i)}) = \sum_{r_l=1}^\infty \sum_{s_l=1}^\infty \frac{1}{r_l! s_l!} G_{r_k^{(i)}}^{(r_l)}(\theta_k^{(i)}) G_{s_k^{(i)}}^{(s_l)}(\theta_k^{(i)}) + G_{c_k^{(i)}}^{(r_l)}(\theta_k^{(i)}) G_{d_k^{(i)}}^{(s_l)}(\theta_k^{(i)}) \partial \delta_{m_l}
\]
(2.84)
\[
= \sum_{r_l=1}^m \frac{1}{m_l!} \sum_{r_l=1}^m C_{r_l} G_{r_k^{(i)}}^{(r_l)}(\theta_k^{(i)}) G_{c_k^{(i)}}^{(m_l-r_l)}(\theta_k^{(i)}) + G_{c_k^{(i)}}^{(r_l)}(\theta_k^{(i)}) G_{d_k^{(i)}}^{(m_l-r_l)}(\theta_k^{(i)})
\]
and

\[
G^{(m)}_{c_k^{(l)}}(\theta^{(i)}_k) = \frac{1}{\Delta} [\Delta_2 G^{(m)}_{d_k^{(l)}}(\theta^{(i)}_k) - \Delta_1 G^{(m)}_{d_k^{(l)}}(\theta^{(i)}_k)],
\]

\[
G^{(m)}_{d_k^{(l)}}(\theta^{(i)}_k) = \frac{1}{\Delta} [\Delta_1 G^{(m)}_{c_k^{(l)}}(\theta^{(i)}_k) - \Delta_2 G^{(m)}_{c_k^{(l)}}(\theta^{(i)}_k)].
\]  

(2.85)

From the foregoing definition, consider the first-order terms of G-function

\[
G^{(1)}_{c_k^{(l)}}(x_k, p) = G^{(1)}_{c_k^{(l)},1}(x_k, p) + G^{(1)}_{c_k^{(l)},2}(x_k, p),
\]

(2.86)

\[
G^{(1)}_{d_k^{(l)}}(x_k, p) = G^{(1)}_{d_k^{(l)},1}(x_k, p) + G^{(1)}_{d_k^{(l)},2}(x_k, p)
\]

where

\[
G^{(1)}_{c_k^{(l)},1}(x_k, p) = u^T_i \cdot D_{x_i} f(x_k, p) \partial_{c_k^{(l)}} x_k = u^T_i \cdot D_{x_i} f(x_k, p) u_i
\]

\[
= u^T_i \cdot (-\beta_i v_i + \alpha_i u_i) = \alpha_i \Delta_1 - \beta_i \Delta_2,
\]

\[
G^{(1)}_{c_k^{(l)},2}(x_k, p) = u^T_i \cdot D_{x_i} f(x_k, p) \partial_{c_k^{(l)}} x_k = u^T_i \cdot D_{x_i} f(x_k, p) v_i
\]

\[
= u^T_i \cdot (\beta_i u_i + \alpha_i v_i) = \alpha_i \Delta_2 + \beta_i \Delta_1;
\]  

(2.87)

and

\[
G^{(1)}_{d_k^{(l)},1}(x_k, p) = v^T_i \cdot D_{x_i} f(x_k, p) \partial_{d_k^{(l)}} x_k = v^T_i \cdot D_{x_i} f(x_k, p) u_i
\]

\[
= v^T_i \cdot (-\beta_i v_i + \alpha_i u_i) = -\beta_i \Delta_2 + \alpha_i \Delta_2,
\]

\[
G^{(1)}_{d_k^{(l)},2}(x, p) = v^T_i \cdot D_{x_i} f(x_k, p) \partial_{d_k^{(l)}} x_k = v^T_i \cdot D_{x_i} f(x_k, p) v_i
\]

\[
= v^T_i \cdot (\beta_i u_i + \alpha_i v_i) = \alpha_i \Delta_2 + \beta_i \Delta_2.
\]  

(2.88)

Substitution of Eqs. (2.86)–(2.88) into Eq. (2.82) gives

\[
G^{(1)}_{c_k^{(l)}}(\theta^{(i)}_k) = G^{(1)}_{c_k^{(l)},1}(x_k, p) \cos \theta^{(i)}_k + G^{(1)}_{c_k^{(l)},2}(x_k, p) \sin \theta^{(i)}_k
\]

\[
= (\alpha_i \Delta_1 - \beta_i \Delta_2) \cos \theta^{(i)}_k + (\alpha_i \Delta_2 + \beta_i \Delta_1) \sin \theta^{(i)}_k,
\]

\[
G^{(1)}_{d_k^{(l)}}(\theta^{(i)}_k) = G^{(1)}_{d_k^{(l)},1}(x_k, p) \cos \theta^{(i)}_k + G^{(1)}_{d_k^{(l)},2}(x_k, p) \sin \theta^{(i)}_k
\]

\[
= (-\beta_i \Delta_2 + \alpha_i \Delta_2) \cos \theta^{(i)}_k + (\alpha_i \Delta_2 + \beta_i \Delta_1) \sin \theta^{(i)}_k.
\]  

(2.89)

From Eq. (2.85), we have
We know that
\[
\frac{G^{(2)}_{\theta_k^{(i)}}}{\theta_k^{(i)}} = \frac{1}{\Delta}[\Delta_{2}G^{(1)}_{\theta_k^{(i)}} - \Delta_{12}G^{(1)}_{\theta_k^{(i)}}(\theta_{k+1}^{(i)})] = \beta_{1} \sin \theta_{k}^{(i)}.
\]
\[
\frac{G^{(2)}_{d_{k+1}}}{d_{k+1}} = \frac{1}{\Delta}[\Delta_{12}G^{(1)}_{\theta_k^{(i)}} - \Delta_{12}G^{(1)}_{\theta_k^{(i)}}(\theta_{k+1}^{(i)})] = \beta_{1} \sin \theta_{k}^{(i)} + \beta_{1} \cos \theta_{k}^{(i)}.
\]
Thus,
\[
\frac{G^{(2)}_{\theta_k^{(i)}}}{\theta_k^{(i)}} = \frac{1}{\Delta}[\Delta_{2}G^{(1)}_{\theta_k^{(i)}}(\theta_{k+1}^{(i)}) - \Delta_{12}G^{(1)}_{\theta_k^{(i)}}(\theta_{k+1}^{(i)})] = \beta_{1} \sin \theta_{k}^{(i)}.
\]
Furthermore, Eq. (2.83) gives
\[
r^{(i)}_{k+1} = \rho_{1}r^{(i)}_{k} + o(r^{(i)}_{k}) \quad \text{and} \quad \theta^{(i)}_{k+1} = \vartheta^{(i)}_{k} - \tau_{i} + o(r^{(i)}_{k}).
\]
where
\[
\vartheta_{i} = \arctan(\beta_{i}/\zeta_{i}) \quad \text{and} \quad \rho_{1} = \sqrt{\zeta_{i}^{2} + \beta_{i}^{2}}.
\]
As \( r^{(i)}_{k} \ll 1 \) and \( r^{(i)}_{k} \to 0 \), we have
\[
r^{(i)}_{k+1} = \rho_{1}r^{(i)}_{k} \quad \text{and} \quad \theta^{(i)}_{k+1} = \vartheta_{i} - \theta_{k}^{(i)}.
\]
With an initial condition of \( r^{(i)}_{k} = r^{(i)}_{0} \) and \( \theta^{(i)}_{k} = \theta^{(i)}_{0} \), the corresponding solution of Eq. (2.94) is
\[
r^{(i)}_{k+1} = (\rho_{1})^{(i)}r^{(i)}_{k} \quad \text{and} \quad \theta^{(i)}_{k+1} = j\vartheta_{i} - \theta_{k}^{(i)}.
\]
From Eqs. (2.80), (2.81), and (2.90), we have
\[
c^{(i)}_{k+1} = \zeta_{i}r^{(i)}_{k} \cos \theta^{(i)}_{k} + \beta_{i}r^{(i)}_{k} \sin \theta^{(i)}_{k} = \zeta_{i}c^{(i)}_{k} + \beta_{i}d^{(i)}_{k},
\]
\[
d^{(i)}_{k+1} = \zeta_{i}r^{(i)}_{k} \sin \theta^{(i)}_{k} - \beta_{i}r^{(i)}_{k} \cos \theta^{(i)}_{k} = -\beta_{i}c^{(i)}_{k} + \zeta_{i}d^{(i)}_{k}.
\]
That is,
\[
\begin{align*}
\begin{cases}
c^{(i)}_{k+1} \\
d^{(i)}_{k+1}
\end{cases} = \left[
\begin{array}{cc}
\zeta_{i} & \beta_{i} \\
-\beta_{i} & \zeta_{i}
\end{array}
\right] \begin{cases}
c^{(i)}_{k} \\
d^{(i)}_{k}
\end{cases} = \rho_{1} \begin{cases}
\cos \vartheta_{i} & \sin \vartheta_{i} \\
-\sin \vartheta_{i} & \cos \vartheta_{i}
\end{cases} \begin{cases}
c^{(i)}_{k} \\
d^{(i)}_{k}
\end{cases}.
\end{align*}
\]
From the foregoing equation, we have

\[
\begin{bmatrix}
  c_{k+j}^{(i)} \\
  d_{k+j}^{(i)}
\end{bmatrix} =
\begin{bmatrix}
  \alpha_i & \beta_i \\
  -\beta_i & \alpha_i
\end{bmatrix}^{-1}
\begin{bmatrix}
  c_k^{(i)} \\
  d_k^{(i)}
\end{bmatrix} = (\rho_i)^{-1}
\begin{bmatrix}
  \cos j\vartheta_i & \sin j\vartheta_i \\
  -\sin j\vartheta_i & \cos j\vartheta_i
\end{bmatrix}
\begin{bmatrix}
  c_k^{(i)} \\
  d_k^{(i)}
\end{bmatrix}
\] (2.98)

**Definition 2.22** Consider a discrete, nonlinear dynamical system \( x_{k+1} = f(x_k, p) \) in Eq. (2.4) with a fixed point \( x_k^* \). The corresponding solution is given by \( x_{k+j} = f(x_{k+j-1}, p) \) with \( j \in \mathbb{Z} \). Suppose there is a neighborhood of the fixed point \( x_k^* \) (i.e., \( U_k(x_k^*) \subset \Omega \)), and \( f(x_k, p) \) is \( C^r \) \((r \geq 1)\)-continuous in \( U_k(x_k^*) \) with Eq. (2.28). The linearized system is \( y_{k+j+1} = Df(x_k^*, p)y_{k+j} (y_{k+j} = x_{k+j} - x_k^*) \) in \( U_k(x_k^*) \). Consider a pair of complex eigenvalues \( \alpha_i \pm i\beta_i \) \((i \in N = \{1, 2, \ldots, n\}, \ i = \sqrt{-1})\) of matrix \( Df(x_k^*, p) \) with a pair of eigenvectors \( u_i \pm iv_i \). On the invariant plane of \((u_i, v_i)\), consider \( r_k^{(i)} = y_k^{(i)} = y_k^{(i)} + y_k^{(i)} \) with Eqs. (2.73) and (2.75). For any arbitrarily small \( \varepsilon > 0 \), the stability of the fixed point \( x_k^* \) on the invariant plane of \((u_i, v_i)\) can be determined.

(i) \( x_k^{(i)} \) at the fixed point \( x_k^* \) on the plane of \((u_i, v_i)\) is spirally stable if

\[
r_k^{(i)} - r_k^{(i)} < 0.
\] (2.99)

(ii) \( x_k^{(i)} \) at the fixed point \( x_k^* \) on the plane of \((u_i, v_i)\) is spirally unstable if

\[
r_k^{(i)} - r_k^{(i)} > 0.
\] (2.100)

(iii) \( x_k^{(i)} \) at the fixed point \( x_k^* \) on the plane of \((u_i, v_i)\) is spirally stable with the \( m_i \)-th-order singularity if for \( \theta_k^{(i)} \in [0, 2\pi] \)

\[
\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1,
\]

\[
G_{s_k^{(i)}}^{(i)}(\theta_k) = 0 \quad \text{for} \quad s_k^{(i)} = 1, 2, \ldots, m_i - 1,
\] (2.101)

\[
r_k^{(i)} - r_k^{(i)} < 0.
\]

(iv) \( x_k^{(i)} \) at the fixed point \( x_k^* \) on the plane of \((u_i, v_i)\) is spirally unstable with the \( m_i \)-th-order singularity if for \( \theta_k^{(i)} \in [0, 2\pi] \)

\[
\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1,
\]

\[
G_{s_k^{(i)}}^{(i)}(\theta_k) = 0 \quad \text{for} \quad s_k^{(i)} = 1, 2, \ldots, m_i - 1,
\] (2.102)

\[
r_k^{(i)} - r_k^{(i)} > 0.
\]
(v) \( x_k^{(i)} \) at the fixed point \( x_k^* \) on the plane of \((u_i, v_i)\) is circular if for \( \theta_k^{(i)} \in [0, 2\pi] \)

\[
\theta_{k+1}^{(i)} - \theta_k^{(i)} = 0. \tag{2.103}
\]

(vi) \( x_k^{(i)} \) at the fixed point \( x_k^* \) on the plane of \((u_i, v_i)\) is degenerate in the direction of \( u_i \) if

\[
\beta_i = 0 \quad \text{and} \quad \theta_{k+1}^{(i)} - \theta_k^{(i)} = 0. \tag{2.104}
\]

**Theorem 2.7** Consider a discrete, nonlinear dynamical system \( x_{k+1} = f(x_k, p) \in \mathbb{R}^n \) in Eq. (2.4) with a fixed point \( x_k^* \). The corresponding solution is given by \( x_{k+j} = f(x_{k+j-1}, p) \) with \( j \in \mathbb{Z} \). Suppose there is a neighborhood of the fixed point \( x_k^* \) (i.e., \( U_k(x_k^*) \subset \Omega \)), and \( f(x_k, p) \) is \( C^r \) (\( r \geq 1 \))-continuous in \( U_k(x_k^*) \) with Eq. (2.28). The linearized system is \( y_{k+j} = Df(x_k^*, p)y_{k+j} \) \((y_{k+j} = x_{k+j} - x_k^*)\) in \( U_k(x_k^*) \). Consider a pair of complex eigenvalues \( \lambda_i \pm i\beta_i \) \((i \in N = \{1, 2, \ldots, n\}, \ i = \sqrt{-1})\) of matrix \( Df(x^*, p) \) with a pair of eigenvectors \( u_i \pm iv_i \). On the invariant plane of \((u_i, v_i)\), consider \( r_k^{(i)} = y_k^{(i)} = y_{k+}^{(i)} + y_{k-}^{(i)} \) with Eqs. (2.73) and (2.75). For any arbitrarily small \( \varepsilon > 0 \), the stability of the equilibrium \( x_k^* \) on the invariant plane of \((u_i, v_i)\) can be determined.

(i) \( x_k^{(i)} \) at the fixed point \( x_k^* \) on the plane of \((u_k, v_k)\) is spirally stable if and only if

\[
\rho_i < 1. \tag{2.105}
\]

(ii) \( x_k^{(i)} \) at the fixed point \( x_k^* \) on the plane of \((u_i, v_i)\) is spirally unstable if and only if

\[
\rho_i > 1. \tag{2.106}
\]

(iii) \( x_k^{(i)} \) at the fixed point \( x_k^* \) on the plane of \((u_i, v_i)\) is stable with the \( m_i \)th-order singularity if and only if for \( \theta_k^{(i)} \in [0, 2\pi] \)

\[
\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1,
\]

\[
G_{r_k^{(i)}}^{(s_k)} (\theta_k^{(i)}) = 0 \quad \text{for} \ s_k = 1, 2, \ldots, m_i - 1,
\]

\[
G_{r_k^{(i)}}^{(m_i)} (\theta_k^{(i)}) < 0. \tag{2.107}
\]

(iv) \( x_k^{(i)} \) at the fixed point \( x_k^* \) on the plane of \((u_i, v_i)\) is spirally unstable with the \( m_i \)th-order singularity if and only if for \( \theta_k^{(i)} \in [0, 2\pi] \)
\[ \rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1, \]
\[ G_{\theta_k}^{(s)}(\theta_k^{(i)}) = 0 \text{ for } s_k^{(i)} = 0, 1, 2, \ldots, m_i - 1, \]
\[ G_{\theta_k}^{(m)}(\theta_k^{(i)}) > 0. \]

(v) \( \mathbf{x}_k^{(i)} \) at the fixed point \( \mathbf{x}_k^* \) on the plane of \( (u_i, v_i) \) is circular if and only if for \( \theta_k^{(i)} \in [0, 2\pi] \)
\[ \rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1, \]
\[ G_{\theta_k}^{(s)}(\theta_k^{(i)}) = 0 \text{ for } s_k^{(i)} = 0, 1, 2, \ldots. \]  

Proof The proof can be referred to Luo (2011). □

2.4 Bifurcation Theory

**Definition 2.23** Consider a discrete, nonlinear dynamical system \( \mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathcal{H}^n \) in Eq. (2.4) with a fixed point \( \mathbf{x}_k^* \). The corresponding solution is given by \( \mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p}) \) with \( j \in \mathbb{Z} \). Suppose there is a neighborhood of the fixed point \( \mathbf{x}_k^* \) (i.e., \( U_k(\mathbf{x}_k^*) \subset \Omega \), and \( \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \) is \( C^r (r \geq 1) \)-continuous in \( U_k(\mathbf{x}_k^*) \) with Eq. (2.28). The linearized system is \( \mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j} \) \( (\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*) \) in \( U_k(\mathbf{x}_k^*) \) and there are \( n \) linearly independent vectors \( \mathbf{v}_i \) \( (i = 1, 2, \ldots, n) \). For a perturbation of fixed point \( \mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^* \), let \( \mathbf{y}_k^{(i)} = c_k^{(i)} \mathbf{v}_i \) and \( \mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)} \mathbf{v}_i \).

\[ s_k^{(i)} = v_i^T \cdot \mathbf{y}_k = v_i^T \cdot (\mathbf{x}_k - \mathbf{x}_k^*) \]  

(2.110)

where \( s_k^{(i)} = c_k^{(i)} \| \mathbf{v}_i \|^2 \).

\[ s_{k+1}^{(i)} = v_i^T \cdot \mathbf{y}_{k+1} = v_i^T \cdot [\mathbf{f}(\mathbf{x}_k, \mathbf{p}) - \mathbf{x}_k^*]. \]  

(2.111)

In the vicinity of point \( (\mathbf{x}_{k(0)}, \mathbf{p}_0) \), \( v_i^T \cdot \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \) can be expanded for \( (0 < \theta < 1) \) as

\[ v_i^T \cdot [\mathbf{f}(\mathbf{x}_k, \mathbf{p}) - \mathbf{x}_{k(0)}^*] = a_i(s_k^{(i)} - s_{k(0)}^{(i)}) + b_i^T \cdot (\mathbf{p} - \mathbf{p}_0) \\
+ \sum_{q=2}^{m} \frac{1}{q!} \sum_{r=0}^{q} C^r_q a_i^{(q-r, r)} (s_k^{(i)} - s_{k(0)}^{(i)})^{q-r} (\mathbf{p} - \mathbf{p}_0)^r \\
+ \frac{1}{(m+1)!} [(s_k^{(i)} - s_{k(0)}^{(i)}) \partial_{s_{k(0)}} + (\mathbf{p} - \mathbf{p}_0) \partial_{\mathbf{p}}]^{m+1} \\
\times (v_i^T \cdot \mathbf{f}(\mathbf{x}_k(0) + \theta \Delta \mathbf{x}_k, \mathbf{p}_0 + \theta \Delta \mathbf{p})) \]  

(2.112)
where

\[
\begin{align*}
    a_i &= v_i^T \cdot \partial f(x_k, p) \bigg|_{(x_k(0), p_0)}, \quad b_i^T = v_i^T \cdot \partial_p f(x_k, p) \bigg|_{(x_k(0), p_0)}, \\
    a_i^{(r,s)} &= v_i^T \cdot \partial_{x_k}^{(r)} \partial_p^{(s)} f(x_k, p) \bigg|_{(x_k(0), p_0)}. \tag{2.113}
\end{align*}
\]

If \( a_i = 1 \) and \( p = p_0 \), the stability of the fixed point \( x_k^* \) on an eigenvector \( v_i \) changes from stable to unstable state (or from unstable to stable state). The bifurcation manifold on the direction of \( v_i \) is determined by

\[
b_i^T \cdot (p - p_0) + \sum_{q=2}^{m} \frac{1}{q!} \sum_{r=0}^{q} C_q^r a_i^{(q-r)} (s^{(i)*}_k - s^{(i)*}_{k(0)})^{q-r}(p - p_0)^r = 0. \tag{2.114}
\]

In the neighborhood of \( (x_k^*(0), p_0) \), when other components of fixed point \( x_k^* \) on the eigenvector of \( v_j \) for all \( j \neq i, (i, j \in N) \) do not change their stability states, Eq. (2.114) possesses \( l \)-branch solutions of equilibrium \( s^{(i)*}_k \) \((0 < i < m)\) with \( l_1 \)-stable and \( l_2 \)-unstable solutions \((l_1, l_2 \in \{0, 1, 2, \ldots, l\})\). Such \( l \)-branch solutions are called the bifurcation solutions of fixed point \( x_k^* \) on the eigenvector of \( v_i \) in the neighborhood of \( (x_k^*(0), p_0) \). Such a bifurcation at point \( (x_k^*(0), p_0) \) is called the hyperbolic bifurcation of \( m \)-order on the eigenvector of \( v_i \). Consider two special cases herein.

(i) If

\[
    a_i^{(1,1)} = 0 \text{ and } b_i^T \cdot (p - p_0) + \frac{1}{2!} a_i^{(2,1)} (s^{(i)*}_k - s^{(i)*}_{k(0)})^2 = 0 \tag{2.115}
\]

where

\[
\begin{align*}
    a_i^{(2,0)} &= v_i^T \cdot \partial_{x_k}^{(2)} \partial_p^{(0)} f(x_k, p) \bigg|_{(x_k(0), p_0)} = v_i^T \cdot \partial_{x_k}^{(2)} f(x_k, p) \bigg|_{(x_k(0), p_0)} \tag{2.116} \\
    b_i^T &= v_i^T \cdot \partial_p f(x_k, p) \bigg|_{(x_k(0), p_0)} \\
    \end{align*}
\]

such a bifurcation at point \( (x_k^*(0), p_0) \) is called the saddle–node bifurcation on the eigenvector of \( v_i \).

(ii) If

\[
    b_i^T \cdot (p - p_0) = 0 \text{ and } a_i^{(1,1)} \cdot (p - p_0) (s^{(i)*}_k - s^{(i)*}_{k(0)}) + \frac{1}{2!} a_i^{(2,1)} (s^{(i)*}_k - s^{(i)*}_{k(0)})^2 = 0 \tag{2.117}
\]
where
\[
a_i^{(2,0)} = v_i^T \cdot \partial_{x_k}^T f(x_k, p) \bigg|_{(x_k^{(0)}, p_0)} = v_i^T \cdot \partial_{x_k}^T f(x_k, p) \bigg|_{(x_k^{(0)}, p_0)}
\]
\[
= v_i^T \cdot \partial_{x_k}^T f(x_k, p) (v_i v_i) \bigg|_{(x_k^{(0)}, p_0)} = C_{2}^{(0)} x_k^{(0)} (p_0) \neq 0,
\]
\[
a_i^{(1,1)} = v_i^T \cdot \partial_{x_k}^T f(x_k, p) \bigg|_{(x_k^{(0)}, p_0)} = v_i^T \cdot \partial_{x_k}^T \partial_{p} f(x_k, p) \bigg|_{(x_k^{(0)}, p_0)}
\]
\[
= v_i^T \cdot \partial_{x_k}^T \partial_{p} f(x_k, p) v_i \bigg|_{(x_k^{(0)}, p_0)} \neq 0,
\]
\[
a_i^{(2,0)} \times [a_i^{(1,1)} \cdot (p - p_0)] < 0,
\]
\[
(2.119)
\]

such a bifurcation at point \((x_k^*, p_0)\) is called the \textit{transcritical} bifurcation on the eigenvector of \(v_i\).

**Definition 2.24** Consider a discrete, nonlinear dynamical system \(x_{k+1} = f(x_k, p) \in \mathcal{H}^n\) in Eq. (2.4) with a fixed point \(x_k^*\). The corresponding solution is given by \(x_{k+j} = f(x_{k+j-1}, p)\) with \(j \in \mathbb{Z}\). Suppose there is a neighborhood of the fixed point \(x_k^*\) (i.e., \(\mathcal{O}(x_k^*) \subseteq \Omega\)), and \(f(x_k, p)\) is \(C^r (r \geq 1)\)-continuous in \(U_k(x_k^*)\) with Eq. (2.28). The linearized system is \(y_{k+j+1} = Df(x_k^*, p) y_{k+j} (y_{k+j} = x_{k+j} - x_k^*)\) in \(U_k(x_k^*)\) and there are \(n\) linearly independent vectors \(v_i (i = 1, 2, \ldots, n)\). For a perturbation of fixed point \(y_k = x_k - x_k^*\), let \(y_k^{(i)} = c_k^{(i)} v_i\) and \(y_{k+1}^{(i)} = c_{k+1}^{(i)} v_i\). Equations (2.110), (2.111), and (2.113) hold. In the vicinity of point \((x_k^*, p_0)\), \(v_i^T \cdot f(x_k, p)\) can be expended for \((0 < \theta < 1)\) as
\[
v_i^T \cdot [f(x_k, p) - x_k^*] = a_i (s_k^{(i)} - s_k^{(i)*} + b_i^T \cdot (p - p_0))
\]
\[
+ \sum_{q=2}^{m} \frac{1}{q!} \sum_{r=0}^{q} C_r^q a_i^{(q-r,r)} (s_k^{(i)} - s_k^{(i)*})^{q-r} (p - p_0)^r
\]
\[
+ \frac{1}{(m+1)!} [(s_k^{(i)} - s_k^{(i)*}) \partial_{x_k} + (p - p_0) \partial_p]^{m+1}
\]
\[
\times (v_k^T \cdot f(x_k^* + \theta \Delta x_k, p_0 + \theta \Delta p))
\]
\[
(2.121)
\]
and
\[
v_i^T \cdot [f(x_{k+1}, p) - x_k^*] = a_i (s_{k+1}^{(i)} - s_{k+1}^{(i)*}) + b_i^T \cdot (p - p_0)
\]
\[
+ \sum_{q=2}^{m} \frac{1}{q!} \sum_{r=0}^{q} C_r^q a_i^{(q-r,r)} (s_k^{(i)} - s_k^{(i)*})^{q-r} (p - p_0)^r
\]
\[
+ \frac{1}{(m+1)!} [(s_k^{(i)} - s_k^{(i)*}) \partial_{x_{k+1}} + (p - p_0) \partial_p]^{m+1}
\]
\[
\times (v_i^T \cdot f(x_{k+1}^* + \theta \Delta x_{k+1}, p_0 + \theta \Delta p)).
\]
\[
(2.122)
\]
If $a_i = -1$ and $p = p_0$, the stability of current equilibrium $x_k^*$ on an eigenvector $v_i$ changes from stable to unstable state (or from unstable to stable state). The bifurcation manifold in the direction of $v_i$ is determined by

$$b_i^T \cdot (p - p_0) + a_i (s_k^{(i)*} - s_k^{(0)*})$$

$$+ \sum_{q=1}^{m-1} \frac{1}{1!} \sum_{r=0}^{q} C_q \cdot a_i^{(q-r,r)} (s_k^{(i)*} - s_k^{(0)*})^{q-r} (p - p_0)^r = (s_k^{(i)*} - s_k^{(0)*});$$

$$b_i^T \cdot (p - p_0) + a_i (s_k^{(0)*} - s_k^{(i)*})$$

$$+ \sum_{q=1}^{m-1} \frac{1}{1!} \sum_{r=0}^{q} C_q \cdot a_i^{(q-r,r)} (s_k^{(i)*} - s_k^{(0)*})^{q-r} (p - p_0)^r = (s_k^{(i)*} - s_k^{(0)*}).$$

(2.123)

In the neighborhood of $(x_k^{(0)*}, p_0)$, when other components of fixed point $x_k^*$ on the eigenvector of $v_j$ for all $j \neq i$, $(i, i \in N)$ do not change their stability states, Eq. (2.123) possesses $l$-branch solutions of equilibrium $s_k^{(i)*} (0 < l \leq m)$ with $l_1$-stable and $l_2$-unstable solutions ($l_1, l_2 \in \{0, 1, 2, \ldots, l\}$). Such $l$-branch solutions are called the bifurcation solutions of fixed point $x_k^*$ on the eigenvector of $v_i$ in the neighborhood of $(x_k^{(0)*}, p_0)$. Such a bifurcation at point $(x_k^{(0)*}, p_0)$ is called the hyperbolic bifurcation of $m$th-order with doubling iterations on the eigenvector of $v_i$. Consider a special case. If

$$b_i^T \cdot (p - p_0) = 0, a_i = -1, a_i^{(2,0)} = 0, a_i^{(2,1)} = 0, a_i^{(1,2)} = 0,$$

$$[a_i^{(1,1)} \cdot (p - p_0) + a_i (s_k^{(i)*} - s_k^{(0)*}) + \frac{1}{3!} a_i^{(3,0)} (s_k^{(i)*} - s_k^{(0)*})^3 = (s_k^{(i)*} - s_k^{(0)*}),$$

$$a_i^{(1,1)} \cdot (p - p_0) + a_i (s_k^{(i)*} - s_k^{(0)*}) + \frac{1}{3!} a_i^{(3,0)} (s_k^{(i)*} - s_k^{(0)*})^3 = s_k^{(i)*} - s_k^{(0)*})$$

(2.124)

where

$$a_i^{(3,0)} = v_i^T \cdot \partial \cdot \partial_s (\partial_{v_i} (x_k, p)) \bigg|_{x_k^{(0)*}, p_0} = v_i^T \cdot \partial \cdot \partial_s (f(x_k, p)) \bigg|_{x_k^{(0)*}, p_0} = v_i^T \cdot \partial \cdot \partial_s (f(x_k, p)) \bigg|_{x_k^{(0)*}, p_0} \neq 0,$$

$$a_i^{(1,1)} = v_i^T \cdot \partial \cdot \partial_s (f(x_k, p)) \bigg|_{x_k^{(0)*}, p_0} = v_i^T \cdot \partial \cdot \partial_s (f(x_k, p)) \bigg|_{x_k^{(0)*}, p_0} \neq 0.$$

(2.125)

$$a_i^{(3,0)} \times [a_i^{(1,1)} \cdot (p - p_0)] < 0,$$

(2.126)
such a bifurcation at point \((x_k^*, p_0)\) is called the *pitchfork* bifurcation (or period-doubling bifurcation) on the eigenvector of \(v_i\).

For the saddle–node bifurcation of the first kind, the \((2m)\)th-order singularity of the fixed point at the bifurcation point exists as a saddle of the \((2m)\)th-order. For the transcritical bifurcation, the \((2m)\)th-order singularity of the fixed point at the bifurcation point exists as a saddle of the \((2m)\)th-order. However, for the stable pitchfork bifurcation (or saddle–node bifurcation of the second kind, or period-doubling bifurcation), the \((2m+1)\)th-order singularity of the fixed point at the bifurcation point exists as an oscillatory sink of the \((2m+1)\)th-order. For the unstable pitchfork bifurcation (or unstable saddle–node bifurcation of the second kind, or unstable period-doubling bifurcation), the \((2m+1)\)th-order singularity of the fixed point at the bifurcation point exists as an oscillatory source of the \((2m+1)\)th-order.

**Definition 2.25** Consider a discrete, nonlinear dynamical system \(x_{k+1} = f(x_k, p) \in \mathcal{B}^n\) in Eq. (2.4) with a fixed point \(x_k^*\). The corresponding solution is given by \(x_{k+j} = f(x_{k+j-1}, p)\) with \(j \in \mathbb{Z}\). Suppose there is a neighborhood of the fixed point \(x_k^*\) (i.e., \(U_k(x_k^*) \subset \Omega\)), and \(f(x_k, p)\) is \(C^r (r \geq 1)\)-continuous in \(U_k(x_k^*)\) with Eq. (2.28). The linearized system is \(y_{k+j+1} = Df(x_k^*, p)y_{k+j} = y_{k+j} = x_{k+j} - x_k^*\) in \(U_k(x_k^*)\). Consider a pair of complex eigenvalues \(x_i \pm i\beta_i\) \((i \in N = \{1, 2, \ldots, n\}, i = \sqrt{-1})\) of matrix \(Df(x^*, p)\) with a pair of eigenvectors \(u_i, v_i\). On the invariant plane of \((u_i, v_i)\), consider \(r_k^i = y_{k}^i = y_{k+}^i + y_{k-}^i\) with

\[
\begin{align*}
    r_k^i &= c_k^{(i)} u_i + d_k^{(i)} v_i, \quad \text{and} \quad r_{k+1}^i = c_{k+1}^{(i)} u_i + d_{k+1}^{(i)} v_i. 
\end{align*}
\]

and

\[
\begin{align*}
    c_k^{(i)} &= \frac{1}{\Delta} [\Delta_2 (u_i^T \cdot y_k) - \Delta_{12} (v_i^T \cdot y_k)], \\
    d_k^{(i)} &= \frac{1}{\Delta} [\Delta_1 (v_i^T \cdot y_k) - \Delta_{12} (u_i^T \cdot y_k)]; \\
    \Delta_1 &= ||u_i||^2, \quad \Delta_2 = ||v_i||^2, \quad \Delta_{12} = u_i^T \cdot v_i; \\
    \Delta &= \Delta_1 \Delta_2 - \Delta_{12}^2.
\end{align*}
\]  

Consider a polar coordinate of \((r_k, \theta_k)\) defined by

\[
\begin{align*}
    c_k^{(i)} &= r_k^{(i)} \cos \theta_k^{(i)}, \quad \text{and} \quad d_k^{(i)} = r_k^{(i)} \sin \theta_k^{(i)}; \\
    r_k^{(i)} &= \sqrt{(c_k^{(i)})^2 + (d_k^{(i)})^2}, \quad \text{and} \quad \theta_k^{(i)} = \arctan(d_k^{(i)}/c_k^{(i)}).
\end{align*}
\]

Thus,

\[
\begin{align*}
    c_{k+1}^{(i)} &= \frac{1}{\Delta} [\Delta_2 G_{c_k}^{(i)}(x_k, p) - \Delta_{12} G_{d_k}^{(i)}(x_k, p)], \\
    d_{k+1}^{(i)} &= \frac{1}{\Delta} [\Delta_1 G_{d_k}^{(i)}(x_k, p) - \Delta_{12} G_{c_k}^{(i)}(x_k, p)].
\end{align*}
\]
where
\[
G_{c_k}^{(i)}(x_k, p) = u_i^T \cdot [f(x_k, p) - x^*_{k(0)}]
\]
\[
= a^T_i \cdot (p - p_0) + a_{i11}(c_k^{(i)} - c_{k(0)}^{(i)}) + a_{i12}(d_k^{(i)} - d_{k(0)}^{(i)})
\]
\[
+ \sum_{q=2}^{m_i} \frac{1}{q!} \sum_{r=0}^{q} \binom{m_i}{q} G_{c_k}^{(m_i-q,r)}(x_k^*, p_0)(p - p_0)^r (x^*_{k(0)})^{m_i-r} + 
\]
\[
\frac{1}{(m_i + 1)!} [(c_k^{(i)} - c_{k(0)}^{(i)}) \partial_{x_k}^{(i)} + (d_k^{(i)} - d_{k(0)}^{(i)}) \partial_{x_k}^{(i)} + (p - p_0) \partial_p]^{m_i+1} 
\]
\[
\times (u_i^T \cdot f(x^*_{k(0)} + \theta \Delta x, p_0 + \theta \Delta p)),
\]
\[
G_{d_k}^{(i)}(x_k, p) = v_i^T \cdot [f(x_k, p) - x^*_{k(0)}]
\]
\[
= b^T_i \cdot (p - p_0) + a_{i21}(c_k^{(i)} - c_{k(0)}^{(i)}) + a_{i22}(d_k^{(i)} - d_{k(0)}^{(i)})
\]
\[
+ \sum_{q=2}^{m_i} \frac{1}{q!} \sum_{r=0}^{q} \binom{m_i}{q} G_{d_k}^{(m_i-q,r)}(x_k^*, p_0)(p - p_0)^r (x^*_{k(0)})^{m_i-r} + 
\]
\[
\frac{1}{(m_i + 1)!} [(c_k^{(i)} - c_{k(0)}^{(i)}) \partial_{x_k}^{(i)} + (d_k^{(i)} - d_{k(0)}^{(i)}) \partial_{x_k}^{(i)} + (p - p_0) \partial_p]^{m_i+1} 
\]
\[
\times (v_i^T \cdot f(x^*_{k(0)} + \theta \Delta x, p_0 + \theta \Delta p)) \tag{2.131}
\]

and
\[
G_{c_k}^{(s,r)}(x_k^{*}(0), p_0) = u_i^T \cdot [\partial_{\theta_k^{(i)}} u_i \cos \theta_k^{(i)} + \partial_{\theta_k^{(i)}} v_i \sin \theta_k^{(i)}] \partial_p^{s-r} f(x_k, p) \bigg|_{(x_k^{*}(0), p_0)},
\]
\[
G_{d_k}^{(s,r)}(x_k^{*}(0), p_0) = v_i^T \cdot [\partial_{\theta_k^{(i)}} u_i \cos \theta_k^{(i)} + \partial_{\theta_k^{(i)}} v_i \sin \theta_k^{(i)}] \partial_p^{s-r} f(x_k, p) \bigg|_{(x_k^{*}(0), p_0)} \tag{2.132}
\]

\[
a^T_i = u^T_i \cdot \partial_p f(x_k, p), \quad b^T_i = v^T_i \cdot \partial_p f(x_k, p);
\]
\[
a_{i11} = u^T_i \cdot \partial_{x_k} f(x_k, p) u_i, \quad a_{i12} = u^T_i \cdot \partial_{x_k} f(x_k, p) u_i;
\]
\[
a_{i21} = v^T_i \cdot \partial_{x_k} f(x_k, p) u_i, \quad a_{i22} = v^T_i \cdot \partial_{x_k} f(x_k, p) v_i \tag{2.133}
\]

Suppose
\[
a_i = 0 \quad \text{and} \quad b_i = 0 \tag{2.134}
\]
then

\[ r_{k+1}^{(i)} = \sqrt{(c_{k+1}^{(i)})^2 + (d_{k+1}^{(i)})^2} = \sqrt{\sum_{m=2}^{\infty} (r_k^{(i)})^m G_{r_k^{(i)}}^{(m)}} \]

\[ = \frac{G_{r_k^{(i)}}^{(2,0)}}{r_k^{(i)}} \sqrt{1 + \lambda^{(i)} + \sum_{m=3}^{\infty} \chi_m^{(i)} (r_k^{(i)})^{m-2}} \]

\[ \theta_{k+1}^{(i)} = \arctan\left(\frac{d_{k+1}^{(i)}}{c_{k+1}^{(i)}}\right) \]

where

\[ G_{r_k^{(i)}}^{(2,0)} = G_{r_k^{(i)}}^{(2,0)} + G_{r_k^{(i)}}^{(1,1)} \quad \text{and} \quad \lambda^{(i)} = \frac{G_{r_k^{(i)}}^{(1,1)}}{G_{r_k^{(i)}}^{(2,0)}} \]

\[ G_{r_k^{(i)}}^{(2,0)} = \left[ G_{r_k^{(i)}}^{(1,0)} \left( \theta_k^{(i)}, p_0 \right) \right]^2 + \left[ G_{r_k^{(i)}}^{(1,0)} \left( \theta_k^{(i)}, p_0 \right) \right]^2, \]

\[ G_{r_k^{(i)}}^{(1,1)} = \sum_{r=1}^{M} \sum_{s=1}^{M} \left[ G_{r_k^{(i)}}^{(1,0)} \left( \theta_k^{(i)}, p_0 \right) \cdot G_{r_k^{(i)}}^{(1,0)} \left( \theta_k^{(i)}, p_0 \right) \cdot (p - p_0)^{r+s} \right] \]

and

\[ \lambda_m^{(i)} = G_{r_k^{(i)}}^{(m)} / G_{r_k^{(i)}}^{(2,0)} \]

\[ G_{r_k^{(i)}}^{(m)} = \sum_{m_1=0}^{M} \sum_{m_2=0}^{M} \frac{1}{m_1! m_2!} \left[ G_{r_k^{(i)}}^{(m-r_1-r_2)} \left( \theta_k^{(i)}, p_0 \right) \cdot (p - p_0)^{m_1-r_1} \right] \]

\[ \times \left[ G_{r_k^{(i)}}^{(m-r_1-r_2)} \left( \theta_k^{(i)}, p_0 \right) \cdot (p - p_0)^{m_2-r_2} \right] \]

\[ + \left[ G_{r_k^{(i)}}^{(m-r_1-r_2)} \left( \theta_k^{(i)}, p_0 \right) \cdot (p - p_0)^{m_1-r_1} \right] \]

\[ \times \left[ G_{r_k^{(i)}}^{(m-r_1-r_2)} \left( \theta_k^{(i)}, p_0 \right) \cdot (p - p_0)^{m_2-r_2} \right] \]

\[ = \frac{1}{m!} \sum_{r=1}^{m-1} \sum_{s=1}^{M} \frac{1}{s! (2M - m)} \left[ G_{r_k^{(i)}}^{(r,s)} \left( \theta_k^{(i)}, p_0 \right) \cdot G_{r_k^{(i)}}^{(m-r,2M-m-r)} \left( \theta_k^{(i)}, p_0 \right) \right] \]

\[ + \left[ G_{r_k^{(i)}}^{(r,s)} \left( \theta_k^{(i)}, p_0 \right) \cdot G_{r_k^{(i)}}^{(m-r,2M-m-r)} \left( \theta_k^{(i)}, p_0 \right) \right] \cdot (p - p_0)^{M-m}, \]

\[ G_{r_k^{(i)}}^{(m-r,r)} \left( \theta_k, p_0 \right) = \frac{1}{\Delta} \left[ \Delta_2 G_{r_k^{(i)}}^{(m-r,r)} \left( x_k^{(0)}, p_0 \right) - \Delta_{12} G_{r_k^{(i)}}^{(m-r,r)} \left( x_k^{(0)}, p_0 \right) \right] \]

\[ G_{r_k^{(i)}}^{(m-r,r)} \left( \theta_k, p_0 \right) = \frac{1}{\Delta} \left[ \Delta_1 G_{r_k^{(i)}}^{(m-r,r)} \left( x_k^{(0)}, p_0 \right) - \Delta_{12} G_{r_k^{(i)}}^{(m-r,r)} \left( x_k^{(0)}, p_0 \right) \right], \]

\[ G_{r_k^{(i)}}^{(m-r,r)} \left( \theta_k, p_0 \right) = \frac{1}{\Delta} \left[ \Delta_2 G_{r_k^{(i)}}^{(m-r,r)} \left( x_k^{(0)}, p_0 \right) - \Delta_{12} G_{r_k^{(i)}}^{(m-r,r)} \left( x_k^{(0)}, p_0 \right) \right]. \]
If $G^{(2,0)}_{r_{k+1}} = 1$ and $p = p_0$, the stability of current fixed point $x_k^*$ on an eigenvector plane of $(u_i, v_i)$ changes from stable to unstable state (or from unstable to stable state). The bifurcation manifold in the direction of $v_i$ is determined by

$$\lambda^{(i)} + \sum_{m=3}^{\infty} \lambda_m^{(i)} \left( r_k^{(i)} \right)^{m-2} = 0. \quad (2.139)$$

Such a bifurcation at the fixed point $(x_{k(0)}^*, p_0)$ is called the generalized Neimark bifurcation on the eigenvector plane of $(u_i, v_i)$.

For a special case, if

$$\lambda^{(i)} + \lambda_4^{(i)} \left( r_k^{(i)} \right)^2 = 0, \quad \text{for} \quad \lambda^{(i)} \times \lambda_4^{(i)} < 0 \quad \text{and} \quad \lambda_3^{(i)} = 0 \quad (2.140)$$

such a bifurcation at point $(x_0^*, p_0)$ is called the Neimark bifurcation on the eigenvector plane of $(u_i, v_i)$.

For the repeating eigenvalues of $DP(x_k^*, p)$, the bifurcation of fixed point $x_k^*$ can be similarly discussed in the foregoing Theorems 2.5 and 2.6. Herein, such a procedure will not be repeated. From the foregoing analysis of the Neimark bifurcation, the Neimark bifurcation points possess the higher-order singularity of the flow in discrete dynamical system in the radial direction. For the stable Neimark bifurcation, the $m$th-order singularity of the flow at the bifurcation point exists as a sink of the $m$th-order in the radial direction. For the unstable Neimark bifurcation, the $m$th-order singularity of the flow at the bifurcation point exists as a source of the $m$th-order in the radial direction.

Consider a 2D map

$$P : x_k \rightarrow x_{k+1} \quad \text{with} \quad x_{k+1} = f(x_k, p) \quad (2.141)$$

where $x_k = (x_k, y_k)^T$ and $f = (f_1, f_2)^T$ with a parameter vector $p$. The period-$n$ fixed point for Eq. (2.141) is $(x_k^*, p)$, i.e., $P^{(n)} x_k^* = x_{k+n}^*$ where $P^{(n)} = P \circ P^{(n-1)}$ and $P^{(0)} = 1$, and its stability and bifurcation conditions are given as follows.

(i) period-doubling (flip or pitchfork) bifurcation

$$\text{tr}(DP^{(n)}) + \text{det}(DP^{(n)}) + 1 = 0, \quad (2.142)$$

(ii) saddle–node bifurcation

$$\text{det}(DP^{(n)}) + 1 = \text{tr}(DP^{(n)}), \quad (2.143)$$

(iii) Neimark bifurcation

$$\text{det}(DP^{(n)}) = 1. \quad (2.144)$$
The bifurcation and stability conditions for the solution of period-$n$ for Eq. (2.141) are summarized in Fig. 2.8 with $\det(DP(n)) = \det(DP(n)(x_{k(0)}, p_0))$ and $\text{tr}(DP(n)) = \text{tr}(DP(n)(x_{k(0)}^*, p_0))$. The thick dashed lines are bifurcation lines. The stability of the fixed point is given by the eigenvalues in complex plane. The stability of the fixed point for higher-dimensional systems can be identified by using a naming of stability for linear dynamical systems in Luo (2011, 2012). The saddle–node bifurcation possesses stable saddle–node bifurcation (critical) and unstable saddle–node bifurcation (degenerate).

**Fig. 2.8** Stability and bifurcation diagrams through the complex plane of eigenvalues for 2D discrete dynamical systems
References


Discretization and Implicit Mapping Dynamics
Luo, A.C.J.
2015, X, 310 p. 46 illus., 26 illus. in color., Hardcover
ISBN: 978-3-662-47274-3