Chapter 2
Thermoelasticity

Abstract This chapter contains the foundations of the theory of thermoelasticity required for its implementation for calculating strength of structural members as well as for studying the following chapters of the monograph. It is demonstrated that thermoelastic deformations are independent of loading and heating sequence and are always reversible, including the case of temperature-dependent elasticity moduli and simultaneously varying temperatures and stresses. The "dynamic" elasticity moduli evaluating from measured resonance frequencies are more informative than the "static" ones originating from deformation diagrams and depending on loading rate. The governing equations in the theory of thermoelasticity for steady-state and transient non-uniform temperature fields are presented. Formation of temperature stresses in both statically determinate and indeterminate structures is described. Efficient methods for solving non-isothermal boundary value problems with temperature-dependent elasticity moduli are analyzed. The energy principles in the theory of elasticity usually underlying the FEM algorithms and specific features of the variational equations in thermoelasticity are subject to detailed discussion.

2.1 Foundations of Thermoelasticity

2.1.1 Thermal Stresses in an Elastic Body

2.1.1.1 Moduli of Elasticity

An elementary example of phenomenological description of a material property is the Hooke's law. Tensile tests of a rod demonstrated that its elastic elongation Δl can be defined as $\Delta l = Fl_0/EA_0$, where F is a longitudinal force applied to a rod, l_0 and A_0 are the initial length and cross-sectional area of a rod, respectively, and the constant E is a characteristic of the material named the *elasticity modulus* (also known as Young's modulus). Using the definition of stress $\sigma = F/A_0$ and relative

deformation (strain) $\varepsilon = \Delta l/l_0$, we obtain a basic formula of material elastic deformation $\varepsilon = \sigma/E$.

The elasticity modulus basically depends on temperature, but the refined measurements show that it is also affected by loading conditions (tension or compression, increasing force or unloading, etc.). At elevated temperatures, there is a distinction between *dynamic* and *static* moduli depending on the loading speed. The deviation between those increases with the growth of temperature.

The dynamic modulus of elasticity $E_{dyn}(T)$ is evaluated from the resonance frequencies measured at standard tests at a given temperature (see Chap. 11). The accuracy of its definition can be rather high depending not only on the accuracy of processing of the vibrating signal, but also on an adopted dynamic model of the specimen.

The static modulus of elasticity can be determined as $E_{st,0}(T) = \tan \varphi_1$, where φ_1 is an angle of inclination of a tangent to any point of the initial part of the experimental curve $\sigma(\varepsilon, T)$ as shown by line 1 in Fig. 2.1.

The accuracy of modulus $E_{st}(T)$ definition is influenced by, at least, two factors.

- The initial part of a curve $\sigma(\varepsilon, T)$ is not quite rectilinear, and to eliminate subjectivity while evaluating $E_{st}(T)$ by a line 2, it should be calculated under some "contractual" conditions; and
- At elevated temperatures, the deformation curve $\sigma(\varepsilon, T)$ is influenced by the duration of loading which can lead to creep effects (line 3).

As a result, the values of static modulus $E_{st}(T)$, defined at higher temperatures and normal speeds of the static loading of a specimen, usually appear to be underestimated in comparison with the dynamic modulus $E_{dyn}(T)$. The area, representing ratio $E_{dyn}(T)/E_{st}(T)$ at various temperatures for selected four heat-resistant alloys, is demonstrated in Fig. 2.2 (Sizova's data in Birger and Shorr 1975).

For static calculations at high temperature, it is recommended to use the values $E_{dyn}(T)$.

The tests with unloading usually go fast, so creep does not occur even at high temperatures, and the modulus $E_{st}(T)$ defined on unloading line 4 in Fig. 2.1 differs slightly from the modulus $E_{dyn}(T)$.

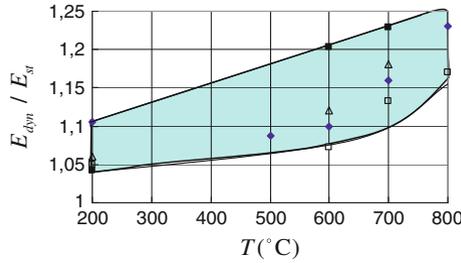**Fig. 2.1** Definition of the static modulus of elasticity
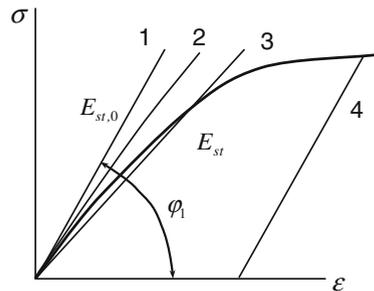
Fig. 2.2 Temperature dependence of the ratio of dynamic and static moduli of elasticity

Similar torsion tests define correlations between the temperature and the shear modulus $G(T)$.

2.1.1.2 Thermal Stresses

The terms corresponding to the temperature expansion have been introduced into the equations of elasticity by Duhamel (1837).

The classical linear theory of thermoelasticity is limited to small deformations and constant values of elasticity moduli E, G and coefficient of linear thermal expansion α_T. The theory is based on the following assumptions confirmed by experiments:

- When calculating the body stress–strain state (SSS), its temperature field is considered to be known on the basis of experimental measurements or calculations and does not change due to the stresses arising in the body. This approach is usually referred to as "the uncoupled theory of thermoelasticity."
- The elastic "mechanical" strain ε_σ induced by the action of the stress σ and temperature expansion of the material $\varepsilon_T = \alpha_T \Delta T$ may be summarized, resulting in a total strain $\varepsilon = \varepsilon_\sigma + \varepsilon_T$, according to the "the principle of strain superposition" (see Chap. 1). The total strain ε connected with displacements by geometric relations (1.14a) does not depend on material's mechanical characteristics.

The application of the linear theory remains valid within the moderate ranges of temperature variation.

The given stationary temperature field $T(x_i)$ differs from the initial one $T_0(x_i)$ as $\Delta T(x_i) = T(x_i) - T_0(x_i)$.

For the *uniaxial* (1D) stress state ($\sigma_1 = \sigma_0$, $\sigma_2 = \sigma_3 = 0$), we have

$$\varepsilon_1 = \frac{\sigma_0}{E} + \varepsilon_T, \quad \varepsilon_2 = \varepsilon_3 = -\mu \frac{\sigma_0}{E} + \varepsilon_T, \tag{2.1}$$

where μ is the Poisson's ratio. Its value for most of the engineering materials lies within the limits $\mu \approx (0.26-0.34)$. The Poisson's ratio μ typically varies slightly within temperature and is therefore usually assumed constant.

The inverse relation connecting the stress σ_0 and thermal expansion is given by

$$\sigma_0 = E(\varepsilon_1 - \varepsilon_T). \tag{2.2}$$

Alongside linear strains, a volume deformation is also of interest. The relative volume change of a small element of a body is

$$\theta = 3\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \tag{2.3}$$

where ε is the mean strain. Volume deformation may also be expressed as a sum of mechanical and thermal components $\theta = \theta_\sigma + \theta_T$, or

$$\theta = (1 - 2\mu)\varepsilon_\sigma + 3\varepsilon_T = \frac{\sigma_0}{3K_\theta} + 3\varepsilon_T, \tag{2.4}$$

where

$$K_\theta = \frac{E}{3(1 - 2\mu)} \tag{2.5}$$

is a *modulus of volumetric elasticity* (also known as a modulus of volumetric compression or a bulk modulus).

Thermal stresses in a homogeneous statically determinate uniformly heated system can emerge only due to restrictions in displacements along a boundary by the other constructive elements. In all other cases, thermal stresses occur when the temperature field is non-uniform.

If the connected parts of a structure are composed of materials with different characteristics of elasticity and thermal expansion, thermal stresses can be provoked by uniform heating of the whole structure.

Thermal Stresses in a Fixed Rod

As an example of arising thermal stresses, let us consider an elastic rod of length L with constant cross-sectional area A and fixed borders, heated in ΔT (Fig. 2.3). If one of the borders was free, the rod length would increase by $\Delta L = \alpha \Delta T L$ (Fig. 2.3a). If a rigid wall restricts expansion of a rod (Fig. 2.3b), the reaction force F causes its compression in $\Delta L_F = FL/EA$ (Fig. 2.3c).

Having utilized $\Delta L = \Delta L_F$, we find $F = E\alpha\Delta TA$ or $\sigma = F/A = -E\alpha\Delta T$ where the negative sign specifies compression.

The same result can be obtained, setting a rod strain to zero $\varepsilon = \sigma/E + \varepsilon_T = 0$ that leads to $\sigma = -E\varepsilon_T = -E\alpha\Delta T$.
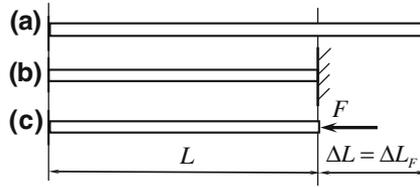
Fig. 2.3 Thermal stresses arising in a heated rod with fixed borders

2.1.2 Thermoelastic Strains

2.1.2.1 Temperature-Dependent Mechanical Properties

Under considerably non-uniform temperature field, it is necessary to treat the moduli of elasticity E, G, K and thermal expansion α_T as functions of temperature. For $\mu \approx \text{const}$, the temperature variation of all elastic moduli is proportional to the change in the Young's modulus $E(T)$.

Different procedures for the thermal expansion coefficient defining in dependence on temperature $\alpha_T(T)$ have been described in Sect. 2.1.2.3.

Since Eqs. (2.1)–(2.5) correspond to any particular point of the body, they remain valid under non-uniform temperature field.

2.1.2.2 Independence of Thermoelastic Strains of the Order of Loading and Heating

An important feature of the linear theory of thermoelasticity is independence of a body SSS from the sequence of loading and heating. This applies also to mechanical characteristics depending on temperature. The influence of the sequence of loading and heating on energy relations is discussed below in Sect. 2.3.

Figure 2.4 demonstrates the deformation diagrams for a material at two temperatures T_1 and T_2, with corresponding elastic moduli E_1 and E_2. Under transition

Fig. 2.4 Independence of elastic strains from loading and heating order
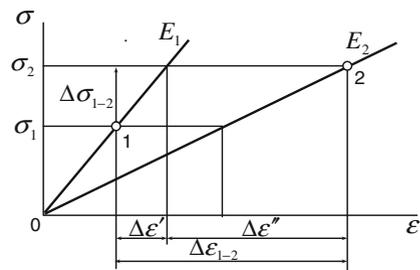
from one state $\sigma_1, \varepsilon_1, T_1$ to another state $\sigma_2, \varepsilon_2, T_2$, the total increment of elastic strain is

$$\Delta\varepsilon_{1-2} = \frac{\sigma_2}{E_2} - \frac{\sigma_1}{E_1}. \tag{2.6}$$

This transition can be achieved by various ways.

- By stress change $\Delta\sigma_{1-2} = \sigma_2 - \sigma_1$ at the initial temperature T_1 and strain increase

$$\Delta\varepsilon' = \frac{\Delta\sigma_{1-2}}{E_1}, \tag{2.7a}$$

followed by temperature change $\Delta T_{1-2} = T_2 - T_1$ at the new stress σ_2 with the additional increase in strain

$$\Delta\varepsilon'' = \sigma_2 \left(\frac{1}{E_2} - \frac{1}{E_1} \right). \tag{2.7b}$$

- By temperature change $\Delta T_{1-2} = T_2 - T_1$ at the initial stress σ_1, and then change of the stress $\Delta\sigma_{1-2} = \sigma_2 - \sigma_1$ at a new temperature T_2 and corresponding increasing strain.
- By complete unloading $\Delta\sigma_{1-0} = -\sigma_1$ at the initial temperature T_1 and then by new full loading $\Delta\sigma_{0-2} = \sigma_2$ at the new temperature T_2. It is easy to check that all three ways will lead to the same value of the elastic strain due to Eq. (2.6).

2.1.2.3 Increments of Thermoelastic Strains

Under smooth variations of loads and temperature, equations of thermoelasticity can be differentiated with respect to operating or formal time t (see Chap. 1) and represented in differential form. This form is required, in particular, to solve contact problems with altering boundary conditions and in combination with plastic and creep strains.

Equation (2.1) yields

$$d\varepsilon_1 = \frac{d\sigma_0}{E} + \left(\frac{d\varepsilon_T}{dT} - \frac{\sigma_0}{E^2} \frac{dE}{dT} \right) dT, \tag{2.8}$$

where

$$\frac{d\varepsilon_T}{dT} = \frac{d(\alpha_T T)}{dT} = \alpha_T + T \frac{d\alpha_T}{dT}. \tag{2.9}$$

Instead of the modulus E, it is more convenient to use its reciprocal—the compliance coefficient

$$a(T) = [E(T)]^{-1}, \tag{2.10}$$

which has to be reconstructed beforehand from the temperature dependence $E(T)$.

Equation (2.8) then takes the form

$$d\varepsilon_1 = a d\sigma_0 + \left(\sigma_0 \frac{da}{dT} + \frac{d\varepsilon_T}{dT} \right) dT. \tag{2.11a}$$

During calculations, the differentials $d\sigma, dT, \ldots$, are replaced by small final increments $\Delta\sigma, \Delta T$, etc; then, Eq. (2.11a) yields

$$\Delta\varepsilon_1 = a(T)\Delta\sigma_0 + \sigma_0 \Delta a(T) + \Delta\varepsilon_T(T). \tag{2.11b}$$

Non-isothermal loading scenario with simultaneous changes of stresses, strains, and temperatures can have a sophisticated character. In Fig. 2.5, "a deformation chart" for the four-stage consecutive increase of stresses, strains, and temperatures (without thermal expansion) is shown. The solid and dotted broken lines assume that stepped temperature transitions occur at constant stresses or strain, respectively.

If a number of transitions are increased, the specified dependence becomes smooth, which is shown by dashed line. Under unloading according to any scenario, residual stresses do not arise.

Reproducing such non-isothermal elastic process at elevated temperatures experimentally is rather difficult because of possible occurrence of plastic strains and creep.
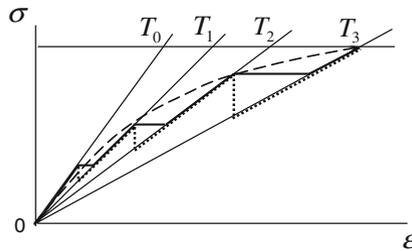

Fig. 2.5 A "deformation chart" at simultaneous changes of stresses, strains, and temperatures

2.2 The Theory of Thermoelasticity

2.2.1 Constitutive Equations

The equations of thermoelasticity are given in numerous books and are used in some software codes, incorporating final element method. Below, a concise description of the basic equations of thermoelasticity is presented with some comments useful for their correct application in engineering.

2.2.1.1 Equations of Combined Stress–Strain State

In case of a combined (multi-axial) SSS, the following basic equations of thermoelasticity yield:

$$\left.\begin{array}{l} \varepsilon_x = \dfrac{1}{E}\left[\sigma_x - \mu(\sigma_y + \sigma_z)\right] + \varepsilon_T, \\[2mm] \varepsilon_y = \dfrac{1}{E}\left[\sigma_y - \mu(\sigma_z + \sigma_x)\right] + \varepsilon_T, \\[2mm] \varepsilon_z = \dfrac{1}{E}\left[\sigma_z - \mu(\sigma_x + \sigma_y)\right] + \varepsilon_T, \\[2mm] \gamma_{xy} = \dfrac{1}{G}\tau_{xy}, \gamma_{yz} = \dfrac{1}{G}\tau_{yz}, \gamma_{zx} = \dfrac{1}{G}\tau_{zx} \end{array}\right\}, \tag{2.12}$$

where $\sigma_x, \sigma_y, \sigma_z$ are normal (direct) stresses, $\tau_{xy}, \tau_{yz}, \tau_{zx}$ are tangential stresses, $\varepsilon_x, \varepsilon_y, \varepsilon_z$ are linear strains (elongations), $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ are angular strains (shears), and $G = E/2(1 + \mu)$ is the shear modulus.

Equation (2.12) may be represented in other forms, using the expressions for the mean strain ε (see Eq. 2.3) and mean stress, namely

$$\sigma = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right), \tag{2.13}$$

resulting in

$$\varepsilon_x = \frac{1}{E}[(1 + \mu)\sigma_x - 3\mu\sigma] + \varepsilon_T. \tag{2.14}$$

Similar expressions can be obtained for the strain components $\varepsilon_y, \varepsilon_z$. Relations for the shear strains are independent of the mean stress.

The relative change of a small volume in the general case, according to Eq. (2.4), is equal to

$$\theta = \frac{\sigma}{K_\theta} + \theta_T, \tag{2.15}$$

where $\theta_T = 3\varepsilon_T = 3\alpha_T\Delta T = \beta_T\Delta T$.

In case of a hydrostatic pressure p, the mean stress is $\sigma = -p$, so that the bulk modulus K_θ characterizes the relative decrease of the elementary volume under compression.

Using index notation (see Chap. 1), Eq. (2.12) becomes

$$\varepsilon_{ij} = \frac{1}{E}\left[(1+\mu)\sigma_{ij} - 3\mu\delta_{ij}\sigma\right] + \delta_{ij}\varepsilon_T, \tag{2.16}$$

where for $i \neq j$ ($\delta_{ij} = 0$), we arrive at $\varepsilon_{ij} = \sigma_{ij}/2G = \gamma_{ij}/2$.

The constitutive Eq. (2.16) is also used in the form

$$\varepsilon_{ij} = \frac{1}{2G}\left(\sigma_{ij} - \delta_{ij}\sigma\right) + \delta_{ij}\left(\frac{\sigma}{K} + \varepsilon_T\right), \tag{2.17}$$

where $K = 3K_\theta$.

Inverse expressions for Eqs. (2.16) and (2.17) are given by

$$\sigma_{ij} = \frac{E}{1+\mu}\left[\varepsilon_{ij} + \frac{3\mu}{1-2\mu}\delta_{ij}\varepsilon + \frac{1+\mu}{1-2\mu}\delta_{ij}\varepsilon_T\right] \tag{2.18}$$

and

$$\sigma_{ij} = 2G(\varepsilon_{ij} - \delta_{ij}\varepsilon) + \delta_{ij}K(\varepsilon - \varepsilon_T). \tag{2.19}$$

Equations (2.16)–(2.19) are valid for both normal and tangential components of stresses and strains.

Thermoelasticity equations in the form (2.17) and (2.19) are used in the theories of thermoplasticity and creep. In these theories (see Chaps. 3 and 6), the stress tensor σ_{ij} is represented as a sum of spherical part $\delta_{ij}\sigma$ and deviator s_{ij}, and analogously for the strain tensor, namely

$$\begin{aligned}\sigma_{ij} &= \delta_{ij}\sigma + s_{ij}, \\ \varepsilon_{ij} &= \delta_{ij}\varepsilon + e_{ij},\end{aligned} \tag{2.20}$$

where σ and ε are mean stress and strain, respectively.

Components of the deviators at $i \neq j$ coincide with the corresponding components of the tensors, and mean stresses associated with the mean strain and thermal expansion are given by Eq. (2.15). The deviator components are related to the equation

$$s_{ij} = 2Ge_{ij}, \tag{2.21}$$

arising from Eqs. (2.16) to (2.20).

2.2.1.2 Differential Form of the General Relations

Presentation of the general equations of thermoelasticity in differential form is necessary for theoretical study of thermoplasticity and creep problems. Thus, differentiating Eq. (2.17) gives

$$d\varepsilon_{ij} = 0.5a_G ds_{ij} + \delta_{ij}a_K d\sigma + \delta_{ij}d\varepsilon_T + 0.5s_{ij}da_G + \delta_{ij}\sigma\, da_K. \qquad (2.22a)$$

In the inverse form,

$$d\sigma_{ij} = 2Gde_{ij} + \delta_{ij}Kd\varepsilon + 2e_{ij}dG + \delta_{ij}[(\varepsilon - \varepsilon_T)dK - Kd\varepsilon_T)], \qquad (2.22b)$$

where the compliance coefficients are

$$a_G(T) = G^{-1}(T), \quad a_K(T) = K^{-1}(T). \qquad (2.23)$$

Temperature increment of the elasticity modulus G is defined as

$$dG = \frac{dG}{dT}dT.$$

The latter may similarly be applied to other material characteristics.

Derivative dG/dT has to be obtained from the experimental curve $G(T)$ using numerical differentiation. Since the dependence $\mu(T)$ is often insignificant, then the functions $\bar{E}(T) = E(T)/E_0$, $\bar{G}(T) = G(T)/G_0$, and $\bar{K}(T) = K(T)/K_0$, where E_0, G_0, and K_0 refer to the normal temperature, are identical.

In the literature, equations, analogous to (2.22a), are sometimes expressed in terms of the modulus of elasticity $E(T)$, Poisson's ratio $\mu(T)$, and their derivatives in temperature. Thus, the differentiation of Eqs. (2.16) and (2.17) gives (Birger 1957):

$$d\varepsilon_{ij} = \frac{1}{E}\left[(1 + \mu)d\sigma_{ij} - 3\mu\delta_{ij}d\sigma\right]$$
$$- \left\{\frac{1}{E^2}\left[(1 + \mu)\sigma_{ij} - 3\mu\delta_{ij}\sigma\right]\frac{dE}{dT} - \frac{\sigma_{ij} - 3\delta_{ij}\sigma}{E}\frac{d\mu}{dT} - \delta_{ij}\frac{d\varepsilon_T}{dT}\right\}dT.$$

2.2.2 Solution Methods

2.2.2.1 Constant Parameters of Elasticity

For the given surface and volume loads and temperature field, the constitutive equations in terms of Eqs. (2.16) or (2.17) together with Eqs. (1.12)–(1.14) of Chap. 1 form a complete system of equations of the linear theory of thermoelasticity, which

has been considered by Duhamel (1837), Lebedev (1934), Maysel (1951), Boley and Weiner (1960), and many others. The material properties—the modulus of elasticity E and the Poisson's ratio μ are assumed to be constant for the whole body, whereas the displacements and deformations are considered to be small. If instead of the field ΔT one deals with a field of thermal expansion $\varepsilon_T = \alpha_T(T)\Delta T$, the nature of the problem does not change.

Common analytic approaches to the problems of linear thermoelasticity are mainly identical to the approaches of linear theory of isothermal elasticity and are presented in a series of monographs, for example, Timoshenko and Goodier (1970). A summary of methods for solving thermoelastic problems is presented in Boley and Weiner (1960).

A number of approximate analytical solutions obtained by methods of strength of materials, variational methods, etc., were provided by Melan and Parkus (1953), Gatewood (1957), Zhou (2008), and many others.

2.2.2.2 Variable Parameters of Elasticity

Analytical solutions, containing the variable parameters of elasticity, can be obtained for limited number of problems. General approaches are proposed in Birger (1957), Boley and Weiner (1960), and also in Birger and Shorr (1975). Calculation methods for unevenly heated pre-twisted rods are stated in Shorr (1960a) and Vorob'ev and Shorr (1983), for thin-walled rods—in Shorr (1960b), for stretched strings—in Birger and Mavljutov (1986), etc. (some examples see in App.).

In finite element method (FEM), the temperature dependence of the material elastic characteristics is used in the form of tables.

In calculations, solutions are usually established by successive approximations approach. These methods, in particular, the method of additional strains, have been developed mainly in relation to problems of thermoplasticity (see Chap. 3), but can also be applied to the problems of thermoelasticity.

The Method of Additional Strains

The method of additional strains allows consideration of an equivalent problem with constant elastic parameters together with the so-called additional strains (Birger and Shorr 1975 and others).

Along with the actual strains ε_{ij} of the original elastic body with variable values E and μ (or K, and G), let us consider "reduced" strains ε_{ij}^r, corresponding to the equivalent elastic body with constant values E_r and μ_r (or K_r and G_r) under the same loads and boundary conditions and at the same actual stresses σ_{ij}. The difference between these strains is referred to as the additional strain

$$\varepsilon_{ij}^{\circ} = \varepsilon_{ij} - \varepsilon_{ij}^{r}. \tag{2.24}$$

The strains ε_{ij}° at the moduli G_r and K_r must satisfy the equalities

$$s_{ij} = 2G(\varepsilon_{ij} - \delta_{ij}\varepsilon) = 2G_r\left(\varepsilon_{ij}^{r} - \delta_{ij}\varepsilon^{r}\right). \tag{2.25}$$

and

$$\sigma = K(\varepsilon - \varepsilon_T) = K_r(\varepsilon^{r} - \varepsilon_T). \tag{2.26}$$

Equations (2.24)–(2.26) yield

$$\varepsilon_{ij}^{\circ} = \left(1 - \frac{G}{G_r}\right)(\varepsilon_{ij} - \delta_{ij}\varepsilon) + \delta_{ij}\left(1 - \frac{K}{K_r}\right)(\varepsilon - \varepsilon_T). \tag{2.27a}$$

Usually, the difference in Poisson's ratio can be neglected, then

$$\varepsilon_{ij}^{\circ} \approx \left(1 - \frac{E}{E_r}\right)(\varepsilon_{ij} - \delta_{ij}\varepsilon_T). \tag{2.27b}$$

Due to Eq. (2.24), the stresses in the equivalent body will be the same as in the original one. After the strain ε_{ij}° is determined, Eq. (2.16) with the additional term ε_{ij}° in the right-hand side corresponding to the constant value E_r leads to the exact solution of the original elastic problem with variable modulus $E(T)$.

The strain ε_{ij}° is calculated using successive approximations

$$(\varepsilon_{ij}^{\circ})^{(k)} \approx \left(1 - \frac{E}{E^{r}}\right)\left[(\varepsilon_{ij})^{(k-1)} - \delta_{ij}\varepsilon_T\right], \tag{2.28}$$

starting with the initial value $(\varepsilon_{ij}^{\circ})^{(0)} = 0$ or $(\varepsilon_{ij})^{(0)} = (\varepsilon_{ij}^{r})^{(0)}$.

To ensure rapid convergence of solutions, the reduced modulus E_r has to satisfy the condition $E_r(T_r) \geq E(T)$. Once calculated, stresses and strains corresponding to external loads and thermal expansion remain constant for every approximation. The self-equilibrated stress system and the corresponding elastic strains caused by the field of additional strains ε_{ij}° are calculated at each step.

Due to the development of the FEM, the majority of modern practical problems of thermoelasticity can be solved numerically by FEM (see, i.e., Zienkiewicz and Taylor 2000).

2.2.3 Interpretation of Calculation Results

Calculations of thermoelastic stresses and strains which take into account temperature-dependent moduli of elasticity can be used in a variety of ways, for example:

- To reveal the most stressed points of a structure and estimate the loads and temperatures at which plastic deformation in these points can start;
- To estimate a critical point of a structure of a poorly plastic material, where fracture may occur without visible plastic strain. In this case, the maximum principle tensile stress is often used as a criterion of material failure;
- To calculate deformations, which can sometimes essentially influence working capacity of a structure (e.g., in case of elimination of a backlash between stator and rotating parts of a turbomachine or due to distortion of a telescope lens profile);
- To calculate dynamic stress and strain distribution for natural oscillation modes of non-uniformly heated machine parts; and
- To calculate the ranges of cyclic stress defining structure endurance (HCF limit) at forced vibrations.

There are various ways applied in engineering to decrease the thermal stresses.

For example, in gas turbines disks, due to the hot rim at rather cold hub, considerable temperature difference along the radius leads to large tensile stresses at the central part of the disk. The temperature field of the disk is adjusted by heating the disk nave with hot air by the compressor.

Various compensating joints (also known as compensators) are used in the long pipelines in order to absorb longitudinal stresses along the pipes.

Frame structures, in particular, bridge supports, should be rigidly fixed only at one end in order to allow free expansion at heating, etc.

2.3 Energy Principles

2.3.1 Work and Internal Energy of Elastic Deformation

2.3.1.1 Deformation Work

Let us consider a small cubic element of a solid of side dx, hence the face area is given by $dA = dx^2$. The cubic element is affected by the principal stresses σ_k where index $k = 1, 2, 3$. Displacement speeds for each of the main directions $\dot{u}_k = du_k/dt$ vary along the length of the element as $\dot{u}_{k,k}dx$, so that the stresses σ_k in the absence of mass forces (when $\dot{\sigma}_{k,k} = 0$) perform the work per unit time

$$d\dot{W}_{in} = \big[(\sigma_k + \sigma_{k,k}dx)(\dot{u}_k + \dot{u}_{k,k}dx) - \sigma_k\dot{u}_k\big]dA = \sigma_k\dot{u}_{k,k}dxdA = \sigma_k\dot{\varepsilon}_k dV,$$

where dV is the volume of a small element of the body.

Here the time, as usually in statics problems, is understood as the parameter characterizing the scenario of the processes of loading and heating.

In general case work over entire element per unit time, i.e., power will be

$$\dot{W} = \sigma_{ij}\dot{\varepsilon}_{ij}. \tag{2.29a}$$

Having integrated Eq. (2.29a) in time at *constant stresses*, we obtain the work

$$W = \sigma_{ij}\varepsilon_{ij}. \tag{2.29b}$$

Note that the strain ε_{ij}, as well as its speed $\dot{\varepsilon}_{ij}$, includes temperature expansion. Therefore, the work W is equal to all mechanical work, which is done on the element from other parts of the body during loading and heating.

The total work of deformation in case of *proportional increase* of stresses and full strains is

$$W = 0.5\sigma_{ij}\varepsilon_{ij} = 0.5(\delta_{ij}\sigma + s_{ij})(\delta_{ij}\varepsilon + e_{ij}) = 0.5(\sigma\theta + s_{ij}e_{ij}). \tag{2.30}$$

It consists of work $W_\theta = 0.5\sigma\theta$ which occurs when the volume of a body changes and a work $W_s = 0.5s_{ij}e_{ij}$ that corresponds to the changes of a body shape.

2.3.1.2 Internal Mechanical Energy of Elastic Deformation

According to the law of energy conservation, the total work done on an element during a small step loading, in the absence of a heat supply, is equal to the increase of its internal mechanical energy

$$dP_{mech} = dW = \sigma_{ij}d\varepsilon_{ij}. \tag{2.31a}$$

The tensor of full strain increment $d\varepsilon_{ij}$ does not depend on the physical nature of the process, being a purely geometric characteristic of the deformable continuous medium.

Within a finite loading step transferring a body from state 1 to state 2, the work done is

$$P_{mech} = W = \int_1^2 \sigma_{ij}d\varepsilon_{ij}. \tag{2.31b}$$

The integral (2.31b) also does not depend on the physical nature of the process and the type of loading $\sigma_{ij} = f(\varepsilon_{ij})$. In the particular case, when the stress increases proportionally to strain, Eq. (2.31b) becomes

$$P_{mech} = 0.5\sigma_{ij}\varepsilon_{ij}. \tag{2.31c}$$

For *isothermal* loading of an elastic body ($\varepsilon_T = 0$), the internal energy is equal to the potential energy of the deformation P which, according to the equations of elasticity (2.17) and (2.19), can be presented in two equivalent forms:

$$P_{mech} = P = G\varepsilon_{ij}\varepsilon_{ij} + \frac{3}{2}(K - 2G)\varepsilon^2, \tag{2.32a}$$

or

$$P_{mech} = P = \frac{1}{4G}\sigma_{ij}\sigma_{ij} + \frac{3}{2}\left(\frac{1}{K} - \frac{1}{2G}\right)\sigma^2. \tag{2.32b}$$

Obviously, the potential energy of elastic deformation can also be expressed in terms of the Young's modulus E and Poisson's factor μ. The magnitude of P_{mech} is unequivocally defined by the elastic strain reached to the moment and does not depend on the loading path.

For *non-isothermal* loading of an elastic body ($\varepsilon_T \neq 0$) when the heat is supplied to the system, the potential energy of elastic deformation P includes a work done during the temperature expansion, depending on conditions of heating or cooling in loading process, and differs from internal mechanical energy P_{mech}. This fact shall be taken into account while treating the law of energy conservation. The problems of thermoelasticity concerning the energy were considered in detail by Boley and Weiner (1960), Birger (1959a, b), and others.

Basic features of energy relations of the uncoupled theory of thermoelasticity will be illustrated below on simple examples.

2.3.1.3 Work of Thermal Expansion

Let us consider three situations for an elastic rod being loaded with the stress σ_1 and heated with the thermal expansion $\varepsilon_T = \alpha_T\Delta T$. Here, we assume that $\alpha_T = \text{const}$ and $E = \text{const}$.

In the first process, a rod is stretched by stress σ_1, and then, it is heated up; in the second, on the contrary, the rod is first heated up and then stretched; and in the third situation, the heating and stretching processes occur simultaneously. In all three cases, the final quantity of the full strain will be identical and equal to $\varepsilon_1 = \sigma_1/E + \varepsilon_T$.

Work increment will be defined according to Eq. (2.32b) as

$$dW = \sigma_1(t)\left(\frac{d\sigma_1}{E} + d\varepsilon_T\right).$$

In order to find the total work W, it is required to integrate this equation in time parameter t.

In the first case, consisting of steps 0–1 and 1–2

$$W = \int_0^1 \sigma_1(t)\frac{d\sigma_1}{E} + \sigma_1 \int_1^2 d\varepsilon_T = \frac{\sigma_1^2}{2E} + \sigma_1\varepsilon_T.$$

In the second case, there is no work done during the first step, resulting in the total work

$$W = \int_1^2 \sigma_1(t)\frac{d\sigma_1}{E} = \frac{\sigma_1^2}{2E}.$$

In the third case, under the assumption of identical rate of loading and heating, we arrive at

$$W = \int_0^2 \sigma_1(t)\left(\frac{d\sigma_1}{E} + d\varepsilon_T\right) = \frac{\sigma_1^2}{2E} + \frac{\sigma_1\varepsilon_T}{2}.$$

Thus, the independence of thermoelastic deformations of loading and heating order that has been specified above in Sect. 2.1.2.2, concerns only the estimation of total strain, but not the energy characteristics.

In a specific case of pure shear, the tangential stresses do not done the work under thermal expansion. This results in the identical amount of work at any sequence of loading and heating.

In papers on non-isothermal elasticity, the process of proportional heating and loading is usually assumed though the authors do not mention this explicitly. In this case, the work done on an element under uniaxial loading and heating can be presented in the following equivalent forms:

$$W = 0.5\sigma_1\varepsilon_1 = 0.5\left(\sigma_1^2/E + \sigma_1\varepsilon_T\right) = 0.5\left(E\varepsilon_1^2 - \varepsilon_T\varepsilon_1\right). \tag{2.33}$$

The work of the *constant* stress σ_1 at the thermal expansion ε_T will be further denoted as W_T and referred to as *the work of thermal expansion*. In the given example,

$$W_T = \sigma_1 \varepsilon_T. \tag{2.34}$$

It follows from comparison of Eqs. (2.33) and (2.34) that the potential energy of deformation, equal to work of elastic strain and expressed in terms of stresses, is defined by the difference

$$P = \frac{\sigma_1^2}{2E} = W - \frac{1}{2} W_T. \tag{2.35}$$

In the theories of thermoelasticity and thermoplasticity, the *additional work* of deformation is defined by the sum

$$P^* = W + \frac{1}{2} W_T. \tag{2.36}$$

It is obvious that $P + P^* = 2W = \sigma_1 \varepsilon_1$. Isothermal loading, characterized by $\varepsilon_T = 0$, yields $P = P^*$; for non-isothermal loading ($\varepsilon_T \neq 0$), these characteristics are different.

Having expressed the works W and W_T consecutively in terms of stresses and strains, and differentiating the obtained expressions accordingly by σ_1 and ε_1, we find that independently from thermal expansion

$$\frac{\partial P^*}{\partial \sigma_1} = \varepsilon_1, \quad \frac{\partial P}{\partial \varepsilon_1} = \sigma_1. \tag{2.37}$$

This justifies the introduction of those two functions.

Energy Balance

The following evident interpretation of maintaining energy balance is demonstrated in Fig. 2.6. The doubled work $2W = \sigma_1 \varepsilon_1$ done on an element is equal to the area of the rectangle consisting of four parts. Parts I and II, each of the area $P = 0.5 \sigma_1^2 / E$, are equal to the potential energy of elastic deformation of an element; parts III and IV,
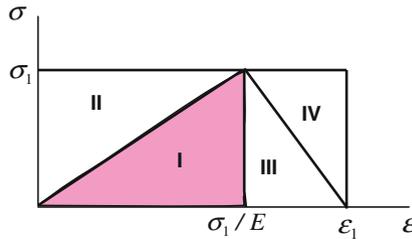

Fig. 2.6 Interpretation of energy balance under non-isothermal loading

each of the area $0.5W_T = 0.5\sigma_1\varepsilon_T$, are equal to half of the work of thermal loading. The sum of parts I or II and III or IV (in any combination) are equal to the total work W, which is in turn equal to the increase of the total internal energy of an element with heating taken into account.

It is obvious from Fig. 2.6 that for $\varepsilon_T \neq 0$, it is impossible to provide equality of the mechanical energy and work $P = W$ which also follows from the given above relations. But the sum of all parts of the work, shown in Fig. 2.6, gives the correct magnitude $2W$. Using the concept of additional work which compensates energy lack $W - P = 0.5W_T$ in P by increasing the energy in P^* at the same quantity $P^* = W + (W - P) = 2W - P$, we obtain in total the necessary energy balance. The magnitude of energy "lack" corresponds to the area of the part III, and additional work is equal to the sum of areas II + III + IV.

2.3.2 Variational Equations of Isothermal Elasticity

Variational methods based on energy relationships were widely applied in the "pre-computer" period for solving the problems of isothermal elasticity and then were extended to non-isothermal elasticity. Energy relationships are also often used in theoretical substantiations of FEM in software packages applied to non-isothermal problems, including nonelastic deformations. Therefore, it is useful to provide a brief demonstration of such approaches, taking into consideration the above-mentioned feature of energy relationships for both isothermal and non-isothermal cases.

2.3.2.1 Law of Mechanical Energy Conservation

The basic energy equation in isothermal theory connects the power (capacity) of all external forces applied to a body with the power (capacity) of the internal forces affecting all elements (see e.g., Kachanov 1948)

$$\int_S p_k \dot{u}_k dS + \int_V F_k \dot{u}_k dV = \int_V \sigma_{ij} \dot{\varepsilon}_{ij} dV, \tag{2.38}$$

where p_k, F_k are vectors of surface and volume forces, $\dot{u}_k$ is vector of displacements of the corresponding points, and σ_{ij} and $\dot{\varepsilon}_{ij}$ are tensors of stresses and strain speeds.

This formula makes use of the Eq. (2.29a) for work of internal forces. The integrals are taken over the body surface S and volume V.

Integration of the Eq. (2.38) in the time parameter t leads to

$$W_{S\Sigma} = W_{\Sigma}, \tag{2.39}$$

where

$$W_{S\Sigma} = \int\limits_0^t \left(\int\limits_S p_k \dot{u}_k dS + \int\limits_V F_k \dot{u}_k dV \right) dt \tag{2.40a}$$

is the total work of external forces, and

$$W_{\Sigma} = \int\limits_0^t \int\limits_V \sigma_{ij} \dot{\varepsilon}_{ij} dV dt \tag{2.40b}$$

is the total work of internal forces.

The important feature of Eqs. (2.38) and (2.39) is that they rest only on the conditions of continuity and body equilibrium and therefore remain valid for any mechanical properties of a material.

In statics, assuming $\dot{u}_k dt = du_k$ and $\dot{\varepsilon}_{ij} dt = d\varepsilon_{ij}$, Eq. (2.38) is represented in the form

$$\int\limits_S p_k u_k dS + \int\limits_V F_k u_k dV = \int\limits_V \sigma_{ij} \varepsilon_{ij} dV, \tag{2.41}$$

where the left-hand and right-hand sides of the equation correspond to the double value of the work of external and internal forces, respectively.

The total potential energy of elastic deformation of the body in case of $\varepsilon_T = 0$ is equal to the total work of internal forces, i.e., it is defined as

$$P_{\Sigma} = W_{\Sigma} = \int\limits_V P dV = 0.5 \int\limits_V \sigma_{ij} \varepsilon_{ij} dV. \tag{2.42}$$

Having expressed P according to Eqs. (2.16)–(2.19) in terms of strains or stresses, we arrive to the well-known relations

$$\frac{\partial P}{\partial \varepsilon_{ij}} = \sigma_{ij}, \quad \frac{\partial P}{\partial \sigma_{ij}} = \varepsilon_{ij}. \tag{2.43}$$

In the theory of elasticity, two variation types of Eq. (2.41) are considered.

2.3.2.2 Principle of Admissible Displacements

According to the principle of possible displacements, the kinematically possible (i.e., compatible to boundary conditions) displacements δu_k and deformations $\delta \varepsilon_{ij}$ vary, while external forces p_k, F_k and stresses σ_{ij} remain constant. Then, Eq. (2.41) yields

$$\int_S p_k \delta u_k dS + \int_V F_k \delta u_k dV = \int_V \sigma_{ij} \delta \varepsilon_{ij} dV. \tag{2.44a}$$

or

$$\delta W_{S\Sigma} = \delta W_\Sigma, \tag{2.44b}$$

where $\delta W_{S\Sigma}$ and δW_Σ are variations of the total work of external and internal forces, respectively.

For $\varepsilon_T = 0$, the magnitude δW_Σ is equal to the variation of the body potential energy of elastic deformation δP_Σ, expressed in terms of strains according to Eq. (2.32a).

Thus, $\delta W_{S\Sigma} = \delta P_\Sigma$ or

$$\delta(P_\Sigma - W_{S\Sigma}) = 0. \tag{2.45}$$

The difference $P_\Sigma - W_{S\Sigma}$ can be considered as the total potential energy of a body. Equation (2.45) shows that among kinematically admissible displacements of the points of the body, the real displacements correspond to minimum of the total potential energy.

2.3.2.3 Principle of Admissible Changes of a Stress State

According to the principle of possible changes of a stress state, statically possible (i.e., compatible to equilibrium and boundary conditions) stresses $\delta \sigma_{ij}$ and external forces δp_i, δF_i vary, while displacements u_i and strains ε_{ij} remain constant. Then,

$$\int_S u_i \delta p_i dS + \int_V u_i \delta F_i dV = \int_V \varepsilon_{ij} \delta \sigma_{ij} dV \tag{2.46a}$$

or

$$\delta W_{S\Sigma}^* = \delta W_\Sigma^*. \tag{2.46b}$$

Equations (2.46a, b) show that variations of "additional" works of external $\delta W_{S\Sigma}^*$ and internal δW_Σ^* forces are equal.

For $\varepsilon_T = 0$, variation δW_Σ^* standing in the right-hand side of Eqs. (2.46a, b) is equal to a variation of potential energy δP_Σ. Analysis similar to that carried out above shows that among statically possible stresses, the real stresses correspond to equality of variations of potential energy of a body and additional work of external forces.

2.3.3 Variational Equations of Thermoelasticity

Variational equations of thermoelasticity incorporating the temperature terms were considered in Kachanov (1948), Maysel (1951), and Argyris (1953) and were further developed by Hemp (1954), Birger (1959a), and others.

2.3.3.1 Potential Functions of Thermoelasticity

In general, the *work of thermal expansion* represented in case of uniaxial extension and heating by Eq. (2.34) is defined as

$$W_{T\Sigma} = \int_V \sigma_{ij}\delta_{ij}\varepsilon_T dV. \tag{2.47}$$

Temperature expansion ε_T does not vary, as the temperature field is assumed to be known, but with variation of stress σ_{ij}, the work of thermal expansion varies.

The basic energy Eq. (2.39) for $\varepsilon_T \neq 0$ may be presented as

$$W_{S\Sigma} = W_\Sigma = 0.5\left(P_\Sigma' + P_\Sigma''\right). \tag{2.48}$$

The functions P_Σ' and P_Σ'' are typically quoted as "potential functions of thermoelasticity" (Birger 1959b) defined as

$$P_\Sigma' = W_\Sigma - 0.5W_{T\Sigma} = 0.5 \int_V \sigma_{ij}\left(\varepsilon_{ij} - \delta_{ij}\varepsilon_T\right)dV, \tag{2.49a}$$

$$P_\Sigma'' = W_\Sigma + 0.5W_{T\Sigma} = P_\Sigma' + W_{T\Sigma} = 0.5 \int_V \sigma_{ij}\left(\varepsilon_{ij} + \delta_{ij}\varepsilon_T\right)dV. \tag{2.49b}$$

Expressing the function P_Σ' in terms of the strains by means of Eqs. (2.19), we obtain from Eq. (2.49a)

$$P'_\Sigma == \int\limits_V P' dV = \int\limits_V \left[G\varepsilon_{ij}\varepsilon_{ij} + \frac{3}{2}(K-2G)\varepsilon^2 - 3K\varepsilon\varepsilon_T \right] dV, \qquad (2.50)$$

hence

$$\frac{\partial P'}{\partial \varepsilon_{ij}} = \sigma_{ij}. \qquad (2.51)$$

Similarly, the function P''_Σ may be expressed in terms of the stresses through Eq. (2.17)

$$P''_\Sigma = \int\limits_V P'' dV = \frac{1}{2}\int\limits_V \left[\frac{1}{2G}\sigma_{ij}\sigma_{ij} + 3\left(\frac{1}{K} - \frac{1}{2G}\right)\sigma^2 + 3\sigma\varepsilon_T \right] dV, \qquad (2.52)$$

resulting in

$$\frac{\partial P''}{\partial \sigma_{ij}} = \varepsilon_{ij}. \qquad (2.53)$$

For $\varepsilon_T = 0$, the potential functions of thermoelasticity are $P' = P$, $P'' = P^*$. However, for $\varepsilon_T \neq 0$, these functions contain not only elastic, but also the strains including thermal expansion.

Further, Eq. (2.47) together with Eqs. (2.48) and (2.49a) varies either in strains or stresses.

2.3.3.2 Non-isothermal Principle of Admissible Displacements

According to the principle of admissible displacements, let us consider the variations of kinematically possible displacements δu_i and strains $\delta \varepsilon_{ij}$ under fixed external forces and stresses σ_{ij}. Here, the variation of work of thermal expansion is $\delta W_{T\Sigma} = 0$ and $\delta P'_\Sigma = \delta P_\Sigma$. Then,

$$\delta W_{S\Sigma} = \delta W_\Sigma = \delta P'_\Sigma = 0.5 \int\limits_V \sigma_{ij}\delta\varepsilon_{ij} dV$$

or

$$\delta(P_\Sigma - W_{S\Sigma}) = 0. \qquad (2.54)$$

This equation looks similar to Eq. (2.45) and states the condition for minimum of the full potential energy, whereas the stress in the expression for potential energy P_Σ is connected with the full deformations by Eq. (2.19).

2.3.3.3 The Non-isothermal Principle of Admissible Changes of Stress States

As in case of $\varepsilon_T = 0$, we consider the variation of statically admissible stress $\delta\sigma_{ij}$ and external forces δp_i, δF_i, while displacements u_i and strains ε_{ij} remain constant. The variation of additional work of external forces is defined as

$$\delta W_{S\Sigma} = \int_0^t \left(\int_S u_i \delta p_i dS + \int_V u_i \delta F_i dV \right) dt, \tag{2.55}$$

and the variation of the work of thermal expansion is given by

$$\delta W_{T\Sigma} = \int_V \delta_{ij} \delta\sigma_{ij} \varepsilon_T dV. \tag{2.56}$$

The stress variation $\delta\sigma_{ij}$ in Eq. (2.52) yields

$$\delta W_{S\Sigma} = \delta W_\Sigma = \delta P_\Sigma'' = 0.5 \int_V \varepsilon_{ij} \delta\sigma_{ij} dV$$

or

$$\delta\left(P_\Sigma'' - W_{S\Sigma} \right) = 0, \tag{2.57}$$

Thus, among statically admissible thermoelastic stresses, the real stresses satisfy the equality of variations of additional work (or potential thermoelasticity function P_Σ'' of a body) and additional work of external forces.

Under heating, in the absence of external forces when $W_{S\Sigma} = 0$, Eqs. (2.54) and (2.57) enable to determine the thermal SSS. If the state is expressed in strains, the condition of potential energy minimum $\delta P_\Sigma = 0$ is used; if the state is expressed in stresses, then the condition of a minimum of additional work $\delta P_\Sigma'' = 0$ is imposed.

Thermoelasticity energy correlations are used in a number of works, see for example, Birger (1959b) and Boley and Weiner (1960).

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