Tutorial on Admissible Rules in Gudauri

Rosalie Iemhoff(✉)

Department of Philosophy, Utrecht University,
Utrecht, The Netherlands
R.Iemhoff@uu.nl
http://www.phil.uu.nl/~iemhoff

Keywords: Logic · Admissible rules · Consequence relations

1 Introduction

Most theorems have more than one proof and most theories have more than one axiomatization. Certain proofs or axiomatizations are preferable to others because they are shorter or more transparent or for some other reason. Our aim is to describe or study the possible proofs of a theorem or the possible axiomatizations of a theory. As the former is a special instance of the latter, by considering a theory consisting of one theorem, it suffices to consider theories.

To describe the possible axiomatizations of a theory we first have to specify what we mean by a theory and what counts as an axiomatization of it. We assume that theories are given by consequence relations, and consider an arbitrary consequence relation to be an axiomatization of the theory if it has the same theorems as the consequence relation of the theory.

In [1] Avron argues convincingly that in general a logic is more than its set of theorems, meaning that there exist logics which have the same set of theorems but which nevertheless do not seem to be equal. For example, because the proofs of certain theorems differ with the logic. Then the question what counts as an axiomatization of a certain theory becomes more complex in that one wishes to axiomatize certain other characteristics of the theory, such as certain inference steps, rather than just its theorems.

In this paper, however, we restrict ourselves to the set of theorems as that part of a theory that an axiomatization has to capture. And as we will see, already in this case the variety of possible axiomatizations of a theory can be quite complicated and is in many cases not yet well-understood.

Thus our main aim is a description of the consequence relations that have the same theorems as a given consequence relation. As it turns out, admissible rules are the central notion here, where a rule is admissible in a theory if it can be added to a theory but no new theorems can be proved in the extension. Clearly, such extensions are axiomatizations of the original theory, which is why admissible rules are so important in this setting.

R. Iemhoff—Support by the Netherlands Organisation for Scientific Research under grant 639.032.918 is gratefully acknowledged.

M. Aher et al. (Eds.): TbiLLC 2013, LNCS 8984, pp. 10–17, 2015.
DOI: 10.1007/978-3-662-46906-4_2

The notion of admissibility, although sometimes under a different name, goes back to the 1930's, but a systematic study of the subject was first undertaken by Rybakov in the 1980's [24] and is continued by him and many others till today (see the bibliography for references). The first major results on this subject concerned the decidability of admissibility in certain intermediate and modal propositional logics, such as intuitionistic logic, modal logic K4, GL and S4. Later, the description of admissible rules in terms of bases was obtained for many of these logics and their fragments. Nowadays there are many aspects of admissibility that are studied. The work of Ghilardi [6] established a firm connection between admissibility and unification theory, and provided an algebraic approach to the issues discussed above. This algebraic approach to admissibility has flourished over the last decade and has been especially successful in the setting of substructural logics.

This paper is organized as follows. In Sect. 2 consequence relations and admissible rules are defined, and the main aim is formulated in these terms. Section 3 contains some of the main results in the area, a summary that, because of lack of space, is by no means complete. The paper ends with a brief discussion of topics that have been omitted in the main exposition. I thank Emil Jeřábek for useful comments on an earlier draft of this note.

2 Framework

To maintain a certain level of generality we assume that there is a language $\mathcal{L}$, which contains *propositional variables* or *atoms* $p, q, r, \dots$, and possibly some connectives, constants or operators. There is a set of *expressions* $\mathcal{F}_{\mathcal{L}}$ in this language that at least contains the propositional variables. In this way, what we discuss below applies to various consequence relation, such as consequence relations for propositional intermediate and modal logics, to mention the main examples. But also consequence relations that are relations on sequents rather than formulas are captured by this approach. Although some of what we are going to say also applies to predicate logics, we restrict ourselves in this paper to propositional logics. *Substitutions* σ are maps from $\mathcal{F}_{\mathcal{L}}$ to $\mathcal{F}_{\mathcal{L}}$ that commute with all logical symbols in the language.

2.1 Consequence Relations

Multi-conclusion consequence relations are relations $\vdash$ between sets of expressions. We write $\Gamma \vdash \Delta$ if the pair (Γ, Δ) belongs to the relation. We also write Γ/Δ for the pair (Γ, Δ), and A, Γ for $\{A\} \cup \Gamma$, and Γ, Π for $\Gamma \cup \Pi$. A *finitary multi-conclusion structural consequence relation (mcr)* is a relation $\vdash$ between finite sets of expressions that satisfies, for all finite sets of expressions $\Gamma, \Gamma', \Delta, \Delta'$ and expressions A:

reflexivity $A \vdash A$,
weakening if $\Gamma \vdash \Delta$, then $\Gamma', \Gamma \vdash \Delta, \Delta'$,

transitivity if $\Gamma \vdash \Delta, A$ and $\Gamma', A \vdash \Delta'$, then $\Gamma', \Gamma \vdash \Delta, \Delta'$,
structurality if $\Gamma \vdash \Delta$, then $\sigma\Gamma \vdash \sigma\Delta$ for all substitutions σ.

A *finitary single-conclusion consequence relation (scr)* is a relation between finite sets of expressions and expressions satisfying the variants of the three properties above where there is a singleton to the right of $\vdash$, and $\Gamma \vdash \{A\}$ is replaced by $\Gamma \vdash A$. We often omit the word "finitary" in what follows, and when we speak about "consequence relations" we refer to both multi-conclusion and single-conclusion ones.

Although most logics we discuss can be represented via a single-conclusion consequence relation, the multi-conclusion analogue allows us to express certain properties more naturally, such as the disjunction property. It follows from Proposition 1 below that an intermediate logic has the disjunction property if and only if $\{p \vee q\}/\{p, q\}$ is admissible, and similarly for modal logic and the modal disjunction property, expressed by the admissibility of $\{\Box p \vee \Box q\}/\{p, q\}$.

The minimal single-conclusion and multi-conclusion consequence relations $\vdash_{\overline{\text{m}}}$ and $\vdash_{\overline{\text{mm}}}$ are defined as follows.

$$\Gamma \vdash_{\overline{\text{m}}} A \equiv_{def} A \in \Gamma \qquad \Gamma \vdash_{\overline{\text{mm}}} \Delta \equiv_{def} \Gamma \cap \Delta \neq \emptyset.$$

A is a *theorem* if $\emptyset \vdash A$, which we write as $\vdash A$. The set of all theorems of a consequence relation is denoted by $\mathsf{Th}(\vdash)$. Δ is a *multi-conclusion theorem* if $\vdash \Delta$, which is short for $\emptyset \vdash \Delta$. The set of all multi-conclusion theorems is denoted by $\mathsf{Thm}(\vdash)$. When we speak about consequence relations in general we use the word *theorem*, meaning *theorem* in case the relation is single-conclusion and *multi-conclusion theorem* in case the relation is multi-conclusion.

Given a logic L with set of theorems $\mathsf{Th}(\mathsf{L})$, there are in general many multi-conclusion consequence relations $\vdash$ such that $\mathsf{Th}(\vdash) = \mathsf{Th}(\mathsf{L})$. Natural examples are

$$\Gamma \vdash \Delta \equiv_{def} \Delta \cap \mathsf{Th}(\mathsf{L}) \neq \emptyset,$$

or, in case the language contains implication and conjunction,

$$\Gamma \vdash \Delta \equiv_{def} \exists A \in \Delta \, (\bigwedge \Gamma \to A) \in \mathsf{Th}(\vdash).$$

Both these consequence relations are *saturated*, meaning that

$$\Gamma \vdash \Delta \Rightarrow \exists A \in \Delta \, \Gamma \vdash A.$$

Clearly, every single-conclusion consequence relation is saturated. And if one starts with a single-conclusion consequence relation or logic and wishes to associate a saturated multi-conclusion consequence relation with it (meaning with the same theorems as the single-conclusion consequence relation or logic), then the two consequence relations given in the previous paragraph provide examples. In the next section we encounter multi-conclusion consequence relations that are no longer saturated, such as the admissibility relation.

2.2 Admissible and Derivable Rules

A *(multi-conclusion) rule* is an ordered pair of finite sets of expressions, written Γ/Δ or $\frac{\Gamma}{\Delta}$. It is *single-conclusion* if $|\Delta| = 1$, in which case we also write Γ/A for $\Gamma/\{A\}$. For $R = \Gamma/\Delta$ and a substitution σ, σR is short for $\sigma\Gamma/\sigma\Delta$, and similarly for sets of rules.

Given a multi-conclusion consequence relation $\vdash$ and a set of rules $\mathcal{R}$, $\vdash^{\mathcal{R}}$ is the smallest consequence relation extending $\vdash$ for which $\Gamma \vdash \Delta$ holds for all Γ/Δ in $\mathcal{R}$. Similarly for single-conclusion rules and single-conclusion consequence relations. In case of a single rule R we write $\vdash^R$ for $\vdash^{\{R\}}$. Given a consequence relation $\vdash$, a set of rules $\mathcal{R}$ is a *basis* for a consequence relation $\vdash' \supseteq \vdash$ or *axiomatizes* $\vdash'$ *over* $\vdash$ if $\vdash' = \vdash^{\mathcal{R}}$. A rule $R = \Gamma/\Delta$ is *derivable* if $\Gamma \vdash \Delta$. It is *admissible*, written $\Gamma \mathrel{|\!\!\sim} \Delta$, if $\mathsf{Thm}(\vdash) = \mathsf{Thm}(\vdash^R)$, and $\mathsf{Th}(\vdash) = \mathsf{Th}(\vdash^R)$ in case $\vdash$ and R are single-conclusion. A set of rules is admissible if all of its members are.

As can be seen from the definition, a rule is admissible when one can add it to the consequence relation without obtaining new theorems, just (possibly) new derivations. This shows that admissibility solely depends on the theorems of a consequence relation, while derivability does not. The admissibility relation $\mathrel{|\!\!\sim}$ itself is a consequence relation, namely the largest consequence relation with the same theorems as $\vdash$. Therefore, the main topic of this paper, the possible axiomatizations of a theory, can now be reformulated in exact terms as the admissible rules of consequence relations.

The following proposition provides the link between admissibility and unification.

Proposition 1. For every saturated consequence relation $\vdash$,

$$\Gamma \mathrel{|\!\!\sim} \Delta \;\Leftrightarrow\; \forall\sigma:\; \forall A \in \Gamma\,(\vdash \sigma A) \;\Rightarrow\; \exists B \in \Delta\,(\vdash \sigma B).$$

Therefore every single-conclusion consequence relation satisfies

$$\Gamma \mathrel{|\!\!\sim} A \;\Leftrightarrow\; \forall\sigma:\; \forall B \in \Gamma\,(\vdash \sigma B) \;\Rightarrow\; \vdash \sigma A.$$

In the literature admissibility is often defined via the equivalence above.

A single-conclusion consequence relation $\vdash$ is *structurally complete* [19] if all proper extensions in the same language have new theorems. It is not difficult to see that $\vdash$ is structurally complete if and only if it coincides with $\mathrel{|\!\!\sim}$. Thus structural completeness means that there are no “hidden” principles of inference, no underivable admissible rules, all valid inferences are already captured by the consequence relation itself.

3 Results

Classical propositional logic as well as a certain formulation of classical predicate logic in which substitution is an explicit rule, are structurally complete [19,20]. Or, to be precise, for any rule Γ/A admissible in classical logic, $(\bigwedge\Gamma \to A)$ is a theorem of classical logic, and therefore Γ/A is derivable in any consequence relation for classical logic in which the deduction theorem holds. Nonderivable admissible rules appear as soon as one turns from classical logic to extensions such as modal logic or weaker logics such as intermediate logics. There do exist, though, some proper intermediate and modal logics that are structurally complete, Gödel-Dummett logic LC being an example [5].

3.1 Decidability

Rybakov proved numerous results on admissibility, most importantly the decidability of the admissibility relation of intuitionistic propositional logic IPC, the modal logics K4, GL, S4 and several other intermediate and modal logics [24]. He thereby answered a question by Harvey Friedman from 1975 about the decidability of admissibility in intuitionistic logic positively. Rybakov's method can be adapted to many other logics, as has been done in [2,18,25,26], where the decidability of admissibility in various temporal logics and minimal logic is established. Ghilardi constructed a transparent algorithm for deciding admissibility in IPC [7], and Metcalfe and the author developed proof systems for admissibility for several well-known intermediate and modal logics, from which decision algorithms can be obtained as well [11,12]. Jeřábek proved that the complexity of the admissibility relation is coNEXP-complete in many modal and intermediate logics such as K4, S4, GL and IPC [15], thus showing that in these logics checking admissibility is strictly more complex than checking derivability.

Derivability is a special case of admissibility, and therefore decidability of the latter implies the decidability of theoremhood in the former. That the other direction does not hold has been shown in [3], and later also in [34], where certain modal logics are shown to be instances of this phenomenon.

3.2 Bases

An explicit description of the admissible rules is a next step in the investigation of logics for which the admissibility relation is decidable. Even in the case that admissibility is undecidable it cannot be excluded that there exists a useful description of them, but until now the logics for which such an explicit description has been found all have a decidable admissibility relation.

Rybakov in [24] showed that various modal and intermediate logics, including IPC and K4, cannot have a finite basis for their admissible rules. This, of course, does not imply that these logics do not have an infinite basis that still can be described in a compact way. As we will see, they often do.

Roziére [23] was the first to provide a concrete basis for the admissible rules for a logic for which the problem is not trivial, by proving that the set V of the so-called *Visser rules* is a basis for the admissible rules of IPC. This result was not published and was independently but later obtained by the author, who, using techniques from [6], strengthened it by showing that in every intermediate logic in which these rules are admissible they form a basis [10]. This theorem has implications for several intermediate logics. It implies, for example, that the rules are a basis for the admissible rules in the logics of frames with exactly n maximal nodes. In particular, they are a basis for KC.

The Visser rules also appeared in the work of Visser [30,31], who proved that the admissible rules of IPC and Heyting Arithmetic are equal, and Skura [27], who used them in the context of refutation systems. Examples of intermediate logics in which not all Visser Rules are admissible are the Gabbay–de Jongh logics [9] and Medvedev logic, which is structurally complete [10,22,33].

Using similar techniques, Jeřábek provided bases for many transitive modal logics, including well-known logics such as K4, S4 and GL [14]. For modal logics below K4 much less is known about admissibility. Some partial answers can be found in [16,32].

As one would expect, admissibility is very sensitive to the language one uses. It has long been known that the implicational fragment of IPC is hereditarily structurally complete [21]. The same holds for the implication–conjunction and some other fragments of IPC [17,29]. In [17] Mints showed that any admissible underivable rule of IPC must contain both implication and disjunction. Interestingly, the implication–negation fragment of IPC is not structurally complete, as was first observed by Wroński. In [4] Cintula and Metcalfe proved that the so-called *Wroński Rules* are a basis for the admissible rules of this fragment. A nontrivial example of a logic for which the implication–negation fragment is structurally complete is relevant logic [28].

4 Furthermore

The above is but a brief summary of some of the highlights in the area of admissibility. I have mainly covered the topics that I have treated in my tutorial in beautiful Gudauri. Several equally important aspects of admissibility have been omitted due to lack of space. Over the last twenty years, admissibility has been studied in various other contexts than the ones mentioned above, such as substructural logics, canonical rules and predicate logic. Unification theory has been central in some of the results described above. Also, the algebraic view on admissibility has been explored and lead to various beautiful results. I hope that the exposition above has made the reader wish to know more about this field and that the bibliography may provide a guideline towards that aim.

References

1. Avron, A.: Simple consequence relations. Inf. Comput. **92**(1), 105–139 (1991)
2. Babenyshev, S., Rybakov, V.V.: Linear temporal logic LTL: basis for admissible rules. J. Log. Comput. **21**(2), 157–177 (2011)
3. Chagrov, A.: A decidable modal logic with undecidable admissibility problem. Algebra Log. **31**(1), 53–61 (1992). In Russian
4. Cintula, P., Metcalfe, G.: Admissible rules in the implication-negation fragment of intuitionistic logic. Ann. Pure Appl. Log. **162**(2), 162–171 (2010)
5. Dzik, W., Wroński, A.: Structural completeness of Gödel and Dummett's propositional calculi. Stud. Log. **32**, 69–73 (1973)

6. Ghilardi, S.: Unification in intuitionistic logic. J. Symb. Log. **64**(2), 859–880 (1999)
7. Ghilardi, S.: A resolution/tableaux algorithm for projective approximations in IPC. Log. J. IGPL **10**(3), 227–241 (2002)
8. Ghilardi, S.: Unification, finite duality and projectivity in varieties of Heyting algebras. Ann. Pure Appl. Log. **127**(1–3), 99–115 (2004)
9. Goudsmit, J.G., Iemhoff, R.: On unification and admissible rules in Gabbay-de Jongh logics. Ann. Pure Appl. Log. **165**(2), 652–672 (2014)
10. Iemhoff, R.: Intermediate logics and Visser's rules. Notre Dame J. Formal Log. **46**(1), 65–81 (2005)
11. Iemhoff, R., Metcalfe, G.: Proof theory for admissible rules. Ann. Pure Appl. Log. **159**(1–2), 171–186 (2009)
12. Iemhoff, R., Metcalfe, G.: Hypersequent systems for the admissible rules of modal and intermediate logics. In: Artemov, S., Nerode, A. (eds.) LFCS 2009. LNCS, vol. 5407, pp. 230–245. Springer, Heidelberg (2008)
13. Iemhoff, R., Rozière, P.: Unification in fragments of intermediate logics. J. Symb. Logic (to appear)
14. Jeřábek, E.: Admissible rules of modal logics. J. Log. Comput. **15**(4), 411–431 (2005)
15. Jeřábek, E.: Complexity of admissible rules. Arch. Math. Log. **46**, 73–92 (2007)
16. Jeřábek, E.: Blending margins: the modal logic K has nullary unification type. J. Logic Comput. (to appear)
17. Mints, G.: Derivability of admissible rules. In: Studies in Constructive Mathematics and Mathematical Logic. Zap. Nauchn. Sem. LOMI, part V, vol. 32, pp. 85–89. Nauka, Leningrad (1972)
18. Odintsov, S., Rybakov, V.V.: Unification and admissible rules for paraconsistent minimal Johansson's logic J and positive intuitionistic logic IPC^+. Ann. Pure Appl. Log. **164**(7–8), 771–784 (2013)
19. Pogorzelski, W.A.: Structural completeness of the propositional calculus. Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques **19**, 349–351 (1971)
20. Pogorzelski, W.A., Prucnal, T.: Structural completeness of the first-order predicate calculus. Zeitschrift für mathematische Logik und Grundlagen der Mathematik **21**(1), 315–320 (1975)
21. Prucnal, T.: On the structural completeness of some pure implicational propositional calculi. Stud. Log. **32**(1), 45–50 (1973)
22. Prucnal, T.: Structural completeness of Medvedev's propositional calculus. Rep. Math. Log. **6**, 103–105 (1976)
23. Rozière, P.: Règles admissibles en calcul propositionnel intuitionniste. Ph.D. thesis, Université Paris VII (1992)
24. Rybakov, V.: Admissibility of Logical Inference Rules. Elsevier, Amsterdam (1997)
25. Rybakov, V.: Rules admissible in transitive temporal logic T_{S4}, sufficient condition. Theoret. Comput. Sci. **411**(50), 4323–4332 (2010)
26. Rybakov, V.: Writing out unifiers in linear temporal logic. J. Log. Comput. **22**(5), 1199–1206 (2012)
27. Skura, R.: A complete syntactical characterization of the intuitionistic logic. Rep. Math. Log. **23**, 75–80 (1989)
28. Slaney, J., Meyer, R.: A structurally complete fragment of relevant logic. Notre Dame J. Formal Log. **33**(4), 561–566 (1992)
29. Słomczyńska, K.: Algebraic semantics for the $(\leftrightarrow, \neg\neg)$-fragment of IPC. Math. Log. Q. **58**(12), 29–37 (2012)

30. Visser, A.: Rules and arithmetics. Notre Dame J. Formal Log. **40**(1), 116–140 (1999)
31. Visser, A.: Substitutions of Σ^0_1-sentences: explorations between intuitionistic propositional logic and intuitionistic arithmetic. Ann. Pure Appl. Log. **114**(1–3), 227–271 (2002)
32. Williamson, T.: An alternative rule of disjunction in modal logic. Notre Dame J. Formal Log. **33**(1), 89–100 (1992)
33. Wojtylak, P.: On a problem of H. Friedman and its solution by T. Prucnal. Rep. Math. Log. **38**, 69–86 (2004)
34. Wolter, F., Zakharyaschev, M.: Undecidability of the unification and admissibility problems for modal and description logics. ACM Trans. Comput. Log. **9**(4), 1–20 (2008). Article 25

http://www.springer.com/978-3-662-46905-7

Logic, Language, and Computation
10th International Tbilisi Symposium on Logic, Language, and Computation, TbiLLC 2013, Gudauri, Georgia, September 23-27, 2013. Revised Selected Papers
Aher, M.; Hole, D.; Jeřábek, E.; Kupke, C. (Eds.)
2015, XIV, 333 p. 47 illus., Softcover
ISBN: 978-3-662-46905-7