1 Arithmetics ...................................................... 1
  1.1 Elementary Rules for Calculations ....................... 1
    1.1.1 Numbers ............................................. 1
      1.1.1.1 Natural, Integer, and Rational Numbers .............. 1
      1.1.1.2 Irrational and Transcendental Numbers .......... 2
      1.1.1.3 Real Numbers .................................. 2
      1.1.1.4 Continued Fractions .......................... 3
      1.1.1.5 Commensurability ............................. 4
    1.1.2 Methods for Proof ................................ 4
      1.1.2.1 Direct Proof .................................. 5
      1.1.2.2 Indirect Proof or Proof by Contradiction ....... 5
      1.1.2.3 Mathematical Induction ....................... 5
      1.1.2.4 Constructive Proof ........................... 6
    1.1.3 Sums and Products ................................ 6
      1.1.3.1 Sums .......................................... 6
      1.1.3.2 Products ..................................... 7
    1.1.4 Powers, Roots, and Logarithms ....................... 7
      1.1.4.1 Powers ........................................ 7
      1.1.4.2 Roots ......................................... 8
      1.1.4.3 Logarithms .................................... 9
      1.1.4.4 Special Logarithms ............................ 9
    1.1.5 Algebraic Expressions ................................ 10
      1.1.5.1 Definitions .................................... 10
      1.1.5.2 Algebraic Expressions in Detail ............... 11
    1.1.6 Integral Rational Expressions ....................... 11
      1.1.6.1 Representation in Polynomial Form .............. 11
      1.1.6.2 Factoring Polynomials ......................... 11
      1.1.6.3 Special Formulas .............................. 12
      1.1.6.4 Binomial Theorem ............................. 12
      1.1.6.5 Determination of the Greatest Common Divisor of Two Polynomials 14
    1.1.7 Rational Expressions ................................ 14
      1.1.7.1 Reducing to the Simplest Form .................. 14
      1.1.7.2 Determination of the Integral Rational Part .... 15
      1.1.7.3 Partial Fraction Decomposition ................. 15
      1.1.7.4 Transformations of Proportions ................. 17
    1.1.8 Irrational Expressions ................................ 17
  1.2 Finite Series .............................................. 18
    1.2.1 Definition of a Finite Series ....................... 18
    1.2.2 Arithmetic Series .................................. 18
    1.2.3 Geometric Series .................................. 19
    1.2.4 Special Finite Series ............................. 19
    1.2.5 Mean Values ....................................... 19
      1.2.5.1 Arithmetic Mean or Arithmetic Average .......... 19
      1.2.5.2 Geometric Mean or Geometric Average .......... 20
      1.2.5.3 Harmonic Mean ................................ 20
      1.2.5.4 Quadratic Mean ............................... 20
1.2.5.5 Relations Between the Means of Two Positive Values

1.3 Business Mathematics

1.3.1 Calculation of Interest or Percentage

1.3.1.1 Percentage or Interest

1.3.1.2 Increment

1.3.1.3 Discount or Reduction

1.3.2 Calculation of Compound Interest

1.3.2.1 Interest

1.3.2.2 Compound Interest

1.3.3 Amortization Calculus

1.3.3.1 Amortization

1.3.3.2 Equal Principal Repayments

1.3.3.3 Equal Annuities

1.3.4 Annuity Calculations

1.3.4.1 Annuities

1.3.4.2 Future Amount of an Ordinary Annuity

1.3.4.3 Balance after $n$ Annuity Payments

1.3.5 Depreciation

1.3.5.1 Methods of Depreciation

1.3.5.2 Straight-Line Method

1.3.5.3 Arithmetically Declining Balance Depreciation

1.3.5.4 Digital Declining Balance Depreciation

1.3.5.5 Geometrically Declining Balance Depreciation

1.3.5.6 Depreciation with Different Types of Depreciation Account

1.4 Inequalities

1.4.1 Pure Inequalities

1.4.1.1 Definitions

1.4.1.2 Properties of Inequalities of Type I and II

1.4.2 Special Inequalities

1.4.2.1 Triangle Inequality for Real Numbers

1.4.2.2 Triangle Inequality for Complex Numbers

1.4.2.3 Inequalities for Absolute Values of Differences of Real and Complex Numbers

1.4.2.4 Inequality for Arithmetic and Geometric Means

1.4.2.5 Inequality for Arithmetic and Quadratic Means

1.4.2.6 Inequalities for Different Means of Real Numbers

1.4.2.7 Bernoulli’s Inequality

1.4.2.8 Binomial Inequality

1.4.2.9 Cauchy-Schwarz Inequality

1.4.2.10 Chebyshev Inequality

1.4.2.11 Generalized Chebyshev Inequality

1.4.2.12 Hölder Inequality

1.4.2.13 Minkowski Inequality

1.4.3 Solution of Linear and Quadratic Inequalities

1.4.3.1 General Remarks

1.4.3.2 Linear Inequalities

1.4.3.3 Quadratic Inequalities

1.4.3.4 General Case for Inequalities of Second Degree

1.5 Complex Numbers

1.5.1 Imaginary and Complex Numbers

1.5.1.1 Imaginary Unit

1.5.1.2 Complex Numbers
1.5.2 Geometric Representation ........................................... 34
1.5.2.1 Vector Representation ......................................... 34
1.5.2.2 Equality of Complex Numbers ................................. 35
1.5.2.3 Trigonometric Form of Complex Numbers ..................... 35
1.5.2.4 Exponential Form of a Complex Number ....................... 36
1.5.2.5 Conjugate Complex Numbers .................................. 36
1.5.3 Calculation with Complex Numbers ............................... 36
1.5.3.1 Addition and Subtraction ...................................... 36
1.5.3.2 Multiplication .................................................... 37
1.5.3.3 Division .......................................................... 37
1.5.3.4 General Rules for the Basic Operations ....................... 37
1.5.3.5 Taking Powers of Complex Numbers ............................ 38
1.5.3.6 Taking the $n$-th Root of a Complex Number ................. 38
1.6 Algebraic and Transcendental Equations .......................... 38
1.6.1 Transforming Algebraic Equations to Normal Form ............. 38
1.6.1.1 Definition ....................................................... 38
1.6.1.2 Systems of $n$ Algebraic Equations .......................... 39
1.6.1.3 Extraneous Roots .............................................. 39
1.6.2 Equations of Degree at Most Four ............................... 39
1.6.2.1 Equations of Degree One (Linear Equations) ................. 39
1.6.2.2 Equations of Degree Two (Quadratic Equations) ............. 40
1.6.2.3 Equations of Degree Three (Cubic Equations) ............... 40
1.6.2.4 Equations of Degree Four .................................... 42
1.6.2.5 Equations of Higher Degree .................................. 43
1.6.3 Equations of Degree $n$ ........................................... 43
1.6.3.1 General Properties of Algebraic Equations .................. 43
1.6.3.2 Equations with Real Coefficients ............................ 44
1.6.4 Reducing Transcendental Equations to Algebraic Equations ... 45
1.6.4.1 Definition ....................................................... 45
1.6.4.2 Exponential Equations ....................................... 46
1.6.4.3 Logarithmic Equations ....................................... 46
1.6.4.4 Trigonometric Equations .................................... 46
1.6.4.5 Equations with Hyperbolic Functions ........................ 47

2 Functions 48
2.1 Notion of Functions .................................................... 48
2.1.1 Definition of a Function ......................................... 48
2.1.1.1 Function ....................................................... 48
2.1.1.2 Real Functions ............................................... 48
2.1.1.3 Functions of Several Variables ............................... 48
2.1.1.4 Complex Functions .......................................... 48
2.1.1.5 Further Functions ............................................ 48
2.1.1.6 Functionals .................................................... 48
2.1.1.7 Functions and Mappings ..................................... 49
2.1.2 Methods for Defining a Real Function .......................... 49
2.1.2.1 Defining a Function ......................................... 49
2.1.2.2 Analytic Representation of a Function ....................... 49
2.1.3 Certain Types of Functions ..................................... 50
2.1.3.1 Monotone Functions ......................................... 50
2.1.3.2 Bounded Functions .......................................... 51
2.1.3.3 Extreme Values of Functions ............................... 51
2.1.3.4 Even Functions ................................................ 51
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1.3.5</td>
<td>Odd Functions</td>
<td>51</td>
</tr>
<tr>
<td>2.1.3.6</td>
<td>Representation with Even and Odd Functions</td>
<td>52</td>
</tr>
<tr>
<td>2.1.3.7</td>
<td>Periodic Functions</td>
<td>52</td>
</tr>
<tr>
<td>2.1.3.8</td>
<td>Inverse Functions</td>
<td>52</td>
</tr>
<tr>
<td>2.1.4</td>
<td>Limits of Functions</td>
<td>53</td>
</tr>
<tr>
<td>2.1.4.1</td>
<td>Definition of the Limit of a Function</td>
<td>53</td>
</tr>
<tr>
<td>2.1.4.2</td>
<td>Definition by Limit of Sequences</td>
<td>53</td>
</tr>
<tr>
<td>2.1.4.3</td>
<td>Cauchy Condition for Convergence</td>
<td>53</td>
</tr>
<tr>
<td>2.1.4.4</td>
<td>Infinity as a Limit of a Function</td>
<td>53</td>
</tr>
<tr>
<td>2.1.4.5</td>
<td>Left-Hand and Right-Hand Limit of a Function</td>
<td>54</td>
</tr>
<tr>
<td>2.1.4.6</td>
<td>Limit of a Function as ( x ) Tends to Infinity</td>
<td>54</td>
</tr>
<tr>
<td>2.1.4.7</td>
<td>Theorems About Limits of Functions</td>
<td>55</td>
</tr>
<tr>
<td>2.1.4.8</td>
<td>Calculation of Limits</td>
<td>55</td>
</tr>
<tr>
<td>2.1.4.9</td>
<td>Order of Magnitude of Functions and Landau Order Symbols</td>
<td>57</td>
</tr>
<tr>
<td>2.1.5</td>
<td>Continuity of a Function</td>
<td>58</td>
</tr>
<tr>
<td>2.1.5.1</td>
<td>Notion of Continuity and Discontinuity</td>
<td>58</td>
</tr>
<tr>
<td>2.1.5.2</td>
<td>Definition of Continuity</td>
<td>58</td>
</tr>
<tr>
<td>2.1.5.3</td>
<td>Most Frequent Types of Discontinuities</td>
<td>59</td>
</tr>
<tr>
<td>2.1.5.4</td>
<td>Continuity and Discontinuity of Elementary Functions</td>
<td>60</td>
</tr>
<tr>
<td>2.1.5.5</td>
<td>Properties of Continuous Functions</td>
<td>60</td>
</tr>
<tr>
<td>2.2</td>
<td>Elementary Functions</td>
<td>62</td>
</tr>
<tr>
<td>2.2.1</td>
<td>Algebraic Functions</td>
<td>62</td>
</tr>
<tr>
<td>2.2.1.1</td>
<td>Polynomials</td>
<td>62</td>
</tr>
<tr>
<td>2.2.1.2</td>
<td>Rational Functions</td>
<td>62</td>
</tr>
<tr>
<td>2.2.1.3</td>
<td>Irrational Functions</td>
<td>62</td>
</tr>
<tr>
<td>2.2.2</td>
<td>Transcendental Functions</td>
<td>62</td>
</tr>
<tr>
<td>2.2.2.1</td>
<td>Exponential Functions</td>
<td>62</td>
</tr>
<tr>
<td>2.2.2.2</td>
<td>Logarithmic Functions</td>
<td>63</td>
</tr>
<tr>
<td>2.2.2.3</td>
<td>Trigonometric Functions</td>
<td>63</td>
</tr>
<tr>
<td>2.2.2.4</td>
<td>Inverse Trigonometric Functions</td>
<td>63</td>
</tr>
<tr>
<td>2.2.2.5</td>
<td>Hyperbolic Functions</td>
<td>63</td>
</tr>
<tr>
<td>2.2.2.6</td>
<td>Inverse Hyperbolic Functions</td>
<td>63</td>
</tr>
<tr>
<td>2.2.3</td>
<td>Composite Functions</td>
<td>63</td>
</tr>
<tr>
<td>2.3</td>
<td>Polynomials</td>
<td>63</td>
</tr>
<tr>
<td>2.3.1</td>
<td>Linear Function</td>
<td>63</td>
</tr>
<tr>
<td>2.3.2</td>
<td>Quadratic Polynomial</td>
<td>64</td>
</tr>
<tr>
<td>2.3.3</td>
<td>Cubic Polynomials</td>
<td>64</td>
</tr>
<tr>
<td>2.3.4</td>
<td>Polynomials of ( n )-th Degree</td>
<td>65</td>
</tr>
<tr>
<td>2.3.5</td>
<td>Parabola of ( n )-th Degree</td>
<td>66</td>
</tr>
<tr>
<td>2.4</td>
<td>Rational Functions</td>
<td>66</td>
</tr>
<tr>
<td>2.4.1</td>
<td>Special Fractional Linear Function (Inverse Proportionality)</td>
<td>66</td>
</tr>
<tr>
<td>2.4.2</td>
<td>Linear Fractional Function</td>
<td>66</td>
</tr>
<tr>
<td>2.4.3</td>
<td>Curves of Third Degree, Type I</td>
<td>67</td>
</tr>
<tr>
<td>2.4.4</td>
<td>Curves of Third Degree, Type II</td>
<td>67</td>
</tr>
<tr>
<td>2.4.5</td>
<td>Curves of Third Degree, Type III</td>
<td>68</td>
</tr>
<tr>
<td>2.4.6</td>
<td>Reciprocal Powers</td>
<td>70</td>
</tr>
<tr>
<td>2.5</td>
<td>Irrational Functions</td>
<td>71</td>
</tr>
<tr>
<td>2.5.1</td>
<td>Square Root of a Linear Binomial</td>
<td>71</td>
</tr>
<tr>
<td>2.5.2</td>
<td>Square Root of a Quadratic Polynomial</td>
<td>71</td>
</tr>
<tr>
<td>2.5.3</td>
<td>Power Function</td>
<td>71</td>
</tr>
<tr>
<td>2.6</td>
<td>Exponential Functions and Logarithmic Functions</td>
<td>72</td>
</tr>
<tr>
<td>2.6.1</td>
<td>Exponential Functions</td>
<td>72</td>
</tr>
</tbody>
</table>
2.6.2 Logarithmic Functions ............................................. 73
2.6.3 Error Curve ......................................................... 73
2.6.4 Exponential Sum ................................................... 74
2.6.5 Generalized Error Function ........................................ 74
2.6.6 Product of Power and Exponential Functions ................. 75
2.7 Trigonometric Functions (Functions of Angles) .................... 76
2.7.1 Basic Notions ....................................................... 76
  2.7.1.1 Definition and Representation ............................. 76
  2.7.1.2 Range and Behavior of the Functions .................... 79
2.7.2 Important Formulas for Trigonometric Functions ............... 81
  2.7.2.1 Relations Between the Trigonometric Functions ......... 81
  2.7.2.2 Trigonometric Functions of the Sum and Difference of Two Angles (Addition Theorems) .................. 81
  2.7.2.3 Trigonometric Functions of an Integer Multiple of an Angle .......... 81
  2.7.2.4 Trigonometric Functions of Half-Angles ................... 82
  2.7.2.5 Sum and Difference of Two Trigonometric Functions .... 83
  2.7.2.6 Products of Trigonometric Functions ....................... 83
  2.7.2.7 Powers of Trigonometric Functions ........................ 83
2.7.3 Description of Oscillations ...................................... 84
  2.7.3.1 Formulation of the Problem ................................. 84
  2.7.3.2 Superposition of Oscillations .............................. 84
  2.7.3.3 Vector Diagram for Oscillations ........................... 85
  2.7.3.4 Damping of Oscillations .................................. 85
2.8 Cyclometric or Inverse Trigonometric Functions ................... 85
  2.8.1 Definition of the Inverse Trigonometric Functions ........... 85
  2.8.2 Reduction to the Principal Value ................................ 86
  2.8.3 Relations Between the Principal Values .................... 87
  2.8.4 Formulas for Negative Arguments ............................. 87
  2.8.5 Sum and Difference of arcsin x and arcsin y ............... 88
  2.8.6 Sum and Difference of arccos x and arccos y ............... 88
  2.8.7 Sum and Difference of arctan x and arctan y ............... 88
  2.8.8 Special Relations for arcsin x, arccos x, arctan x ........ 88
2.9 Hyperbolic Functions ............................................... 89
  2.9.1 Definition of Hyperbolic Functions ........................... 89
  2.9.2 Graphical Representation of the Hyperbolic Functions ....... 89
    2.9.2.1 Hyperbolic Sine ........................................... 89
    2.9.2.2 Hyperbolic Cosine ........................................ 89
    2.9.2.3 Hyperbolic Tangent ....................................... 90
    2.9.2.4 Hyperbolic Cotangent .................................... 90
  2.9.3 Important Formulas for the Hyperbolic Functions ............. 91
    2.9.3.1 Hyperbolic Functions of One Variable ................... 91
    2.9.3.2 Expressing a Hyperbolic Function by Another One with the Same Argument ..................... 91
    2.9.3.3 Formulas for Negative Arguments .......................... 91
    2.9.3.4 Hyperbolic Functions of the Sum and Difference of Two Arguments (Addition Theorems) ............ 91
    2.9.3.5 Hyperbolic Functions of Double Arguments .............. 92
    2.9.3.6 De Moivre Formula for Hyperbolic Functions ............ 92
    2.9.3.7 Hyperbolic Functions of Half-Argument .................. 92
    2.9.3.8 Sum and Difference of Hyperbolic Functions ............. 92
    2.9.3.9 Relation Between Hyperbolic and Trigonometric Functions with Complex Arguments z ............... 92
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.10</td>
<td>Area Functions</td>
<td>93</td>
</tr>
<tr>
<td>2.10.1</td>
<td>Definitions</td>
<td>93</td>
</tr>
<tr>
<td>2.10.1.1</td>
<td>Area Sine</td>
<td>93</td>
</tr>
<tr>
<td>2.10.1.2</td>
<td>Area Cosine</td>
<td>93</td>
</tr>
<tr>
<td>2.10.1.3</td>
<td>Area Tangent</td>
<td>94</td>
</tr>
<tr>
<td>2.10.1.4</td>
<td>Area Cotangent</td>
<td>94</td>
</tr>
<tr>
<td>2.10.2</td>
<td>Determination of Area Functions Using Natural Logarithm</td>
<td>94</td>
</tr>
<tr>
<td>2.10.3</td>
<td>Relations Between Different Area Functions</td>
<td>94</td>
</tr>
<tr>
<td>2.10.4</td>
<td>Sum and Difference of Area Functions</td>
<td>95</td>
</tr>
<tr>
<td>2.10.5</td>
<td>Formulas for Negative Arguments</td>
<td>95</td>
</tr>
<tr>
<td>2.11</td>
<td>Curves of Order Three (Cubic Curves)</td>
<td>95</td>
</tr>
<tr>
<td>2.11.1</td>
<td>Semicubic Parabola</td>
<td>95</td>
</tr>
<tr>
<td>2.11.2</td>
<td>Witch of Agnesi</td>
<td>95</td>
</tr>
<tr>
<td>2.11.3</td>
<td>Cartesian Folium (Folium of Descartes)</td>
<td>96</td>
</tr>
<tr>
<td>2.11.4</td>
<td>Cissoid</td>
<td>96</td>
</tr>
<tr>
<td>2.11.5</td>
<td>Strophoide</td>
<td>97</td>
</tr>
<tr>
<td>2.12</td>
<td>Curves of Order Four (Quartics)</td>
<td>97</td>
</tr>
<tr>
<td>2.12.1</td>
<td>Conchoid of Nicomedes</td>
<td>97</td>
</tr>
<tr>
<td>2.12.2</td>
<td>General Conchoid</td>
<td>98</td>
</tr>
<tr>
<td>2.12.3</td>
<td>Pascal’s Limaçon</td>
<td>98</td>
</tr>
<tr>
<td>2.12.4</td>
<td>Cardioid</td>
<td>99</td>
</tr>
<tr>
<td>2.12.5</td>
<td>Cassinian Curve</td>
<td>100</td>
</tr>
<tr>
<td>2.12.6</td>
<td>Lemniscate</td>
<td>101</td>
</tr>
<tr>
<td>2.13</td>
<td>Cycloids</td>
<td>101</td>
</tr>
<tr>
<td>2.13.1</td>
<td>Common (Standard) Cycloid</td>
<td>101</td>
</tr>
<tr>
<td>2.13.2</td>
<td>Prolate and Curtate Cycloids or Trochoids</td>
<td>102</td>
</tr>
<tr>
<td>2.13.3</td>
<td>Epicycloid</td>
<td>102</td>
</tr>
<tr>
<td>2.13.4</td>
<td>Hypocycloid and Astroid</td>
<td>103</td>
</tr>
<tr>
<td>2.13.5</td>
<td>Prolate and Curtate Epicycloid and Hypocycloid</td>
<td>104</td>
</tr>
<tr>
<td>2.14</td>
<td>Spirals</td>
<td>105</td>
</tr>
<tr>
<td>2.14.1</td>
<td>Archimedean Spiral</td>
<td>105</td>
</tr>
<tr>
<td>2.14.2</td>
<td>Hyperbolic Spiral</td>
<td>105</td>
</tr>
<tr>
<td>2.14.3</td>
<td>Logarithmic Spiral</td>
<td>106</td>
</tr>
<tr>
<td>2.14.4</td>
<td>Evolute of the Circle</td>
<td>106</td>
</tr>
<tr>
<td>2.14.5</td>
<td>Clothoid</td>
<td>107</td>
</tr>
<tr>
<td>2.15</td>
<td>Various Other Curves</td>
<td>107</td>
</tr>
<tr>
<td>2.15.1</td>
<td>Catenary Curve</td>
<td>107</td>
</tr>
<tr>
<td>2.15.2</td>
<td>Tractrix</td>
<td>108</td>
</tr>
<tr>
<td>2.16</td>
<td>Determination of Empirical Curves</td>
<td>108</td>
</tr>
<tr>
<td>2.16.1</td>
<td>Procedure</td>
<td>108</td>
</tr>
<tr>
<td>2.16.1.1</td>
<td>Curve-Shape Comparison</td>
<td>108</td>
</tr>
<tr>
<td>2.16.1.2</td>
<td>Rectification</td>
<td>108</td>
</tr>
<tr>
<td>2.16.1.3</td>
<td>Determination of Parameters</td>
<td>109</td>
</tr>
<tr>
<td>2.16.2</td>
<td>Useful Empirical Formulas</td>
<td>109</td>
</tr>
<tr>
<td>2.16.2.1</td>
<td>Power Functions</td>
<td>109</td>
</tr>
<tr>
<td>2.16.2.2</td>
<td>Exponential Functions</td>
<td>110</td>
</tr>
<tr>
<td>2.16.2.3</td>
<td>Quadratic Polynomial</td>
<td>111</td>
</tr>
<tr>
<td>2.16.2.4</td>
<td>Rational Linear Functions</td>
<td>111</td>
</tr>
<tr>
<td>2.16.2.5</td>
<td>Square Root of a Quadratic Polynomial</td>
<td>111</td>
</tr>
<tr>
<td>2.16.2.6</td>
<td>General Error Curve</td>
<td>112</td>
</tr>
<tr>
<td>2.16.2.7</td>
<td>Curve of Order Three, Type II</td>
<td>112</td>
</tr>
<tr>
<td>2.16.2.8</td>
<td>Curve of Order Three, Type III</td>
<td>112</td>
</tr>
</tbody>
</table>
3.1.5 Angles Measured in Degrees and in Radians .................................. 131
3.1.2 Geometrical Definition of Circular and Hyperbolic Functions .......... 131
3.1.2.1 Definition of Circular or Trigonometric Functions .................. 131
3.1.2.2 Definitions of the Hyperbolic Functions ................................ 132
3.1.3 Plane Triangles ............................................................................. 132
3.1.3.1 Statements about Plane Triangles ........................................... 132
3.1.3.2 Symmetry .............................................................................. 133
3.1.4 Plane Quadrangles ....................................................................... 135
3.1.4.1 Parallelogram ....................................................................... 135
3.1.4.2 Rectangle and Square ............................................................. 136
3.1.4.3 Rhombus .............................................................................. 136
3.1.4.4 Trapezoid ............................................................................ 136
3.1.4.5 General Quadrangle ............................................................... 136
3.1.4.6 Inscribed Quadrangle ............................................................. 137
3.1.4.7 Circumscribing Quadrangle .................................................. 137
3.1.5 Polygons in the Plane ................................................................. 138
3.1.5.1 General Polygon ................................................................... 138
3.1.5.2 Regular Convex Polygons ....................................................... 138
3.1.5.3 Some Regular Convex Polygons ............................................ 139
3.1.6 The Circle and Related Shapes ..................................................... 139
3.1.6.1 Circle ................................................................................ 139
3.1.6.2 Circular Segment and Circular Sector ..................................... 141
3.1.6.3 Annulus .............................................................................. 141
3.2 Plane Trigonometry ......................................................................... 142
3.2.1 Triangles ................................................................................... 142
3.2.1.1 Calculations in Right-Angled Triangles in the Plane .............. 142
3.2.1.2 Calculations in General (Oblique) Triangles in the Plane ........ 142
3.2.2 Geodesic Applications .................................................................. 144
3.2.2.1 Geodesic Coordinates ............................................................ 144
3.2.2.2 Angles in Geodesy ................................................................. 146
3.2.2.3 Applications in Surveying ....................................................... 148
3.3 Stereometry ..................................................................................... 151
3.3.1 Lines and Planes in Space ............................................................ 151
3.3.2 Edge, Corner, Solid Angle .......................................................... 152
3.3.3 Polyeder or Polyhedron ............................................................... 153
3.3.4 Solids Bounded by Curved Surfaces ............................................ 156
3.4 Spherical Trigonometry ................................................................... 160
3.4.1 Basic Concepts of Geometry on the Sphere .................................. 160
3.4.1.1 Curve, Arc, and Angle on the Sphere .................................... 160
3.4.1.2 Special Coordinate Systems .................................................. 162
3.4.1.3 Spherical Lune or Biangle ....................................................... 163
3.4.1.4 Spherical Triangle ................................................................. 163
3.4.1.5 Polar Triangle ..................................................................... 164
3.4.1.6 Euler Triangle and Non-Euler Triangles ............................... 164
3.4.1.7 Trihedral Angle .................................................................. 164
3.4.2 Basic Properties of Spherical Triangles ...................................... 165
3.4.2.1 General Statements ............................................................... 165
3.4.2.2 Fundamental Formulas and Applications .............................. 165
3.4.2.3 Further Formulas ................................................................. 168
3.4.3 Calculation of Spherical Triangles ............................................. 169
3.4.3.1 Basic Problems, Accuracy Observations ............................... 169
3.4.3.2 Right-Angled Spherical Triangles ......................................... 169
3.4.3.3 Spherical Triangles with Oblique Angles ........................................ 171
3.4.3.4 Spherical Curves .................................................................................. 174
3.5 Vector Algebra and Analytical Geometry ..................................................... 181
  3.5.1 Vector Algebra ......................................................................................... 181
    3.5.1.1 Definition of Vectors ........................................................................ 181
    3.5.1.2 Calculation Rules for Vectors .......................................................... 182
    3.5.1.3 Coordinates of a Vector .................................................................... 183
    3.5.1.4 Directional Coefficient ...................................................................... 184
    3.5.1.5 Scalar Product and Vector Product .................................................. 184
    3.5.1.6 Combination of Vector Products ....................................................... 185
    3.5.1.7 Vector Equations ............................................................................... 188
    3.5.1.8 Covariant and Contravariant Coordinates of a Vector ....................... 188
    3.5.1.9 Geometric Applications of Vector Algebra ........................................ 190
  3.5.2 Analytical Geometry of the Plane ............................................................. 190
    3.5.2.1 Basic Concepts, Coordinate Systems in the Plane ............................ 190
    3.5.2.2 Coordinate Transformations ............................................................. 191
    3.5.2.3 Special Notations and Points in the Plane .......................................... 192
    3.5.2.4 Areas .................................................................................................. 194
    3.5.2.5 Equation of a Curve .......................................................................... 195
    3.5.2.6 Line .................................................................................................... 195
    3.5.2.7 Circle ................................................................................................. 198
    3.5.2.8 Ellipse ............................................................................................... 199
    3.5.2.9 Hyperbola ......................................................................................... 201
    3.5.2.10 Parabola .......................................................................................... 204
    3.5.2.11 Quadratic Curves (Curves of Second Order or Conic Sections) .......... 206
  3.5.3 Analytical Geometry of Space .................................................................. 209
    3.5.3.1 Basic Concepts ................................................................................... 209
    3.5.3.2 Spatial Coordinate Systems ............................................................... 210
    3.5.3.3 Transformation of Orthogonal Coordinates ....................................... 212
    3.5.3.4 Rotations with Direction Cosines ....................................................... 213
    3.5.3.5 Cardan Angles .................................................................................. 214
    3.5.3.6 Euler’s angles .................................................................................... 215
    3.5.3.7 Special Quantities in Space ............................................................... 216
    3.5.3.8 Equation of a Surface ........................................................................ 217
    3.5.3.9 Equation of a Space Curve ............................................................... 218
    3.5.3.10 Line and Plane in Space ................................................................. 218
    3.5.3.11 Lines in Space .................................................................................. 221
    3.5.3.12 Intersection Points and Angles of Lines and Planes in Space ............ 223
    3.5.3.13 Surfaces of Second Order, Equations in Normal Form ..................... 224
    3.5.3.14 Surfaces of Second Order or Quadratic Surfaces, General Theory ...... 228
  3.5.4 Geometric Transformations and Coordinate Transformations .................. 229
    3.5.4.1 Geometric 2D Transformations ......................................................... 229
    3.5.4.2 Homogeneous Coordinates, Matrix Representation .......................... 231
    3.5.4.3 Coordinate Transformation ............................................................... 231
    3.5.4.4 Composition of Transformations ....................................................... 232
    3.5.4.5 3D–Transformations .......................................................................... 233
    3.5.4.6 Deformation Transformations ............................................................ 236
  3.5.5 Planar Projections ..................................................................................... 237
    3.5.5.1 Classification of the projections ......................................................... 237
    3.5.5.2 Local or Projection Coordinate System ............................................. 238
    3.5.5.3 Principal Projections .......................................................................... 238
    3.5.5.4 Axonometric Projection ..................................................................... 238
3.5.5.5 Isometric Projection ........................................... 239
3.5.5.6 Oblique Parallel Projection ................................... 240
3.5.5.7 Perspective Projection ........................................ 241

3.6 Differential Geometry ............................................. 243
3.6.1 Plane Curves ...................................................... 243
3.6.1.1 Ways to Define a Plane Curve ................................. 243
3.6.1.2 Local Elements of a Curve .................................... 243
3.6.1.3 Special Points of a Curve ..................................... 249
3.6.1.4 Asymptotes of Curves ........................................ 252
3.6.1.5 General Discussion of a Curve Given by an Equation ...... 253
3.6.1.6 Evolutes and Evolvents ....................................... 254
3.6.1.7 Envelope of a Family of Curves ............................... 255

3.6.2 Space Curves ....................................................... 256
3.6.2.1 Ways to Define a Space Curve ................................. 256
3.6.2.2 Moving Trihedral ................................................ 256
3.6.2.3 Curvature and Torsion ........................................ 258

3.6.3 Surfaces .............................................................. 261
3.6.3.1 Ways to Define a Surface ..................................... 261
3.6.3.2 Tangent Plane and Surface Normal ............................ 262
3.6.3.3 Line Elements of a Surface .................................... 263
3.6.3.4 Curvature of a Surface ......................................... 265
3.6.3.5 Ruled Surfaces and Developable Surfaces .................... 268
3.6.3.6 Geodesic Lines on a Surface .................................. 268

4 Linear Algebra .......................................................... 269
4.1 Matrices ............................................................... 269
4.1.1 Notion of Matrix .................................................. 269
4.1.2 Square Matrices ................................................... 270
4.1.3 Vectors ............................................................... 271
4.1.4 Arithmetical Operations with Matrices ......................... 272
4.1.5 Rules of Calculation for Matrices ............................... 275
4.1.6 Vector and Matrix Norms ....................................... 276
4.1.6.1 Vector Norms .................................................. 277
4.1.6.2 Matrix Norms .................................................. 277

4.2 Determinants .......................................................... 278
4.2.1 Definitions .......................................................... 278
4.2.2 Rules of Calculation for Determinants ......................... 278
4.2.3 Evaluation of Determinants ..................................... 279

4.3 Tensors ................................................................. 280
4.3.1 Transformation of Coordinate Systems ......................... 280
4.3.2 Tensors in Cartesian Coordinates .............................. 281
4.3.3 Tensors with Special Properties ................................. 283
4.3.3.1 Tensors of Rank 2 .............................................. 283
4.3.3.2 Invariant Tensors .............................................. 283

4.3.4 Tensors in Curvilinear Coordinate Systems .................... 284
4.3.4.1 Covariant and Contravariant Basis Vectors ................. 284
4.3.4.2 Covariant and Contravariant Coordinates of Tensors of Rank 1 285
4.3.4.3 Covariant, Contravariant and Mixed Coordinates of Tensors of Rank 2 ...... 286

4.3.5 Pseudotensors ....................................................... 287
4.3.5.1 Symmetry with Respect to the Origin ......................... 287
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.3.5.2</td>
<td>Introduction to the Notion of Pseudotensors</td>
<td>288</td>
</tr>
<tr>
<td>4.4</td>
<td>Quaternions and Applications</td>
<td>289</td>
</tr>
<tr>
<td>4.4.1</td>
<td>Quaternions</td>
<td>290</td>
</tr>
<tr>
<td>4.4.1.1</td>
<td>Definition and Representation</td>
<td>290</td>
</tr>
<tr>
<td>4.4.1.2</td>
<td>Matrix Representation of Quaternions</td>
<td>291</td>
</tr>
<tr>
<td>4.4.1.3</td>
<td>Calculation Rules</td>
<td>292</td>
</tr>
<tr>
<td>4.4.2</td>
<td>Representation of Rotations in $\mathbb{R}^3$</td>
<td>294</td>
</tr>
<tr>
<td>4.4.2.1</td>
<td>Rotations of an Object About the Coordinate Axes</td>
<td>295</td>
</tr>
<tr>
<td>4.4.2.2</td>
<td>Cardan-Angles</td>
<td>295</td>
</tr>
<tr>
<td>4.4.2.3</td>
<td>Euler Angles</td>
<td>296</td>
</tr>
<tr>
<td>4.4.2.4</td>
<td>Rotation Around an Arbitrary Zero Point Axis</td>
<td>296</td>
</tr>
<tr>
<td>4.4.2.5</td>
<td>Rotation and Quaternions</td>
<td>297</td>
</tr>
<tr>
<td>4.4.2.6</td>
<td>Quaternions and Cardan Angles</td>
<td>298</td>
</tr>
<tr>
<td>4.4.2.7</td>
<td>Efficiency of the Algorithms</td>
<td>301</td>
</tr>
<tr>
<td>4.4.3</td>
<td>Applications of Quaternions</td>
<td>302</td>
</tr>
<tr>
<td>4.4.3.1</td>
<td>3D Rotations in Computer Graphics</td>
<td>302</td>
</tr>
<tr>
<td>4.4.3.2</td>
<td>Interpolation by Rotation matrices</td>
<td>303</td>
</tr>
<tr>
<td>4.4.3.3</td>
<td>Stereographic Projection</td>
<td>303</td>
</tr>
<tr>
<td>4.4.3.4</td>
<td>Satellite navigation</td>
<td>304</td>
</tr>
<tr>
<td>4.4.3.5</td>
<td>Vector Analysis</td>
<td>305</td>
</tr>
<tr>
<td>4.4.3.6</td>
<td>Normalized Quaternions and Rigid Body Motion</td>
<td>306</td>
</tr>
<tr>
<td>4.5</td>
<td>Systems of Linear Equations</td>
<td>307</td>
</tr>
<tr>
<td>4.5.1</td>
<td>Linear Systems, Pivoting</td>
<td>307</td>
</tr>
<tr>
<td>4.5.1.1</td>
<td>Linear Systems</td>
<td>307</td>
</tr>
<tr>
<td>4.5.1.2</td>
<td>Pivoting</td>
<td>307</td>
</tr>
<tr>
<td>4.5.1.3</td>
<td>Linear Dependence</td>
<td>308</td>
</tr>
<tr>
<td>4.5.1.4</td>
<td>Calculation of the Inverse of a Matrix</td>
<td>308</td>
</tr>
<tr>
<td>4.5.2</td>
<td>Solution of Systems of Linear Equations</td>
<td>308</td>
</tr>
<tr>
<td>4.5.2.1</td>
<td>Definition and Solvability</td>
<td>308</td>
</tr>
<tr>
<td>4.5.2.2</td>
<td>Application of Pivoting</td>
<td>310</td>
</tr>
<tr>
<td>4.5.2.3</td>
<td>Cramer’s Rule</td>
<td>311</td>
</tr>
<tr>
<td>4.5.2.4</td>
<td>Gauss’s Algorithm</td>
<td>312</td>
</tr>
<tr>
<td>4.5.3</td>
<td>Overdetermined Linear Systems of Equations</td>
<td>313</td>
</tr>
<tr>
<td>4.5.3.1</td>
<td>Overdetermined Linear Systems of Equations and Linear Least Squares Problems</td>
<td>313</td>
</tr>
<tr>
<td>4.5.3.2</td>
<td>Suggestions for Numerical Solutions of Least Squares Problems</td>
<td>314</td>
</tr>
<tr>
<td>4.6</td>
<td>Eigenvalue Problems for Matrices</td>
<td>314</td>
</tr>
<tr>
<td>4.6.1</td>
<td>General Eigenvalue Problem</td>
<td>314</td>
</tr>
<tr>
<td>4.6.2</td>
<td>Special Eigenvalue Problem</td>
<td>315</td>
</tr>
<tr>
<td>4.6.2.1</td>
<td>Characteristic Polynomial</td>
<td>315</td>
</tr>
<tr>
<td>4.6.2.2</td>
<td>Real Symmetric Matrices, Similarity Transformations</td>
<td>316</td>
</tr>
<tr>
<td>4.6.2.3</td>
<td>Transformation of Principal Axes of Quadratic Forms</td>
<td>317</td>
</tr>
<tr>
<td>4.6.2.4</td>
<td>Suggestions for the Numerical Calculations of Eigenvalues</td>
<td>319</td>
</tr>
<tr>
<td>4.6.3</td>
<td>Singular Value Decomposition</td>
<td>321</td>
</tr>
<tr>
<td>5</td>
<td>Algebra and Discrete Mathematics</td>
<td>323</td>
</tr>
<tr>
<td>5.1</td>
<td>Logic</td>
<td>323</td>
</tr>
<tr>
<td>5.1.1</td>
<td>Propositional Calculus</td>
<td>323</td>
</tr>
<tr>
<td>5.1.2</td>
<td>Formulas in Predicate Calculus</td>
<td>326</td>
</tr>
<tr>
<td>5.2</td>
<td>Set Theory</td>
<td>327</td>
</tr>
<tr>
<td>5.2.1</td>
<td>Concept of Set, Special Sets</td>
<td>327</td>
</tr>
<tr>
<td>5.2.2</td>
<td>Operations with Sets</td>
<td>328</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>--------------------------------------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>5.4.3 Congruences and Residue Classes</td>
<td>377</td>
<td></td>
</tr>
<tr>
<td>5.4.4 Theorems of Fermat, Euler, and Wilson</td>
<td>381</td>
<td></td>
</tr>
<tr>
<td>5.4.5 Prime Number Tests</td>
<td>382</td>
<td></td>
</tr>
<tr>
<td>5.4.6 Codes</td>
<td>383</td>
<td></td>
</tr>
<tr>
<td>5.4.6.1 Control Digits</td>
<td>383</td>
<td></td>
</tr>
<tr>
<td>5.4.6.2 Error correcting codes</td>
<td>385</td>
<td></td>
</tr>
<tr>
<td>5.5 Cryptology</td>
<td>386</td>
<td></td>
</tr>
<tr>
<td>5.5.1 Problem of Cryptology</td>
<td>386</td>
<td></td>
</tr>
<tr>
<td>5.5.2 Cryptosystems</td>
<td>387</td>
<td></td>
</tr>
<tr>
<td>5.5.3 Mathematical Foundation</td>
<td>387</td>
<td></td>
</tr>
<tr>
<td>5.5.4 Security of Cryptosystems</td>
<td>388</td>
<td></td>
</tr>
<tr>
<td>5.5.4.1 Methods of Conventional Cryptography</td>
<td>388</td>
<td></td>
</tr>
<tr>
<td>5.5.4.2 Linear Substitution Ciphers</td>
<td>389</td>
<td></td>
</tr>
<tr>
<td>5.5.4.3 Vigenère Cipher</td>
<td>389</td>
<td></td>
</tr>
<tr>
<td>5.5.4.4 Matrix Substitution</td>
<td>389</td>
<td></td>
</tr>
<tr>
<td>5.5.5 Methods of Classical Cryptanalysis</td>
<td>389</td>
<td></td>
</tr>
<tr>
<td>5.5.5.1 Statistical Analysis</td>
<td>390</td>
<td></td>
</tr>
<tr>
<td>5.5.5.2 Kasiski-Friedman Test</td>
<td>390</td>
<td></td>
</tr>
<tr>
<td>5.5.6 One-Time Pad</td>
<td>390</td>
<td></td>
</tr>
<tr>
<td>5.5.7 Public Key Methods</td>
<td>391</td>
<td></td>
</tr>
<tr>
<td>5.5.7.1 Diffie-Hellman Key Exchange</td>
<td>391</td>
<td></td>
</tr>
<tr>
<td>5.5.7.2 One-Way Function</td>
<td>391</td>
<td></td>
</tr>
<tr>
<td>5.5.7.3 RSA Codes and RSA Method</td>
<td>392</td>
<td></td>
</tr>
<tr>
<td>5.5.8 DES Algorithm (Data Encryption Standard)</td>
<td>393</td>
<td></td>
</tr>
<tr>
<td>5.5.9 IDEA Algorithm (International Data Encryption Algorithm)</td>
<td>393</td>
<td></td>
</tr>
<tr>
<td>5.6 Universal Algebra</td>
<td>394</td>
<td></td>
</tr>
<tr>
<td>5.6.1 Definition</td>
<td>394</td>
<td></td>
</tr>
<tr>
<td>5.6.2 Congruence Relations, Factor Algebras</td>
<td>394</td>
<td></td>
</tr>
<tr>
<td>5.6.3 Homomorphism</td>
<td>394</td>
<td></td>
</tr>
<tr>
<td>5.6.4 Homomorphism Theorem</td>
<td>395</td>
<td></td>
</tr>
<tr>
<td>5.6.5 Varieties</td>
<td>395</td>
<td></td>
</tr>
<tr>
<td>5.6.6 Term Algebras, Free Algebras</td>
<td>395</td>
<td></td>
</tr>
<tr>
<td>5.7 Boolean Algebras and Switch Algebra</td>
<td>395</td>
<td></td>
</tr>
<tr>
<td>5.7.1 Definition</td>
<td>395</td>
<td></td>
</tr>
<tr>
<td>5.7.2 Duality Principle</td>
<td>396</td>
<td></td>
</tr>
<tr>
<td>5.7.3 Finite Boolean Algebras</td>
<td>397</td>
<td></td>
</tr>
<tr>
<td>5.7.4 Boolean Algebras as Orderings</td>
<td>397</td>
<td></td>
</tr>
<tr>
<td>5.7.5 Boolean Functions, Boolean Expressions</td>
<td>397</td>
<td></td>
</tr>
<tr>
<td>5.7.6 Normal Forms</td>
<td>399</td>
<td></td>
</tr>
<tr>
<td>5.7.7 Switch Algebra</td>
<td>399</td>
<td></td>
</tr>
<tr>
<td>5.8 Algorithms of Graph Theory</td>
<td>401</td>
<td></td>
</tr>
<tr>
<td>5.8.1 Basic Notions and Notation</td>
<td>401</td>
<td></td>
</tr>
<tr>
<td>5.8.2 Traverse of Undirected Graphs</td>
<td>404</td>
<td></td>
</tr>
<tr>
<td>5.8.2.1 Edge Sequences or Paths</td>
<td>404</td>
<td></td>
</tr>
<tr>
<td>5.8.2.2 Euler Trails</td>
<td>405</td>
<td></td>
</tr>
<tr>
<td>5.8.2.3 Hamiltonian Cycles</td>
<td>406</td>
<td></td>
</tr>
<tr>
<td>5.8.3 Trees and Spanning Trees</td>
<td>407</td>
<td></td>
</tr>
<tr>
<td>5.8.3.1 Trees</td>
<td>407</td>
<td></td>
</tr>
<tr>
<td>5.8.3.2 Spanning Trees</td>
<td>408</td>
<td></td>
</tr>
<tr>
<td>5.8.4 Matchings</td>
<td>409</td>
<td></td>
</tr>
<tr>
<td>5.8.5 Planar Graphs</td>
<td>410</td>
<td></td>
</tr>
<tr>
<td>5.8.6 Paths in Directed Graphs</td>
<td>410</td>
<td></td>
</tr>
</tbody>
</table>
5.8.7 Transport Networks .......................................................... 411
5.9 Fuzzy Logic ............................................................................. 413
  5.9.1 Basic Notions of Fuzzy Logic .............................................. 413
    5.9.1.1 Interpretation of Fuzzy Sets ........................................ 413
    5.9.1.2 Membership Functions on the Real Line ..................... 414
    5.9.1.3 Fuzzy Sets ............................................................. 416
  5.9.2 Connections (Aggregations) of Fuzzy Sets ......................... 418
    5.9.2.1 Concepts for Aggregations of Fuzzy Sets .................... 418
    5.9.2.2 Practical Aggregation Operations of Fuzzy Sets ............ 419
    5.9.2.3 Compensatory Operators ........................................ 421
    5.9.2.4 Extension Principle ............................................... 421
    5.9.2.5 Fuzzy Complement ................................................ 421
  5.9.3 Fuzzy-Valued Relations ................................................... 422
    5.9.3.1 Fuzzy Relations ................................................... 422
    5.9.3.2 Fuzzy Product Relation $R \circ S$ .................................. 424
  5.9.4 Fuzzy Inference (Approximate Reasoning) ......................... 425
  5.9.5 Defuzzification Methods ................................................ 426
  5.9.6 Knowledge-Based Fuzzy Systems ....................................... 427
    5.9.6.1 Method of Mamdani .............................................. 427
    5.9.6.2 Method of Sugeno ............................................... 428
    5.9.6.3 Cognitive Systems ............................................... 428
    5.9.6.4 Knowledge-Based Interpolation Systems ..................... 430

6 Differentiation ......................................................................... 432
  6.1 Differentiation of Functions of One Variable .......................... 432
    6.1.1 Differential Quotient .................................................. 432
    6.1.2 Rules of Differentiation for Functions of One Variable ....... 433
      6.1.2.1 Derivatives of the Elementary Functions ................. 433
      6.1.2.2 Basic Rules of Differentiation ............................... 433
    6.1.3 Derivatives of Higher Order ......................................... 438
      6.1.3.1 Definition of Derivatives of Higher Order ............... 438
      6.1.3.2 Derivatives of Higher Order of some Elementary Functions 438
      6.1.3.3 Leibniz’s Formula .............................................. 438
      6.1.3.4 Higher Derivatives of Functions Given in Parametric Form 440
      6.1.3.5 Derivatives of Higher Order of the Inverse Function .... 440
    6.1.4 Fundamental Theorems of Differential Calculus ............... 441
      6.1.4.1 Monotonicity ................................................... 441
      6.1.4.2 Fermat’s Theorem ............................................. 441
      6.1.4.3 Rolle’s Theorem ............................................... 441
      6.1.4.4 Mean Value Theorem of Differential Calculus ........... 442
      6.1.4.5 Taylor’s Theorem of Functions of One Variable .......... 442
      6.1.4.6 Generalized Mean Value Theorem of Differential Calculus (Cauchy’s Theorem) 443
    6.1.5 Determination of the Extreme Values and Inflection Points .... 443
      6.1.5.1 Maxima and Minima ........................................... 443
      6.1.5.2 Necessary Conditions for the Existence of a Relative Extreme Value of a Differentiable, Explicit Function $y = f(x)$ ........ 444
      6.1.5.3 Determination of the Relative Extreme Values and the Inflection Points of a Differentiable, Explicit Function $y = f(x)$ ........ 444
      6.1.5.4 Determination of Absolute Extrema ........................ 445
      6.1.5.5 Determination of the Extrema of Implicit Functions ..... 445
  6.2 Differentiation of Functions of Several Variables .................... 445
    6.2.1 Partial Derivatives ................................................... 445
6.2.1.1 Partial Derivative of a Function ............................................. 445
6.2.1.2 Geometrical Meaning for Functions of Two Variables .............. 446
6.2.1.3 Differentials of $x$ and $f(x)$ .............................................. 446
6.2.1.4 Basic Properties of the Differential ..................................... 447
6.2.1.5 Partial Differential .......................................................... 447

6.2.2 Total Differential and Differentials of Higher Order .................. 447
6.2.2.1 Notion of Total Differential of a Function of Several Variables (Complete Differential) ............................................. 447
6.2.2.2 Derivatives and Differentials of Higher Order ....................... 448
6.2.2.3 Taylor’s Theorem for Functions of Several Variables ............. 449

6.2.3 Rules of Differentiation for Functions of Several Variables ........... 450
6.2.3.1 Differentiation of Composite Functions .................................. 450
6.2.3.2 Differentiation of Implicit Functions ..................................... 451

6.2.4 Substitution of Variables in Differential Expressions and Coordinate Transformations .......................................................... 452
6.2.4.1 Function of One Variable ...................................................... 452
6.2.4.2 Function of Two Variables ...................................................... 453

6.2.5 Extreme Values of Functions of Several Variables ..................... 454
6.2.5.1 Definition of a Relative Extreme Value ................................... 454
6.2.5.2 Geometric Representation ..................................................... 454
6.2.5.3 Determination of Extreme Values of Differentiable Functions of Two Variables ............................................. 455
6.2.5.4 Determination of the Extreme Values of a Function of $n$ Variables 455
6.2.5.5 Solution of Approximation Problems ...................................... 456
6.2.5.6 Extreme Value Problem with Side Conditions .......................... 456

7 Infinite Series ............................................................................. 457

7.1 Sequences of Numbers ............................................................... 457
7.1.1 Properties of Sequences of Numbers ......................................... 457
7.1.1.1 Definition of Sequence of Numbers ....................................... 457
7.1.1.2 Monotone Sequences of Numbers ........................................... 457
7.1.1.3 Bounded Sequences of Numbers ............................................. 457
7.1.2 Limits of Sequences of Numbers .............................................. 458

7.2 Number Series ........................................................................... 459
7.2.1 General Convergence Theorems .............................................. 459
7.2.1.1 Convergence and Divergence of Infinite Series ...................... 459
7.2.1.2 General Theorems about the Convergence of Series .............. 460
7.2.2 Convergence Criteria for Series with Positive Terms ................. 460
7.2.2.1 Comparison Criterion ............................................................ 460
7.2.2.2 D’Alembert’s Ratio Test ......................................................... 461
7.2.2.3 Root Test of Cauchy .............................................................. 461
7.2.2.4 Integral Test of Cauchy .......................................................... 462

7.2.3 Absolute and Conditional Convergence ...................................... 462
7.2.3.1 Definition ............................................................... 462
7.2.3.2 Properties of Absolutely Convergent Series ............................. 463
7.2.3.3 Alternating Series ............................................................... 463

7.2.4 Some Special Series ............................................................... 464
7.2.4.1 The Values of Some Important Number Series ..................... 464
7.2.4.2 Bernoulli and Euler Numbers ............................................... 465

7.2.5 Estimation of the Remainder ................................................... 466
7.2.5.1 Estimation with Majorant ...................................................... 466
7.2.5.2 Alternating Convergent Series .............................................. 467
7.2.5.3 Special Series .................................................. 467

7.3 Function Series .................................................. 467
  7.3.1 Definitions .................................................. 467
  7.3.2 Uniform Convergence ........................................ 468
    7.3.2.1 Definition, Weierstrass Theorem ...................... 468
    7.3.2.2 Properties of Uniformly Convergent Series ........ 468
  7.3.3 Power series ................................................ 469
    7.3.3.1 Definition, Convergence ............................... 469
    7.3.3.2 Calculations with Power Series ....................... 470
    7.3.3.3 Taylor Series Expansion, Maclaurin Series .......... 471
  7.3.4 Approximation Formulas .................................. 472
  7.3.5 Asymptotic Power Series ................................. 472
    7.3.5.1 Asymptotic Behavior .................................. 472
    7.3.5.2 Asymptotic Power Series ............................... 472

7.4 Fourier Series .................................................. 474
  7.4.1 Trigonometric Sum and Fourier Series .................. 474
    7.4.1.1 Basic Notions ......................................... 474
    7.4.1.2 Most Important Properties of the Fourier Series .. 475
  7.4.2 Determination of Coefficients for Symmetric Functions 476
    7.4.2.1 Different Kinds of Symmetries ....................... 476
    7.4.2.2 Forms of the Expansion into a Fourier Series ...... 477
  7.4.3 Determination of the Fourier Coefficients with Numerical Methods 477
  7.4.4 Fourier Series and Fourier Integrals ................... 478
  7.4.5 Remarks on the Table of Some Fourier Expansions ...... 479

8 Integral Calculus ................................................. 480
  8.1 Indefinite Integrals .......................................... 480
    8.1.1 Primitive Function or Antiderivative .................. 480
      8.1.1.1 Indefinite Integrals ................................. 481
      8.1.1.2 Integrals of Elementary Functions ................ 481
    8.1.2 Rules of Integration .................................... 482
    8.1.3 Integration of Rational Functions ..................... 485
      8.1.3.1 Integrals of Integer Rational Functions (Polynomials) 485
      8.1.3.2 Integrals of Fractional Rational Functions ........ 485
      8.1.3.3 Four Cases of Partial Fraction Decomposition .... 485
    8.1.4 Integration of Irrational Functions .............. 488
      8.1.4.1 Substitution to Reduce to Integration of Rational Functions 488
      8.1.4.2 Integration of Binomial Integrands ................. 489
      8.1.4.3 Elliptic Integrals .................................. 490
    8.1.5 Integration of Trigonometric Functions .............. 491
      8.1.5.1 Substitution ........................................ 491
      8.1.5.2 Simplified Methods ................................ 491
    8.1.6 Integration of Further Transcendental Functions .... 492
      8.1.6.1 Integrals with Exponential Functions ............... 492
      8.1.6.2 Integrals with Hyperbolic Functions ............... 493
      8.1.6.3 Application of Integration by Parts ............... 493
      8.1.6.4 Integrals of Transcendental Functions ............ 493
  8.2 Definite Integrals ............................................ 493
    8.2.1 Basic Notions, Rules and Theorems .................... 493
      8.2.1.1 Definition and Existence of the Definite Integral 493
      8.2.1.2 Properties of Definite Integrals ................... 494
      8.2.1.3 Further Theorems about the Limits of Integration 496
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.2.1.4</td>
<td>Evaluation of the Definite Integral</td>
<td>498</td>
</tr>
<tr>
<td>8.2.2</td>
<td>Applications of Definite Integrals</td>
<td>500</td>
</tr>
<tr>
<td>8.2.2.1</td>
<td>General Principle for Applications of the Definite Integral</td>
<td>500</td>
</tr>
<tr>
<td>8.2.2.2</td>
<td>Applications in Geometry</td>
<td>501</td>
</tr>
<tr>
<td>8.2.2.3</td>
<td>Applications in Mechanics and Physics</td>
<td>504</td>
</tr>
<tr>
<td>8.2.3</td>
<td>Improper Integrals, Stieltjes and Lebesgue Integrals</td>
<td>506</td>
</tr>
<tr>
<td>8.2.3.1</td>
<td>Generalization of the Notion of the Integral</td>
<td>506</td>
</tr>
<tr>
<td>8.2.3.2</td>
<td>Integrals with Infinite Integration Limits</td>
<td>507</td>
</tr>
<tr>
<td>8.2.3.3</td>
<td>Integrals with Unbounded Integrand</td>
<td>509</td>
</tr>
<tr>
<td>8.2.4</td>
<td>Parametric Integrals</td>
<td>512</td>
</tr>
<tr>
<td>8.2.4.1</td>
<td>Definition of Parametric Integrals</td>
<td>512</td>
</tr>
<tr>
<td>8.2.4.2</td>
<td>Differentiation Under the Symbol of Integration</td>
<td>512</td>
</tr>
<tr>
<td>8.2.4.3</td>
<td>Integration Under the Symbol of Integration</td>
<td>512</td>
</tr>
<tr>
<td>8.2.5</td>
<td>Integration by Series Expansion, Special Non-Elementary Functions</td>
<td>513</td>
</tr>
<tr>
<td>8.3</td>
<td>Line Integrals</td>
<td>515</td>
</tr>
<tr>
<td>8.3.1</td>
<td>Line Integrals of the First Type</td>
<td>516</td>
</tr>
<tr>
<td>8.3.1.1</td>
<td>Definitions</td>
<td>516</td>
</tr>
<tr>
<td>8.3.1.2</td>
<td>Existence Theorem</td>
<td>516</td>
</tr>
<tr>
<td>8.3.1.3</td>
<td>Evaluation of the Line Integral of the First Type</td>
<td>516</td>
</tr>
<tr>
<td>8.3.1.4</td>
<td>Application of the Line Integral of the First Type</td>
<td>517</td>
</tr>
<tr>
<td>8.3.2</td>
<td>Line Integrals of the Second Type</td>
<td>517</td>
</tr>
<tr>
<td>8.3.2.1</td>
<td>Definitions</td>
<td>517</td>
</tr>
<tr>
<td>8.3.2.2</td>
<td>Existence Theorem</td>
<td>519</td>
</tr>
<tr>
<td>8.3.2.3</td>
<td>Calculation of the Line Integral of the Second Type</td>
<td>519</td>
</tr>
<tr>
<td>8.3.3</td>
<td>Line Integrals of General Type</td>
<td>519</td>
</tr>
<tr>
<td>8.3.3.1</td>
<td>Definition</td>
<td>519</td>
</tr>
<tr>
<td>8.3.3.2</td>
<td>Properties of the Line Integral of General Type</td>
<td>520</td>
</tr>
<tr>
<td>8.3.3.3</td>
<td>Integral Along a Closed Curve</td>
<td>521</td>
</tr>
<tr>
<td>8.3.4</td>
<td>Independence of the Line Integral of the Path of Integration</td>
<td>521</td>
</tr>
<tr>
<td>8.3.4.1</td>
<td>Two-Dimensional Case</td>
<td>521</td>
</tr>
<tr>
<td>8.3.4.2</td>
<td>Existence of a Primitive Function</td>
<td>521</td>
</tr>
<tr>
<td>8.3.4.3</td>
<td>Three-Dimensional Case</td>
<td>522</td>
</tr>
<tr>
<td>8.3.4.4</td>
<td>Determination of the Primitive Function</td>
<td>522</td>
</tr>
<tr>
<td>8.3.4.5</td>
<td>Zero-Valued Integral Along a Closed Curve</td>
<td>523</td>
</tr>
<tr>
<td>8.4</td>
<td>Multiple Integrals</td>
<td>523</td>
</tr>
<tr>
<td>8.4.1</td>
<td>Double Integrals</td>
<td>524</td>
</tr>
<tr>
<td>8.4.1.1</td>
<td>Notion of the Double Integral</td>
<td>524</td>
</tr>
<tr>
<td>8.4.1.2</td>
<td>Evaluation of the Double Integral</td>
<td>524</td>
</tr>
<tr>
<td>8.4.1.3</td>
<td>Applications of the Double Integral</td>
<td>527</td>
</tr>
<tr>
<td>8.4.2</td>
<td>Triple Integrals</td>
<td>527</td>
</tr>
<tr>
<td>8.4.2.1</td>
<td>Notion of the Triple Integral</td>
<td>527</td>
</tr>
<tr>
<td>8.4.2.2</td>
<td>Evaluation of the Triple Integral</td>
<td>529</td>
</tr>
<tr>
<td>8.4.2.3</td>
<td>Applications of the Triple Integral</td>
<td>531</td>
</tr>
<tr>
<td>8.5</td>
<td>Surface Integrals</td>
<td>532</td>
</tr>
<tr>
<td>8.5.1</td>
<td>Surface Integral of the First Type</td>
<td>532</td>
</tr>
<tr>
<td>8.5.1.1</td>
<td>Notion of the Surface Integral of the First Type</td>
<td>532</td>
</tr>
<tr>
<td>8.5.1.2</td>
<td>Evaluation of the Surface Integral of the First Type</td>
<td>533</td>
</tr>
<tr>
<td>8.5.1.3</td>
<td>Applications of the Surface Integral of the First Type</td>
<td>535</td>
</tr>
<tr>
<td>8.5.2</td>
<td>Surface Integral of the Second Type</td>
<td>535</td>
</tr>
<tr>
<td>8.5.2.1</td>
<td>Notion of the Surface Integral of the Second Type</td>
<td>535</td>
</tr>
<tr>
<td>8.5.2.2</td>
<td>Evaluation of Surface Integrals of the Second Type</td>
<td>537</td>
</tr>
<tr>
<td>8.5.3</td>
<td>Surface Integral in General Form</td>
<td>537</td>
</tr>
</tbody>
</table>
9 Differential Equations

9.1 Ordinary Differential Equations

9.1.1 First-Order Differential Equations

9.1.1.1 Existence Theorems, Direction Field

9.1.1.2 Important Solution Methods

9.1.1.3 Implicit Differential Equations

9.1.1.4 Singular Integrals and Singular Points

9.1.1.5 Approximation Methods for Solution of First-Order Differential Equations

9.1.2 Differential Equations of Higher Order and Systems of Differential Equations

9.1.2.1 Basic Results

9.1.2.2 Lowering the Order

9.1.2.3 Linear $n$-th Order Differential Equations

9.1.2.4 Solution of Linear Differential Equations with Constant Coefficients

9.1.2.5 Systems of Linear Differential Equations with Constant Coefficients

9.1.2.6 Linear Second-Order Differential Equations

9.1.3 Boundary Value Problems

9.1.3.1 Problem Formulation

9.1.3.2 Fundamental Properties of Eigenfunctions and Eigenvalues

9.1.3.3 Expansion in Eigenfunctions

9.1.3.4 Singular Cases

9.2 Partial Differential Equations

9.2.1 First-Order Partial Differential Equations

9.2.1.1 Linear First-Order Partial Differential Equations

9.2.1.2 Non-Linear First-Order Partial Differential Equations

9.2.2 Linear Second-Order Partial Differential Equations

9.2.2.1 Classification and Properties of Second-Order Differential Equations with Two Independent Variables

9.2.2.2 Classification and Properties of Linear Second-Order Differential Equations with more than two Independent Variables

9.2.2.3 Integration Methods for Linear Second-Order Partial Differential Equations

9.2.3 Some further Partial Differential Equations From Natural Sciences and Engineering

9.2.3.1 Formulation of the Problem and the Boundary Conditions

9.2.3.2 Wave Equation

9.2.3.3 Heat Conduction and Diffusion Equation for Homogeneous Media

9.2.3.4 Potential Equation

9.2.4 Schroedinger’s Equation

9.2.4.1 Notion of the Schroedinger Equation

9.2.4.2 Time-Dependent Schroedinger Equation

9.2.4.3 Time-Independent Schroedinger Equation

9.2.4.4 Statistical Interpretation of the Wave Function

9.2.4.5 Force-Free Motion of a Particle in a Block

9.2.4.6 Particle Movement in a Symmetric Central Field (see 13.1.2.2, p. 702)

9.2.4.7 Linear Harmonic Oscillator

9.2.5 Non-Linear Partial Differential Equations: Solitons, Periodic Patterns, Chaos

9.2.5.1 Formulation of the Physical-Mathematical Problem

9.2.5.2 Korteweg de Vries Equation (KdV)
## 11.5.2 Singular Integral Equation with Cauchy Kernel

- 11.5.2.1 Formulation of the Problem
- 11.5.2.2 Existence of a Solution
- 11.5.2.3 Properties of Cauchy Type Integrals
- 11.5.2.4 The Hilbert Boundary Value Problem
- 11.5.2.5 Solution of the Hilbert Boundary Value Problem (in short: Hilbert Problem)
- 11.5.2.6 Solution of the Characteristic Integral Equation

## 12 Functional Analysis

### 12.1 Vector Spaces
- 12.1.1 Notion of a Vector Space
- 12.1.2 Linear and Affine Linear Subsets
- 12.1.3 Linearly Independent Elements
- 12.1.4 Convex Subsets and the Convex Hull
  - 12.1.4.1 Convex Sets
  - 12.1.4.2 Cones
- 12.1.5 Linear Operators and Functionals
  - 12.1.5.1 Mappings
  - 12.1.5.2 Homomorphism and Endomorphism
  - 12.1.5.3 Isomorphic Vector Spaces
- 12.1.6 Complexification of Real Vector Spaces

### 12.2 Metric Spaces
- 12.2.1 Notion of a Metric Space
  - 12.2.1.1 Balls, Neighborhoods and Open Sets
  - 12.2.1.2 Convergence of Sequences in Metric Spaces
  - 12.2.1.3 Closed Sets and Closure
  - 12.2.1.4 Dense Subsets and Separable Metric Spaces
- 12.2.2 Complete Metric Spaces
  - 12.2.2.1 Cauchy Sequences
  - 12.2.2.2 Complete Metric Spaces
  - 12.2.2.3 Some Fundamental Theorems in Complete Metric Spaces
  - 12.2.2.4 Some Applications of the Contraction Mapping Principle
  - 12.2.2.5 Completion of a Metric Space
- 12.2.3 Continuous Operators

### 12.3 Normed Spaces
- 12.3.1 Notion of a Normed Space
  - 12.3.1.1 Axioms of a Normed Space
  - 12.3.1.2 Some Properties of Normed Spaces
- 12.3.2 Banach Spaces
  - 12.3.2.1 Series in Normed Spaces
  - 12.3.2.2 Examples of Banach Spaces
  - 12.3.2.3 Sobolev Spaces
- 12.3.3 Ordered Normed Spaces
- 12.3.4 Normed Algebras

### 12.4 Hilbert Spaces
- 12.4.1 Notion of a Hilbert Space
12.9.2 Measurable Functions

12.9.2.1 Measurable Function

12.9.2.2 Properties of the Class of Measurable Functions

12.9.3 Integration

12.9.3.1 Definition of the Integral

12.9.3.2 Some Properties of the Integral

12.9.3.3 Convergence Theorems

12.9.4 $L^p$ Spaces

12.9.5 Distributions

12.9.5.1 Formula of Partial Integration

12.9.5.2 Generalized Derivative

12.9.5.3 Distributions

12.9.5.4 Derivative of a Distribution

13 Vector Analysis and Vector Fields

13.1 Basic Notions of the Theory of Vector Fields

13.1.1 Vector Functions of a Scalar Variable

13.1.1.1 Definitions

13.1.1.2 Derivative of a Vector Function

13.1.1.3 Rules of Differentiation for Vectors

13.1.1.4 Taylor Expansion for Vector Functions

13.1.2 Scalar Fields

13.1.2.1 Scalar Field or Scalar Point Function

13.1.2.2 Important Special Cases of Scalar Fields

13.1.2.3 Coordinate Representation of Scalar Fields

13.1.2.4 Level Surfaces and Level Lines of a Field

13.1.3 Vector Fields

13.1.3.1 Vector Field or Vector Point Function

13.1.3.2 Important Cases of Vector Fields

13.1.3.3 Coordinate Representation of Vector Fields

13.1.3.4 Transformation of Coordinate Systems

13.1.3.5 Vector Lines

13.2 Differential Operators of Space

13.2.1 Directional and Space Derivatives

13.2.1.1 Directional Derivative of a Scalar Field

13.2.1.2 Directional Derivative of a Vector Field

13.2.1.3 Volume Derivative

13.2.2 Gradient of a Scalar Field

13.2.2.1 Definition of the Gradient

13.2.2.2 Gradient and Directional Derivative

13.2.2.3 Gradient and Volume Derivative

13.2.2.4 Further Properties of the Gradient

13.2.2.5 Gradient of the Scalar Field in Different Coordinates

13.2.2.6 Rules of Calculations

13.2.3 Vector Gradient

13.2.4 Divergence of Vector Fields

13.2.4.1 Definition of Divergence

13.2.4.2 Divergence in Different Coordinates

13.2.4.3 Rules for Evaluation of the Divergence

13.2.4.4 Divergence of a Central Field

13.2.5 Rotation of Vector Fields

13.2.5.1 Definitions of the Rotation
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>14.1.1.4 Differentiability of a Complex Function</td>
<td>731</td>
</tr>
<tr>
<td>14.1.2 Analytic Functions</td>
<td>732</td>
</tr>
<tr>
<td>14.1.2.1 Definition of Analytic Functions</td>
<td>732</td>
</tr>
<tr>
<td>14.1.2.2 Examples of Analytic Functions</td>
<td>732</td>
</tr>
<tr>
<td>14.1.2.3 Properties of Analytic Functions</td>
<td>732</td>
</tr>
<tr>
<td>14.1.2.4 Singular Points</td>
<td>733</td>
</tr>
<tr>
<td>14.1.3 Conformal Mapping</td>
<td>734</td>
</tr>
<tr>
<td>14.1.3.1 Notion and Properties of Conformal Mappings</td>
<td>734</td>
</tr>
<tr>
<td>14.1.3.2 Simplest Conformal Mappings</td>
<td>735</td>
</tr>
<tr>
<td>14.1.3.3 Schwarz Reflection Principle</td>
<td>741</td>
</tr>
<tr>
<td>14.1.3.4 Complex Potential</td>
<td>741</td>
</tr>
<tr>
<td>14.1.3.5 Superposition Principle</td>
<td>744</td>
</tr>
<tr>
<td>14.1.3.6 Arbitrary Mappings of the Complex Plane</td>
<td>745</td>
</tr>
<tr>
<td>14.2 Integration in the Complex Plane</td>
<td>745</td>
</tr>
<tr>
<td>14.2.1 Definite and Indefinite Integral</td>
<td>745</td>
</tr>
<tr>
<td>14.2.1.1 Definition of the Integral in the Complex Plane</td>
<td>745</td>
</tr>
<tr>
<td>14.2.1.2 Properties and Evaluation of Complex Integrals</td>
<td>746</td>
</tr>
<tr>
<td>14.2.2 Cauchy Integral Theorem</td>
<td>747</td>
</tr>
<tr>
<td>14.2.2.1 Cauchy Integral Theorem for Simply Connected Domains</td>
<td>747</td>
</tr>
<tr>
<td>14.2.2.2 Cauchy Integral Theorem for Multiply Connected Domains</td>
<td>748</td>
</tr>
<tr>
<td>14.2.3 Cauchy Integral Formulas</td>
<td>748</td>
</tr>
<tr>
<td>14.2.3.1 Analytic Function on the Interior of a Domain</td>
<td>748</td>
</tr>
<tr>
<td>14.2.3.2 Analytic Function on the Exterior of a Domain</td>
<td>749</td>
</tr>
<tr>
<td>14.3 Power Series Expansion of Analytic Functions</td>
<td>749</td>
</tr>
<tr>
<td>14.3.1 Convergence of Series with Complex Terms</td>
<td>749</td>
</tr>
<tr>
<td>14.3.1.1 Convergence of a Number Sequence with Complex Terms</td>
<td>749</td>
</tr>
<tr>
<td>14.3.1.2 Convergence of an Infinite Series with Complex Terms</td>
<td>749</td>
</tr>
<tr>
<td>14.3.1.3 Power Series with Complex Terms</td>
<td>750</td>
</tr>
<tr>
<td>14.3.2 Taylor Series</td>
<td>751</td>
</tr>
<tr>
<td>14.3.3 Principle of Analytic Continuation</td>
<td>751</td>
</tr>
<tr>
<td>14.3.4 Laurent Expansion</td>
<td>752</td>
</tr>
<tr>
<td>14.3.5 Isolated Singular Points and the Residue Theorem</td>
<td>752</td>
</tr>
<tr>
<td>14.3.5.1 Isolated Singular Points</td>
<td>752</td>
</tr>
<tr>
<td>14.3.5.2 Meromorphic Functions</td>
<td>753</td>
</tr>
<tr>
<td>14.3.5.3 Elliptic Functions</td>
<td>753</td>
</tr>
<tr>
<td>14.3.5.4 Residue</td>
<td>753</td>
</tr>
<tr>
<td>14.3.5.5 Residue Theorem</td>
<td>754</td>
</tr>
<tr>
<td>14.4 Evaluation of Real Integrals by Complex Integrals</td>
<td>754</td>
</tr>
<tr>
<td>14.4.1 Application of Cauchy Integral Formulas</td>
<td>754</td>
</tr>
<tr>
<td>14.4.2 Application of the Residue Theorem</td>
<td>755</td>
</tr>
<tr>
<td>14.4.3 Application of the Jordan Lemma</td>
<td>755</td>
</tr>
<tr>
<td>14.4.3.1 Jordan Lemma</td>
<td>755</td>
</tr>
<tr>
<td>14.4.3.2 Examples of the Jordan Lemma</td>
<td>756</td>
</tr>
<tr>
<td>14.5 Algebraic and Elementary Transcendental Functions</td>
<td>758</td>
</tr>
<tr>
<td>14.5.1 Algebraic Functions</td>
<td>758</td>
</tr>
<tr>
<td>14.5.2 Elementary Transcendental Functions</td>
<td>758</td>
</tr>
<tr>
<td>14.5.3 Description of Curves in Complex Form</td>
<td>760</td>
</tr>
<tr>
<td>14.6 Elliptic Functions</td>
<td>762</td>
</tr>
<tr>
<td>14.6.1 Relation to Elliptic Integrals</td>
<td>762</td>
</tr>
<tr>
<td>14.6.2 Jacobian Functions</td>
<td>763</td>
</tr>
<tr>
<td>14.6.3 Theta Functions</td>
<td>764</td>
</tr>
<tr>
<td>14.6.4 Weierstrass Functions</td>
<td>765</td>
</tr>
</tbody>
</table>
15 Integral Transformations

15.1 Notion of Integral Transformation
   15.1.1 General Definition of Integral Transformations
   15.1.2 Special Integral Transformations
   15.1.3 Inverse Transformations
   15.1.4 Linearity of Integral Transformations
   15.1.5 Integral transformations for functions of several variables
   15.1.6 Applications of Integral Transformations

15.2 Laplace Transformation
   15.2.1 Properties of the Laplace Transformation
      15.2.1.1 Laplace Transformation, Original and Image Space
      15.2.1.2 Rules for the Evaluation of the Laplace Transformation
      15.2.1.3 Transforms of Special Functions
      15.2.1.4 Dirac δ Function and Distributions
   15.2.2 Inverse Transformation into the Original Space
      15.2.2.1 Inverse Transformation with the Help of Tables
      15.2.2.2 Partial Fraction Decomposition
      15.2.2.3 Series Expansion
      15.2.2.4 Inverse Integral
   15.2.3 Solution of Differential Equations using Laplace Transformation
      15.2.3.1 Ordinary Linear Differential Equations with Constant Coefficients
      15.2.3.2 Ordinary Linear Differential Equations with Coefficients Depending on the Variable
      15.2.3.3 Partial Differential Equations

15.3 Fourier Transformation
   15.3.1 Properties of the Fourier Transformation
      15.3.1.1 Fourier Integral
      15.3.1.2 Fourier Transformation and Inverse Transformation
      15.3.1.3 Rules of Calculation with the Fourier Transformation
      15.3.1.4 Transforms of Special Functions
   15.3.2 Solution of Differential Equations using the Fourier Transformation
      15.3.2.1 Ordinary Linear Differential Equations
      15.3.2.2 Partial Differential Equations

15.4 Z-Transformation
   15.4.1 Properties of the Z-Transformation
      15.4.1.1 Discrete Functions
      15.4.1.2 Definition of the Z-Transformation
      15.4.1.3 Rules of Calculations
      15.4.1.4 Relation to the Laplace Transformation
      15.4.1.5 Inverse of the Z-Transformation
   15.4.2 Applications of the Z-Transformation
      15.4.2.1 General Solution of Linear Difference Equations
      15.4.2.2 Second-Order Difference Equations (Initial Value Problem)
      15.4.2.3 Second-Order Difference Equations (Boundary Value Problem)

15.5 Wavelet Transformation
   15.5.1 Signals
   15.5.2 Wavelets
   15.5.3 Wavelet Transformation
   15.5.4 Discrete Wavelet Transformation
      15.5.4.1 Fast Wavelet Transformation
      15.5.4.2 Discrete Haar Wavelet Transformation
   15.5.5 Gabor Transformation
### 16.3.4.2 Linear Regression for two Measurable Characters

Page 841

### 16.3.4.3 Multidimensional Regression

Page 842

### 16.3.5 Monte Carlo Methods

Page 843

#### 16.3.5.1 Simulation

Page 843

#### 16.3.5.2 Random Numbers

Page 843

#### 16.3.5.3 Example of a Monte Carlo Simulation

Page 845

#### 16.3.5.4 Application of the Monte Carlo Method in Numerical Mathematics

Page 845

#### 16.3.5.5 Further Applications of the Monte Carlo Method

Page 847

### 16.4 Calculus of Errors

Page 848

#### 16.4.1 Measurement Error and its Distribution

Page 848

##### 16.4.1.1 Qualitative Characterization of Measurement Errors

Page 848

##### 16.4.1.2 Density Function of the Measurement Error

Page 848

##### 16.4.1.3 Quantitative Characterization of the Measurement Error

Page 850

##### 16.4.1.4 Determining the Result of a Measurement with Bounds on the Error

Page 853

##### 16.4.1.5 Error Estimation for Direct Measurements with the Same Accuracy

Page 853

##### 16.4.1.6 Error Estimation for Direct Measurements with Different Accuracy

Page 854

#### 16.4.2 Error Propagation and Error Analysis

Page 854

##### 16.4.2.1 Gauss Error Propagation Law

Page 855

##### 16.4.2.2 Error Analysis

Page 856

### 17 Dynamical Systems and Chaos

Page 857

#### 17.1 Ordinary Differential Equations and Mappings

Page 857

##### 17.1.1 Dynamical Systems

Page 857

###### 17.1.1.1 Basic Notions

Page 857

###### 17.1.1.2 Invariant Sets

Page 859

##### 17.1.2 Qualitative Theory of Ordinary Differential Equations

Page 860

###### 17.1.2.1 Existence of Flows, Phase Space Structure

Page 860

###### 17.1.2.2 Linear Differential Equations

Page 861

###### 17.1.2.3 Stability Theory

Page 863

###### 17.1.2.4 Invariant Manifolds

Page 866

###### 17.1.2.5 Poincaré Mapping

Page 868

###### 17.1.2.6 Topological Equivalence of Differential Equations

Page 870

##### 17.1.3 Discrete Dynamical Systems

Page 871

###### 17.1.3.1 Steady States, Periodic Orbits and Limit Sets

Page 871

###### 17.1.3.2 Invariant Manifolds

Page 872

###### 17.1.3.3 Topological Conjugation of Discrete Systems

Page 873

##### 17.1.4 Structural Stability (Robustness)

Page 873

###### 17.1.4.1 Structurally Stable Differential Equations

Page 873

###### 17.1.4.2 Structurally Stable Time Discrete Systems

Page 874

###### 17.1.4.3 Generic Properties

Page 874

#### 17.2 Quantitative Description of Attractors

Page 876

##### 17.2.1 Probability Measures on Attractors

Page 876

###### 17.2.1.1 Invariant Measure

Page 876

###### 17.2.1.2 Elements of Ergodic Theory

Page 877

##### 17.2.2 Entropies

Page 879

###### 17.2.2.1 Topological Entropy

Page 879

###### 17.2.2.2 Metric Entropy

Page 879

##### 17.2.3 Lyapunov Exponents

Page 880

##### 17.2.4 Dimensions

Page 882

###### 17.2.4.1 Metric Dimensions

Page 882

###### 17.2.4.2 Dimensions Defined by Invariant Measures

Page 884

###### 17.2.4.3 Local Hausdorff Dimension According to Douady and Oesterlé

Page 886
### 17.2.4.4 Examples of Attractors
- Strange Attractors and Chaos
- Chaos in One-Dimensional Mappings
- Reconstruction of Dynamics from Time Series
  - Foundations, Reconstruction with Basic Properties
  - Reconstructions with Prevalent Properties

### 17.2.5 Strange Attractors and Chaos

### 17.2.6 Chaos in One-Dimensional Mappings

### 17.2.7 Reconstruction of Dynamics from Time Series
  - Foundations, Reconstruction with Basic Properties
  - Reconstructions with Prevalent Properties

### 17.3 Bifurcation Theory and Routes to Chaos
- Bifurcations in Morse-Smale Systems
  - Local Bifurcations in Neighborhoods of Steady States
  - Local Bifurcations in a Neighborhood of a Periodic Orbit
  - Global Bifurcation
- Transitions to Chaos
  - Cascade of Period Doublings
  - Intermittency
  - Global Homoclinic Bifurcations
  - Destruction of a Torus

### 18 Optimization

#### 18.1 Linear Programming
- Formulation of the Problem and Geometrical Representation
- The Form of a Linear Programming Problem
- Examples and Graphical Solutions
- Basic Notions of Linear Programming, Normal Form
  - Extreme Points and Basis
  - Normal Form of the Linear Programming Problem
- Simplex Method
  - Simplex Tableau
  - Transition to the New Simplex Tableau
  - Determination of an Initial Simplex Tableau
  - Revised Simplex Method
  - Duality in Linear Programming
- Special Linear Programming Problems
  - Transportation Problem
  - Assignment Problem
  - Distribution Problem
  - Travelling Salesman
  - Scheduling Problem

#### 18.2 Non-linear Optimization
- Formulation of the Problem, Theoretical Basis
- Formulation of the Problem
- Optimality Conditions
- Duality in Optimization
- Special Non-linear Optimization Problems
- Convex Optimization
- Quadratic Optimization
- Solution Methods for Quadratic Optimization Problems
  - Wolfe’s Method
  - Hildreth-d’Esopo Method
- Numerical Search Procedures
  - One-Dimensional Search
  - Minimum Search in \( n \)-Dimensional Euclidean Vector Space
- Methods for Unconstrained Problems
## 20.2 Patterns
- Functional Operations ........................................ 1032
- Programming .................................................. 1034
- Supplement about Syntax, Information, Messages
  - Contexts, Attributes ................................... 1035
  - Information .................................................... 1035
  - Messages ....................................................... 1035

## 20.3 Important Applications with Mathematica
- Manipulation of Algebraic Expressions
  - Multiplication of Expressions .......................... 1036
  - Factorization of Polynomials ......................... 1037
  - Operations with Polynomials .......................... 1037
  - Partial Fraction Decomposition ...................... 1037
  - Manipulation of Non-Polynomial Expressions .......... 1038
- Solution of Equations and Systems of Equations
  - Equations as Logical Expressions .................... 1038
  - Solution of Polynomial Equations .................... 1039
  - Solution of Transcendental Equations ................ 1039
  - Solution of Systems of Equations .................... 1040
- Linear Systems of Equations and Eigenvalue Problems
- Differential and Integral Calculus
  - Calculation of Derivatives ....................... 1042
  - Indefinite Integrals ........................................ 1043
  - Definite Integrals and Multiple Integrals ........... 1044
  - Solution of Differential Equations .................. 1044

## 20.4 Graphics with Mathematica
- Basic Elements of Graphics ................................. 1045
- Graphics Primitives ........................................ 1046
- Graphical Options .......................................... 1047
- Syntax of Graphical Representation ..................... 1047
  - Building Graphic Objects ............................... 1047
  - Graphical Representation of Functions ............... 1048
  - Exponential Functions ................................. 1049
  - Function $y = x + \text{Arccoth } x$ ................... 1049
  - Bessel Functions (see 9.1.2.6, 2., p. 562) .......... 1050
- Parametric Representation of Curves .................... 1050
- Representation of Surfaces and Space Curves ............. 1051
  - Graphical Representation of Surfaces ................. 1051
  - Options for 3D Graphics ............................... 1051
  - Three-Dimensional Objects in Parametric Representation 1051

## 21 Tables
- Frequently Used Mathematical Constants .................. 1053
- Important Natural Constants ............................... 1053
- Metric Prefixes ............................................. 1054
- International System of Physical Units (SI Units) .... 1055
- Important Series Expansions .............................. 1057
- Fourier Series ................................................ 1062
- Indefinite Integrals ......................................... 1065
  - Integral Rational Functions .......................... 1065
    - Integrals with $X = ax + b$ ......................... 1065
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>21.7.1.2 Integrals with $X = ax^2 + bx + c$</td>
<td>1067</td>
</tr>
<tr>
<td>21.7.1.3 Integrals with $X = a^2 + x^2$</td>
<td>1068</td>
</tr>
<tr>
<td>21.7.1.4 Integrals with $X = a^3 + x^2$</td>
<td>1070</td>
</tr>
<tr>
<td>21.7.1.5 Integrals with $X = a^4 + x^4$</td>
<td>1071</td>
</tr>
<tr>
<td>21.7.1.6 Integrals with $X = a^4 - x^4$</td>
<td>1071</td>
</tr>
<tr>
<td>21.7.1.7 Some Cases of Partial Fraction Decomposition</td>
<td>1071</td>
</tr>
<tr>
<td>21.7.2 Integrals of Irrational Functions</td>
<td></td>
</tr>
<tr>
<td>21.7.2.1 Integrals with $\sqrt{x}$ and $a^2 \pm b^2 x$</td>
<td>1072</td>
</tr>
<tr>
<td>21.7.2.2 Other Integrals with $\sqrt{x}$</td>
<td>1072</td>
</tr>
<tr>
<td>21.7.2.3 Integrals with $\sqrt{ax + b}$</td>
<td>1073</td>
</tr>
<tr>
<td>21.7.2.4 Integrals with $\sqrt{ax + b}$ and $\sqrt{fx + g}$</td>
<td>1074</td>
</tr>
<tr>
<td>21.7.2.5 Integrals with $\sqrt{a^2 - x^2}$</td>
<td>1075</td>
</tr>
<tr>
<td>21.7.2.6 Integrals with $\sqrt{x^2 + a^2}$</td>
<td>1076</td>
</tr>
<tr>
<td>21.7.2.7 Integrals with $\sqrt{x^2 - a^2}$</td>
<td>1078</td>
</tr>
<tr>
<td>21.7.2.8 Integrals with $\sqrt{ax^2 + bx + c}$</td>
<td>1080</td>
</tr>
<tr>
<td>21.7.2.9 Integrals with other Irrational Expressions</td>
<td>1082</td>
</tr>
<tr>
<td>21.7.2.10 Recursion Formulas for an Integral with Binomial Differential</td>
<td>1082</td>
</tr>
<tr>
<td>21.7.3 Integrals of Trigonometric Functions</td>
<td></td>
</tr>
<tr>
<td>21.7.3.1 Integrals with Sine Function</td>
<td>1083</td>
</tr>
<tr>
<td>21.7.3.2 Integrals with Cosine Function</td>
<td>1085</td>
</tr>
<tr>
<td>21.7.3.3 Integrals with Sine and Cosine Function</td>
<td>1087</td>
</tr>
<tr>
<td>21.7.3.4 Integrals with Tangent Function</td>
<td>1091</td>
</tr>
<tr>
<td>21.7.3.5 Integrals with Cotangent Function</td>
<td>1091</td>
</tr>
<tr>
<td>21.7.4 Integrals of other Transcendental Functions</td>
<td></td>
</tr>
<tr>
<td>21.7.4.1 Integrals with Hyperbolic Functions</td>
<td>1092</td>
</tr>
<tr>
<td>21.7.4.2 Integrals with Exponential Functions</td>
<td>1093</td>
</tr>
<tr>
<td>21.7.4.3 Integrals with Logarithmic Functions</td>
<td>1095</td>
</tr>
<tr>
<td>21.7.4.4 Integrals with Inverse Trigonometric Functions</td>
<td>1096</td>
</tr>
<tr>
<td>21.7.4.5 Integrals with Inverse Hyperbolic Functions</td>
<td>1097</td>
</tr>
<tr>
<td>21.8 Definite Integrals</td>
<td></td>
</tr>
<tr>
<td>21.8.1 Definite Integrals of Trigonometric Functions</td>
<td>1098</td>
</tr>
<tr>
<td>21.8.2 Definite Integrals of Exponential Functions</td>
<td>1099</td>
</tr>
<tr>
<td>21.8.3 Definite Integrals of Logarithmic Functions</td>
<td>1100</td>
</tr>
<tr>
<td>21.8.4 Definite Integrals of Algebraic Functions</td>
<td>1101</td>
</tr>
<tr>
<td>21.9 Elliptic Integrals</td>
<td></td>
</tr>
<tr>
<td>21.9.1 Elliptic Integral of the First Kind $F(\varphi, k), k = \sin \alpha$</td>
<td></td>
</tr>
<tr>
<td>21.9.2 Elliptic Integral of the Second Kind $E(\varphi, k), k = \sin \alpha$</td>
<td></td>
</tr>
<tr>
<td>21.9.3 Complete Elliptic Integral, $k = \sin \alpha$</td>
<td>1104</td>
</tr>
<tr>
<td>21.10 Gamma Function</td>
<td></td>
</tr>
<tr>
<td>21.11 Bessel Functions (Cylindrical Functions)</td>
<td>1106</td>
</tr>
<tr>
<td>21.12 Legendre Polynomials of the First Kind</td>
<td>1108</td>
</tr>
<tr>
<td>21.13 Laplace Transformation</td>
<td>1109</td>
</tr>
<tr>
<td>21.14 Fourier Transformation</td>
<td></td>
</tr>
<tr>
<td>21.14.1 Fourier Cosine Transformation</td>
<td>1114</td>
</tr>
<tr>
<td>21.14.2 Fourier Sine Transformation</td>
<td>1120</td>
</tr>
<tr>
<td>21.14.3 Fourier Transformation</td>
<td>1125</td>
</tr>
<tr>
<td>21.14.4 Exponential Fourier Transformation</td>
<td>1127</td>
</tr>
<tr>
<td>21.15 Z Transformation</td>
<td></td>
</tr>
<tr>
<td>21.16 Poisson Distribution</td>
<td>1131</td>
</tr>
<tr>
<td>21.17 Standard Normal Distribution</td>
<td>1133</td>
</tr>
<tr>
<td>21.17.1 Standard Normal Distribution for $0.00 \leq x \leq 1.99$</td>
<td>1133</td>
</tr>
</tbody>
</table>
21.17.2 Standard Normal Distribution for $2.00 \leq x \leq 3.90$ .......................... 1134
21.18 $\chi^2$ Distribution ................................................................................. 1135
21.19 Fisher $F$ Distribution ............................................................................ 1136
21.20 Student $t$ Distribution ............................................................................. 1138
21.21 Random Numbers .................................................................................... 1139

22 Bibliography 1140

Index 1152

Mathematic Symbols A
Handbook of Mathematics
Bronshstein, I.N.; Semendyayev, K.A.; Musiol, G.; Mühlig, H.
ISBN: 978-3-662-46220-1