Preface

The language of geometry has changed drastically in the last decades. New and fundamental ideas such as the language of categories, sheaves, and cohomology are now indispensable in many incarnations of geometry, such as the theory of complex analytic spaces, algebraic geometry, or non-archimedean geometry. This book is intended as an introduction to these ideas illustrating them by example with the most ubiquitous branch of geometry, the theory of manifolds.

In its contemporary form, a "geometric object" is usually defined as an "object" that "locally" "looks like" a "standard geometric object". Depending on the geometry that one is interested in, there will be very different "standard geometric objects" as the basic building blocks. For the theory of (finite-dimensional) manifolds one chooses open subsets of finite-dimensional $\mathbb{R}$- or $\mathbb{C}$-vector spaces together with their "differentiable structure".

To make the notion of a geometric object precise, one proceeds in general as follows. First one introduces the language of categories yielding the notions of objects and the precise meaning of "looks like" as being isomorphic in that category. Next one has to find a (maybe very large) category C that contains the "standard geometric objects" as a subcategory and in which it makes sense to use the word "locally". Then finally one can give the precise definition of a *geometric object* as an object of C that is locally isomorphic to an object in the subcategory of standard geometric objects.

In this textbook we choose for such a category C the category of ringed spaces, although other choices such as ringed topoi might even be more natural. But for the sake of an introduction, ringed spaces seem to be the most accessible choice and they are also adequate for many geometric theories such as differential geometry, complex geometry, or the theory of schemes in algebraic geometry. Moreover a good grasp of ringed spaces will help a great deal in understanding more abstract concepts such as ringed topoi.

For the theory of manifolds the basic building blocks are open subsets X of finite-dimensional $\mathbb{R}$- or $\mathbb{C}$-vector spaces together with the collection of their "functions", where a "function" will be an α-fold continuously differentiable function for $\alpha \in \mathbb{N}_0 \cup \{\infty\}$ or an analytic function (also called C^ω-function) on some open subspace of X. Then a continuous map between standard geometric objects is a C^α-map ($\alpha \in \mathbb{N}_0 \cup \{\infty, \omega\}$) if and only if the composition with f sends a C^α-function to a C^α-function. This allows one to view such standard geometric objects and their structure preserving maps as a subcategory of (locally) ringed spaces over $\mathbb{R}$ or $\mathbb{C}$. Hence one obtains the notion of a geometric object by the general procedure explained above. These are called premanifolds.

A manifold will then be a premanifold whose underlying topological space has certain good properties (being Hausdorff and second countable). Let me briefly digress on this choice of terminology. First of all I follow the classical terminology. But an even more compelling reason to restrict the class of manifolds by asking for these topological properties is a multitude of techniques and results where these topological properties are indispensable hypotheses (such as embedding results or the theory of integration on manifold). In this textbook this is not that visible as many of such results are not covered here. Hence we will more often encounter premanifolds than manifolds and it would have been tempting to change terminology, if only to get rid of the annoying "pre-" everywhere. But I decided against this to remind the reader that the contents of this book are only the very beginning of a journey into the wondrous world of differential geometry – a world in which very often manifolds and not premanifolds are the central objects.

A fundamental idea in modern mathematics is the notion of a sheaf. Sheaves are needed to define the notion of a ringed space but their usefulness goes far beyond this. Sheaves embody the principle for passing from local to global situations – a central topic in mathematics. In the theory of smooth manifolds one can often avoid the use of sheaves as there is another powerful tool for local-global constructions, namely partitions of unity. Their existence corresponds to the fact that the sheaf of smooth functions is soft (see Chap. 9). But this is particular to the case of real C^α-manifolds with $\alpha \leq \infty$. In all other geometries mentioned before the theory of sheaves is indispensable. Hence sheaves will be a central topic of this book.

Together with sheaves and manifolds (as ringed spaces), the third main topic is the cohomology of sheaves. It is the main tool to make use of sheaves for local-global problems. As a rule, it allows one to consider the obstruction for the passage from local to global objects as an element in an algebraic cohomology object of a sheaf, usually a group. Moreover, the formalism of cohomology also yields a powerful tool to calculate such obstructions. In addition, many interesting objects (such as fiber bundles) are classified by the cohomology of certain sheaves.

It is not the goal of this book – and also would have been clearly beyond my abilities – to give a new quick and streamlined introduction to differential geometry using clever arguments to obtain deep results with minimal technical effort. Quite the contrary, the focus of this book is on the technical methods necessary to work with modern theories of geometry. As a principle I tried to explain these techniques in their "correct generality" (which is certainly a very subjective notion) to provide a reliable point of departure towards geometry.

There are some instances where I deviate from this principle, either because of lack of space or because I think that the natural generality and abstractness would seriously conceal the underlying simple idea. This includes the following subjects:

1. One might argue that the natural framework for sheaves are sheaves on an arbitrary site (i.e., a category endowed with a Grothendieck topology) or even general topos theory. But in my opinion this would have seriously hampered the accessibility of the theory.
2. Instead of working with manifolds modeled on open subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ one might argue that it is more natural (and not much more difficult) to model them on open subsets of arbitrary Banach spaces. I decided against this because the idea of the book is to demonstrate general techniques used in geometry in the most accessible example: finite-dimensional real and complex manifolds.
3. In the chapter on cohomology I do not use derived categories, although I tried to formulate the theory in such a way that a reader familiar with the notion of a derived category can easily transfer the results into this language[1].

Moreover, there are several serious omissions due to lack of space, among them Lie algebras, manifolds with corners (or, more generally, singular spaces), and integration theory – just to name a few. The educated reader will find many more such omissions.

Prerequisits

The reader should have knowledge of basic algebraic notions such as groups, rings and vector spaces, basic analytic and topological notions such as differentialbility in several variables and metric spaces.

Further prerequisites are assembled in five appendix chapters. It is assumed that the reader knows some but not all of the results here. Therefore many proofs and many examples are given in the appendices. These appendices give (Chap. 12) a complete if rather brisk treatment of basic concepts of point set topology, (Chap. 13) a quick introduction to the language of categories focused on examples, (Chap. 14) some basic definitions and results of abstract algebra, (Chap. 15) those notions of homological algebra necessary to cope with the beginning of cohomology theory, and (Chap. 16) a reminder on the notion of differential and analytic functions on open subsets of finite-dimensional $\mathbb{R}$- and $\mathbb{C}$-vector spaces.

Outline of contents

The main body of the text starts with two preliminary chapters. The first chapter (Chap. 1) introduces more advanced concepts from point set topology. The main notions of the first three sections are paracompact and normal spaces, covering important techniques like Urysohn's theorem, the Tietze extension theorem and the Shrinking Lemma for paracompact Hausdorff spaces. The last two sections of this chapter focus on separated and proper maps. In Chap. 2 basic notions of algebraic topology used in the sequel are introduced. Here we restrict the contents to those absolutely necessary (but with complete proofs) and ignore all progress made in the last decades.

[1] Of course, for most readers that are familiar with derived categories the cohomology chapter will not contain that many new results anyway.

In the third chapter (Chap. 3) we introduce the first main topic of the book: sheaves. We introduce two (equivalent) definitions of sheaves. The first one is that of a rule attaching to every open set of a topological space a set of so-called sections such that these sections can be glued from local to global objects. This is also the definition that generalizes from topological spaces to more abstract geometric objects such as sites. There is another point of view of sheaves that works just fine for sheaves on topological spaces, namely étalé spaces. It is proved that both concepts are equivalent and explained that some constructions for sheaves are more accessible via the first definition (such as direct images) and some are more accessible via étalé spaces (such as inverse images).

In Chap. 4 we introduce the class of geometric objects that will be studied in this book: manifolds. As explained above, we start by defining the very general category of ringed spaces over a fixed ring. Then we explain how to consider open subsets of real or complex finite-dimensional vector spaces as ringed spaces over $\mathbb{R}$ or $\mathbb{C}$. This yields our class of standard geometric objects. Then a premanifold is by definition a ringed space that is locally isomorphic to such a standard geometric object.

The central topic of Chap. 5 is that of linearization. We start by linearizing manifolds by introducing tangent spaces. Then the derivative of a morphism at a point is simply the induced map on tangent spaces and we can think of it as a pointwise linearization of the morphism. Next we study morphisms of (pre)manifolds that can even be locally linearized. These are precisely the morphisms whose derivatives have a locally constant rank. Examples are immersions, submersions, and locally constant maps. These linearization techniques are then used in the remaining sections of the chapter to study submanifolds, their intersections (or, more generally, fiber products of manifolds), and quotients of manifolds by an equivalence relation.

Chapter 6 introduces the symmetry groups in the theory of manifolds, the Lie groups. It focuses on actions of Lie groups on manifolds. The construction of quotients of manifolds in Chap. 5 then yields the existence of quotients for proper free actions of Lie groups.

In Chap. 7 we start to study local-global problems by introducing the first cohomology of a sheaf of (not necessarily abelian) groups via the language of torsors or, equivalently, by the language of Čech cocycles.

This is used in Chap. 8 to classify fiber bundles that are important examples of locally but not globally trivial objects. We start the chapter by introducing the general notion of a morphism that looks locally like a given morphisms p. Very often it is useful to restrict the classes of local isomorphisms. This yields the notion of a twist of p with a structure sheaf that is a subsheaf of the sheaf of all automorphisms of p. Specializing to the case that p is a projection and the structure sheaf is given by the faithful action of a Lie group G on the fiber then yields the notion of a fiber bundle with structure group G. Specializing further we obtain the important notions of G-principal bundles and vector bundles. We explain that vector bundles can also be described as certain modules over the sheaf of functions of the manifold. In the last two sections we study the most important examples of vector bundles on a manifold, the tangent bundle and the bundles of differential forms. In particular we will obtain the de Rham complex of a manifold.

As mentioned above, for real C^α-manifolds with $\alpha \le \infty$ it is often possible to use other techniques than sheaves for solving local-global problems. The sheaf-theoretic reason for this lies in the softness of the sheaf of C^α-functions. This notion is introduced in Chap. 9. We deduce from the softness of the structure sheaf the existence of arbitrary fine partitions of unity. In the last section we show that the first cohomology of a soft sheaf is trivial and deduce immediately some local-global principles. For real C^α-manifolds with $\alpha \le \infty$ all these results can also be obtained via arguments with a partition of unity. But these examples illustrate how to use the triviality of certain cohomology classes also in cases where partitions of unity are not available, for instance for certain complex manifolds.

After giving the rather ad hoc definition of the first cohomology in Chap. 7 (that had the advantage also to work for sheaves of not necessarily abelian groups) we now introduce cohomology in arbitrary degree in Chap. 10. After a quick motivation on how to do this, it becomes clear that it is more natural not to work with a single sheaf but with a whole complex of sheaves. This is carried out in the first three sections. Applying the whole formalism of cohomology to the de Rham complex we obtain de Rham's theorem relating de Rham cohomology and cohomology of constant sheaves in Sect. 10.4. We conclude the chapter by proving an other important result: the theorem of proper base change, either in the case of arbitrary topological spaces and proper separated maps (giving the theorem its name) or for metrizable spaces and closed maps.

In the last chapter (Chap. 11) we focus on the cohomology of constant sheaves. We show that for locally contractible spaces this is the same as singular cohomology, which is quickly introduced in Sect. 11.1. In particular we obtain the corollary that we can describe the de Rham cohomology on a manifold also via singular cohomology. Then we use the proper base change theorem to prove the main result of this chapter, the homotopy invariance of the cohomology of locally constant sheaves for arbitrary topological spaces. We conclude with some quick applications.

As with almost every mathematical text, this book contains a myriad of tiny exercises in the form of statements where the reader has to make some straightforward checks to convince herself (or himself) that the statement is correct. Beyond this, all chapters and appendices end with a group of problems. Some of the problems are sketching further important results that were omitted from the main text due to lack of space, and the reader should feel encouraged to use these problems as a motivation to study further literature on the topic.

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