2 Algebraic Topological Preliminaries

In this chapter we briefly introduce some elementary notions and results on homotopy, fundamental groups, and covering spaces that are used throughout the book.

2.1 Homotopy

We start with the notion of homotopy that is a central notion in topology.

Definition 2.1 (Homotopy). Two continuous maps $f, g\colon X \to Y$ of topological spaces are called *homotopic* if there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. Then H is called a *homotopy between f and g*. We then write $f \simeq g$ or $H\colon f \simeq g$.

One should think of $H\colon X \times [0,1] \to Y$ as a continuously varying family of maps $H_t\colon X \to Y$, $x \mapsto H(x,t)$ parametrized by $t \in [0,1]$ interpolating between $f = H_0$ and $g = H_1$.

Remark and Definition 2.2. Let X and Y be topological spaces. Then the homotopy relation $\simeq$ is an equivalence relation on the set of all continuous maps $X \to Y$:

1. Clearly, one has $f \simeq f$ via the homotopy $H\colon X \times [0,1] \to Y$, $(x,t) \mapsto f(x)$.
2. Given a homotopy $H\colon f \simeq g$, then $H^-\colon g \simeq f$ via the *inverse homotopy* $H^-\colon (x,t) \mapsto H(x, 1-t)$.
3. Let $H\colon f \simeq g$ and $K\colon g \simeq h$ be given. Then $H * K\colon f \simeq h$ via the *product homotopy*

$$(H * K)(x,t) := \begin{cases} H(x,2t), & 0 \le t \le 1/2; \\ K(x,2t-1), & 1/2 \le t \le 1. \end{cases}$$

T. Wedhorn, *Manifolds, Sheaves, and Cohomology*, Springer Studium Mathematik – Master,
DOI 10.1007/978-3-658-10633-1_2

The equivalence class of $f: X \to Y$ is denoted by $[f]$ and called the *homotopy class of* f. We denote by $[X, Y]$ the set of homotopy classes of continuous maps $X \to Y$.

A continuous map $X \to Y$ is called *null homotopic* if it is homotopic to a constant map.

Remark and Definition 2.3 (Homotopy category). We define the *homotopy category* (h-Top) as follows:

(a) Objects are topological spaces.
(b) For two objects X and Y we define $\mathrm{Hom}_{(\text{h-Top})}(X, Y) := [X, Y]$, the set of homotopy classes of continuous maps $X \to Y$.
(c) The identity of a topological space in (h-Top) is the homotopy class of id_X.
(d) For $[f] \in [X, Y]$ and $[g] \in [Y, Z]$ we define the composition by $[g] \circ [f] := [g \circ f]$. This is well defined. Indeed, let $f, f': X \to Y$ and $g, g': Y \to Z$ be continuous maps, and let $H: f \simeq f'$ and $K: g \simeq g'$ be homotopies. Then $g \circ f \simeq g' \circ f'$ via the homotopy $(x, t) \mapsto K(H(x, t), t)$.

A continuous map $f: X \to Y$ whose homotopy class is an isomorphism in (h-Top) is called a *homotopy equivalence*. This means that there exists a continuous map $g: Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. Topological spaces X and Y are called *homotopy equivalent* if they are isomorphic in (h-Top).

Example 2.4. Let $n \geq 1$ be an integer and let $\| \ \|$ be the Euclidean norm on $\mathbb{R}^n$. Then the $(n-1)$-dimensional sphere

$$S^{n-1} := \{ x \in \mathbb{R}^n \ ; \ \|x\| = 1 \}$$

is homotopy equivalent to $\mathbb{R}^n \setminus \{0\}$: The inclusion $i: S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ and the map $p: \mathbb{R}^n \setminus \{0\} \to S^{n-1}$, $x \mapsto \frac{x}{\|x\|}$ are mutually inverse homotopy equivalences. Indeed $p \circ i = \mathrm{id}_{S^{n-1}}$ and $i \circ p \simeq \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ via the homotopy

$$(\mathbb{R}^n \setminus \{0\}) \times [0, 1] \to \mathbb{R}^n \setminus \{0\}, \qquad (x, t) \mapsto (1-t)x/\|x\| + tx.$$

This is in particular an example of homotopy equivalent spaces that are not homeomorphic (S^{n-1} is compact, $\mathbb{R}^n \setminus \{0\}$ not).

Definition 2.5 ((Locally) contractible spaces). A topological space X is called *contractible*, if it is homotopy equivalent to a point (i.e., to the unique topological space consisting of a single point). It is called *locally contractible* if every point has a fundamental system of open contractible neighborhoods.

In other words, X is contractible if and only if there exists a point $x_0 \in X$ such that the constant map $X \to X$, $x \mapsto x_0$ and id_X are homotopy equivalent, i.e., if and only if id_X is null homotopic.

Example 2.6. Let V be a normed real or complex vector space.

1. Recall that a subset S of V is called *star-shaped* with *star center* $s_0 \in S$ if for all $s \in S$ the line segment $\{ts + (1-t)s_0 ; 0 \leq t \leq 1\}$ from s to s_0 is contained in S. Then the subspace S of V is contractible: $H\colon (s,t) \mapsto ts + (1-t)s_0$ is a homotopy between the constant map $s \mapsto s_0$ and id_S.
2. Every open ball in V is star-shaped. As the open balls form a basis of the topology on V, we see that every open subspace of V is locally contractible.

2.2 Paths

We quickly introduce some notation for paths. In this section, let $a, b \in \mathbb{R}$ with $a < b$.

Definition 2.7 (Paths). Let X be a topological space.

1. A continuous map $\gamma\colon [a,b] \to X$ is called *path in* X. The point $\gamma(a)$ is called the *start point*, $\gamma(b)$ is called the *end point* of γ. We say that γ is a *path from* $\gamma(a)$ *to* $\gamma(b)$. We also set:
$$\{\gamma\} := \gamma([a,b]) \subseteq X.$$
2. A path $\gamma\colon [a,b] \to X$ is called *closed* or a *loop* if $\gamma(a) = \gamma(b)$.

Let $\gamma\colon [a,b] \to X$ be a map. Let $\varphi\colon [0,1] \to [a,b]$, $\varphi(t) = a + (b-a)t$. Then γ is continuous if and only if $\gamma \circ \varphi$ is continuous and $\{\gamma\} = \{\gamma \circ \varphi\}$. This allows us usually to assume that $[a,b] = [0,1]$.

Definition 2.8. Let X be a topological space.

1. Let $\gamma\colon [0,1] \to X$ be a path. Define
$$\gamma^{-}\colon [0,1] \to X, \gamma^{-}(t) = \gamma(1-t)$$
the *inverse path*.
2. Let $\gamma, \delta\colon [0,1] \to X$ be two paths with $\gamma(1) = \delta(0)$. Define the *product* or the *concatenation of paths* by
$$\gamma \cdot \delta\colon [0,1] \to X, \qquad t \mapsto \begin{cases} \gamma(2t), & 0 \leq t \leq 1/2; \\ \delta(2t-1), & 1/2 \leq t \leq 1, \end{cases}$$
i.e., $\gamma \cdot \delta$ is the path where one first "walks along γ and then along δ each time with double velocity".

2.3 Path Connected Spaces

Let X be a topological space. We write $x \sim y$ if there exists a path with start point x and end point y. Then $\sim$ is an equivalence relation on X as shown by the construction in Definition 2.8.

Definition 2.9. An equivalence class with respect to $\sim$ is called a *path component of* X. The set of path components of X is denoted by $\pi_0(X)$. The space X is called *path connected* if there exists a single path component, i.e., $\pi_0(X)$ consists of a single point.

A topological space X is called *locally path connected* if every point of X has fundamental system of path connected neighborhoods.

The empty space is not path connected by definition.

Remark 2.10. Let $f\colon X \to Y$ be a surjective continuous map. If X is path connected, then Y is path connected.

Indeed, let $y, y' \in Y$. Choose $x, x' \in X$ with $f(x) = y$ and $f(x') = y'$ and a path γ in X connecting x and x'. Then $f \circ \gamma$ is a path connecting y and y'.

Example 2.11.
1. A discrete space X is locally path connected, but it is not path connected if it contains more than one point.
2. Conversely, there are path connected topological spaces that are not locally path connected (Problem 2.7).
3. Every open subspace of of a normed $\mathbb{R}$-vector space is locally path connected.

Proposition 2.12. *Every path connected topological space X is connected.*

Proof. Assume that X is not connected. Then X is the disjoint union of two non-empty open subsets U and V. Choose $x \in U$ and $y \in V$ and let $\gamma\colon [0,1] \to X$ be a path with $\gamma(0) = x$ and $\gamma(1) = y$. Then $[0,1]$ is the disjoint union of the open non-empty subsets $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$. This is a contradiction, because $[0,1]$ is connected (Appendix Proposition 12.43). □

In general there exist connected spaces that are not path connected (Problems 2.6 and 2.7). But for locally path connected spaces this cannot happen. More precisely we have:

Proposition 2.13. *Let X be a topological space such that every point has a path connected neighborhood. Then a subset of X is a path component if and only if it is a connected component. All path components are open and closed in X, i.e., X is the sum of its path components.*

Proof. By Proposition 2.12 every path component A of X is connected (and hence contained in a connected component). Let $x \in X$ and let A be the path component of X containing x. By hypothesis there exists a path connected neighborhood V of x. Then $A \cup V$ is path connected and hence contained in A. This shows that A is open in X. As the complement of A is a union of path components, A is also closed in X.

Proposition 2.12 shows that every connected component B of X is a union of path components. Each of theses path components is open and closed in X and hence in B. As B is connected, B can only contain a single path component. □

Remark 2.14. Suppose that X is a locally path connected space. Then Proposition 2.13 holds also for every open subspace of X. This shows in particular that every point of X has a fundamental system of *open* path connected neighborhoods.

Remark 2.15. Let $f: X \to Y$ be a continuous map. If $x, x' \in X$ are connected by some path γ, then $f(x)$ and $f(x')$ are connected by the path $f \circ \gamma$. Hence f induces a map $\pi_0(f): \pi_0(X) \to \pi_0(Y)$. We obtain a functor π_0: (Top) $\longrightarrow$ (Sets). Moreover, if $f, g: X \to Y$ are homotopic continuous maps, then $\pi_0(f) = \pi_0(g)$ and we even obtain a functor

$$\pi_0\colon (\text{h-Top}) \longrightarrow (\text{Sets}).$$

Hence every homotopy equivalence $f: X \to Y$ induces a bijection $\pi_0(X) \xrightarrow{\sim} \pi_0(Y)$. In particular, we see that every (locally) contractible space is (locally) path connected.

2.4 Fundamental Group

Definition 2.16. Let X and Y be topological spaces, let $A \subseteq X$ be a subspace, and let $f, g: X \to Y$ be continuous maps. A homotopy $H: f \simeq g$ is said to be *relative to* A if $H(a,t) = f(a) = g(a)$ for all $a \in A$, $t \in [0,1]$. We write $H: f \simeq g$ (rel A) in this case.

The existence of such a homotopy implies in particular that $f_{|A} = g_{|A}$. The same constructions as in Remark 2.2 show that "homotopy relative to A" is an equivalence relation on the set of all continuous maps $X \to Y$.

Definition 2.17. Let X be a topological space and let $\gamma, \delta: [0,1] \to X$ be two paths with the same start and end points. A *homotopy of paths* between γ and δ is a homotopy $H: \gamma \simeq \delta$ (rel $\{0,1\}$), i.e., H is a homotopy that leaves the start point and the end point fixed.

Remark 2.18. Let $\gamma, \gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2: [0,1] \to X$ be paths in X.

1. Let $\varphi: [0,1] \to [0,1]$ be continuous with $\varphi(0) = 0$ and $\varphi(1) = 1$. Then $\gamma \simeq \gamma \circ \varphi$ (rel $\{0,1\}$).

2. Set $x_0 := \gamma(0)$, $x_1 := \gamma(1)$, and let ε_i be the constant path $t \mapsto x_i$. Then

$$\varepsilon_0 \cdot \gamma \simeq \gamma \simeq \gamma \cdot \varepsilon_1 \text{ (rel } \{0,1\}).$$

3. Assume $\gamma_i(1) = \delta_i(0)$ for $i = 1, 2$. Then $\gamma_1 \simeq \gamma_2$ (rel $\{0,1\}$) and $\delta_1 \simeq \delta_2$ (rel $\{0,1\}$) imply that $\gamma_1 \cdot \delta_1 \simeq \gamma_2 \cdot \delta_2$ (rel $\{0,1\}$).
4. If one has $\gamma_1 \simeq \gamma_2$ (rel $\{0,1\}$), then $\gamma_1 \cdot \gamma_2^-$ is null-homotopic.
5. Assume $\gamma_1(1) = \gamma_2(0)$ and $\gamma_2(1) = \gamma_3(0)$. Then

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3) \simeq (\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \text{ (rel } \{0,1\}).$$

Proof. *1*. A homotopy $\gamma \circ \varphi \simeq \gamma$ is given by $(s,t) \mapsto \gamma(ts + (1-t)\varphi(s))$.

2.,*5*. One has $\varepsilon_0 \cdot \gamma = \gamma \circ \varphi$ with $\varphi(t) = 0$ for $0 \le t \le \frac{1}{2}$ and $\varphi(t) = 2t - 1$ for $\frac{1}{2} \le t \le 1$. Hence $\varepsilon_0 \cdot \gamma \simeq \gamma$ (rel $\{0,1\}$) by 1. Similarly one shows $\gamma \simeq \gamma \cdot \varepsilon_1$ (rel $\{0,1\}$) and 5.

3. If $H\colon \gamma_1 \simeq \gamma_2$ (rel $\{0,1\}$) and $G\colon \delta_1 \simeq \delta_2$ (rel $\{0,1\}$), then $\gamma_1 \cdot \delta_1 \simeq \gamma_2 \cdot \delta_2$ (rel $\{0,1\}$). via the homotopy $(s,t) \mapsto H(2s,t)$ for $0 \le s \le \frac{1}{2}$ and $(s,t) \mapsto G(2s-1,t)$ for $\frac{1}{2} \le s \le 1$.

4. One has $\gamma_1 \simeq \gamma_2$ (rel $\{0,1\}$) if and only if $\gamma_1^- \simeq \gamma_2^-$ (rel $\{0,1\}$). Hence we may assume that $\gamma_1 = \gamma_2$ by 3. Then $(s,t) \mapsto \gamma_1(2s(1-t))$ for $0 \le s \le \frac{1}{2}$ and $(s,t) \mapsto \gamma_1(2(1-s)(1-t))$ for $\frac{1}{2} \le s \le 1$ is a homotopy between $\gamma \cdot \gamma^-$ and the constant path with value $\gamma(0)$. □

Definition 2.19. Let X be a topological space, $x \in X$. Define the *fundamental group of X with base point x* by

$$\pi_1(X,x) := \{\, \gamma\colon [0,1] \to X \,;\, \gamma \text{ path with } \gamma(0) = \gamma(1) = x \} / (\simeq \text{ (rel } \{0,1\}),$$

i.e., $\pi_1(X,x)$ is the set of homotopy classes $[\gamma]$ of closed paths γ starting (and ending) in x. Define a multiplication on $\pi_1(X,x)$ by

$$[\gamma][\delta] := [\gamma \cdot \delta].$$

By Remark 2.18 this is well defined and yields a group structure on $\pi_1(X,x)$. The neutral element is the constant path with value x and the inverse of $[\gamma] \in \pi_1(X,x)$ is $[\gamma^-]$.

In the sequel we will often write γ instead of $[\gamma]$ for elements in $\pi_1(X,x)$.

Remark 2.20. Let (Toppt) be the category whose objects are pointed topological spaces, i.e., pairs (X,x) consisting of a topological space X and a point $x \in X$. Morphisms $(X,x) \to (Y,y)$ are continuous maps $f\colon X \to Y$ with $f(x) = y$. Composition in (Toppt) is given by composition of maps.

If $f\colon (X,x) \to (Y,y)$ is a morphism in (Toppt), then $\gamma \mapsto f \circ \gamma$ defines a group homomorphism $\pi_1(f)\colon \pi_1(X,x) \to \pi_1(Y,y)$. It is easy to check that we obtain a functor

$$\pi_1\colon (\text{Toppt}) \to (\text{Grp}).$$

Remark 2.21. Let X be a topological space, $x_0, x_1 \in X$ be points and let σ be a path in X with start point x_0 and end point x_1. Then

$$\sigma_*\colon \pi_1(X, x_0) \to \pi_1(X, x_1), \qquad \gamma \mapsto \sigma^- \cdot \gamma \cdot \sigma$$

is an isomorphism of groups: Remark 2.18 shows that σ_* is well defined and that for $\gamma, \gamma' \in \pi_1(X, x_0)$ one has

$$\sigma_*(\gamma)\sigma_*(\gamma') = \sigma^- \cdot \gamma \cdot \sigma \cdot \sigma^- \cdot \gamma' \cdot \sigma \simeq \sigma^- \cdot \gamma \cdot \gamma' \cdot \sigma \simeq \sigma^- \cdot \gamma \cdot \gamma' \cdot \sigma = \sigma_*(\gamma\gamma').$$

Hence σ_* is a group homomorphism. An inverse is given by $\sigma_*^-\colon \delta \mapsto \sigma \cdot \delta \cdot \sigma^-$.

In particular we see that for a path connected space X the fundamental group $\pi_1(X, x_0)$ is up to isomorphism independent of the chosen point $x_0 \in X$.

Definition 2.22. A topological space X is called *simply connected* if X is path connected and if $\pi_1(X, x) = 1$ for all (equivalently, for one) $x \in X$.

In other words, X is simply connected if and only if every closed path in X is homotopic relative $\{0, 1\}$ to a given constant path.

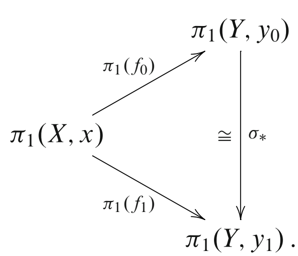
Remark 2.23. Let X and Y be topological spaces, $x_0 \in X$, $y_0 \in Y$, let p and q be the projections from $X \times Y$ to X and Y, respectively. Then

$$(\pi_1(p), \pi_1(q))\colon \pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

is an isomorphism of groups. An inverse is given by $\pi_1(i)\pi_1(j)$, where $i\colon X \to X \times Y$, $x \mapsto (x, y_0)$ and $j\colon Y \to X \times Y$, $y \mapsto (x_0, y)$.

We conclude this section by showing that homotopic maps induce isomorphic maps on fundamental groups.

Proposition 2.24. *Let $f_0, f_1\colon X \to Y$ be continuous maps and let $H\colon f_0 \simeq f_1$ be a homotopy. Let $x \in X$ and define $y_i := f_i(x)$ for $i = 0, 1$. Then $\sigma(t) := H(x, t)$ is a path from y_0 to y_1 and the following diagram commutes:*



Proof. Let $[\gamma] \in \pi_1(X, x)$. We have to show that

$$(f_0 \circ \gamma) \cdot \sigma \simeq \sigma \cdot (f_1 \circ \gamma) \text{ (rel } \{0, 1\}).$$

Let $h\colon [0,1] \times [0,1] \to Y$, $(s,t) \mapsto H(\gamma(s),t)$. Let $a, b, c, d\colon [0,1] \to [0,1] \times [0,1]$, $a(t) = (t, 0)$, $b(t) = (1, t)$, $c(t) = (0, t)$, $d(t) = (t, 1)$ be the sides of the square. Then $(f_0 \circ \gamma) \cdot \sigma = h \circ (a \cdot b)$ and $\sigma \cdot (f_1 \circ \gamma) = h \circ (c \cdot d)$. Clearly there is a homotopy $a \cdot b \simeq c \cdot d$ in $[0,1] \times [0,1]$ relative $\{0, 1\}$ (e.g., a linear homotopy $(s,t) \mapsto (1-t)(a \cdot b)(s) + t(c \cdot d)(s)$) and this homotopy induces the desired homotopy by composition. □

Corollary 2.25. *Let $f\colon X \to Y$ be a homotopy equivalence.*

1. *Let $x \in X$. Then $\pi_1(f)\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ is a group isomorphism.*
2. *The topological space X is simply connected if and only if Y is simply connected.*

Proof. Let $g\colon Y \to X$ be a homotopy inverse map of f. As $g \circ f$ (respectively $f \circ g$) is homotopic to the identity, Proposition 2.24 implies that $\pi_1(f)$ is injective (respectively surjective). This shows (1).

Assertion (2) follows from (1) and because X is path connected if and only if Y is path connected (Remark 2.15). □

Corollary 2.26. *Every contractible space is simply connected.*

Altogether we obtain the following implications for various notions of connectivity (where the first two notions have been only defined for subspaces of normed vector spaces):

$$\text{“convex”} \Rightarrow \text{“star shaped”} \Rightarrow \text{“contractible”} \Rightarrow \text{“simply connected”}$$
$$\Rightarrow \text{“path connected”} \Rightarrow \text{“connected”}.$$

Each of these implications is a proper implication: The union of the coordinate axes $\{(x, y) \in \mathbb{R}^2 \,;\, xy = 0\}$ is star shaped but not convex, $\mathbb{R}^2 \setminus \{(x, y) \,;\, y = x^2, x \geq 0\}$ is not star shaped but contractible (Problem 2.8), $\mathbb{R}^n \setminus \{0\}$ is simply connected for $n \geq 3$ (Problem 2.15) but not contractible (see Corollary 11.21 below), $\mathbb{R}^2 \setminus \{0\}$ is path connected (Problem 2.5) but not simply connected (see Corollary 2.44 below), and Problems 2.6 and 2.7 give examples of connected spaces that are not path connected.

2.5 Covering Spaces

Definition 2.27 (Covering space). Let X be a topological space.

1. A continuous map $p\colon \tilde{X} \to X$ is called a *covering space of* X or a *covering map* if for all $x \in X$ there exists $x \in U \subseteq X$ open such that

$$p^{-1}(U) = \coprod_{i \in I} \tilde{U}_i \quad \text{sum of topological spaces, } I \neq \emptyset \text{ some set} \tag{2.1}$$

   for open sets $\tilde{U}_i \subseteq X$ such that $p_{|\tilde{U}_i}\colon \tilde{U}_i \to U$ is a homeomorphism for all $i \in I$ (see Fig. 2.1).
2. A covering map $p\colon \tilde{X} \to X$ is called a *universal covering map* if $\tilde{X}$ is simply connected.
3. Define the category (CovSp(X)) of covering spaces of X as follows. Objects are covering spaces $p\colon \tilde{X} \to X$. A morphism between two covering spaces $p_1\colon \tilde{X}_1 \to X$ and $p_2\colon \tilde{X}_2 \to X$ is a continuous map $\alpha\colon \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 \circ \alpha = p_1$.

For a justification of the notion of a "universal covering" see Remark 2.38 below.

A covering map p is always a surjective map: For every U as in (2.1) and for every $y \in U$ we have a bijection $p^{-1}(\{y\}) \leftrightarrow I$ and $I \neq \emptyset$ by definition. The cardinality of the set $p^{-1}(x)$ is locally constant in x. If it is constant (e.g., if X is connected), we call the cardinality of a fiber the *degree of the covering* p.

Sometimes we consider *pointed covering spaces* $p\colon (\tilde{X}, \tilde{x}) \to (X, x)$ of pointed topological spaces. By this we mean a covering map $p\colon \tilde{X} \to X$ with $p(\tilde{x}) = x$. One has the obvious notion of a *morphism of pointed covering spaces*.

Remark 2.28. Let X be a topological space. Let $E \neq \emptyset$ be a set considered as a discrete topological space. Then the projection $X \times E \to X$ is a covering map. Covering maps that are isomorphic to a covering map of this form are called *trivial covering maps*.

A continuous map $p\colon \tilde{X} \to X$ is a covering map if and only if it is locally on X a trivial covering map.

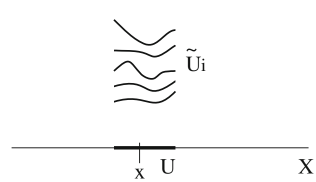


Figure 2.1 Triviality of the covering over U

Example 2.29.

1. The function $p\colon \mathbb{R} \to S^1 = \{\, z \in \mathbb{C} \; ; \; |z| = 1 \,\}$, $x \mapsto e^{2\pi i x}$ is a universal covering. Indeed, $\mathbb{R}$ is simply connected because it is convex. The map is a covering: For $j = -1, 1$ let $U_j = S^1 \setminus \{j\}$. Then $U_{-1} \cup U_1 = S^1$. One has $p^{-1}(U_1) = \coprod_{n\in\mathbb{Z}}(n, n+1)$ and $p_{|(n,n+1)}$ is a homeomorphism $(n, n+1) \xrightarrow{\sim} U_1$; similarly for $p^{-1}(U_{-1}) = \coprod_{n\in\mathbb{Z}}(n - \frac{1}{2}, n + \frac{1}{2})$.
2. The function $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is a universal covering (use (1) and the polar decomposition of complex numbers).
3. For $n \geq 2$, the map $f\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto z^n$ is surjective, but there exists no open neighborhood U of 0 such that $\# f^{-1}(\{z_0\}) = \# f^{-1}(\{0\}) = 1$ for all $z_0 \in U$. Hence f is not a covering. However its restriction to $\mathbb{C} \setminus \{0\}$ is a covering of degree n (Problem 2.17).

Remark 2.30. The fibers of a covering map $p\colon \tilde{X} \to X$ are relatively Hausdorff, hence every covering map is separated (Proposition 1.25). In particular, if X is Hausdorff, then $\tilde{X}$ is Hausdorff (Corollary 1.27).

Let $p\colon \tilde{X} \to X$ and let $f\colon Z \to X$ be a continuous map. A continuous map $\tilde{f}\colon Z \to \tilde{X}$ such that $p \circ \tilde{f} = f$ is called a *lifting of f along p*.

Proposition 2.31 (Uniqueness of liftings). *For $i = 1, 2$ let $\tilde{f}_i\colon Z \to \tilde{X}$ be liftings of f along a covering map $p\colon \tilde{X} \to X$. Suppose that Z is connected and that there exists $z_0 \in Z$ with $\tilde{f}_1(z_0) = \tilde{f}_2(z_0)$. Then $\tilde{f}_1 = \tilde{f}_2$.*

Proof. As $p \circ \tilde{f}_i = f$, $(\tilde{f}_1, \tilde{f}_2)$ yields a map $\tilde{f}\colon Z \to \tilde{X} \times_X \tilde{X}$. Let $\Delta_p \subseteq \tilde{X} \times_X \tilde{X}$ be the diagonal. As p is a separated by Remark 2.30 and a local homeomorphism, Δ_p is closed and open in $\tilde{X} \times_X \tilde{X}$ (Proposition 1.25 and Appendix Remark 12.37). Hence $\tilde{f}^{-1}(\Delta_p) = \left\{ z \in Z \; ; \; \tilde{f}_1(z) = \tilde{f}_2(z) \right\}$ is closed and open in Z. It contains z_0 and hence is equal to Z because Z is connected. □

The proof shows that the same assertion remains true for separated local homeomorphisms p.

Proposition 2.32 (Lifting homotopies). *Let $p\colon \tilde{X} \to X$ be a covering map. Let Z be a topological space, let $H\colon Z \times [0,1] \to X$ be a homotopy and let $\tilde{f}\colon Z \to \tilde{X}$ be a continuous map such that $p(\tilde{f}(z)) = H(z, 0)$. Then there exists a unique homotopy $\tilde{H}\colon Z \times [0,1] \to \tilde{X}$ such that $p \circ \tilde{H} = H$ and such that $\tilde{f}(z) = \tilde{H}(z, 0)$ for all $z \in Z$.*

The proof will show that if H is a homotopy relative to a subset $A \subseteq Z$, then $\tilde{H}$ is also a homotopy relative to A.

Springer

http://www.springer.com/978-3-658-10632-4

Manifolds, Sheaves, and Cohomology
Wedhorn, T.
2016, XVI, 354 p. 9 illus., Softcover
ISBN: 978-3-658-10632-4