Chapter 2
Conductive Heat Transport Systems

Thermodynamics gives me two strong impressions: first of a subject not yet complete or at least one of whose ultimate possibilities have not yet been explored, so that perhaps there may still be further generalizations awaiting discovery; and secondly and even more strongly as a subject whose fundamental and elementary operations have never been subject to adequate analysis.

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In this chapter, we directly apply the law of motive force in place of variational formulation as well as optimal control theory for a class of problems pertaining to conductive heat transport mode in the realm of thermal insulation design. From the physics of the principle it has been deduced that a truly minimum exists for such class of problems. To start with, the optimum distribution of limited amount of insulating material on one side of a plane surface as well as a curved wall is obtained assuming that the amount of insulating material does not affect the imposed temperature gradient. Next, we apply the same physical theory for a more general case when a stream of fluid is suspended in a different temperature, and where the volume of insulation material does affect the temperature distribution. Finally, it has been argued that Schmidt's criterion for the fin design, tangent law of conductive heat transport and the Fermat's principle in geometrical optics are but special stipulations of the proposed law of nature, whereas the constructal law is a stand-alone principle where the proposed law of motive force is manifested through the competition of backward and forward motivation of slower (diffusion-like) and faster (convection-like) processes.

2.1 The Problem

The problem of optimization is the very essence of reality [1–4]. It is well known that many physical theories naturally give rise to the variational optimization principle from which the governing equations of the system can be deduced; the class of theories that do not yield a spontaneous variational formulation on account of nonlinearity, or else can be modified to admit a variational form [5]. Inversely, it also follows at once that the laws of physical theories when expressed as differential equations, the possibility of their reduction to a variational principle is evident from purely mathematical reasoning and does not depend on certain

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attributes intrinsic to the theory [6]. Despite these mathematical assertions, remarkably the classical thermodynamics [7–9] usually formulated is devoid of variational principles. However, it can be shown that as far as the implications for quasistatic transitions are concerned, the second law of thermodynamics can be formulated as a variational principle [10]. In classical mechanics, it can be established that by means of Gauss's principle [11], all problems may be reduced to those pertaining to maxima and minima and, hence possibly, to a problem of variational calculus. Thus, the variational technique as an optimization procedure has undergone tremendous upsurge both in science and engineering [12–19]. However, physicists and engineers often seem to disagree about the meaning of a variational principle [20]. For physicists, the fundamental element is generally the existence of a Lagrangian function through which the governing equations of the system are obtained by taking the functional derivatives. The main appeal of the Lagrange function is its power of synthesis. The whole physics of the problem is expressed in terms of a single function. But the Lagrangian in our extended sense exists only for dissipative systems. On the other hand, for engineers the main point often seems to be the existence of a variational technique, as clearly indicated by the type of approximation methods [21] employed in engineering optimization, which are largely independent of the existence of a Lagrange function. Variational principle can also be formulated [22] outside the postulate of minimum entropy production [23] and the concept of local potential [23]. Quite apart from variational formulation, a wide class of practical optimization problems can be expressed in the form of the Pontryagin maximum principle [24]. It is reported that attempts to solve these problems by the method of classical calculus of variations are not attractive [25].

An optimization procedure, such as variational method, is usually carried out halfway, that is, the values of the parameters of a trial function are found for which a property of the system under consideration, such as the energy, reaches its optimum value [26]. Thus, the current research methodology emphasizes the physical understanding of the problem in thermodynamic optimization of systems with particular examples in mind.

The present contribution explores the proposed law of motive force, a physical principle, mainly for the design of conductive insulation systems, which was recently analyzed by the formal method of calculus of variations [27]. Thermodynamic optimization of insulation system is also historically important and remains an active research frontier in the contemporary heat transfer research. It is historically important because the chronology of the entropy generation minimization field [28] began with the design of insulation systems subject to finite-size constraint [29]. It is an active area of research since power plant and refrigeration unit can be regarded as thermal insulation system [30], while accepting the general definition that thermal insulation is a system that prevents two surfaces of different temperatures from coming into direct thermal communication. From the physical perspective of the problem, it is demonstrated that for such a class of optimization problems a truly minimum exists. Finally, it has been argued that there should be some basis for analogies among physical theories [31]. Being persuaded by such basis the manifestation of law of motive force in Fermat's principle [32, 33] and constructal law [34, 35] from which geometric forms [36] can be deduced out of a single physics principle is sought. The current contribution examines the result obtained by the author [37] in the light of the proposed law of motive force.

2.2 A Physical Principle in Heat Transport

To engineer nature is to understand her first. In this endeavor we seek continually a more general principle than the existing till an all-encompassing theory is established. The speculative way of seeking a new law is but to guess it first [38]. In a nutshell, the law of motive force enunciates to identify the "conservation" of some physical quantities as a physical principle of thermodynamic optimization. Existence of such "isolines" is one of the most fundamental characteristics of extremality. Guided by this line of thought we proceed to identify the contributing competing mechanisms that constitute the locus of the physical process path describing the isoline.

To illustrate the rudimental feature of this principle, we first consider a plane wall of length L and width W perpendicular to the plane of the paper as shown in Fig. 2.1. The wall temperature variation $T(x)$ is only along the longitudinal direction x. The fundamental question corners around how to distribute a finite amount of insulating material either with constant or varying thickness $t(x)$ on the wall for minimum heat loss.

The insulated wall can be thought of being pieced into m equal or unequal length of sections. The more the local distribution of unit insulation material ΔV, the less the local heat transfer rate Δq in general. On the other hand, making a particular segment of the wall more effective leads other parts of the wall to be less effective in insulation. Thus, we identify heat transfer and insulation volume to be two competing physical factors (forces, motives) in insulation design. Here, the incidence of heat transfer acts as a forward motivation, whereas insulation volume plays the role of backward motivation. Hence, following the proposition of law of motive force, the legitimate postulate should be the uniform (equal) effectiveness of the insulation. This natural law translates mathematically into

$$\Delta q_i + \lambda_i \Delta V = \Delta q + \lambda \Delta V = C_{sv} \tag{2.1a}$$

for $i = 1, 2, 3, \ldots, m$ and where C_{sv} is a constant. We drop the subscript i for equal segmentation. Here, λ is a numerical and dimensional factor which makes the volume, a physical quantity, to be dimensionally homogeneous with another physical entity heat. The far reaching consequences of this parameter in a greater perspective are to be realized [39]. The order of magnitude of the parameter λ is such that for which the problem of optimization is nontrivial. Hypothetically, there may be some portion of the wall not covered with insulation at all, meaning $\lambda = 0$ as in the leading and trailing edges of the wall. On the contrary, all insulation

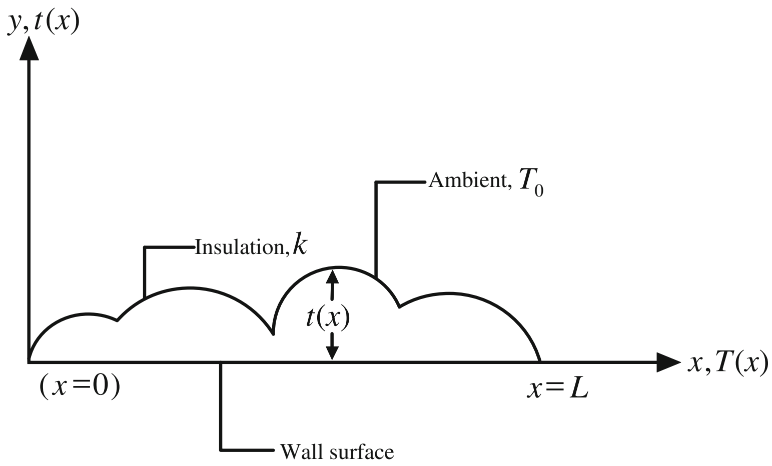

Fig. 2.1 A flat plate with arbitrary variation in insulation thickness

material can be applied onto a limited spot, leading to $\lambda > 0$. Thus, from the physical point of view the dimensional scale factor λ is bounded only in the domain $[0, \infty]$. To realize in another way the role played by λ, Eq. (2.1a) may be written in an alternative fashion when one of the constituents leads to a constant as

$$\frac{\Delta q_i}{\lambda_i \Delta V} = \frac{\Delta q}{\lambda \Delta V} = C_{rv} \tag{2.1b}$$

for $i = 1, 2, 3, \ldots, m$ and where C_{rv} is another constant. Notationally, the subscript is dropped for equal segmentations as before. It can be seen that for $\lambda \to 0$, the constant on the right side of Eq. (2.1b) tends to a very high value, meaning a very high rate of heat transfer as also indicated by Eq. (2.1a) and thus not a desirable feature for modeling. On the other hand, for $\lambda \to \infty$ the constant on the right side of Eq. (2.1b) runs to a very low value implying a very low heat transfer as also implied by Eq. (2.1a), and thus ensures a favorable modeling feature. But at the same time for the optimization problem to be nontrivial, the material volume cannot be unlimited or scarce posing a restriction to the upper and lower bounds for the value of λ too.

Either Eq. (2.1a) or Eq. (2.1b) can be employed, as the case may be for the ease of computation or applicability, to obtain optimal profile of insulation in connection with minimum heat transfer from the wall with definite curvature and temperature profile.

If we consider insulating a line element instead of a plane wall, Eqs. (2.1a) and (2.1b) transform, respectively, into

$$\Delta q_i + \mu_i \Delta A = \Delta q + \lambda \Delta A = C_{sa} \tag{2.2a}$$

and

$$\frac{\Delta q_i}{\mu_i \Delta A} = \frac{\Delta q}{\mu \Delta A} = C_{ra} \tag{2.2b}$$

where the volume element ΔV is replaced by the surface area element ΔA and the dimensional role of μ has been changed to that of λ.

It is interesting to report that Eq. (2.2b) resembles that of Schmidt's idea [40] of optimum profile shape for cooling fin with minimum weight. At the same time, it is to be noted that Schmidt's criterion was obtained on a different heuristic logic. The intuitive logic of Schmidt was confirmed through rigorous variational formulation by Duffin [41]. Jany and Bejan [42] came to the conclusion that the idea of fin shape optimization has an important analog in the design of long ducts for fluid flow.

It is thought-provoking that the problem for maximum heat transfer objective resembles the challenge of insulation design for minimum heat transfer. It truly reflects the opposing action of the motive forces [43] as forward and backward motivation in apparently two antagonistic arrangements. The physical factor that transcribes a problem of insulation into a question of fin is the curvature of the surface in consideration. For example, critical insulation thickness [44] exists only in reality for the design of cylindrical and spherical layers, but not in the sizing of plane or nearly plane layers. Thus, we repeat the symmetric appearance of a physical principle [45–47] with respect to its foundation in mathematical terms [31].

2.3 The Physical Basis for Extremum Heat Transfer

The criteria for distinguishing between the maximum and minimum values of the functional have been investigated by many eminent mathematicians [48]. A rigorous mathematical discussion of the discriminating conditions may be found from the fundamental principle alone [49]. In our present endeavor we will however, provide a physical basis for the existence of the extremum. To be specific with the domain of application of this analysis, we take the example of purely conductive insulation system.

From the physical perspective, heat transfer and insulation volume are both nonnegative quantities. It is to be noted that we did not adopt here a control volume approach so as to regard heat transfer as positive or negative with respect to the system in a conventional manner. Again, Eq. (2.1a) truly represents a competition between two opposing tendencies of the system: backward motivation and forward motivation. Further, their constancy of summation leads to the fact that increment of one quantity drives to the decrement of the other in numerical estimate. These logics translate into the following mathematical prescriptions

$$\Delta q = \Delta Q^2,\ \Delta V = \Delta v^2,\ \text{and}\ \lambda = -\psi. \tag{2.3}$$

Thus, Eq. (2.1a) transforms into

$$\Delta Q^2 - \psi \Delta v^2 = C_{sv}. \tag{2.4}$$

Since we are interested in global extremum, integrating upon Eq. (2.4) the entire length of the plate, we have

$$\int_0^L \left(\Delta Q^2 - \psi \Delta v^2\right) \mathrm{d}x = C_{sv} L. \tag{2.5}$$

As indicated by the first example of the use of trigonometric series in the theory of heat [50], we adopt Fourier expansion [51] for the pattern of distribution of insulating material in primitive variables to be

$$\Delta v = \sum_{m=1}^{\infty} a_m \sin \frac{m\pi}{L} x \tag{2.6}$$

where a_m's are some constants compatible with the convergence of the series. Rearranging Eq. (2.1a) in the following form:

$$\Delta q = C_{sv} - \lambda \Delta V \tag{2.7}$$

and recognizing that heat transfer takes place in a normal direction to the plane under consideration [52], we find a compatible [53, 54] Fourier series as

$$\Delta Q = \sum_{m=1}^{\infty} \frac{m\pi}{L} a_m \cos \frac{m\pi}{L} x. \tag{2.8}$$

Invoking Parseval's theorem [55] to the relations (2.6) and (2.8) we arrive, respectively, at

$$\int_0^L \Delta v^2 \mathrm{d}x = \frac{L}{2} \sum_{m=1}^{\infty} a_m^2 \tag{2.9}$$

and

$$\int_0^L \Delta Q^2 \mathrm{d}x = \frac{L}{2} \sum_{m=1}^{\infty} \frac{m^2 \pi^2}{L^2} a_m^2. \tag{2.10}$$

The mathematical prescription for the applicability of Parseval's theorem is that

$$\Delta v(0) = \Delta v(L) = 0 \tag{2.11}$$

and ΔQ whose square is Lebesgue integrable [56] over the interval [0, L]. From the physics of the problem these criteria are quite recognizable. Thus, incorporating Eqs. (2.9) and (2.10) into Eq. (2.5) we have

$$C_{sv} = \frac{1}{2}\sum_{m=1}^{\infty}\left(\frac{m^2\pi^2}{L^2} - \psi\right)a_m^2. \tag{2.12}$$

Noting that the left-hand side is a finite positive quantity and hence there exists a minimum for the parameter ψ in the range

$$\psi \leq \frac{\pi^2}{L^2}. \tag{2.13}$$

It is to be remarked that the physical role [57] played by the parameter ψ here is different from λ in Eq. (2.1a). The physical argument presented above easily extends to the Sturm–Liouville theory [58]. Thus, we conclude that a truly minimum exists for this class of problems of insulation design. Next, we will calculate only the optimum profile for different geometries and temperature distributions. Once thus obtained optimum profile tallies with the established results, the minimum heat transfer quantity follows at once.

2.4 Temperature Distribution and Heat Transfer from an Insulated Wall

In many engineering applications [27], a nonlinear temperature variation $T(x)$ in the longitudinal direction x of the wall of finite length L arises with definite curvature $\mathrm{d}^2T/\mathrm{d}^2x$. When the curvature of the wall temperature function is positive, temperature profile of the wall can be outlined as

$$\frac{T(x) - T_0}{T_L - T_0} = \frac{\exp\left(n\frac{x}{L}\right) - 1}{e^n - 1} \tag{2.14a}$$

where T_0 and T_L are wall temperatures at $x = 0$ and $x = L$, respectively, and the nondimensional parameter n bears the same sign as the curvature of the wall temperature function. Here, T_0 is also the ambient temperature. For the curvature of the temperature function of the wall to be negative, temperature distribution of the wall can be expressed algebraically as

$$\frac{T(x) - T_0}{T_L - T_0} = \frac{1 - \exp\left(n\frac{x}{L}\right)}{1 - e^n}. \tag{2.14b}$$

In case of vanishingly small curvature of temperature function, passing to the limit $n \to 0$ either from Eq. (2.14a) or Eq. (2.14b), we obtain by applying L'Hospital's theorem a linear temperature distribution as

$$\frac{T(x) - T_0}{T_L - T_0} = \frac{x}{L}. \tag{2.14c}$$

On the other hand, the mathematical advantage of the exponential representation of temperature is that it can be readily treated in resulting differential equations [59]. Hence, it may be possible to cast the whole exercise as a control problem of differential equation alone [60].

Recognizing the local temperature gradient $\Delta T = T(x) - T_0$ to be the cause of spontaneous heat transfer effect Δq in a coupled conductive–convective formulation the expression for heat transfer stands as

$$\Delta q = \frac{\Delta T}{\frac{F[t(x)]}{k\Delta A} + \frac{1}{h\Delta A}} \tag{2.15a}$$

where ΔA is the elemental heat transferring area, k is the constant thermal conductivity of the insulating material, h is the local convective heat transfer coefficient, $t(x)$ is the local thickness of insulation, and $F[t(x)]$ is the function of insulation thickness. Passing to the limit $h \to \infty$ in Eq. (2.15a) we arrive at

$$\lim_{h\to\infty} \Delta q = \lim_{h\to\infty} \frac{\Delta T}{\frac{F[t(x)]}{k\Delta A} + \frac{1}{h\Delta A}} = \frac{\Delta T}{\frac{F[t(x)]}{k\Delta A}}. \tag{2.15b}$$

Equation (2.15a) is connected to Eq. (2.15b) in the same manner as the two-dimensional problem of heat transfer is related to the one-dimension when either of the dimensions is very great in comparison with the other.

In mathematical modeling of the problem we have both the choices: either to consider or not the effect of local insulation thickness on the driving force (ΔT) for the heat transfer.

2.5 Insulation on Plane Surface with Static Wall Temperature Condition

By static wall temperature condition we mean that the temperature distribution on the wall will not be affected by the amount of insulation mounted. We consider here a plane wall of length L and width W. The average insulation thickness $\bar{t}$ can be defined on total volume V as

$$\bar{t} = \frac{V}{WL} = \frac{1}{L}\int_0^L t(x)\mathrm{d}x. \tag{2.16}$$

In Eq. (2.15b) we recognize for a plane wall that

$$\frac{F[t(x)]}{\Delta A} = \frac{t(x)}{W\mathrm{d}x}. \tag{2.17}$$

Now, employing the law of motive force (2.1b) in Eq. (2.15b) along with Eq. (2.17) we directly obtain

$$t(x) = \left(\frac{\lambda}{k}\right)^{1/2} (\Delta T)^{1/2}. \tag{2.18}$$

For linear temperature distribution, invoking Eq. (2.14c) in Eq. (2.18) we arrive at

$$t(x) = \Lambda_1 \left(\frac{x}{L}\right)^{1/2} \tag{2.19}$$

where Λ_1 is the shorthand for the constant $\left[\frac{\lambda}{k}(T_L - T_0)\right]^{1/2}$. Integrating Eq. (2.19) between 0 and L and employing Eq. (2.16) for the definition of average thickness we have

$$\Lambda_1 = \frac{3}{2}\bar{t}. \tag{2.20}$$

Optimal insulation profile is obtained by eliminating the constant Λ_1 between Eqs. (2.19) and (2.20) as

$$t_{1l}(x) = \frac{3}{2}\bar{t}\left(\frac{x}{L}\right)^{1/2}. \tag{2.21}$$

When the curvature of the wall temperature function is positive, employing Eq. (2.14a) in Eq. (2.18) and adopting similar procedure, we get optimal insulation thickness as function

$$t_{2l}(x) = \frac{n}{2}\bar{t}\frac{\sqrt{\exp\left(n\frac{x}{L}\right) - 1}}{\sqrt{e^n - 1} - \tan^{-1}\sqrt{e^n - 1}}. \tag{2.22}$$

For the curvature of the wall temperature profile to be negative, recruiting Eq. (2.14b) to Eq. (2.18), similarly we obtain the optimal insulation profile as

$$t_{3l}(x) = \frac{n}{2}\bar{t}\frac{\sqrt{1 - \exp\left(n\frac{x}{L}\right)}}{\sqrt{1 - e^n} - \tanh^{-1}\sqrt{1 - e^n}}. \tag{2.23}$$

2.6 Insulation on Cylindrical Surface with Static Wall Temperature Condition

As stated before, static wall temperature condition implies that the temperature of the wall is not a function of insulation volume. We now consider a cylinder of radius r and length L. Geometrically, we mean a situation with the surface of revolution of the plane wall mounted with arbitrary insulation volume along with a translation in the vertical direction. Such a description bears easy extension to the fundamental problem presented in Fig. 2.1. Then the fixed volume V of insulation is rendered by

$$V = \int_0^L \pi r^2 \left\{ \left[1 + \frac{t(x)}{r} \right]^2 - 1 \right\} \mathrm{d}x. \tag{2.24a}$$

The relative thickness of insulation material is obtained in dimensionless form as

$$\bar{V} = \frac{V}{\pi r^2 L} = \frac{1}{L} \int_0^L \left\{ \left[1 + \frac{t(x)}{r} \right]^2 - 1 \right\} \mathrm{d}x. \tag{2.24b}$$

When the wall thickness is not negligibly relative to the radius of the curvature of the wall surface, the problem must be analyzed by a method that takes the curvature into account. In Eq. (2.15b) we identify for a cylindrical wall [61]

$$\frac{F[t(x)]}{\Delta A} = \frac{\ln\left[1 + \frac{t(x)}{r}\right]}{2\pi \mathrm{d}x}. \tag{2.25}$$

Plugging Eq. (2.25) in lieu of Eq. (2.15b) along with Eq. (2.24a) into the law of motive force (2.1a), we get the optimal insulation profile to comply with the following condition:

$$\delta \ln \delta = \left(\frac{k}{\lambda r^2} \right)^{1/2} (\Delta T)^{1/2} \tag{2.26a}$$

where

$$\delta(x) = 1 + \frac{t(x)}{r}. \tag{2.26b}$$

Assuming a linear temperature distribution (2.14c) in Eq. (2.26a) we obtain

$$\delta \ln \delta = \Lambda_2 \left(\frac{x}{L} \right)^{1/2} \tag{2.27a}$$

where Λ_2 is the notation for the parameter $\left[\frac{k}{\lambda r^2}(T_L - T_0)\right]^{1/2}$. The constant Λ_2 is determined from the definition (2.24b) as

$$\Lambda_2 = \left[\frac{2}{\bar{V}}\int_1^{\Delta} \delta\left(\delta^2 - 1\right) \ln\delta \ln(e\delta)\mathrm{d}\delta\right]^{1/2} \tag{2.27b}$$

where

$$\Delta = 1 + \frac{t_{\mathrm{opt}}(L)}{r}. \tag{2.27c}$$

Eliminating the constant Λ_2 between Eqs. (2.27a) and (2.27b) optimal insulation profile is obtained as

$$\delta\ln\delta = \left[\frac{2}{\bar{V}}\int_1^{\Delta} \delta\left(\delta^2 - 1\right) \ln\delta \ln(e\delta)\mathrm{d}\delta\right]^{1/2}\left(\frac{x}{L}\right)^{1/2}. \tag{2.28}$$

In the event of positive wall temperature curvature recruiting Eq. (2.14a) in Eq. (2.26a) we have

$$\delta\ln\delta = \Lambda_3\left[\exp\left(n\frac{x}{L}\right) - 1\right]^{1/2} \tag{2.29a}$$

where Λ_3 is the shorthand for the group $\left[\frac{k}{\lambda r^2}\frac{T_L - T_0}{e^n - 1}\right]^{1/2}$. The constant Λ_3 is implicitly determined using definition (2.24b) as

$$\frac{2}{n}\int_1^{\Delta} \frac{\delta\left(\delta^2 - 1\right) \ln\delta \ln(e\delta)}{(\delta\ln\delta)^2 + \Lambda_3}\mathrm{d}\delta = \bar{V}. \tag{2.29b}$$

Eliminating the constant term Λ_3 between Eqs. (2.29a) and (2.29b) we obtain the required optimum insulation profile.

Similarly, for negative curvature of the wall temperature function employing Eq. (2.14b) in Eq. (2.26a) and exercising the same procedure, we obtain the optimal profile of insulation as the eliminant of the parametric constant Λ_4 between the following equations:

$$\delta\ln\delta = \Lambda_4\left[1 - \exp\left(n\frac{x}{L}\right)\right]^{1/2} \tag{2.30a}$$

and

$$\frac{2}{n}\int_1^{\Delta} \frac{\delta(\delta^2 - 1) \ln\delta \ln(e\delta)}{(\delta\ln\delta)^2 - \Lambda_4}\mathrm{d}\delta = \bar{V} \tag{2.30b}$$

where Λ_4 is the shorthand for the constant $\left[\frac{k}{\lambda r^2}\frac{T_L - T_0}{1 - e^n}\right]^{1/2}$.

2.7 Insulation on Cylindrical Surface with Dynamic Wall Temperature Condition

Unlike in Sects. 2.5 and 2.6, we consider here a dynamic local temperature gradient situation for the wall. In other words, we do not neglect the effect of local insulation thickness on the local temperature distribution. Rather, we impose the more realistic condition that the local temperature distribution is affected by the amount of insulation. Now, as a modeling feature we are at liberty to apply insulation in such a way that the local temperature potential remains piecewise constant, that is, $\Delta T \neq \Delta T(x)$. This makes in turn the local overall heat transfer coefficient U to be independent of longitudinal spatial position [62], that is, again $U \neq U(x)$. This idea of equipartitioned (uniformed) potential difference is due to the author [63].

Let us consider a stream of fluid with local temperature distribution $T_f(x)$ passing through an insulated cylindrical tube whose outer surface is exposed to a constant environment temperature T_0 such that

$$\Delta T = T_f(x) - T_0 = C_T \tag{2.31}$$

where C_T is a constant.

The expression for overall heat transfer coefficient U between the local bulk temperature of the stream $T_f(x)$ and the environment at T_0 can be readily obtained from any standard heat transfer textbook [61] as

$$\frac{1}{U 2\pi r \mathrm{d}x} = \frac{1}{h_0 2\pi [r + t(x)]\mathrm{d}x} + \frac{\ln\left[1 + \frac{t(x)}{r}\right]}{k_i 2\pi \mathrm{d}x} + \frac{t_w}{k_w 2\pi r \mathrm{d}x} + \frac{1}{h_i 2\pi r \mathrm{d}x} \tag{2.32a}$$

where h_i and h_0 are the local convective heat transfer coefficients for the inner fluid and the outer fluid, k_i and k_w are the conductivities of the insulating material and cylinder wall, respectively, r is the inner radius of the wall, t_w is the thickness of the wall. Recognizing the fact that $h_i, h_0 \to \infty$ and $\frac{t_w}{r} \to 0$, we pass on to these limits in Eq. (2.32a) to obtain

$$\frac{1}{U 2\pi r \mathrm{d}x} = \frac{\ln\left[1 + \frac{t(x)}{r}\right]}{k_i 2\pi \mathrm{d}x}. \tag{2.32b}$$

Putting Eq. (2.32b) into Eq. (2.25) along with Eq. (2.15b) into the law of motive force (2.1a) we obtain

$$\left[\frac{\Delta T}{U 2\pi r} + \lambda \pi r^2 \left\{\left[1 + \frac{t(x)}{r}\right]^2 - 1\right\}\right]\mathrm{d}x = \text{constant}. \tag{2.33a}$$

By definition ΔT and U are constants and since $\mathrm{d}x$ can be arbitrarily small, the bracketed quantity on the right side vanishes identically, i.e.,

$$\frac{\Delta T}{U2\pi r} + \lambda \pi r^2 \left\{ \left[1 + \frac{t(x)}{r} \right]^2 - 1 \right\} = 0. \tag{2.33b}$$

Since $t(x)$ is the only variable on the left side, the physical solution of the equation leads to the fact that

$$t(x) = \text{constant}. \tag{2.34}$$

The constant of the right side of Eq. (2.34) is determined from Eq. (2.24b) as

$$t_{\text{opt}} = r\left[(1 + \bar{V})^{1/2} - 1 \right]. \tag{2.35}$$

Equation (2.35) is an important result and was obtained using the calculus of variations [27] and optimal control theory [64] as reported in literature as well as traditionally practiced by engineers.

2.8 Law of Motive Force, Tangent Law, Fermat's Principle, and Constructal Law

For two different materials of the wall and the insulating volume to be in perfect thermal contact, the interfacial boundary conditions demand that [65]

$$-k_1 \left(\frac{\partial T_1}{\partial y} \right)_{0^+} = -k_2 \left(\frac{\partial T_2}{\partial y} \right)_{0^-} \tag{2.36a}$$

and

$$T_1 = T_2 \tag{2.36b}$$

where the subscripts 1 and 2 refer to the general wall and the insulating material, respectively. Equation (2.36a) can be written as

$$\frac{\left(\frac{\partial T_1}{\partial y} \right)_{0^+}}{\left(\frac{\partial T_2}{\partial y} \right)_{0^-}} = \frac{k_2}{k_1} \tag{2.37a}$$

which readily admits the following form:

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{k_2}{k_1} \tag{2.37b}$$

where θ_1 and θ_2 are the angles of incidence and refraction, respectively. In turn, for small angles Eq. (2.37b) can also be written as

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{k_2'}{k_1'} \tag{2.37c}$$

where k_1' and k_2' can be thought of as modified thermal conductivities. However, the approximate form of Eq. (2.37b) reads as

$$\frac{\sin\theta_1}{\sin\theta_2} \approx \frac{k_2}{k_1} \tag{2.37d}$$

for small angles of incidence and refraction. For constant thermal conductivities, each of the forms contained in Eqs. (2.37a), (2.37b), and (2.37c) can be represented, respectively, as

$$\left(\frac{\partial T_1}{\partial y}\right)_{0^+} + \left(\frac{\partial T_2}{\partial y}\right)_{0^-} = \text{constant}, \tag{2.38a}$$

$$\tan\theta_1 + \tan\theta_2 = \text{constant}, \tag{2.38b}$$

and

$$\sin\theta_1 + \sin\theta_2 = \text{constant}. \tag{2.38c}$$

It is to be noted that the message contained in Eqs. (2.37a), (2.37b), and (2.37c) are but principally one and the same: the very proposition of law of motive force. Further, it is to be noted that Eq. (2.37b) is a consequence of tangent law in heat conduction [66], whereas Eq. (2.37c) is an outcome of Fermat's principle and also modeled through dynamic programming approach [67, 68].

Comparing Eq. (2.37b) with (2.37d) we observe that there is a sacrifice in the degree of accuracy. This criterion of accuracy is to be judged from the pertinent application in question. For example, let us consider the more generalized situation of coupled conductive–convective heat transport mechanism [69]. Approximation of surface heat flux at the solid surface of the form

$$-k_1\left(\frac{\partial T_1}{\partial y}\right)_{0^+} \approx \frac{k_1(\Delta T_1)_{\bar{t}}}{\bar{t}} \tag{2.39}$$

is valid for a linear temperature distribution across the wall according to the following relation:

$$T_1 = T_w(x) + \frac{(\Delta T_1)_{\bar{t}}}{\bar{t}} \tag{2.40}$$

where the subscript w refers to the interfacial condition based upon average thickness $\bar{t}$ of the insulation volume. According to the theory of similarity [70] for a nonlinear temperature variation across the wall we may write

$$\left(\frac{\partial T_1}{\partial y}\right)_{0^+} \approx \varepsilon \frac{(\Delta T_1)_{\bar{t}}}{\bar{t}} \tag{2.41}$$

where ε is a correction factor for the distorted temperature profile. The slope on the right side of Eq. (2.41) is a single-valued function of $\frac{(\Delta T_1)_{\bar{t}}}{\bar{t}}$. Equations (2.36a) and (2.41) can be rearranged in the form

$$\frac{(\Delta T_1)_{\bar{t}}}{(\Delta T_2)_{\delta_T}} = \varepsilon \frac{k_2}{k_1} \frac{t}{x} \frac{x}{\delta_T} \tag{2.42}$$

where δ_T is the thermal boundary thickness of the medium. Approximate general local Nusselt number correlation can be expressed in the form [71]

$$Nu_x = \frac{x}{\delta_T} = C\mathrm{Pr}^{\mathrm{a}}\mathrm{Re}_x^b \tag{2.43}$$

where a, b, and C are constants. Thus, the relative temperature drop term $(\Delta T_r)_{\bar{t}}$ contained in Eq. (2.42) is expressible as

$$(\Delta T_r)_{\bar{t}} = \frac{(\Delta T_1)_{\bar{t}}}{(\Delta T_2)_{\delta_T}} = \varepsilon C f\left(\frac{k_2}{k_1} \frac{t}{x} \mathrm{Pr}^{\mathrm{a}}\mathrm{Re}_x^b\right) \tag{2.44}$$

where Pr is the Prandtl number and Re_x is the local Reynolds number of the flow arrangement. Clearly, $(\Delta T_r)_{\bar{t}}$ is a single-valued function of the parametric group

$$Br_x = \frac{k_2}{k_1} \frac{t}{x} \mathrm{Pr}^{\mathrm{a}}\mathrm{Re}_x^b \tag{2.45}$$

known as local Brun number. This local Brun number criterion [72] determines the degree of accuracy surrendered on solving a conjugate problem as a nonconjugate one. In view of this engineering approximation either of Eqs. (2.37a), (2.37b), (2.37c), or (2.37d) can be quantitatively treated to comply with the law of motive force expressed in its fundamental form as

$$\theta_1 + \theta_2 = \text{constant}. \tag{2.46}$$

However, qualitatively ordinary optical rays obey Riemannian geometry, while thermal rays are described by Finslerian geometry [73]. In Eqs. (2.37b) and (2.37d), it is revealed that between tangent law of heat conduction and Fermat's principle in optics there exists a difference only in the degree of accuracy. Philosophically, they are but one and the same: the unique optimization strategy of nature—the law of motive force. Comparing Eqs. (2.37b) and (2.37c) it can be perceived the tangent law of conductive heat transfer pertaining to a combination of media (k_1, k_2) is equivalent to Fermat's principle of optics to an altered combination of media (k_1', k_2'). Unlike point-to-point flow, the demarcation between Fermat type flow and the constructal law is well established in the relevant literature [63, 74]. The Fermat type principle can be readily recognized as a demonstration of the law of motive force, whereas in constructal law the competition of forward and backward motivation is manifested through slower (diffusion-like) and faster (convection-like) processes. Further, it is also to be observed that the law of motive force, while observed in nature or artificial systems, exhibits a category of equipartition [63, 75–79] principle in some macroscopic domains with finite time and length scale.

2.9 Discussions

A number of mathematical studies on nonstandard methods in the calculus of variations [80] are available. However, the present study is under the proposition of a natural law: the law of motive force. It has been suggested in some authoritative treatises that in many problems where we only want a few values of the nonlinear partial differential equation, we can solve the associated variational problems instead [81]. Application of the law of motive force is a justification of the physical basis in this direction.

Specifically, the law of motive force has been exploited for a class of purely conductive systems, where a limited amount of insulating material is to be distributed over a plane wall or curved surface with arbitrary temperature distributions for minimum heat transfer. The method is also extended to a more generalized situation of a stream suspended in an environment of different temperatures and where the wall temperature distribution is affected by the amount of insulation added. The results obtained are in conformity with those reported in the literature [27, 64]. The equivalence of the result obtained in applying the variational principle for a prescribed temperature history to that obtained for a prescribed heat flux is well established in the relevant literature [82].

From the physics of such class of extremum problems, it has been argued that a truly minimum exists. However, the quantification of minimum heat transfer has not been reported here. Once the optimum profile of insulation is obtained, the minimum heat transfer quantity follows readily from the routine procedure and is available in the literature [27]. Since any distribution pattern of insulting material can be represented by a Fourier series, it has been insinuated that such class of conductive minimum heat transfer problems pertain to a category of the Sturm–Liouville system [83–85].

Finally, from a summation form of the law of motive force formulation, a ratio form is derived when one of the constituent competing mechanisms turns out to be a constant. Thus, the ratio form of law of motive force is more restrictive than its corresponding summation counterpart. It turns out to be a mathematical fact that when the ratio form is valid the summation form is spontaneously granted, but not vice versa. In view of this argument Schmidt's criterion for the fin design, the tangent law of conductive heat transport and the Fermat's law of geometrical optics obeys the law of motive force. The constructal law is realized as a competition between slower (diffusion-like) and faster (convection-like) processes and thus complies with the law of motive force. Hence, the basis for analogies among some physical theories is sought. The fundamental feature of this optimization is but a category of macroscopic organization with a class of equipartition principle [63, 75–79].

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