## Contents

1 **Introduction** ................................................................. 1  
   1.1 Motivating Examples ..................................................... 1  
   1.2 Book Structure .......................................................... 4  
   1.3 Useful Notation .......................................................... 9  

2 **Order Relations and Ordering Cones** .................................... 11 
   2.1 Order Relations .......................................................... 11  
   2.2 Cone Properties Related to the Topology and the Order ............ 17  
   2.3 Convexity Notions for Sets and Set-Valued Maps .................... 22  
   2.4 Solution Concepts in Vector Optimization .......................... 28  
   2.5 Vector Optimization Problems with Variable Ordering Structure ................................................. 43 
   2.6 Solution Concepts in Set-Valued Optimization ........................ 45  
      2.6.1 Solution Concepts Based on Vector Approach .......... 45  
      2.6.2 Solution Concepts Based on Set Approach ............ 48  
      2.6.3 Solution Concepts Based on Lattice Structure ........... 55  
      2.6.4 The Embedding Approach by Kuroiwa .......................... 65  
      2.6.5 Solution Concepts with Respect to Abstract Preference Relations ...................................... 67  
      2.6.6 Set-Valued Optimization Problems with Variable Ordering Structure ................................................. 70  
      2.6.7 Approximate Solutions of Set-Valued Optimization Problems .............................................. 73  
   2.7 Relationships Between Solution Concepts ......................... 74  

3 **Continuity and Differentiability** ..................................... 77  
   3.1 Continuity Notions for Set-Valued Maps ................................ 77 
   3.2 Continuity Properties of Set-Valued Maps Under Convexity Assumptions .............................................. 90  
   3.3 Lipschitz Properties for Single-Valued and Set-Valued Maps ....... 96  
   3.4 Clarke’s Normal Cone and Subdifferential .......................... 102
3.5 Limiting Cones and Generalized Differentiability ............. 103
3.6 Approximate Cones and Generalized Differentiability .......... 107

4 Tangent Cones and Tangent Sets ..................................... 109
4.1 First-Order Tangent Cones ........................................... 110
  4.1.1 The Radial Tangent Cone and the Feasible Tangent Cone .... 110
  4.1.2 The Contingent Cone and the Interiorly Contingent Cone ... 112
  4.1.3 The Adjacent Cone and the Interiorly Adjacent Cone ......... 120
4.2 Modified First-Order Tangent Cones ................................ 123
  4.2.1 The Modified Radial and the Modified Feasible Tangent Cones 124
  4.2.2 The Modified Contingent and the Modified Interiorly Contingent Cone 124
  4.2.3 The Modified Adjacent and the Modified Interiorly Adjacent Cones 126
4.3 Miscellaneous Properties of First-Order Tangent Cones ....... 129
4.4 First-Order Tangent Cones on Convex Sets ....................... 132
  4.4.1 Connections Among First-Order Tangent Cones on Convex Sets 132
  4.4.2 Properties of First-Order Tangent Cones on Convex Sets .... 137
4.5 First-Order Local Cone Approximation ............................ 143
4.6 Convex Subcones of the Contingent Cone ......................... 147
4.7 First-Order Inversion Theorems and Intersection Formulas ...... 156
4.8 Expressions of the Contingent Cone on Some Constraint Sets .... 161
4.9 Second-Order Tangent Sets .......................................... 169
  4.9.1 Second-Order Radial Tangent Set and Second-Order Feasible Tangent Set 170
  4.9.2 Second-Order Contingent Set and Second-Order Interiorly Contingent Set 170
  4.9.3 Second-Order Adjacent Set and Second-Order Interiorly Adjacent Set 173
4.10 Generalized Second-Order Tangent Sets ......................... 175
4.11 Second-Order Asymptotic Tangent Cones ......................... 181
  4.11.1 Second-Order Asymptotic Feasible Tangent Cone and Second-Order Asymptotic Radial Tangent Cone 182
  4.11.2 Second-Order Asymptotic Contingent Cone and Second-Order Asymptotic Interiorly Contingent Cone 183
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.11.3</td>
<td>Second-Order Asymptotic Adjacent Cone and Second-Order Asymptotic Interiorly Adjacent Cone</td>
<td>185</td>
</tr>
<tr>
<td>4.12</td>
<td>Miscellaneous Properties of Second-Order Tangent Sets and Second-Order Asymptotic Tangent Cones</td>
<td>187</td>
</tr>
<tr>
<td>4.13</td>
<td>Second-Order Inversion Theorems</td>
<td>192</td>
</tr>
<tr>
<td>4.14</td>
<td>Expressions of the Second-Order Contingent Set on Specific Constraints</td>
<td>197</td>
</tr>
<tr>
<td>4.15</td>
<td>Miscellaneous Second-Order Tangent Cones</td>
<td>202</td>
</tr>
<tr>
<td>4.15.1</td>
<td>Second-Order Tangent Cones of Ledzewicz and Schaettler</td>
<td>202</td>
</tr>
<tr>
<td>4.15.2</td>
<td>Projective Tangent Cones of Second-Order</td>
<td>204</td>
</tr>
<tr>
<td>4.15.3</td>
<td>Second-Order Tangent Cone of N. Pavel</td>
<td>206</td>
</tr>
<tr>
<td>4.15.4</td>
<td>Connections Among the Second-Order Tangent Cones</td>
<td>207</td>
</tr>
<tr>
<td>4.16</td>
<td>Second-Order Local Approximation</td>
<td>207</td>
</tr>
<tr>
<td>4.17</td>
<td>Higher-Order Tangent Cones and Tangent Sets</td>
<td>210</td>
</tr>
<tr>
<td>5</td>
<td><strong>Nonconvex Separation Theorems</strong></td>
<td>213</td>
</tr>
<tr>
<td>5.1</td>
<td>Separating Functions and Examples</td>
<td>213</td>
</tr>
<tr>
<td>5.2</td>
<td>Nonlinear Separation</td>
<td>217</td>
</tr>
<tr>
<td>5.2.1</td>
<td>Construction of Scalarizing Functionals</td>
<td>217</td>
</tr>
<tr>
<td>5.2.2</td>
<td>Properties of Scalarization Functions</td>
<td>219</td>
</tr>
<tr>
<td>5.2.3</td>
<td>Continuity Properties</td>
<td>224</td>
</tr>
<tr>
<td>5.2.4</td>
<td>Lipschitz Properties</td>
<td>225</td>
</tr>
<tr>
<td>5.2.5</td>
<td>The Formula for the Conjugate and Subdifferential of $\varphi_A$ for $A$ Convex</td>
<td>231</td>
</tr>
<tr>
<td>5.3</td>
<td>Scalarizing Functionals by Hiriart-Urruty and Zaffaroni</td>
<td>232</td>
</tr>
<tr>
<td>5.4</td>
<td>Characterization of Solutions of Set-Valued Optimization Problems by Means of Nonlinear Scalarizing Functionals</td>
<td>236</td>
</tr>
<tr>
<td>5.4.1</td>
<td>An Extension of the Functional $\varphi_A$</td>
<td>236</td>
</tr>
<tr>
<td>5.4.2</td>
<td>Characterization of Solutions of Set-Valued Optimization Problems with Lower Set Less Order Relation $\preceq_C$ by Scalarization</td>
<td>240</td>
</tr>
<tr>
<td>5.5</td>
<td>The Extremal Principle</td>
<td>244</td>
</tr>
<tr>
<td>6</td>
<td><strong>Hahn-Banach Type Theorems</strong></td>
<td>249</td>
</tr>
<tr>
<td>6.1</td>
<td>The Hahn–Banach–Kantorovich Theorem</td>
<td>250</td>
</tr>
<tr>
<td>6.2</td>
<td>Classical Separation Theorems for Convex Sets</td>
<td>258</td>
</tr>
<tr>
<td>6.3</td>
<td>The Core Convex Topology</td>
<td>261</td>
</tr>
<tr>
<td>6.4</td>
<td>Yang’s Generalization of the Hahn–Banach Theorem</td>
<td>264</td>
</tr>
<tr>
<td>6.5</td>
<td>A Sufficient Condition for the Convexity of $\mathbb{R}_+ A$</td>
<td>271</td>
</tr>
</tbody>
</table>
7 **Conjugates and Subdifferentials** .......................................................... 275
  7.1 The Strong Conjugate and Subdifferential ........................................ 275
  7.2 The Weak Subdifferential ................................................................ 288
  7.3 Subdifferentials Corresponding to Henig Proper Efficiency ............ 296
  7.4 Exact Formulas for the Subdifferential of the Sum and the Composition .................................................... 298

8 **Duality** .............................................................................................. 307
  8.1 Duality Assertions for Set-Valued Problems Based on Vector Approach ....................................................... 308
    8.1.1 Conjugate Duality for Set-Valued Problems Based on Vector Approach ....................................................... 308
    8.1.2 Lagrange Duality for Set-Valued Optimization Problems Based on Vector Approach ............................ 313
  8.2 Duality Assertions for Set-Valued Problems Based on Set Approach ............................................................... 317
  8.3 Duality Assertions for Set-Valued Problems Based on Lattice Structure .......................................................... 322
    8.3.1 Conjugate Duality for **F**-Valued Problems ................................ 323
    8.3.2 Lagrange Duality for **I**-Valued Problems ................................ 326
  8.4 Comparison of Different Approaches to Duality in Set-Valued Optimization ..................................................... 338
    8.4.1 Lagrange Duality ..................................................................... 339
    8.4.2 Subdifferentials and Stability .................................................. 341
    8.4.3 Duality Statements with Operators as Dual Variables .......... 345

9 **Existence Results for Minimal Points** ............................................. 349
  9.1 Preliminary Notions and Results Concerning Transitive Relations ...................................................................... 349
  9.2 Existence of Minimal Elements with Respect to Transitive Relations ......................................................... 352
  9.3 Existence of Minimal Points with Respect to Cones ....................................................................................... 355
  9.4 Types of Convex Cones and Compactness with Respect to Cones ................................................................... 360
  9.5 Existence of Optimal Solutions for Vector and Set Optimization Problems ...................................................... 362

10 **Ekeland Variational Principle** .......................................................... 369
  10.1 Preliminary Notions and Results ..................................................... 369
  10.2 Minimal Points in Product Spaces .................................................. 373
  10.3 Minimal Points in Product Spaces of Isac–Tammer’s Type ............. 381
  10.4 Ekeland’s Variational Principles of Ha’s Type ................................ 384
  10.5 Ekeland’s Variational Principle for Bi-Set-Valued Maps ............... 390
  10.6 EVP Type Results ........................................................................ 391
  10.7 Error Bounds ................................................................................ 394
11 Derivatives and Epiderivatives of Set-Valued Maps ............................ 399
  11.1 Contingent Derivatives of Set-Valued Maps .............................. 400
    11.1.1 Miscellaneous Graphical Derivatives of Set-valued Maps ........ 407
    11.1.2 Convexity Characterization Using Contingent Derivatives .......... 414
    11.1.3 Proto-Differentiability, Semi-Differentiability, and Related Concepts ........................................ 416
    11.1.4 Weak Contingent Derivatives of Set-Valued Maps ......... 422
    11.1.5 A Lyusternik-Type Theorem Using Contingent Derivatives .......... 426
  11.2 Calculus Rules for Derivatives of Set-Valued Maps ..................... 428
    11.2.1 Calculus Rules by a Direct Approach .......................... 429
    11.2.2 Derivative Rules by Using Calculus of Tangent Cones .................. 432
  11.3 Contingently \( C_a \)-Absorbing Maps .................................. 437
  11.4 Epiderivatives of Set-Valued Maps ......................................... 445
    11.4.1 Contingent Epiderivatives of Set-Valued Maps with Images in \( \mathbb{R} \) ................................... 446
    11.4.2 Contingent Epiderivatives in General Spaces .............. 452
    11.4.3 Existence Theorems for Contingent Epiderivatives .... 457
    11.4.4 Variational Characterization of the Contingent Epiderivatives .......... 464
  11.5 Generalized Contingent Epiderivatives of Set-Valued Maps ......... 470
    11.5.1 Existence Theorems for Generalized Contingent Epiderivatives ................................. 474
    11.5.2 Characterizations of Generalized Contingent Epiderivatives .......... 478
  11.6 Calculus Rules for Contingent Epiderivatives .......................... 482
  11.7 Second-Order Derivatives of Set-Valued Maps ............................. 488
  11.8 Calculus Rules for Second-Order Contingent Derivatives ............. 500
  11.9 Second-Order Epiderivatives of Set-Valued Maps ....................... 504

12 Optimality Conditions in Set-Valued Optimization ............................ 509
  12.1 First-Order Optimality Conditions by the Direct Approach ....... 512
  12.2 First-Order Optimality Conditions by the Dubovitskii-Milyutin Approach .............................. 522
    12.2.1 Necessary Optimality Conditions by the Dubovitskii-Milyutin Approach .......... 523
    12.2.2 Inverse Images and Subgradients of Set-Valued Maps ............. 527
    12.2.3 Separation Theorems and the Dubovitskii-Milyutin Lemma .......... 534
12.2.4 Lagrange Multiplier Rules
   by the Dubovitskii-Milyutin Approach ................. 537

12.3 Sufficient Optimality Conditions in Set-Valued Optimization .... 542
   12.3.1 Sufficient Optimality Conditions Under
   Convexity and Quasi-Convexity .......................... 542
   12.3.2 Sufficient Optimality Conditions Under
   Paraconvexity ............................................. 545
   12.3.3 Sufficient Optimality Conditions Under
   Semidifferentiability ....................................... 549

12.4 Second-Order Optimality Conditions in Set-Valued
   Optimization .................................................. 549
   12.4.1 Second-Order Optimality Conditions
   by the Dubovitskii-Milyutin Approach .................. 550
   12.4.2 Second-Order Optimality Conditions
   by the Direct Approach ................................... 554

12.5 Generalized Dubovitskii-Milyutin Approach
   in Set-Valued Optimization ................................... 557
   12.5.1 A Separation Theorem for Multiple Closed
   and Open Cones ........................................... 559
   12.5.2 First-Order Generalized
   Dubovitskii-Milyutin Approach .......................... 562
   12.5.3 Second-Order Generalized
   Dubovitskii-Milyutin Approach .......................... 567

12.6 Set-Valued Optimization Problems with a Variable
   Order Structure ................................................ 568

12.7 Optimality Conditions for Q-Minimizers
   in Set-Valued Optimization .................................. 572
   12.7.1 Optimality Conditions for Q-Minimizers
   Using Radial Derivatives .................................... 572
   12.7.2 Optimality Conditions for Q-Minimizers
   Using Coderivatives ......................................... 574

12.8 Lagrange Multiplier Rules Based on Limiting Subdifferential ... 578

12.9 Necessary Conditions for Approximate Solutions
   of Set-Valued Optimization Problems .................... 591

12.10 Necessary and Sufficient Conditions for Solution
   Concepts Based on Set Approach .......................... 594

12.11 Necessary Conditions for Solution Concepts
   with Respect to a General Preference Relation ............ 598

12.12 KKT-Points and Corresponding Stability Results ............... 600

13 Sensitivity Analysis in Set-Valued Optimization
   and Vector Variational Inequalities ....................... 605
   13.1 First Order Sensitivity Analysis in Set-Valued Optimization .... 606
   13.2 Second Order Sensitivity Analysis in Set-Valued
   Optimization .................................................. 613
13.3 Sensitivity Analysis in Set-Valued Optimization
Using Coderivatives ............................................... 623
13.4 Sensitivity Analysis for Vector Variational Inequalities .... 634

14 Numerical Methods for Solving Set-Valued Optimization Problems .................................................... 645
14.1 A Newton Method for Set-Valued Maps .................................................................................. 645
14.2 An Algorithm to Solve Polyhedral Convex Set-Valued Optimization Problems ................................. 651
14.2.1 Formulation of the Polyhedral Convex Set-Valued Optimization Problem .............................. 653
14.2.2 An Algorithm for Solving Polyhedral Convex Set-Valued Optimization Problems .................. 655
14.2.3 Properties of the Algorithm ............................................................................................... 658

15 Applications ................................................................................................................................. 663
15.1 Set-Valued Approaches to Duality in Vector Optimization ......................................................... 663
15.1.1 Fenchel Duality for Vector Optimization Problems Using Corresponding Results for \( F \)-Valued Problems ............................................................................................... 667
15.1.2 Lagrange Duality for Vector Optimization Problems Based on Results for \( F \)-Valued Problems ...... 670
15.1.3 Duality Assertions for Linear Vector Optimization Based on Lattice Approach ...................... 677
15.1.4 Further Set-Valued Approaches to Duality in Linear Vector Optimization .................................. 682
15.2 Applications in Mathematical Finance ....................................................................................... 696
15.3 Set-Valued Optimization in Welfare Economics ............................................................................ 701
15.4 Robustness for Vector-Valued Optimization Problems .................................................................... 706
15.4.1 \( \preceq^u \)-Robustness ........................................................................................................... 710
15.4.2 \( \preceq^l \)-Robustness ............................................................................................................ 720
15.4.3 \( \preceq^s \)-Robustness ............................................................................................................ 722
15.4.4 Algorithms for Solving Special Classes of Set-Valued Optimization Problems ....................... 724

Appendix ............................................................................................................................................. 727

References .......................................................................................................................................... 733

Index ............................................................................................................................................... 759
Set-valued Optimization
An Introduction with Applications
Khan, A.A.; Tammer, C.; Zălinescu, C.
2015, XXII, 765 p. 29 illus., Hardcover
ISBN: 978-3-642-54264-0