Chapter 2
Higher Order Drifts and the Parallel Equation of Motion

2.1 General Expression of the Drift Velocity and Higher Order Drifts

We are now in the position of defining a charged particle's guiding center system in a mathematically more rigorous, less qualitative, way. We shall use the three examples discussed in the preceding chapter (force drift, electric field drift, gradient-B drift) and realize that in each one, our implicit "recipe" was to look for a reference frame moving with velocity V in which, in a plane perpendicular to B, the motion-induced electric field force $qV \times B$ cancels the phase-average of the resultant of all other forces acting on the particle as it makes one cyclotron turn. We can now generalize this rule in a more formal way.

First, we should make it clear that all that follows assumes the validity of the adiabatic conditions, which we write here in more general form in terms of the order of magnitude of characteristic space and time variations of any field quantity Q:

$$\rho_C \ll \frac{Q}{\nabla Q} \tag{2.1}$$

$$\tau_C \ll \frac{Q}{dQ/dt} \tag{2.2}$$

It is important to clearly understand the meaning and consequences of these conditions. Condition (2.1) tells us that the quantity Q should vary only very little (in relative magnitude and direction, if it is a vector) along the cyclotron trajectory of the particle (whose dimension depends on the particle's perpendicular velocity for a given magnetic field intensity (1.19)). Concerning the second condition, the time derivative is a total derivative, i.e., it represents change as seen by the particle. Therefore, (2.2) tells us that the quantity Q should change only very little (in relative magnitude or direction) during a cyclotron period (which in the non-relativistic domain does *not* depend on the particle's velocity).

Fig. 2.1 Magnetic field configuration on a cyclotron orbit in the "natural" frame of reference
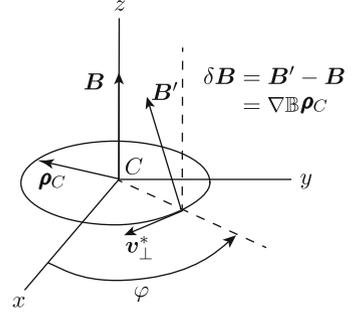
Second, we identify the type of forces that in most general terms may act on a charged particle at point r and time t in its cyclotron motion in the yet-to-be defined GCS. They are: (1) the external electric field force $q E$ (electric field in the OFR); (2) a non-electromagnetic external force F (unimportant in magnetospheric physics); (3) the motion-induced electric field force $q V \times B$ (V is the yet-to-be determined velocity vector of the GCS); (4) the Lorentz force $q v^* \times B$ (v^* is the particle's velocity vector in the GCS); and as a new element, (5) the inertial force $-m d V / d t$ which appears whenever the GCS is an accelerated frame of reference.

Our formal definition can now be formulated in the following way: The guiding center system is a moving (usually non-inertial) frame of reference in which at any given time *the cyclotron phase average of all forces acting on the particle is zero*. Mathematically:

$$\left\langle q E + F + q V \times B + q v^* \times B - m \frac{d V}{d t} \right\rangle \equiv 0 \tag{2.3}$$

Each term of this equation is a cyclotron phase average of the type

$$\langle P \rangle_\varphi = 1/2\pi \int_0^{2\pi} P(\varphi) d\varphi \tag{2.4}$$

Refer to Fig. 2.1, in which the origin is the instantaneous guiding center (position r_C in the OFR, (1.25)) and the axes represent the natural frame defined in Appendix A.1 (z-axis along $e = B / B$; x-axis along binormal b; y-axis along normal n). Since we are assuming the adiabatic condition (2.1) to apply, we can relate any vector $P(r)$ on the particle's cyclotron orbit to its value at the guiding center via a first order Taylor expansion:

$$P(r) = P(r_C) + \delta P = P(r_C) + \nabla \otimes P|^T \rho_C \tag{2.5}$$

where $\nabla \otimes P|^T$ is the transposed tensor gradient of P (see (A.34) and footnote on page 170 of Appendix A.1) and $\rho_C = \rho_C(r_C, \varphi)$ is given by (1.25). The last term has components $\nabla \otimes P|^T \rho_C|_i = \Sigma_k \partial P_i / \partial x_k \rho_{Ck}$.

The first and second term averages in (2.3) are just the electric field and the external force at the position of the guiding center $\boldsymbol{r}_C$, respectively ($\langle \delta \boldsymbol{E} \rangle = \langle \delta \boldsymbol{F} \rangle = 0$ because only $\sin \varphi$ and $\cos \varphi$ would appear in the integral (2.4)). The same applies to the third term (V is a quantity characterizing the whole GCS, not a vector field). In the fourth term, when we replace $\boldsymbol{B}(\boldsymbol{r})$ by an expansion like (2.5), we must take into account that for the vector velocity $\langle \boldsymbol{v}^* \rangle = 0$ in the GCS (1.7), so only the "little Lorentz force" $q\langle \boldsymbol{v}_\perp^* \times \delta \boldsymbol{B} \rangle$ would survive. The fifth term is an inertial force, which also should be considered a characteristic of the whole GCS.

The fourth term, $q\langle \boldsymbol{v}^* \times (\nabla \otimes \boldsymbol{B} \, \boldsymbol{\rho}_C) \rangle$ requires careful and detailed consideration. Consider a particle of mass m and electric charge q gyrating with a 90° pitch angle in a non-uniform magnetic field, as shown (for a positive charge) in Fig. 2.1. First we must find the expression of the tensor product. To that effect we note that by components,

$$\boldsymbol{v}^* = (\frac{q}{|q|} v_\perp^* \sin \varphi \,, \ -\frac{q}{|q|} v_\perp^* \cos \varphi \,, \ 0)$$

$$\boldsymbol{\rho}_C = (\rho_C \, \cos \varphi \,, \ \rho_C \, \sin \varphi \,, \ 0)$$

We inserted the ratio $q/|q|$ in order to put in evidence the effect of the sign of the electric charge.

The components of $\delta \boldsymbol{B} = \nabla \otimes \boldsymbol{B} |^T \boldsymbol{\rho}_C$ are (A.16):

$$\delta B_x = \frac{\partial B_x}{\partial x} \rho_C \cos \varphi + \frac{\partial B_x}{\partial y} \rho_C \sin \varphi$$

$$\delta B_y = \frac{\partial B_y}{\partial x} \rho_C \cos \varphi - (\frac{\partial B}{\partial s} + \frac{\partial B_x}{\partial x}) \rho_C \sin \varphi$$

$$\delta B_z = \nabla_x B \rho_C \cos \varphi + \nabla_y B \rho_C \sin \varphi$$

Remember that in the natural coordinate system $B_z = |B|$, $\partial B_z / \partial z = \partial B / \partial s$, and that $\nabla B = (\partial B / \partial x, \partial B / \partial y, \partial B / \partial s)$ and $\nabla \cdot \boldsymbol{B} = 0$ (Appendix A.1).

For $\langle \boldsymbol{v}^* \times \delta \boldsymbol{B} \rangle$ we need the component products $q v_i^* \delta B_k$ and average over one cyclotron turn. All terms containing $\cos \varphi \sin \varphi$ will average out to zero; in the remaining terms we will have integrals of the type $1/2\pi \int \sin^2 \varphi d\varphi = 1/2\pi \int \cos^2 \varphi d\varphi = 1/2$. The end results are:

$$q\langle \boldsymbol{v}^* \times \delta \boldsymbol{B} \rangle_x = \frac{1}{2} q \big(-\frac{q}{|q|} \big) v_\perp^* \rho_C \nabla_x B = -\frac{1}{2} |q| v_\perp^* \rho_C \nabla_x B$$

$$q\langle \boldsymbol{v}^* \times \delta \boldsymbol{B} \rangle_y = -\frac{1}{2} |q| v_\perp^* \rho_C \nabla_y B$$

$$q\langle \boldsymbol{v}^* \times \delta \boldsymbol{B} \rangle_z = -\frac{1}{2} |q| v_\perp^* \rho_C \frac{\partial B}{\partial s}$$

In vector form, we can write:

$$q\langle \boldsymbol{v}^* \times \boldsymbol{\delta B}\rangle = -\frac{mv_\perp^{*2}}{2B}\boldsymbol{\nabla}B = -M\boldsymbol{\nabla}B \tag{2.6}$$

The projection of this result perpendicular to $\boldsymbol{B}$ represents the *gradient-B force*,

$$\boldsymbol{f}_\perp = \boldsymbol{f}_g = -M\boldsymbol{\nabla}_\perp B \tag{2.7}$$

which can be interpreted as the cause of the gradient-B drift ((1.45)—see also below). The parallel component of the average Lorentz force is what is called the *mirror force*

$$f_\| = -M\nabla_\| B = -M\frac{\partial B}{\partial s} \tag{2.8}$$

This force accelerates and decelerates particles spiralling along a field line into and away from decreasing or increasing B values, respectively. As we shall see later, it governs the bounce motion along a field line and is responsible for particle trapping in the geomagnetic field. Relations (2.7) and (2.8) provide further legitimacy to the model of a virtual guiding center particle and the concept of its magnetic moment: *any* magnetic moment $\boldsymbol{M}$ placed in a non-uniform magnetic field is subjected to a net force $\boldsymbol{f} = \boldsymbol{\nabla}(\boldsymbol{M} \cdot \boldsymbol{B})$ which in our case is $= -M\boldsymbol{\nabla}B$ because of (1.26) and the fact that M is an adiabatic invariant. One cautionary note is in order: we emphasized repeatedly that the velocity $v_\perp^*$ in the definition (1.26) of the magnetic moment is *not* that of the original particle in the OFR, but its transverse velocity in the GCS. When, in the GCS, this velocity is a function of the cyclotron phase the phase-average must be taken (see (1.41) and pertinent discussion). From now on, we shall drop the star supraindex to simplify the aspect of the equations, but we will remind the reader when necessary to distinguish between OFR and GCS variables.[1]

   In summary, the condition (2.3) defining the guiding center system has now become

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[1]It is advisable to revisit the above derivation process starting with Fig. 2.1 and relation (2.3). That process really developed in stages: what the figure intended to show implicitly was a "pre-GCS" in which the particle was circling free of *external* forces—i.e., a system which was moving with a transverse drift velocity, sum of $\boldsymbol{U}$ (1.34) and $\boldsymbol{V}_F$ (1.31) and in which the corresponding motion-induced field force balanced the external field forces. In such a system the particle gyrates with constant speed $v_\perp^*$ (example of Sect. 1.3). However, due to any inhomogeneity of the magnetic field, there was another resultant force, the term (2.6), which leads to an additional drift and the "final GCS". Now the particle's transverse velocity $v_\perp$ is no longer independent of φ (see expression (1.41)), but its cyclotron average is equal to the (constant) velocity in the "pre-GCS" (see expression (1.42)). It is precisely *this* average transverse velocity that enters in the definition of the magnetic moment (1.46). Confused again? Unfortunately, this detail is conceptually important, especially for the fundaments of plasma physics.

$$qE + F + qV \times B - M\nabla B - m\frac{dV}{dt} \equiv 0 \tag{2.9}$$

The quantities involved are evaluated at the GC point, not at the original particle's actual position in its gyromotion. This equation will lead us to a dynamic equation of motion for the guiding center, to be discussed in the next section. At this time we focus on the fact that there is only one drift velocity that satisfies it (multiply the equation vectorially by B/qB^2 and extract $V_\perp$):

$$V_D = V_\perp = \left(qE + F - M\nabla B - m\frac{dV}{dt}\right) \times \frac{B}{qB^2} \tag{2.10}$$

This equation displays the four fundamental drifts of a guiding center particle: (1) the E-cross-B drift (1.34); (2) the force drift (1.31); (3) a gradient-B drift like (1.46); and (4) an inertial force drift.

For the parallel velocity of a guiding center particle, all we can do is repeat relation (1.4):

$$V_\| = \langle v_\| \rangle \cong v_\| \tag{2.11}$$

Our last task is to further analyze the inertial term in (2.10). An apparent problem arises at once: this term contains the unknown vector V which we are trying to determine! But there are "unknown unknowns" and "known unknowns". This particularly applies to adiabatic theory because of its goal of providing useful approximations rather than impractical exactitude. If we divide the velocity into the two vectors $V = V_\| + V_\perp$ we have:

$$\frac{dV}{dt} = \frac{d(V_\| + V_D)}{dt} = \frac{d}{dt}(v_\| e + V_D) = \frac{dv_\|}{dt}e + v_\|\frac{de}{dt} + \frac{dV_D}{dt} \tag{2.12}$$

The second term calls our attention to the fact that the natural coordinate system (Appendix A.1) is a *local* frame of reference, that in an inhomogeneous magnetic field varies from point to point. In the above equation (and many future ones) the operator d/dt represents the total variation per unit time as seen by the particle, always consisting of two parts: (i) a variation in time at a fixed point in space, and (ii) a variation due to the displacement (which can be both perpendicular as well as parallel to B) while the field is frozen in time: $d/dt = \partial/\partial t + V \cdot \nabla$ (the latter operator being $\sum V_k \partial/\partial x_k$). We obtain the following:

$$\begin{aligned}
\frac{dV}{dt} &= \frac{dv_\|}{dt}e + v_\|\left[\frac{\partial e}{\partial t} + (V \cdot \nabla)e\right] + \frac{dV_D}{dt} \\
&= \frac{dv_\|}{dt}e + v_\|\left[\frac{\partial e}{\partial t} + (v_\| \cdot \nabla)e + (V_D \cdot \nabla)e\right] + \frac{dV_D}{dt} \\
&= \frac{dv_\|}{dt}e + v_\|^2\frac{\partial e}{\partial s} + v_\|(V_D \cdot \nabla)e + v_\|\frac{\partial e}{\partial t} + \frac{dV_D}{dt} \tag{2.13}
\end{aligned}$$

Carefully note that this is a purely kinematic/field-geometric equation—it has nothing to do with the dynamics of the particular particle involved, and might as well apply to little balls sliding without friction along bent and moving wires (the field lines)! Since $v_\parallel = ds/dt$, the first term is the "real" guiding center particle acceleration along the field line, measuring how its actual speed along the line varies; it is the one we are really interested in. The other terms are inertial accelerations due to the fact that a particle traveling along a bent and eventually moving field line experiences additional accelerations; they are important through the action of the inertial forces they represent. For instance, we recognize in the second term of the last equality a *centripetal acceleration* governed by the radius of curvature R_C of the field line in question (because according to (A.15) of Appendix A.1 $\partial e/\partial s = -n/R_C$); this indeed represents the fact that the guiding center follows the curved field line in its parallel motion. The third term is an inertial acceleration that appears if the guiding center has a *drift component along the normal n* of the field line; if δl is an element of the GC trajectory, this term can be written as $v_\parallel V_D \partial e/\partial l$. The fourth term is an acceleration due to a *time-change of the direction* of the magnetic field (e.g., in rotating field lines). Finally, concerning the third and fifth term, in most of what follows we will replace V by the electric drift U (1.34) in order to avoid (to first order) the problem of the "unknown unknown" mentioned above. This replacement is more than just a convenience: it is fully justified as a legitimate approximation.

We now insert (2.13) (times the particle mass m) into the corresponding term on the right side of Eq. (2.10) to obtain the most complete expression of the drift velocity. With U and V_F given by (1.34) and (1.31), respectively, we write it in two equivalent forms:

$$
\begin{aligned}
V_D &= \frac{e}{qB} \times \left[-qE - F + M\nabla_\perp B + mv_\parallel^2 \frac{\partial e}{\partial s} + mv_\parallel \frac{\partial e}{\partial t} \right. \\
&\qquad \left. + mv_\parallel (V_D \cdot \nabla)e + m\frac{dV_D}{dt} \right] \\
&= U + V_F + \frac{mv_\perp^2}{2qB^3} B \times \nabla_\perp B + \frac{mv_\parallel^2}{qB^2} B \times \frac{\partial e}{\partial s} \qquad (2.14) \\
&\quad + \frac{m}{qB^2} B \times \left[v_\parallel \frac{\partial e}{\partial t} + v_\parallel (U \cdot \nabla)e + \frac{dU}{dt} \right]
\end{aligned}
$$

The first equality (first two lines) expresses it all as a force drift made up of different contributions. The two lines in the second equality represent three different orders in the adiabatic approximation. Specifically, the first two lines show the zero-order E-cross-B and force drifts (1.34) and (1.31). The third line shows two first-order drifts:

$$
\text{Gradient-B drift:} \quad V_G = \frac{mv_\perp^2}{2qB^3} B \times \nabla_\perp B = -\frac{M \times \nabla_\perp B}{qB} \qquad (2.15)
$$

$$\text{Curvature drift:} \quad \boldsymbol{V}_C = \frac{mv^2_\parallel}{qB^2} \boldsymbol{B} \times \frac{\partial \boldsymbol{e}}{\partial s} \tag{2.16}$$

According to (A.22) in Appendix A.1, *in absence of any currents* $B\partial \boldsymbol{e}/\partial s = \nabla_\perp B$, thus under this special condition, we can combine the last two equations into one:

$$\text{Gradient-curvature drift:} \quad \boldsymbol{V}_{GC} = \frac{1}{2}\frac{m}{qB^3}(v^2_\perp + 2v^2_\parallel)\boldsymbol{B} \times \nabla_\perp B \tag{2.17}$$

The fourth line in the second equality of (2.14) contains second and higher order drifts; in the latter we have replaced $\boldsymbol{V}_D$ with its own first approximation, $\boldsymbol{U}$ (for $\boldsymbol{F} = 0$). The first term represents the effect of time-dependence of the direction of the magnetic field; the second represents the effect of a spatial variation of the $\boldsymbol{e}$ direction.

The last term is of importance in some electric field situations, in particular, for a stationary magnetic field but varying electric field. Taking into account (1.34), we obtain the so-called

$$\text{Polarization drift:} \quad \boldsymbol{V}_P = \frac{m}{qB^2}\frac{\partial \boldsymbol{E}_\perp}{\partial t} \tag{2.18}$$

Finally, a few nitpicking remarks about adiabatic drifts. What exactly are the particle velocities that appear in the general expression (2.14)? We strongly recommend that the reader review again the statements in the footnote on page 38 and all its implications. Another important point is the following. Carefully observe how the various terms in Eqs. (2.14)–(2.18) depend on the velocities $v_\perp$ and $v_\parallel$ of the original (real!) particle. Zero order drifts do not depend on them at all, only on the local $\boldsymbol{B}$, $\boldsymbol{E}$ and force fields. The gradient-B drift depends on M (see its second expression in (2.15)), which is an adiabatic constant of motion, i.e., depends only on the *initial* transverse velocity of the particle. The curvature drift *does* depend on its instantaneous $v_\parallel$. What enters in the remaining higher order drifts is the full drift velocity itself—but as stated above, for a second order approximation, the known drift $\boldsymbol{U}$ suffices.

2.2 Motion Along the Field Line and the Energy Equation

Before we discuss the dynamics of guiding center particles in their field-aligned motion, we should answer a fundamental question: *Why* does the guiding center particle actually follow a curved field line? Usually, this is a fact taken for granted, based on the appearance of a centripetal acceleration term $v_\parallel{}^2(\partial \boldsymbol{e}/\partial s)$ in (2.13) and confirmed by the beautiful pictures of glowing plasma-filled coronal loops. However, as stated before, this equation is a purely kinematic and field-geometric one, having to do with the way we represent our guiding center velocity vector in the

Fig. 2.2 Motion of the GC
along a curved field line:
geometric parameters
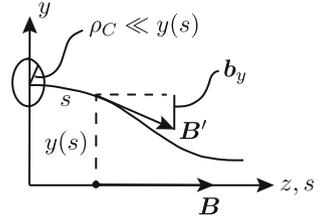
natural frame of reference (2.12). A more legitimate proof is the following. Consider
a magnetic field configuration as shown in Fig. 2.2, with a central rectilinear field
line along axis z and a magnetic intensity that increases toward positive z. A
probe particle is placed on a near-by field line with small perpendicular velocity
$\boldsymbol{v}_\perp$, therefore also with a small gyroradius $\rho_C \ll y(s)$. Assuming cylindrical
symmetry around the z axis, the differential equation of the particle's field line
(see Appendix A.1, footnote on page 164) can be easily derived from the flux tube
property $By^2 = $ const. along z:

$$\frac{dy}{ds} = -\frac{1}{2B}\frac{\partial B}{\partial s}y$$

Now we determine the trajectory of the GC particle under the drift imposed by the
"little Lorentz force" $q(\boldsymbol{v}_\parallel \times \boldsymbol{b}_y)$ directed along x (into the paper):

$$\boldsymbol{V}_D = q(\boldsymbol{v}_\parallel \times \boldsymbol{b}_y) \times \frac{\boldsymbol{B}}{qB^2} \quad \text{Therefore} \quad V_{Dy} = \frac{dy}{dt} = \frac{v_\parallel}{B}b_y$$

But $b_y = (\partial B_y/\partial y)y = -1/2(\partial B/\partial s)y$, because of $\nabla \cdot \boldsymbol{B} = 0$ and the symmetry
around the z axis. Thus, the equation of the GC trajectory will be:

$$\frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = \frac{dy}{ds}v_\parallel = -v_\parallel\frac{1}{2B}\frac{\partial B}{\partial s}y \tag{2.19}$$

Cancelling the particle variable $v_\parallel$ on both sides of the last equality leaves us with
the differential equation of the particle's magnetic field line! (Footnote, page 164).
In summary, under adiabatic conditions the "little Lorentz force" caused by any
small non-uniformity $\boldsymbol{b}$ of the magnetic field acting on the parallel motion of a
charged particle is always such as to impart an acceleration $\perp \boldsymbol{B}$ that makes the
guiding center follow the curved field line.

There is a limit, though, to the property of "following the field line", imposed
by adiabatic condition (2.2). If $\sigma_C = v_\parallel\tau_C$ is the displacement of the GC along a
field line during one cyclotron turn, this condition requires that the magnetic field,
including its direction $\boldsymbol{e}$, change very little along σ_C. This implies $\sigma_C(\partial e/\partial s) \ll 1$
or $\sigma_C \ll R_{fl}$, where R_{fl} is the field line's radius of curvature. For small pitch
angles ($v_\parallel \simeq v$) the particle velocity must then be $v \ll R_{fl}/\tau_C$. Note that rather

than a restriction on pitch angles, this is a restriction on velocity or energy: particles with small pitch angles and too high energy will "fly off" tangentially from their home field line (happens to trapped particles on sharply bent field lines in the neutral sheet)! This is equivalent to what happens to energetic particles with pitch angles $\approx 90°$: when condition (2.1) is violated, they will "fly off" perpendicularly to $\boldsymbol{B}$ (happens to trapped particles in the vicinity of the dayside boundary)!

We now go back to the guiding center system definition (2.9) and extract the time derivative of the GC velocity $\boldsymbol{V}$ in its parallel projection:

$$m\frac{d\boldsymbol{V}}{dt}\cdot\boldsymbol{e} = a_\parallel = qE_\parallel + F_\parallel - M\frac{\partial B}{\partial s} \tag{2.20}$$

Then we turn to (2.13) and consider its parallel projection

$$\frac{dV}{dt}\bigg|_\parallel = (d\boldsymbol{V}/dt)\cdot\boldsymbol{e} = a_\parallel = \frac{dv_\parallel}{dt} + \frac{d\boldsymbol{V}_D}{dt}\cdot\boldsymbol{e} = \frac{dv_\parallel}{dt} - \boldsymbol{V}_D\cdot\frac{d\boldsymbol{e}}{dt}$$

the latter equality because of $\boldsymbol{V}_D\cdot\boldsymbol{e} = 0$. But $d/dt = \partial/\partial t + (\boldsymbol{V}\cdot\boldsymbol{\nabla})$ which leaves the above equation as

$$\frac{dV}{dt}\bigg|_\parallel = a_\parallel = \frac{dv_\parallel}{dt} - v_\parallel\boldsymbol{V}_D\cdot\frac{\partial\boldsymbol{e}}{\partial s} - \boldsymbol{V}_D\cdot\frac{\partial\boldsymbol{e}}{\partial t} - \boldsymbol{V}_D\cdot(\boldsymbol{V}_D\cdot\boldsymbol{\nabla})\boldsymbol{e} \tag{2.21}$$

Note carefully that $a_\parallel$ is *not* d^2s/dt^2! Inserting (2.21) in (2.20), we finally obtain the *parallel equation of motion* of the guiding center:

$$m\frac{dv_\parallel}{dt} = m\frac{d^2s}{dt^2} = qE_\parallel + F_\parallel - M\frac{\partial B}{\partial s}$$

$$+ mv_\parallel\boldsymbol{V}_D\cdot\frac{\partial\boldsymbol{e}}{\partial s} + m\boldsymbol{V}_D\cdot\frac{\partial\boldsymbol{e}}{\partial t} + m\boldsymbol{V}_D\cdot(\boldsymbol{V}_D\cdot\boldsymbol{\nabla})\boldsymbol{e} \tag{2.22}$$

The first line includes zero- and first-order terms in which we recognize the external forces and the mirror force; the second line contains higher-order terms, corrections that are the result of field-geometric effects related to the drift velocity.

Let us give just two quick examples for the action of the second order 4th and 5th terms in (2.22) (quick, because they can be neglected in radiation belt physics), consider Figs. 2.3 and 2.4, respectively. The first one depicts a particle spiralling about the central line in a nearly uniform magnetic field in a long rotating solenoid. As viewed from the OFR (which is fixed to the rotating solenoidal field), the particle is subjected to an induced electric field $\boldsymbol{E}_i = -(\boldsymbol{\Omega}\times\boldsymbol{r})\times\boldsymbol{B}$ which in turn causes a drift $\boldsymbol{V}_D$, responsible for the particle corotating with the solenoid (see discussion on page 19). The term $m\boldsymbol{V}_D\cdot\partial\boldsymbol{e}/\partial t = m\Omega^2 r$ is the centrifugal force leading to a "sling shot effect" on the GC particle. The second example is a particle spiralling along a circular field line around a linear current I and subjected to a uniform electric field parallel to the current, out of the paper. Besides the gradient-B and curvature drifts

Fig. 2.3 Example of particle
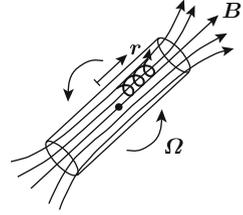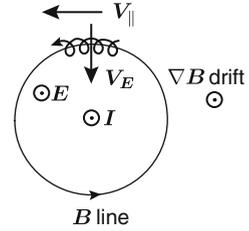drifting in a rotating solenoid
field

Fig. 2.4 Example of particle
drifting in the field of a linear
current I with an electric
field along it

perpendicular to the paper, there will be an E-cross-B drift toward the center line,
in the direction of $\partial e / \partial s$. So the 4th term will be $\neq 0$, representing an acceleration
as the GC particle drifts into higher B-values.

In absence of any drift, or if all three higher order terms can be neglected, and
if the external forces derive from a scalar potential *along* the field line W(s), the
parallel equation of guiding center motion is reduced to

$$m \frac{dv_\parallel}{dt} = -\frac{\partial}{\partial s}\big[MB(s) + W(s)\big] \tag{2.23}$$

The guiding center moves along the field line as if it was subjected to a total scalar
parallel potential $P_\parallel = MB(s) + W(s)$, and one can apply to that parallel motion all
the familiar energy graphs and related methods of mass point mechanics (see next
section).

Finally, there is one more important general equation to be derived. Consider the
instantaneous kinetic energy of a gyrating particle in the OFR:

$$T = \frac{1}{2}mv_\perp^2 + \frac{1}{2}mv_\parallel^2$$

Taking into account relations (1.8), (1.26) and (2.11), we can write for the cyclotron
average:

$$\langle T \rangle_C = MB + \frac{1}{2}mV_\parallel^2 + \frac{1}{2}mV_D^2 \tag{2.24}$$

In the guiding center particle model, the first term represents the GC particle's
"internal energy" (cyclotron energy), the second term represents the kinetic energy

of the GC particle in its motion along the field line, the third term the kinetic energy in its drift motion. The *energy equation* refers to the time rate of change of the GC particle's average kinetic energy, which we write:

$$\frac{d\langle T\rangle_C}{dt} = M\frac{dB}{dt} + mv_\parallel\frac{dv_\parallel}{dt} + m\boldsymbol{V}_D\cdot\frac{d\boldsymbol{V}_D}{dt} \tag{2.25}$$

Setting in the first term $d/dt = \partial/\partial t + v_\parallel\partial/\partial s + \boldsymbol{V}_D\cdot\boldsymbol{\nabla}$ and replacing $dv_\parallel/dt$ with its expression in (2.22), we have:

$$\frac{d\langle T\rangle_C}{dt} = M\frac{\partial B}{\partial t} + V_\parallel(qE_\parallel + F_\parallel) + \boldsymbol{V}_D\cdot\left(M\boldsymbol{\nabla}_\perp B + mv_\parallel^2\frac{\partial\boldsymbol{e}}{\partial s} + mv_\parallel\frac{d\boldsymbol{e}}{dt} + m\frac{d\boldsymbol{V}_D}{dt}\right)$$

The parenthesis appears as part of the expression of $\boldsymbol{V}_D$ in (2.14). Extracting $md\boldsymbol{V}_D/dt$ from (2.13) and then $d\boldsymbol{V}/dt$ from (2.9), the last term of the above equation turns out to be equal to $\boldsymbol{V}_D\cdot(q\boldsymbol{E}_\perp + \boldsymbol{F}_\perp)$, which leaves the energy equation as

$$\frac{d\langle T\rangle}{dt} = M\frac{\partial B}{\partial t} + V_\parallel(qE_\parallel + F_\parallel) + \boldsymbol{V}_D\cdot(q\boldsymbol{E}_\perp + \boldsymbol{F}_\perp)$$

$$= M\frac{\partial B}{\partial t} + \boldsymbol{V}\cdot(q\boldsymbol{E} + \boldsymbol{F}) \tag{2.26}$$

This result is easy to understand intuitively, and again confirms the physical adequacy and conceptual value of the guiding center particle model.[2] The first term is the power delivered by a changing magnetic field to the internal state of the virtual GC particle (the cyclotron motion of the real particle); note that it only includes the *local* time derivative of B. The agent responsible for the delivery or extraction of power is the induced electric field associated with the *locally* changing magnetic field $\boldsymbol{E}_{ind} = -\partial\boldsymbol{A}/\partial t$ (page 176 in Appendix A.1). This induced electric field acts on the real particle in its cyclotron motion, as sketched in Fig. 1.15.[3] Although trivial, we still must point out that changes in B as seen by the guiding center particle (i.e., convective changes in an inhomogeneous field $\boldsymbol{V}\cdot\boldsymbol{\nabla}B$ or $v_\parallel\partial B/\partial s$) do not enter in the first term. For instance, a particle drifting in a static magnetic and electric field like in Fig. 1.21 does experience a varying B-field (in the GCS) because it is driven into it by the electric drift (otherwise the guiding center particle would follow a $B = $ const. contour, like in Fig. 1.19); whenever that electric field has a component

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[2]For the relativistic version, use (1.29) as the relation between the relativistic magnetic moment and kinetic energy.

[3]When the local time derivative $\partial B/\partial t$ is entirely due to changes in the external current intensities, but not their configuration in space, one usually calls this process a *betatron acceleration*; if the current intensities are constant, but their position or distribution changes in space, one calls it a *Fermi acceleration*. This distinction is made mainly in astrophysics. However, the particle doesn't care about *what* causes the local field to change!

in the drift direction ($V_D \cdot E \neq 0$), it will do work on the particle. Another trivial but relevant remark: drifts due to the electric field do not play any role in the second term, because by definition (1.34) their contribution to the total drift V_D is *always perpendicular* to E. This is particularly important to keep in mind when one deals with purely induced electric fields in absence of potential electric fields: we must include the induced electric field in the second term, but *only non-electric drifts* can lead to energy change (if they have a component along E_{ind}). This sometimes is called drift-betatron, in distinction from the above gyro-betatron.

2.3　Particle Trapping and Parallel Electric Fields

In the previous section we have derived three fundamental and most general equations that under the adiabatic conditions (2.1) and (2.2) describe the dynamics of a virtual guiding center particle of given mass, charge, field-aligned velocity and magnetic moment. For the perpendicular motion, the *drift velocity* in its most general form is given by (2.14); we must emphasize again that this drift velocity does not appear as the result of the integration of a dynamic equation but, rather, is defined as the result of an averaging process, which then allows the replacement of the rapidly gyrating original particle by a virtual particle at the guiding center. For the parallel motion, we do have a real dynamic equation (2.22) determining the average acceleration of the particle along a field line. As a corollary, a third equation was derived, giving the average rate of change of the kinetic energy of the guiding center particle (2.26).

In this section we will focus on the parallel motion along a field line and, for that purpose, consider cases in which the particle drift is either a priori zero (say, for symmetry reasons), or negligible with respect to the parallel motion ($V_D \ll v_\parallel$). For instance, consider the field geometry of a "mirror machine", shown at right in Fig. 2.5, with the guiding center particle along the z-axis field line. But here comes a disappointment: instead of using the laboriously derived dynamic parallel equation (2.22), we turn to the following relations based on two simple conservation principles:

$$\text{Conservation of magnetic moment } M\colon \ v_\perp^2(s) = \frac{2M}{m} B(s) \qquad (2.27)$$

$$\text{Conservation of total energy } \mathfrak{E}\colon \ v_\parallel^2(s) + v_\perp^2(s) = \frac{2}{m}\big[\mathfrak{E} - W(s)\big] \quad (2.28)$$

$\mathfrak{E}$ is the total mechanical energy, $W(s)$ is the potential energy. The initial conditions determine the constants: $M = T_i/B_i$ and $\mathfrak{E} = T_i + W_i$, in which the subindex i denotes value of the variable at the initial field line point s_i.

Of more practical significance are the equivalent equations in the variables $v_\perp$, $v_\parallel$ and pitch angle α (1.14), and their respective initial values (see Fig. 2.6):
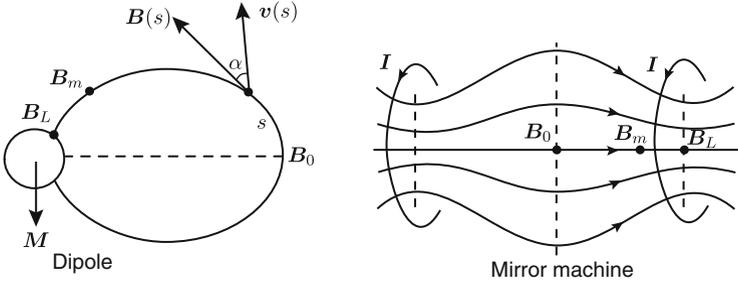
Fig. 2.5 Two fundamental types of trapping magnetic field geometries
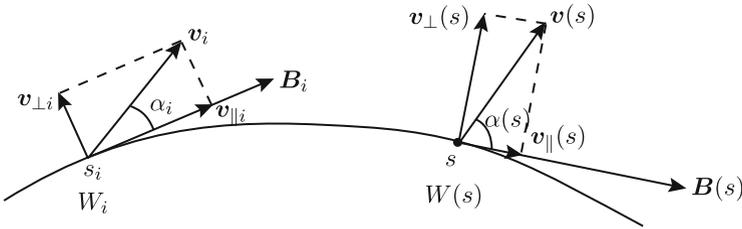

Fig. 2.6 Change of $v_\perp$ and $v_\parallel$ vectors along a magnetic field line

$$v_\perp^2(s) = v_i^2 \frac{\sin^2 \alpha_i}{B_i} B(s) \qquad (2.29)$$

$$v_\parallel^2(s) = v_i^2 \left(1 - \frac{\sin^2 \alpha_i}{B_i} B(s)\right) - \frac{2}{m}\left(W(s) - W_i\right) \qquad (2.30)$$

$$\sin^2 \alpha(s) = \frac{\sin^2 \alpha_i}{B_i} B(s) \left[1 - \frac{W(s) - W_i}{T_i}\right]^{-1} \qquad (2.31)$$

Note that the function $v_\perp(s)$ (and therefore the perpendicular kinetic energy $T_\perp$) is governed *only* by the local magnetic field intensity $B(s)$ (and the initial conditions)—it cannot be influenced by external forces! To change it, the conservation of magnetic moment M has to be violated, e.g., by scattering or resonance with cyclotron-resonant waves.

Our first example will be one in which field-aligned forces are absent, i.e., when magnetic field lines are equipotentials: $W(s) = W_i = $ const., $T = $ const. and $v = \sqrt{v_\perp^2 + v_\parallel^2} = $ const., and with no time variations of $\boldsymbol{B}$. Equation (2.31) is now

$$\sin^2 \alpha(s) = \frac{\sin^2 \alpha_i}{B_i} B(s) \qquad (2.32)$$

The pitch angle $\alpha(s)$ increases toward higher $B(s)$ values and becomes $\pi/2$ when

$$B_m(s_m) = \frac{B_i}{\sin^2 \alpha_i} \tag{2.33}$$

B_m is the *mirror field* and s_m a *mirror point* of the particle on the field line. Field line points on which $B > B_m$ are off limits: the guiding center particle stops instantaneously at s_m ($v_\parallel = 0$ there, with all the motion in cyclotron mode: $v_\perp = v$), and then reverses its sense. To understand why the GC particle doesn't just slow down and stop dead, we must turn to the parallel equation (2.22), which in this particular case is simply $m\, dv_\parallel/dt = -M\, \partial B/\partial s$: it is the mirror force (2.8) that turns the particle around in its parallel motion. If we inject a particle with a 90° initial pitch angle at any point s of a field line, this initial point will also be a mirror point and the mirror force will start accelerating the particle along the field line toward lower B-values. Note that for an equipotential field line, the mirror point field intensity B_m is an adiabatic invariant because at such point the transverse velocity in the definition of M (1.26) is equal to the constant velocity v.

In general, in field configurations like the dipole field or a mirror machine in Fig. 2.5, there are two mirror points on either side of any initial point s_i: as a particle reverses its parallel motion at a mirror point, it will move toward lower B values, which will pass through a minimum (see below) and increase again; as a consequence, another mirror point may eventually be reached. The end result of all this is that a guiding center particle is *trapped*, bouncing back-and-forth between two *conjugate* mirror points. The *bounce period* of a particle trapped between two mirror points s_m and s'_m on an equipotential field line will be, according to (2.30),

$$\tau_b = 2 \int_{s'_m}^{s_m} \frac{ds}{v_\parallel(s)} = \frac{2}{v} \int_{s'_m}^{s_m} \frac{ds}{\left[1 - B(s)/B_m\right]^{\frac{1}{2}}} \tag{2.34}$$

The integral is extended along the field line between mirror points located at s'_m and s_m, where the field intensity B_m is given by (2.33). The integral

$$S_b = \frac{1}{2} v\, \tau_b = \int_{s'_m}^{s_m} \frac{ds}{\left[1 - B(s)/B_m\right]^{\frac{1}{2}}} \tag{2.35}$$

is the *half-bounce path*, i.e., the rectified path of the original cycling particle between one mirror point and its conjugate. Note that it is a purely field-geometric quantity, independent of the particle in question. We can think of it as a function of space, a scalar field $S_b(s_m)$ representing the rectified inter-mirror-point path of a trapped particle mirroring at *that* point. Unfortunately, even for simple field geometries like a dipole field, the integral in (2.34) and (2.35) cannot be expressed in analytical, closed form and in practice must be calculated numerically. In Sect. 3.1 we will use the following relationship which will help overcome the numerically annoying fact that the integrand in the above relations has an integrable singularity at the mirror

points:

$$S_b = I + 2B_m \frac{\partial I}{\partial B_m}\bigg|_{fl} \tag{2.36}$$

where

$$I = \int_{s'_m}^{s_m} \left[1 - B(s)/B_m\right]^{\frac{1}{2}} ds \tag{2.37}$$

This function, whose integrand is the inverse of the integrand in (2.35) is easier to calculate numerically. Moreover, it is directly related to the second adiabatic invariant, as we shall show in the next section. The derivative with respect to B_m in (2.36) is to be taken on I as a function of the mirror point field intensity B_m *on the given field line*. Notice that not only the integrands of (2.35) and (2.37) are functions of B_m but also their integration limits, and that $S_b > I$ always.

Another notable point on a field line is where B has a local minimum: the pitch angle there will be minimum, too, whereas $v_{\parallel}$ will be maximum. In a dipole or dipole-like field, the minimum-B point is called the field-line's *equatorial point* (whether or not its geographic latitude actually is zero.) Note that in absence of parallel forces *all* trapped particles on a field line must transit through it. For this reason it is preferentially chosen as a fundamental reference point and origin for the curvilinear coordinate s (with positive values increasing toward the North in the geomagnetic case). Instead of α we shall work with the pitch angle variable $\mu = \cos\alpha$, commonly used in magnetospheric physics for reasons that will become apparent later. Equations (2.32) and (2.33) referred to the minimum-B point of a field line will now be:

$$\mu^2(s) = 1 - (1 - \mu_0^2)\frac{B(s)}{B_0} \quad \text{and} \quad B_m = \frac{B_0}{1 - \mu_0^2} \tag{2.38}$$

By definition, at a minimum-B point $\partial B/\partial s = 0$; in its neighborhood the function $B(s)$ can be approximated as $B(s) \simeq B_0 + 1/2as^2$, where $a = \partial^2 B/\partial s^2$ at s_0. Near the minimum-B point the parallel equation of motion (2.22) now becomes

$$m\frac{dv_{\parallel}}{dt} = -Ma\,s \tag{2.39}$$

which indeed looks like that of an harmonic oscillator with a constant $k = Ma$. If we inject a GC particle with a 90° pitch angle at the minimum-B point, it will stay there in an equilibrium position (just cyclotron circling—remember that for the time being we are ignoring any drifts, but if we do take them into account, the GC particle will stay on a minimum-B surface even if the latter is slightly warped). If the pitch angle is slightly less than 90°, the particle will bounce in harmonic

field-aligned motion between very close-by mirror points[4] with a bounce period
derived from (2.39):

$$\tau_b = 2\pi \sqrt{\frac{m}{Ma}} \approx \frac{2\pi \sqrt{2}}{v} \sqrt{\frac{B}{a}} \qquad (2.40)$$

The second near-equality is justified because $v_\perp \approx v$ in the expression of the
magnetic moment M. Note that this period (or the bounce angular frequency
$\omega_b = 2\pi/\tau_b$) to first order only depends on the local magnetic field geometry and
the particle's (mostly transverse) velocity. Under the adiabatic condition (2.2), the
bounce period is always much larger than the cyclotron period (1.20), $\tau_b \gg \tau_C$.
The half-bounce path $S_b = 1/2\,v\tau_b$ is independent of the particle and it, too,
is much larger than the Larmor radius under adiabatic condition (2.1). Note the
apparently curious fact that even a 90° equatorial particle does have a finite bounce
period and a half-bounce path—but that's the same thing as a mechanical oscillator
in equilibrium position having a non-zero fundamental frequency despite being at
rest! It can be shown that for near-equatorial particles, $a = B/R_C{}^2$, where R_C is
the field line curvature at the equatorial point. The bounce path (2.35) and bounce
period are then:

$$S_b = \pi \sqrt{2} R_C \quad \text{or} \quad \tau_b = (2\pi \sqrt{2}/v) R_C \qquad (2.41)$$

Finally, there will always be points on a field line beyond which a particle is lost
(intersection with the ionosphere in the geomagnetic field, maximum-B point in the
coils of a mirror machine, Fig. 2.5). This leads to the concept of *loss cone* α_L at
point s, which in absence of field-aligned electric field and forces has an aperture

$$\sin^2 \alpha_L(s) = \frac{B(s)}{B_L} \qquad (2.42)$$

where B_L is the magnetic field intensity at point s_L where the field line intersects
the ionosphere (or passes through the coil in a mirror machine). At any point of a
field line in the examples of Fig. 2.5 there will be always two loss cones, one for
direction $+s$ and one for $-s$. If there is complete hemispheric symmetry of the field
intersections with the ionosphere, or for a mirror machine along the z-axis, the two
loss cones will be equal.

Our second example will consist of a geomagnetic dipole-like field line with
a *parallel electrostatic potential* $V(s)$ (again, please do not confuse this symbol
with drift velocity!). Now we must use the more general Eqs. (2.29)–(2.31). The
right-hand side of the third equation consists of two factors, $(\sin^2 \alpha_i/B_i)B(s)$,

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[4]This justifies the entire discussion in Sect. 1.6 of equatorial particles: even if their pitch angles
deviate a bit from 90°, during their drift they will always be tied to an equilibrium position on the
minimum-B surface.
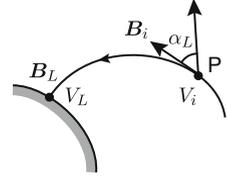
which is the value of $\sin^2\alpha(s)$ in absence of parallel forces, and an energy-dependent correction function $[1 - q(V(s) - V_i)/T_i]^{-1}$. Under these circumstances the expressions of the bounce period (2.34) and half-bounce path (2.35) have to be appropriately modified and the position of the mirror points s_m will be energy-dependent, solution(s) of the following equation directly derived from (2.31):

$$\frac{B(s_m)}{B_m^*} = 1 - q\frac{(V(s_m) - V_i)}{T_i} \tag{2.43}$$

In this expression $B_m^* = B_i/(1 - \mu_i^2)$, the mirror point field intensity in absence of field-aligned forces.

Let us examine the case in detail. The result of Eq. (2.31) (which always should be compromised between 0 and +1) may "go wrong" in two ways, representing an off-limits place for the particle. First, it could be > 1 because of too large values of B(s) (as it happened in our previous discussion of mirror points regarding (2.33)), or because of the energy-dependent correction function. Second, it could turn out < 0 because of this correction function between brackets. We shall discuss these situations in more detail.

First, to the mirror points. Refer to Fig. 2.7; at point P (arc-length s_i from the equatorial point) a particle is injected with kinetic energy T_i and pitch angle α_i traveling toward the Earth. L is the intersection with a loss region where the potential is V_L. First assume that $qV_L > qV_i$, which corresponds to a force directed toward the equator, away from L (for electrons, it would be an electric field directed *toward* the Earth). If $T_i < q(V_L - V_i)$, according to (2.31) the value of $\sin^2(\alpha_L)$ would be negative, i.e. the point s_L would be inaccessible to the particle. This means that *all* particles through P, even those with a 0° pitch angle, would mirror before reaching the loss region—turned around by the *combined* action of the electric force and the mirror force. On the other hand, if $qV_L < qV_i$ (parallel electric force directed toward the loss point L), $\sin^2(\alpha_L) > B/B_L$ and the loss cone at P will be bigger than in absence of the electric field. Thus, in general terms, decelerating potentials will narrow the loss cone (maybe even eliminate it), while accelerating potentials will widen it. This has important consequences for the action of an auroral mechanism which in general involves the generation of a field-aligned electric field directed from the ionosphere toward the equatorial point of a field line ($qV_L > qV(s)$), in both hemispheric branches of the field line.

Under an earthward-directed parallel electric force an interesting situation may arise for particles injected from P toward the equator ($v_{\parallel i} < 0$), i.e., away from

Fig. 2.8 Case of a particle trapped between two mirror points in the same hemisphere, for certain conditions of the field-aligned electric field
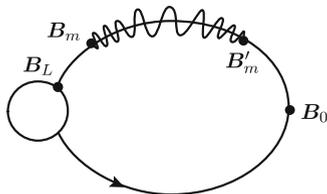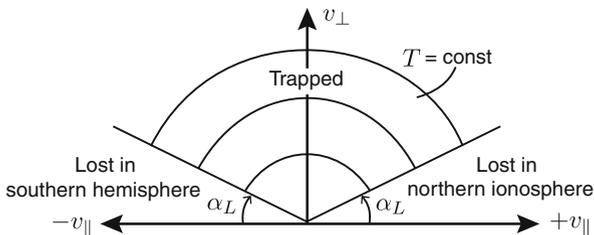

Fig. 2.9 Background of a $v_\perp$, $v_\parallel$ map, for charged particles at a given point P of a centered dipole field line, showing loss-cone angle and constant kinetic energy contours. $+v_\parallel$ points to the northern hemisphere


the ionosphere. An examination of relation (2.31) shows that for certain V's and T_i's, the correction function may win over the other term $(\sin^2 \alpha_i / B_i) B(s)$, and the particle may reach a mirror point *before* it crosses the equatorial point of the field line. In other words, a particle may be trapped on the *same* hemispheric side of a field line (see sketch in Fig. 2.8)! This indeed can happen to electrons under auroral electric field conditions. Positive ions, on the other hand, traveling under these electric field conditions upwards from the ionosphere decrease their pitch angles; bundles of ions from the upper ionosphere accelerated and focused along the field line because of this process are called "conics". In Sect. 3.3 we shall mention a region of the dayside magnetosphere near the boundary where the magnetic field has a slight secondary *maximum* on the equator, where particles can be trapped in high-latitute pockets on the same side of the equator.

A useful representational device are the so-called velocity maps in which a guiding center particle is represented as a point, and on which directional particle fluxes (see Sect. 4.1) can be mapped and key regions and their delimitations identified. For instance, in absence of any external forces, the velocity map of a particle injected at a point P of the field line would look as shown in Fig. 2.9. This figure is drawn for a given guiding center position on a field line of a pure dipole, and shows constant kinetic energy T and pitch angle α contours; trapping and loss regions are also indicated. One point on the map represents a particle with a given pair of velocities $v_\perp$, $v_\parallel$ at that guiding center position on the field line.

This map becomes more interesting when the field line is no longer an equipotential. Let us combine (2.29) and (2.30) into the following equation, for the case of a field-aligned electrostatic potential $V(s)$:

$$v_\parallel^2(s) = v_{\parallel i}^2 + v_{\perp i}^2 \Big[1 - \frac{B(s)}{B_i}\Big] - \frac{2q}{m}\big[V(s) - V_i\big] \tag{2.44}$$

Fig. 2.10 Limiting curves for earthward moving electrons and ions on a $v_\perp$-$v_\parallel$ map
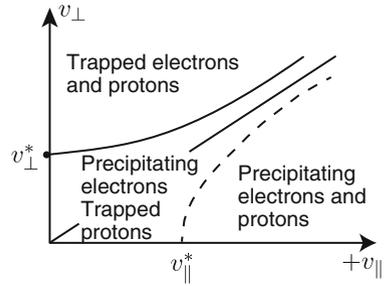
$v_\parallel^2(s)$ will now be equal to zero at mirror points due to the combined action of a mirror force and an electric field force; these points always separate an allowed segment of the field line from an off-limits one. In our example, for particles starting from s_i toward the Earth, we have $B(s) > B_i$ and $qV(s) > qV_i$; for particles starting toward the equator, reverse the inequalities.

The regions of the velocity map representing particles which *either* mirror or precipitate will be separated by a curve of points $v_\perp$, $v_\parallel$ for which at s_L the value of (2.44) is $v_\parallel^2(s_L) = 0$. For electrons, the equation of this separatrix (Fig. 2.10) is given by

$$v_\perp = \sqrt{\frac{v_\parallel^2 + 2|q|/m(V_L - V_i)}{(B_L/B_i - 1)}} \tag{2.45}$$

Notice in the figure the two regions separated by the hyperbola branch given by this equation: (i) the upper region of initial $v_\perp$, $v_\parallel$ values, for which the electron will mirror as it travels toward the Earth before it precipitates into the ionosphere at L; (ii) the lower region of initial values for which the electron accelerated by the electric field will precipitate. The asymptote of the hyperbola branch shown represents the loss cone in absence of any field-aligned electric field. Note that even electrons with an initial 90° pitch angle ($v_\parallel = 0, v_\perp = v_\perp^*$) can be drawn into the ionosphere and precipitate. For positive ions, Eq. (2.45) will have a negative sign in front of $2|q|$ and the separatrix is a conjugate hyperbola, as shown in broken line in Fig. 2.10. Ions that would have been lost in absence of an electric field will now mirror before reaching the ionosphere as the result of a concerted action of electric and mirror forces.

Our next task is to examine what will happen to electrons initially traveling in the opposite direction, i.e., toward the equator (decreasing s, $v_{\parallel i} < 0$). Those are the particles which, as mentioned above, will be decelerated by the external force and may have a chance, under specific conditions, of mirroring *before* reaching the equator, i.e., of being trapped within one hemispheric branch of the field line. The separatrix between these two regions is given by the curve that corresponds to initial $v_\perp$ and $v_\parallel$ values for which the electron mirrors exactly *at* the minimum-B point, i.e., at B_0. Replacing $B(s)$ and $V(s)$ in (2.44) by B_0 and V_0, respectively, and taking
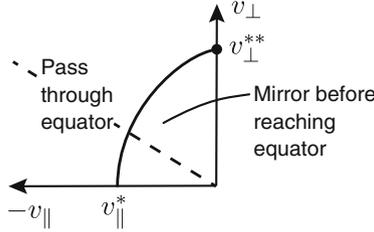
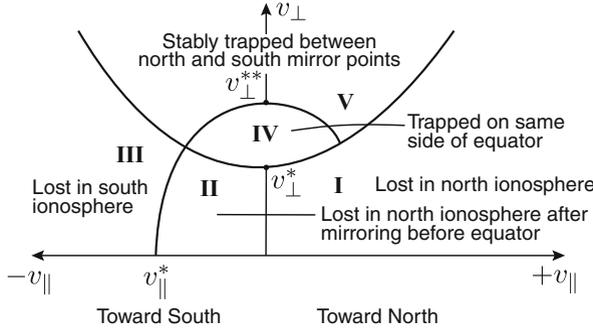**Fig. 2.11** Same as Fig. 2.10, for equatorward moving particles

Fig. 2.12 Sketch of characteristic regions in a $v_\perp, v_\parallel$ map, for electrons injected at a northern hemisphere point of a centered dipole field line with an ionosphere-to-equator directed parallel electric field. The separatrix curves are given by (2.45) (hyperbola) and (2.46) (ellipse), respectively. For explanation, see text

into account that now $B(s) < B_0$ and $V(s) < V_0$, we obtain for the case $v_{\parallel 0} = 0$:

$$
v_\perp = \sqrt{\frac{2|q|/m(V_i - V_0) - v_\parallel^2}{(1 - B_0/B_i)}} \tag{2.46}
$$

The curve is now the quadrant of an ellipse (Fig. 2.11). For initially mirroring electrons ($v_{\parallel i} = 0$) and for which $v_{\perp i} < v_\perp^{**}$, there is another mirror point *before* they reach the equator. For locally field-aligned electrons ($v_{\perp i} = 0$) for which $v_{\parallel i} < v_\parallel^{**}$, there is a mirror point, too, before they reach the equator!

It is very instructive to "play" with these velocity maps and learn how different forms of electrostatic potential, electric charge of particles, degrees of field asymmetries and initial positions and kinetic energies group the particles into different classes with respect to their parallel motion along a field line (see also Fig. 1 in [1]). As a final example, we combine the upward and downward injection cases of electrons in a symmetric dipole-like field under auroral electric field conditions into just one velocity map in Fig. 2.12. This sketch is drawn for electrons passing through a field line point in the northern hemisphere. Carefully observe the properties of five distinct classes. In a real case, regions I, II and III would be empty, except

for backscattered electrons; region IV is a class all by itself, consisting of electrons trapped within the northern half of the field line. Region V includes all stably trapped electrons with mirror points in both hemispheres. It is a good exercise to draw a similar map for positive ions on auroral field lines.

Reference

1. Y.T. Chiu, M. Schulz, Self-consistent particle and parallel electrostatic field distributions in the magnetospheric-ionospheric auroral region. J. Geophys. Res. **83**, 629–642 (1978)